Foreword

Many times throughout the course of their history, theoretical physics and mathematics have been brought together by grand structural ideas which have proved to be a fertile source of inspiration for both subjects.

The importance of the structure of holomorphic functions in several variables became apparent around 1960, with the mathematical formulation of the quantum theory of fields and particles. Indeed, their complex singularities, known as "Landau" singularities, form a whole universe, and their physical interpretation is part of a particle physicist's basic conceptual toolkit. Thus, the presence of a pole in an energy or mass variable indicates the existence of a particle, and singularities of higher complexity are a manifestation of a ubiquitous geometry which underlies classical relativistic multiple collisions and also includes the creation of particles, which lies at the heart of quantum interaction processes.

On the pure mathematics side, one can safely say that this branch of theoretical physics genuinely contributed to the birth of the theory of hyperfunctions and microlocal analysis. (For example, one can mention the initial motivation in the work of M. Sato at the time of the "dispersion relations", which came from physics, and also the work of A. Martineau, B. Malgrange, and M. Zerner on the "edge-of-the-wedge" theorem, which came out of the first meetings in Strasbourg between physicists and mathematicians).

For a mathematical physicist, these holomorphic structures in quantum field theory have a deep meaning, and are inherent in the grand principles of relativistic quantum field theory: Einstein causality, the invariance under the Poincaré group, the positivity of energy, the conservation of probability or "unitarity", and so on. But it is in the "perturbative" approach to quantum field theory (whose relationship with "complete" or "non-perturbative" theory can be compared to the relationship between the study of formal power series and convergent series), that the holomorphic structures which generate Landau singularities appear in their most elementary form: namely, as holomorphic functions defined by integrals of rational functions associated to "Feynman diagrams".

It is to Frédéric Pham's great credit that he undertook a systematic analysis of these mathematical structures, whilst a young physicist at the Service de Physique Théorique at Saclay, using the calculus of residues in several variables as developed by J. Leray, together with R. Thom's isotopy theorem. This fundamental study of the singularities of integrals lies at the interface between analysis and algebraic geometry, and culminated in the Picard–Lefschetz formulae. It was first published in 1967 in the Mémorial des Sciences Mathématiques (edited by Gauthier-Villars), and was subsequently followed by a second piece of work in 1974 (the content of a course given at Hanoi), where the same structures, enriched by the work of Nilsson, were approached using methods from the theory of differential equations and were generalized from the point of view of hyperfunction theory and microlocal analysis.

It seemed to us that, because of the importance of their content and the wide range of different approaches that are adopted, a new edition of these texts could play an extremely important role not only for mathematicians, but also for theoretical physicists, given that the major fundamental problems in the quantum theory of fields and particles still remain unsolved to this day.

First of all, we note that the methods developed by Frédéric Pham have had wide-ranging impact in the non-perturbative approach to this subject, by allowing one to study holomorphic solutions to integral equations in complex varieties with varying cycles. Such equations are inherent in the general formalism of quantum field theory (they are known as equations of "Bethe–Salpeter" type, and are intimately connected to general unitarity relations). In this way, it is possible, for example, in the case of collisions between massive particles which are described in a general manner by the structure functions of a quantum field theory, to disentangle the singularities of a "three-particle threshold", which appears as an accumulation point of "holonomic singularities" (here we refer to a classification of the singularities of collision amplitudes considered by M. Sato).

Some extremely tough problems remain open concerning this type of structure in the case when massless particles are involved in collisions, which is of considerable physical importance. The type of analysis that we have just described above (the case of collisions involving only massive particles) concerns properties of field theory which are independent of "problems of renormalization", which are a different category of crucially important problems. These appear on the perturbative level as the need to redefine primitively-divergent Feynman integrals, and have recently been cast in a new light by the work of Connes and Kreimer, who brought a Hopf algebra structure into the picture. Nonetheless, on a general, non-perturbative level, the problems of renormalization still appear to be at the very source of the problem of the existence of non-trivial field theories in four-dimensional space-time.

This "existential problem" was brought to light by Landau in 1960 in the resummation of renormalized perturbation series in quantum electrodynamics and is exhibited, at a more general level, by the simplest scalar field theory with quartic interaction term. This phenomenon involves the "generic" creation of poles (and other Landau singularities) as a result of renormalization, in a region of complex space where such singularities are forbidden by general principles. *In fact, the existential flaw of these models appears to be related in a deep way to the absence of an important property, known as "asymptotic freedom", for the field structure functions at very high energies.*

Are non-abelian gauge theories, which are conceptually richer and also closer to the experimental complexity of particle physics, capable – as one could hope since they can incorporate asymptotic freedom – of reconciling renormalization with the fundamental holomorphic structures which arise from the grand principles of relativistic quantum physics? In the current state of particle physics, in which the "standard model" of field theory is considered to be very reliable in a vast range of energies, the construction and mathematical analysis of the simplest quantum field theory with non-abelian gauge, namely a Yang–Mills model, is a challenge which cannot be avoided (and the reason why it has been chosen as the subject of one of the major mathematical problems posed by the "Clay Institute" in the year 2000 ...).

In such a conceptually rich subject as quantum relativistic field theory, it seems extremely important to continue to develop the point of view of holomorphic structures and complex singularities in tandem with structures such as symmetries and gauge groups, which have been the most recent driving forces for research. In this light, the present work offers us a panoply of results from which a mathematical physicist should be able to draw great benefit.

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