

Chapter 2

Analysis of Linear Systems

2.1 The Linear Centralizer and Linear Normalizer of a Square Matrix

Assume that systems (1.1a), (1.1b) are linear, i.e., $f(x) = Ax$ (respectively, $F(x) = Ax$),

$$\frac{dx(t)}{dt} = Ax(t), \quad t \in \mathbb{R}, \quad (2.1a)$$

$$x(t+1) = Ax(t), \quad t \in \mathbb{Z}, \quad (2.1b)$$

where $x \in \mathbb{R}^n$, and $A \in \mathbb{R}^{n \times n}$ is said to be the *dynamic matrix* (which is assumed to be constant) of the linear system: a notation common to both (2.1a) and (2.1b) can be adopted:

$$\Delta x(t) = Ax(t), \quad t \in \mathbb{T},$$

where $\Delta x(t) = \frac{dx(t)}{dt}$ if $\mathbb{T} = \mathbb{R}$ and $\Delta x(t) = x(t+1)$ if $\mathbb{T} = \mathbb{Z}$. Symbol $\mathcal{I}_C(Ax)$ (respectively, $\mathcal{I}_D(Ax)$) denotes the set of all first integrals of system (2.1a) (respectively, system (2.1b)).

Assume also that system (1.2) is linear, $g(x) = Bx$,

$$\frac{dx}{d\tau} = Bx = g(x), \quad (2.2)$$

where $B \in \mathbb{R}^{n \times n}$ is constant. As well known,

$$x = e^{B\tau} y \quad (2.3)$$

is the unique solution of system (2.2) at time $\tau \in \mathbb{R}$, starting from the initial condition $x(0) = y$, with $y \in \mathbb{R}^n$.

A *one-parameter group of linear transformations* is given by $x = Q(\tau)y$, if $Q(\tau) \in \mathbb{R}^{n \times n}$ satisfies $Q(0) = E$, $Q(\tau_1)Q(\tau_2) = Q(\tau_1 + \tau_2)$ and $Q^{-1}(\tau) = Q(-\tau)$.

The family of linear transformations given in equation (2.3) qualifies as a one-parameter group of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$: $e^{B0} = E$, $e^{B\tau_1}e^{B\tau_2} = e^{B(\tau_1+\tau_2)}$ and $(e^{B\tau})^{-1} = e^{-B\tau}$. Given any one-parameter group of linear transformations $x = Q(\tau)y$, there exists a constant matrix $B \in \mathbb{R}^{n \times n}$ such that $Q(\tau) = e^{B\tau}$. This matrix can be computed by $B = \frac{dQ(\tau)}{d\tau}|_{\tau=0}$. As a matter of fact, $Q(\tau) = e^{B\tau}$ is obtained by integrating the following differential equation by the initial condition $Q(0) = E$:

$$\begin{aligned} \frac{dQ(\tau)}{d\tau} &= \lim_{T \rightarrow 0^+} \frac{Q(\tau+T) - Q(\tau)}{T} = \left(\lim_{T \rightarrow 0^+} \frac{Q(T) - E}{T} \right) Q(\tau) \\ &= \left(\frac{dQ(\tau)}{d\tau} \Big|_{\tau=0} \right) Q(\tau) = B Q(\tau). \end{aligned}$$

Example 2.1 Consider the one-parameter family of rotations in $\mathbb{R}^2$ given by $Q(\tau) = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}$. Clearly, $Q(0) = E$ and

$$Q(\tau_1)Q(\tau_2) = \begin{bmatrix} \cos(\tau_1 + \tau_2) & -\sin(\tau_1 + \tau_2) \\ \sin(\tau_1 + \tau_2) & \cos(\tau_1 + \tau_2) \end{bmatrix} = Q(\tau_1 + \tau_2);$$

therefore, $Q(\tau)$ is a one-parameter group of linear transformations. Then, $Q(\tau) = e^{B\tau}$, with

$$B = \frac{dQ(\tau)}{d\tau} \Big|_{\tau=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Using (2.3) as a change of coordinates, one can rewrite systems (2.1a), (2.1b) in the new y -coordinates, as follows:

$$\frac{dy(t)}{dt} = e^{-B\tau} A e^{B\tau} y(t), \quad (2.4a)$$

$$y(t+1) = e^{-B\tau} A e^{B\tau} y(t). \quad (2.4b)$$

From Theorem 1 of [37] (see, also, [52]), for $Q \in \mathbb{R}^{n \times n}$, the equation $Q = e^{B\tau}$, with the requirement that B and τ are real, has a solution (not necessarily unique, also when τ is fixed) if and only if $\det(Q) \neq 0$ and each Jordan block of Q corresponding to an eigenvalue with negative real part occurs an even number of times. This shows that only a subset of the linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ can be put into form (2.3), for some real τ . Moreover, since $e^{B\tau} = E + B\tau + O(\tau^2)$, with E being the $n \times n$ identity matrix and $O(\tau^2)$ denoting second and higher order terms, for τ close to 0, transformation (2.3) is close to the *identity transformation* and (see the subsequent Sect. 6.2), for the transformed system (2.4a), (2.4b) one has

$$\begin{aligned} e^{-B\tau} A e^{B\tau} &= (E - B\tau + O(\tau^2))A(E + B\tau + O(\tau^2)) \\ &= A - (BA - AB)\tau + O(\tau^2). \end{aligned} \quad (2.5)$$

Definition 2.1 The linear transformation (2.3) is a *linear symmetry* of systems (2.1a), (2.1b) and system (2.2) is its *infinitesimal generator* if

$$e^{-B\tau} A e^{B\tau} y = Ay, \quad \forall y \in \mathbb{R}^n, \quad \forall \tau \in \mathbb{R}. \quad (2.6)$$

If (2.6) holds, by abuse of notation, also the infinitesimal generator (2.2) is called a *linear symmetry* of systems (2.1a), (2.1b); briefly, Bx is called a *linear symmetry* of Ax .

Remark 2.1 If (2.6) holds, then $e^{-B\tau} e^{At} e^{B\tau} = e^{At}$ (respectively, $e^{-B\tau} A^t e^{B\tau} = A^t$), $\forall \tau \in \mathbb{R}, \forall t \in \mathbb{T}$ ($t \geq 0$, in the discrete-time case if $\det(A) = 0$).

Definition 2.2 Given $x_0 \in \mathbb{R}^n$, the *orbit* of systems (2.1a), (2.1b) passing through x_0 is the set of the points x described by $x = e^{At} x_0$ if $\mathbb{T} = \mathbb{R}$ (respectively, $x = A^t x_0$ if $\mathbb{T} = \mathbb{Z}$), when $t \in \mathbb{T}$ varies from $-\infty$ to $+\infty$ (from 0 to $+\infty$, in the discrete-time case if $\det(A) = 0$).

The meaning of relation (2.6) is that any *orbit* $x(t) = e^{At} x_0$ (respectively, $x(t) = A^t x_0$) of systems (2.1a), (2.1b) is mapped into an *orbit* $y(t) = e^{At} y_0$ (respectively, $y(t) = A^t y_0$) of the same systems (2.1a), (2.1b) by the linear transformation (2.3) generated by system (2.2), $y = e^{-B\tau} x$ and $x_0 = e^{B\tau} y_0$, while preserving the time parameterization along the orbit:

$$\begin{aligned} y(t) &= e^{-B\tau} x(t) = e^{-B\tau} e^{At} x_0 = e^{-B\tau} e^{At} e^{B\tau} y_0 = e^{At} y_0, & \text{if } \mathbb{T} = \mathbb{R}, \\ y(t) &= e^{-B\tau} x(t) = e^{-B\tau} A^t x_0 = e^{-B\tau} A^t e^{B\tau} y_0 = A^t y_0, & \text{if } \mathbb{T} = \mathbb{Z}. \end{aligned}$$

Example 2.2 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$; since $e^{At} = \begin{bmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{bmatrix}$, $A^t = \begin{bmatrix} \cos(\frac{\pi}{2}t) & \sin(\frac{\pi}{2}t) \\ -\sin(\frac{\pi}{2}t) & \cos(\frac{\pi}{2}t) \end{bmatrix}$, $e^{B\tau} = e^\tau \begin{bmatrix} \cos(\tau) & \sin(\tau) \\ -\sin(\tau) & \cos(\tau) \end{bmatrix}$, it is easy to check that $e^{-B\tau} e^{At} e^{B\tau} = e^{At}$ and $e^{-B\tau} A^t e^{B\tau} = A^t$.

The following definition and most of the following properties are standard (see, e.g., [13, 18, 34]).

Definition 2.3 (2.3.1) Given two square matrices $A, B \in \mathbb{R}^{n \times n}$, the *Lie bracket* (it is often called *matrix commutator*) of A and B is

$$[A, B] := BA - AB. \quad (2.7)$$

(2.3.2) The *linear centralizer* $\mathcal{L}_c(B)$ of B is the set of all matrices A such that $[A, B] = -[B, A] = 0$ (see, also, Definition 1.3 at p. 10).

Letting $g(x) = Bx$, $f(x) = Ax$ and $F(x) = Ax$, one finds that $[f(x), g(x)] = [A, B]x$, $[F(x), g(x)] = [A, B]x$; if $[A, B] = 0$, then A and B commute under the matrix product (briefly, A and B are *commuting*), and vice versa. In addition,

$$[A, B] = 0 \iff [A^\top, B^\top] = 0.$$

Remark 2.2 If $[A, B] = 0$, then $e^{At}e^{B\tau} = e^{B\tau}e^{At} = e^{At+B\tau}$, for all $t, \tau \in \mathbb{R}$.

Theorem 2.1 Relation (2.6) holds if and only if $[A, B] = 0$, i.e., if and only if A and B are commuting.

Proof By (2.5), condition $[A, B] = 0$ is certainly necessary for relation (2.6) to hold. Since $e^{-B\tau}Ae^{B\tau}y|_{\tau=0} = Ay, \forall y \in \mathbb{R}^n$, equality (2.6) holds if and only if

$$\frac{\partial}{\partial \tau}(e^{-B\tau}Ae^{B\tau}y) = 0, \quad \forall y \in \mathbb{R}^n, \forall \tau \in \mathbb{R}. \quad (2.8)$$

In this way,

$$\begin{aligned} \frac{\partial}{\partial \tau}(e^{-B\tau}Ae^{B\tau}y) &= -e^{-B\tau}BAe^{B\tau}y + e^{-B\tau}ABe^{B\tau}y = -e^{-B\tau}(BA - AB)e^{B\tau}y \\ &= -e^{-B\tau}[A, B]e^{B\tau}y. \end{aligned}$$

Since $e^{-B\tau}$ is invertible for all B and τ , equality (2.8) holds if and only if $[A, B] = 0$. $\square$

Thanks to Theorem 2.1, the following definition is equivalent to Definition 2.1.

Definition 2.4 The linear transformation (2.3) is a *linear symmetry* of systems (2.1a), (2.1b) and system (2.2) is its *infinitesimal generator* if $[A, B] = 0$.

Remark 2.3 The Lie bracket of two square matrices enjoys the following properties, with $A, B, C \in \mathbb{R}^{n \times n}$ (which can be proven by simple substitution):

- (2.3.1) $[A, B] = -[B, A]$ (*skew-symmetry*);
- (2.3.2) $[\alpha B + \beta C, A] = \alpha[B, A] + \beta[C, A]$ and $[A, \alpha B + \beta C] = \alpha[A, B] + \beta[A, C]$, with $\alpha, \beta \in \mathbb{R}$ being constants (*bi-linearity*);
- (2.3.3) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$ (*the Jacobi identity*).

By the skew-symmetry (Statement (2.3.1) of Remark 2.3),

$$A \in \mathcal{L}_c(B) \iff B \in \mathcal{L}_c(A),$$

although, in general, $\mathcal{L}_c(B) \neq \mathcal{L}_c(A)$.

Another useful property is the *invariance of the matrix Lie bracket to linear transformations*, meaning the fact that, for any invertible $Q \in \mathbb{C}$, letting $\tilde{A} = Q^{-1}AQ$ and $\tilde{B} = Q^{-1}BQ$, one has

$$[\tilde{A}, \tilde{B}] = Q^{-1}[A, B]Q. \quad (2.9)$$

Some key facts about the linear centralizer of a square matrix are reviewed next, since such properties are of great importance in the sequel.

Lemma 2.1 The linear centralizer $\mathcal{L}_c(A)$ of $A \in \mathbb{R}^{n \times n}$ is a finite dimensional vector space over $\mathbb{R}$. The dimension r of $\mathcal{L}_c(A)$ satisfies $n \leq r \leq n^2$.

Proof It is easy to see that the bi-linearity (2.3.2) implies that $\mathcal{L}_c(A)$ is a vector space over $\mathbb{R}$: if $M_1, M_2 \in \mathcal{L}_c(A)$, then $\alpha_1 M_1 + \alpha_2 M_2 \in \mathcal{L}_c(A)$, $\forall \alpha_1, \alpha_2 \in \mathbb{R}$ being constant. The fact that $\mathcal{L}_c(A)$ is finite dimensional is obvious, since $\mathcal{L}_c(A)$ is a subspace of $\mathbb{R}^{n \times n}$. As for its dimension r , the upper bound comes from the case $A = \alpha E$, with α being a (possibly zero) constant and E being the identity matrix; in such a case $\mathcal{L}_c(A) = \mathbb{R}^{n \times n}$, whence $r = n^2$. As for the lower bound, in view of (2.9), assume that $A = \text{block_diag}\{J_1, \dots, J_p\}$ is in the Jordan form, with Jordan blocks J_i of dimension r_i ; then, $J_i^0, \dots, J_i^{r_i-1}$ are linearly independent over $\mathbb{C}$. Therefore, in case of real eigenvalues, the n matrices

$$\begin{aligned} & \text{block_diag}\{J_1^0, 0, \dots, 0\}, \dots, \text{block_diag}\{J_1^{r_1-1}, 0, \dots, 0\}, \dots, \\ & \text{block_diag}\{0, \dots, 0, J_p^0\}, \dots, \text{block_diag}\{0, \dots, 0, J_p^{r_p-1}\} \end{aligned} \quad (2.10)$$

commute with A and are linearly independent, whence $r \geq n$. In case of complex eigenvalues, a similar reasoning can be made by considering the real Jordan form (for a definition of the real Jordan form see the proof of the subsequent Lemma 2.5). $\square$

As for the choice of a basis of $\mathcal{L}_c(A)$, it can be useful to take some of its elements in a simple way; therefore, note that the identity matrix $E = A^0$ can always be included in the basis of $\mathcal{L}_c(A)$, whereas A can be included except for the trivial case $A = 0$. More in general, if $A^0, A^1, \dots, A^{m-1}$, with $m \leq r$, are linearly independent over $\mathbb{R}$, then, with no loss of generality, one can assume that the first m elements of a basis of $\mathcal{L}_c(A)$ are $M_0 = E, M_1 = A^0, \dots, M_{m-1} = A^{m-1}$. A more powerful result is the subsequent Theorem 2.2, which is proven by means of the two lemmas below.

Lemma 2.2 *Let $J = \text{block_diag}\{J_1, \dots, J_p\}$ be a Jordan matrix whose p Jordan blocks J_i , of dimension r_i , have distinct real eigenvalues $\lambda_1, \dots, \lambda_p$. Then, all the matrices commuting with J are of the form*

$$B = \text{block_diag}\{B_1, \dots, B_p\}, \quad (2.11)$$

where $B_i \in \mathbb{R}^{r_i \times r_i}$ and $B_i J_i = J_i B_i$.

Proof The proof of the fact that matrix B in (2.11) commutes with J is trivial. To show the converse, assume that B commutes with J and partition it in blocks according to the dimensions r_i :

$$B = \begin{bmatrix} B_{1,1} & \dots & B_{1,p} \\ \vdots & \ddots & \vdots \\ B_{p,1} & \dots & B_{p,p} \end{bmatrix}, \quad B_{i,j} \in \mathbb{R}^{r_i \times r_j}.$$

By looking at the diagonal blocks of $BJ - JB$, it is easy to see that $BJ = JB$ implies, for each $i \in \{1, \dots, p\}$, that $B_{i,i} J_i = J_i B_{i,i}$, whence that matrices $B_{i,i}$ have

the property of matrices B_i in (2.11). By looking at the off-diagonal blocks, it is easy to see that $BJ = JB$ implies that

$$B_{i,j}J_j = J_iB_{i,j}, \quad \forall i \neq j. \quad (2.12)$$

Consider the chain of generalized right eigenvectors of J_j , $v_1 = \bar{e}_1, \dots, v_{r_j} = \bar{e}_{r_j}$ (with $\bar{e}_h$ being the h th column of the $r_j \times r_j$ identity matrix), that satisfy $J_j v_1 = \lambda_j v_1$, $J_j v_h = \lambda_j v_h + v_{h-1}$, $h = 2, \dots, r_j$, and the chain of generalized left eigenvectors of J_i , namely $u_1^\top = \hat{e}_1^\top, \dots, u_{r_i}^\top = \hat{e}_{r_i}^\top$ that satisfy $u_k^\top J_i = \lambda_i u_k^\top + u_{k+1}^\top$, $k = 1, \dots, r_i - 1$, $u_{r_i}^\top J_i = \lambda_i u_{r_i}^\top$ (with $\hat{e}_k$ being the k th column of the $r_i \times r_i$ identity matrix). Equation (2.12) left multiplied by $u_{r_i}^\top$ and right multiplied by v_1 gives

$$\lambda_j [0 \ \dots \ 0 \ 1] B_{i,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \lambda_i [0 \ \dots \ 0 \ 1] B_{i,j} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad (2.13)$$

which, since $\lambda_i \neq \lambda_j$, implies that the entry of the first column and of the last row of $B_{i,j}$ is zero. The equation similar to (2.13) obtained using v_2 instead of v_1 implies that the entry of the second column and of the last row of $B_{i,j}$ is zero. Using iteratively v_h with increasing h instead of v_1 , one obtains that the last row of $B_{i,j}$ is zero. In the same manner, the equations similar to (2.13) obtained using $u_k^\top$ with decreasing k instead of $u_{r_i}^\top$ imply that the first column of $B_{i,j}$ is zero. The procedure can be repeated using in the proper order all the $u_k^\top$ and v_h to prove that $B_{i,j} = 0$. $\square$

Lemma 2.3 *Let J be a Jordan block of dimension r relative to a real eigenvalue. Then, the dimension of $\mathcal{L}_c(J)$ is r and $\{J^0, J^1, \dots, J^{r-1}\}$ is a basis of $\mathcal{L}_c(J)$.*

Proof The statement is equivalent to saying that the set of the matrices that commute with a Jordan block coincides with the set of all upper triangular *Toeplitz matrices*, i.e., all upper triangular matrices such that, for a given $h \in \{0, \dots, r-2\}$, the entries in position $(k, k+h)$, $k \in \{1, \dots, r-h\}$, are equal. Assume that $A \in \mathbb{R}^{r \times r}$ commutes with J , i.e., $AJ = JA$. Taking into account the two entries in positions $(r-1, 1)$ and $(r, 2)$ of $AJ - JA$, one derives that, for them to be zero, it is necessary and sufficient that the entry $A_{r,1}$ of A is zero. Considering the three entries in positions $(r-2, 1)$, $(r-1, 2)$ and $(r, 3)$ of $AJ - JA$, one concludes that it is necessary and sufficient that $A_{r-1,1}$ and $A_{r,2}$ are zero. Then, iterating on h , which decreases from $r-2$ to 0, considering all the entries in positions $(k+h, k)$, $k \in \{1, \dots, r-h\}$ of $AJ - JA$, one obtains that it is necessary and sufficient that all the entries $A_{i,j}$ such that $i - j = h + 1$ are zero. Therefore, A is upper triangular. Analogously, for each $h \in \{1, \dots, r-1\}$, the entries in positions $(k, k+h)$, $k \in \{1, \dots, r-h\}$, of $AJ - JA$, which must be zero, show that it is necessary and sufficient that all the entries $A_{i,j}$ such that $j - i = h - 1$ are equal, i.e., A is Toeplitz. $\square$

Results analogous to Lemmas 2.2 and 2.3 can be proven for the case of a matrix A having some complex eigenvalues, so that the following theorem holds in the general case.

Theorem 2.2 *Let $A \in \mathbb{R}^{n \times n}$. There exist linearly independent $M_0, \dots, M_{n-1} \in \mathcal{L}_c(A)$ over $\mathbb{R}$, which are pairwise commuting, i.e., $[M_i, M_j] = 0$. If there are no Jordan blocks of A corresponding to the same eigenvalue, then $\mathcal{L}_c(A)$ has dimension n and $\{M_0, \dots, M_{n-1}\}$ is a basis of $\mathcal{L}_c(A)$.*

Proof If A has only real eigenvalues, in view of (2.9), assume that A is in the Jordan form as in the proof of Lemma 2.1. It is easy to see that the matrices in (2.10), which are linearly independent and belong to $\mathcal{L}_c(A)$, are pairwise commuting, whence they constitute the set $\{M_0, \dots, M_{n-1}\}$. To see that, when for each eigenvalue of A there is just one Jordan block, such a set is indeed a basis of $\mathcal{L}_c(A)$, in the case of real eigenvalues it suffices to consider Lemmas 2.2 and 2.3 to see that the n matrices in (2.10) actually generate the whole $\mathcal{L}_c(A)$. In case of complex eigenvalues, a similar reasoning can be made by considering the real Jordan form (see also the proof of the subsequent Lemma 2.5). $\square$

Corollary 2.1 *If the Jordan form of A has not two Jordan blocks corresponding to the same eigenvalue, then $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$.*

Proof The proof follows from the proof of Theorem 2.2, taking into account that the minimal polynomial of A has degree n and, therefore, that $\{A^0, A^1, \dots, A^{n-1}\}$ is a set of n linearly independent matrices that pairwise commute. $\square$

Theorem 2.3 *Assume that $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$. Then, any pair $B_1, B_2 \in \mathcal{L}_c(A)$ is commuting.*

Proof Since $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$, B_1 and B_2 can be written as $B_1 = \sum_{i=0}^{n-1} a_{1,i} A^i$ and $B_2 = \sum_{j=0}^{n-1} a_{2,j} A^j$, for some constants $a_{1,i}, a_{2,i} \in \mathbb{R}$. Therefore, $[B_1, B_2] = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{1,i} a_{2,j} [A^i, A^j] = 0$. $\square$

Example 2.3 Let $J = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$, with J_1 and J_2 being Jordan blocks of dimension two and three, with real eigenvalues λ_1 and λ_2 , respectively. Then, the following matrices belong to $\mathcal{L}_c(J)$, are linearly independent over $\mathbb{R}$, and are commuting:

$$\left\{ \begin{bmatrix} J_1^0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_1^1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & J_2^0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & J_2^1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & J_2^2 \end{bmatrix} \right\}.$$

Furthermore, if $\lambda_1 \neq \lambda_2$, then they constitute a basis of $\mathcal{L}_c(J)$.

Example 2.4 Consider matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix},$$

which is semi-simple with two coincident eigenvalues; $\mathcal{L}_c(A)$ has dimension $r = 5$ and one of its bases is

$$\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

There exist three linearly independent and commuting elements of $\mathcal{L}_c(A)$ over $\mathbb{R}$, which can be constructed from the Jordan form of A ,

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix},$$

as follows:

$$\begin{aligned} M_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix}, \\ M_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix}, \\ M_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix}. \end{aligned}$$

Such three matrices pairwise commute, but they do not constitute a basis of $\mathcal{L}_c(A)$.

A nice consequence of Theorem 2.2 is that if matrix A is semi-simple with distinct (possibly, complex) eigenvalues, then $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$.

Theorem 2.4 Assume that A is semi-simple with distinct eigenvalues; let $Q \in \mathbb{C}^{n \times n}$, $\det(Q) \neq 0$, be such that $\tilde{A} = Q^{-1}AQ$ is diagonal. Then, $\tilde{B} = Q^{-1}BQ$ is diagonal for all $B \in \mathcal{L}_c(A)$; furthermore, any $B = Q\tilde{B}Q^{-1}$, with $\tilde{B}$ diagonal, is an element of $\mathcal{L}_c(A)$.

Proof By (2.9), if $B \in \mathcal{L}_c(A)$, then $\tilde{B} \in \mathcal{L}_c(\tilde{A})$ for all $Q \in \mathbb{C}^{n \times n}$ such that $\det(Q) \neq 0$. If A is semi-simple with distinct eigenvalues, then $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$, whence a basis of $\mathcal{L}_c(\tilde{A})$ is $\{\tilde{A}^0, \tilde{A}^1, \dots, \tilde{A}^{n-1}\}$. If $B \in \mathcal{L}_c(A)$, then there exist $\mu_0, \mu_1, \dots, \mu_{n-1}$ such that $B = \sum_{i=0}^{n-1} \mu_i A^i$, whence $\tilde{B} =$

$\sum_{i=0}^{n-1} \mu_i \tilde{A}^i$, which implies that $\tilde{B}$ is diagonal. Vice versa, if $\tilde{B}$ is diagonal, then $\tilde{B} \in \mathcal{L}_c(\tilde{A})$, whence $B \in \mathcal{L}_c(A)$. $\square$

If A is semi-simple, but with some coincident eigenvalues, and $[A, B] = 0$, then $\tilde{B} = Q^{-1}BQ$ need not be diagonal also if $\tilde{A} = Q^{-1}AQ$ is diagonal (see also Lemma 1.4 at p. 26).

Example 2.5 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; since A is semi-simple with distinct eigenvalues $(\pm i)$, any $B \in \mathcal{L}_c(A)$ can be written as

$$B = \mu_0 A^0 + \mu_1 A^1 = \mu_0 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mu_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix}, \quad \mu_i \in \mathbb{R}.$$

Letting $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2}i \end{bmatrix}$, one has $\tilde{A} = Q^{-1}AQ = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$. Therefore, Q jointly diagonalizes all elements of $\mathcal{L}_c(A)$:

$$\tilde{B} = \begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix} \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}i & -\frac{1}{2}i \end{bmatrix} = \begin{bmatrix} \mu_0 + i\mu_1 & 0 \\ 0 & \mu_0 - i\mu_1 \end{bmatrix}.$$

It is now possible to prove two lemmas that are very important in the sequel.

Lemma 2.4 Any matrix $A \in \mathbb{R}^{n \times n}$ can be decomposed as $A = A_s + A_n$, where $A_s \in \mathbb{R}^{n \times n}$ is semi-simple, $A_n \in \mathbb{R}^{n \times n}$ is nilpotent and A_s, A_n commute under the matrix product. Such matrices A_s and A_n can be expressed as polynomials in A , whence any matrix B that commutes under the matrix product with A also commutes with A_s and A_n .

Proof It is sufficient to bring A into its complex Jordan form, $A = QJQ^{-1}$, where $\det(Q) \neq 0$ and $J = \text{block_diag}\{J_1, \dots, J_p\}$, with J_i being a Jordan block with eigenvalue λ_i ; if A has complex eigenvalues, then matrix Q has to be chosen so that its two block columns Q_i and Q_j containing two corresponding chains of generalized eigenvectors of A relative to λ_i and $\lambda_j = \lambda_i^*$, respectively, satisfy $Q_j = Q_i^*$. Then, in the new coordinates, letting

$$J_{i,s} = \text{diag}\{\lambda_i, \dots, \lambda_i\} \quad \text{and} \quad J_{i,n} = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix},$$

one has $J_i = J_{i,s} + J_{i,n}$, with $J_{i,s}$ semi-simple and $J_{i,n}$ nilpotent. Let $A_s = QJ_sQ^{-1} = Q \text{block_diag}\{J_{1,s}, \dots, J_{p,s}\}Q^{-1}$ and $A_n = QJ_nQ^{-1} = Q \text{block_diag}\{J_{1,n}, \dots, J_{p,n}\}Q^{-1}$. Obviously, A_s is semi-simple and A_n is nilpotent. Moreover, in case of A having complex eigenvalues, thanks to the choice of Q required above, it is easy to verify that A_s and A_n are real. Now, to show that A_s and A_n can be

written as polynomials in A , it is sufficient to show that J_s and J_n can be written as polynomials in J . Taking into account the expression of the k th power of a Jordan block, it is easy to see that if J_a and J_b are two Jordan blocks of dimensions $n_a \geq n_b$, relative to the same eigenvalue λ , then letting $J_{a,s} = \lambda E$ (where E has dimensions $n_a \times n_a$), $J_{a,n} = J_a - J_{a,s}$, $J_{b,s} = \lambda E$ (where E has dimensions $n_b \times n_b$) and $J_{b,n} = J_b - J_{b,s}$, the equations $J_{b,s} = p_s(J_b)$ and $J_{b,n} = p_n(J_b)$ hold necessarily, for any pair of polynomials $p_s(s)$ and $p_n(s)$ such that $J_{a,s} = p_s(J_a)$ and $J_{a,n} = p_n(J_a)$. Now, let $J_{\min}$ be a Jordan matrix with the same minimal polynomial of J , but without repeated eigenvalues (obtained by selecting from J just one Jordan block of highest dimension for each eigenvalue) and let D a diagonal matrix with the same diagonal as $J_{\min}$; let $N = J_{\min} - D$. The structures of J and $J_{\min}$ imply that if there exists a pair of polynomials $p_s(s)$ and $p_n(s)$ such that $D = p_s(J_{\min})$ and $N = p_n(J_{\min})$, then $J_s = p_s(J)$ and $J_n = p_n(J)$ hold necessarily. The existence of $p_s(s)$ and $p_n(s)$ is ensured by Theorem 2.2 with $A = J_{\min}$ and by Lemmas 2.2 and 2.3 (see also the beginning of the proof of Lemma 2.3). Hence, it is proven that A_s and A_n can be written as polynomials in A , and therefore that any matrix B that commutes under the matrix product with A also commutes with A_s and A_n . Clearly, if matrices A_s and A_n can be expressed as polynomials in A , then A_s, A_n commute under the matrix product. $\square$

Remark 2.4 The decomposition $A = A_s + A_n$, with A_s being semi-simple and A_n being nilpotent is not unique; in general, there are many such decompositions with A_s and A_n that do not commute under the matrix product. But, as stated in [34, Lemma 14 at p. 104], if one requires that A_s and A_n commute, then such A_s and A_n are unique. As an example, take

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix};$$

clearly,

$$A_s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

is semi-simple,

$$A_n = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is nilpotent and $A = A_s + A_n$, but

$$A_n A_s - A_s A_n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not zero: it is easy to verify that neither A_s nor A_n can be expressed as a polynomial function of A . Bringing A in the Jordan form,

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

and then letting

$$A_s = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix},$$

$$A_n = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

one concludes that $A = A_s + A_n$, $A_n A_s - A_s A_n = 0$; as a consequence, since

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2,$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = -2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 3 \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^2,$$

one finds that $A_s = 2A^0 - 2A^1 + A^2$ and $A_n = -2A^0 + 3A^1 - A^2$.

Lemma 2.5 *For any matrix A , there exists an invertible $Q \in \mathbb{R}^{n \times n}$, $\det(Q) \neq 0$, such that $\tilde{A} = Q^{-1}AQ$ can be uniquely decomposed as $\tilde{A} = \tilde{A}_{s,n} + \tilde{A}_n$, where $\tilde{A}_{s,n} \in \mathbb{R}^{n \times n}$ is normal, $\tilde{A}_n \in \mathbb{R}^{n \times n}$ is nilpotent and $\tilde{A}_n, \tilde{A}_n^T$ and their powers $\tilde{A}_n^i, (\tilde{A}_n^T)^i, i \in \mathbb{Z}^{\geq}$, commute with $\tilde{A}_{s,n}$ under the matrix product.*

Proof It is sufficient to proceed as in the proof of Lemma 2.4, but considering real Jordan blocks instead of (complex) Jordan blocks. In particular, choose Q so that $\tilde{A}$ is in the *real Jordan form*, i.e., a Jordan form in which each pair J_i, J_j of Jordan blocks of the same dimension $r_i = r_j$ corresponding to $\lambda_i = \alpha + i\beta$ and $\lambda_j = \lambda_i^* = \alpha - i\beta, \alpha, \beta \in \mathbb{R}$, is substituted by a single block of dimension $2r_i$ that is the sum of a block-diagonal matrix whose r_i diagonal blocks are $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ and a matrix having the only $2r_i - 2$ non-zero elements, which are equal to 1, in positions (k, h) , where $h = k + 2$. $\square$

Example 2.6 Let

$$\begin{aligned}
 A &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}i & \frac{1}{2}i & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & i & i & 1+i \\ -1 & -i & -i & 1-i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.
 \end{aligned}$$

Then,

$$\begin{aligned}
 A_s &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}i & \frac{1}{2}i & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1+i & 0 & 0 & 0 \\ 0 & 1-i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & i & i & 1+i \\ -1 & -i & -i & 1-i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

is semi-simple, but it is not normal, and

$$\begin{aligned}
 A_n &= \begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & 0 & 1 \\ -\frac{1}{2}i & \frac{1}{2}i & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & i & i & 1+i \\ -1 & -i & -i & 1-i \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{aligned}$$

is nilpotent, with $A = A_s + A_n$; by construction, such matrices A_s, A_n are commuting. Consider the real transformation represented by

$$Q = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix};$$

then,

$$\begin{aligned}\tilde{A} = Q^{-1}AQ &= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix},\end{aligned}$$

whence $\tilde{A} = \tilde{A}_{s,n} + \tilde{A}_n$, with

$$\tilde{A}_{s,n} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \tilde{A}_n = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

clearly, $\tilde{A}_{s,n}$ is normal, $\tilde{A}_n$ is nilpotent, and matrices $\tilde{A}_{s,n}, \tilde{A}_n$ (respectively, $\tilde{A}_{s,n}, \tilde{A}_n^\top$) are commuting.

The following definition extends the concept of linear symmetry to the concept of linear orbital symmetry, in the continuous-time case.

Definition 2.5 Assume $\mathbb{T} = \mathbb{R}$.

(2.5.1) The linear transformation (2.3) is a *linear orbital symmetry* of system (2.1a) and system (2.2) is its *infinitesimal generator* if

$$[A, B] = \mu A, \quad (2.14)$$

for some constant $\mu \in \mathbb{R}$. When (2.14) holds, by abuse of notation, also the infinitesimal generator (2.2) is called a *linear orbital symmetry* of system (2.1a); briefly, Bx is called a *linear orbital symmetry* of Ax .

(2.5.2) The *linear normalizer* $\mathcal{L}_n(A)$ of A is the set of all matrices B such that $[A, B] = \mu A$, for some constant $\mu \in \mathbb{R}$.

Remark 2.5 The linear orbital symmetry introduced in Definition 2.5 maps an orbit of system (2.1a) into an orbit of the same system, but the time parameterization along the orbit is not preserved when $\mu \neq 0$, as can be seen in the following. If B is a linear orbital symmetry of A , then

$$BA = AB + \mu A = A(B + \mu E),$$

with E being the $n \times n$ identity matrix; left multiplying such an equation by B ,

$$B^2A = BA(B + \mu E) = A(B + \mu E)^2,$$

and iterating such a process, one concludes that

$$B^h A = A(B + \mu E)^h, \quad \forall h \in \mathbb{Z}^{\geq}. \quad (2.15)$$

Hence,

$$\begin{aligned} e^{-B\tau} A &= \sum_{h=0}^{+\infty} \frac{(-\tau)^h}{h!} B^h A = \sum_{h=0}^{+\infty} \frac{(-\tau)^h}{h!} A(B + \mu E)^h \\ &= A e^{-(B+\mu E)\tau}, \end{aligned}$$

from which

$$e^{-B\tau} A e^{B\tau} = A e^{-\mu\tau}. \quad (2.16)$$

This shows that $\frac{dx}{dt} = Ax$ is transformed by the linear change of coordinates $x = e^{B\tau} y$ into $\frac{dy}{ds} = Ay$, where $\frac{ds}{d\tau} = e^{-\mu\tau}$, with s being the new time variable corresponding to the new time parameterization of the system thus transformed.

Example 2.7 Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$; B is a linear orbital symmetry of A , because $[A, B] = -2A$. Then, $e^{-B\tau} A e^{B\tau} = e^{2\tau} A$.

The following theorem, which can be easily proven by means of (2.9), shows that the concepts of linear symmetry and linear orbital symmetry do not depend on the particular coordinates chosen.

Theorem 2.5 *Let $\tilde{A} = Q^{-1} A Q$ and $\tilde{B} = Q^{-1} B Q$, with Q being an invertible square matrix. Then,*

$$[A, B] = \mu A \quad \Longleftrightarrow \quad [\tilde{A}, \tilde{B}] = \mu \tilde{A},$$

namely

$$B \in \mathcal{L}_n(A) \quad \Longleftrightarrow \quad \tilde{B} \in \mathcal{L}_n(\tilde{A}).$$

Theorem 2.6 *Let A be semi-simple with distinct eigenvalues. Then, $\mathcal{L}_n(A) = \mathcal{L}_c(A)$.*

Proof If $n = 1$, then $\mathcal{L}_n(A) = \mathcal{L}_c(A) = \mathbb{R}$, and the theorem holds. Assume $n \geq 2$. By definition $\mathcal{L}_c(A) \subseteq \mathcal{L}_n(A)$; hence, if $\mathcal{L}_n(A) \subseteq \mathcal{L}_c(A)$, then the theorem is proven. By Theorem 2.5, it can be assumed that

$$A = \text{block_diag} \left\{ \lambda_1, \dots, \lambda_{n_r}, \begin{bmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{bmatrix}, \dots, \begin{bmatrix} \alpha_{n_c} & \beta_{n_c} \\ -\beta_{n_c} & \alpha_{n_c} \end{bmatrix} \right\},$$

with $n_r + 2n_c = n$, $\lambda_i \in \mathbb{R}$, $\lambda_i \neq \lambda_j$ if $i \neq j$, and $\alpha_i, \beta_i \in \mathbb{R}$, $\beta_i \neq 0$, and $\alpha_i + i\beta_i \neq \alpha_j + i\beta_j$, if $i \neq j$. Under this assumption, it is now shown that $AM - MA = \mu A$

implies $\mu = 0$, whence that $M \in \mathcal{L}_n(A)$ implies $M \in \mathcal{L}_c(A)$. In the simpler case $n_c = 0$, when A has no complex eigenvalues, it is easily seen that all the diagonal elements of $AM - MA$ are structurally equal to zero; this, in view of the fact that at least one eigenvalue λ_i of A is not zero, implies that $\mu = 0$. As for the general case, if $n_r > 0$ and there is a real eigenvalue $\lambda_i \neq 0$ of A , then the equality $\mu = 0$ follows as above from the fact that the i th diagonal element of $AM - MA$ is zero. Otherwise (i.e., if $n_r = 0$, or $n_r = 1$ and $\lambda_1 = 0$), consider the 2×2 diagonal block of $AM - MA$ whose upper left element is in position $(n_r + 1, n_r + 1)$ and impose that it is equal to the same block of matrix μA , to obtain

$$\begin{bmatrix} \beta_1 L_2 & \beta_1 L_1 \\ \beta_1 L_1 & -\beta_1 L_2 \end{bmatrix} = \begin{bmatrix} \mu \alpha_1 & \mu \beta_1 \\ -\mu \beta_1 & \mu \alpha_1 \end{bmatrix},$$

where $L_1 = M_{n_r+2, n_r+2} - M_{n_r+1, n_r+1}$ and $L_2 = M_{n_r+2, n_r+1} - M_{n_r+1, n_r+2}$. Then, since $\beta_1 \neq 0$, the two equations $\beta_1 L_1 = \mu \beta_1$ and $\beta_1 L_1 = -\mu \beta_1$ imply $\mu = 0$. $\square$

The following Theorem 2.7 shows that the linear normalizer $\mathcal{L}_n(A)$ and the linear centralizer $\mathcal{L}_c(A) \subseteq \mathcal{L}_n(A)$ of A are closed under the Lie bracket operation; in particular, since $B_1, B_2 \in \mathcal{L}_n(A)$ implies $[B_1, B_2] \in \mathcal{L}_c(A)$, by the analysis of the subsequent Sect. 6.2 and taking into account the subsequent Theorem 2.9, one concludes that $\mathcal{L}_n(A)$ is a Lie sub-algebra of the Lie algebra of matrices over $\mathbb{R}$, and $\mathcal{L}_c(A)$ is a Lie ideal of $\mathcal{L}_n(A)$.

Theorem 2.7 *If $B_1 x$ and $B_2 x$ are two linear orbital symmetries (possibly, linear symmetries) of Ax , then $[B_1, B_2]x$ is a linear symmetry of Ax .*

Proof From $[A, B_1] = \mu_1 A$ and $[A, B_2] = \mu_2 A$, with μ_1 and μ_2 being (possibly, equal to zero) real constants, one finds that (taking into account the Jacobi identity (2.3.3) reported in Remark 2.3):

$$\begin{aligned} [A, [B_1, B_2]] &= -[B_1, [B_2, A]] - [B_2, [A, B_1]] = [B_1, \mu_2 A] - [B_2, \mu_1 A] \\ &= -\mu_1 \mu_2 A + \mu_1 \mu_2 A = 0. \end{aligned} \quad \square$$

In particular, the following theorem shows that, if A is semi-simple with distinct eigenvalues and B_1 and B_2 commute with A , then $[B_1, B_2] = 0$ (i.e., B_1 and B_2 are commuting).

Theorem 2.8 *Let A be semi-simple with distinct eigenvalues. Then, all elements of $\mathcal{L}_c(A)$ are commuting, i.e., if $B_1, B_2 \in \mathcal{L}_c(A)$, then $[B_1, B_2] = 0$.*

Proof By the invariance of the matrix Lie bracket to linear transformations, assume that A is diagonal, with distinct eigenvalues. Then, all $B \in \mathcal{L}_c(A)$ are diagonal, but two diagonal matrices B_1, B_2 are necessarily commuting. $\square$

Example 2.8 Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$, which is semi-simple with distinct eigenvalues. A basis of $\mathcal{L}_c(A)$ is given by A^0 and A^1 . Let B_1 and B_2 be two elements of $\mathcal{L}_c(A)$, then

$$B_1 = a_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_2 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & 2a_2 \\ 0 & a_1 + 3a_2 \end{bmatrix},$$

$$B_2 = a_3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + a_4 \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} a_3 + a_4 & 2a_4 \\ 0 & a_3 + 3a_4 \end{bmatrix}.$$

Clearly, $[B_1, B_2] = 0$, for all $a_i \in \mathbb{R}$.

The following theorem shows that $\mathcal{L}_n(A)$ has the structure of a finite dimensional vector space over $\mathbb{R}$, similarly to $\mathcal{L}_c(A)$.

Theorem 2.9 *The linear normalizer $\mathcal{L}_n(A)$ of A is a finite dimensional vector space over $\mathbb{R}$ of dimension $n \leq r \leq n^2$.*

Proof If $B_1, B_2 \in \mathcal{L}_n(A)$, then there exist two constant $\mu_1, \mu_2 \in \mathbb{R}$ such that $[A, B_1] = \mu_1 A$ and $[A, B_2] = \mu_2 A$; by the bi-linearity of the Lie bracket operation, one finds that

$$[A, \alpha_1 B_1 + \alpha_2 B_2] = \alpha_1 [A, B_1] + \alpha_2 [A, B_2] = (\alpha_1 \mu_1 + \alpha_2 \mu_2) A, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R},$$

and, therefore, that $\alpha_1 B_1 + \alpha_2 B_2 \in \mathcal{L}_n(A)$. Since $\mathcal{L}_n(A) \subseteq \mathbb{R}^{n \times n}$, its dimension satisfies $r \leq n^2$. In addition, since $\mathcal{L}_c(A) \subseteq \mathcal{L}_n(A)$, the dimension r of $\mathcal{L}_n(A)$ satisfies $r \geq n$. $\square$

Clearly, $\mathcal{L}_c(A) \subseteq \mathcal{L}_n(A)$. Determining the linear centralizer $\mathcal{L}_c(A)$ (respectively, the linear normalizer $\mathcal{L}_n(A)$) of A is equivalent to solving a set of n^2 algebraic (linear for each fixed μ) equations having the entries of B (respectively, the entries of B and the real number μ) as unknowns, as detailed in the following example.

Example 2.9 Consider matrix $A = \begin{bmatrix} 0 & 1 \\ \alpha & \beta \end{bmatrix}$ with $\alpha, \beta \in \mathbb{R}$; such a matrix is not semi-simple when $\alpha = -\lambda^2, \beta = 2\lambda$ for some constant $\lambda \in \mathbb{R}$. Letting $B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$, from condition

$$0 = [A, B] = \begin{bmatrix} b_2\alpha - b_3 & b_1 + b_2\beta - b_4 \\ b_4\alpha - \alpha b_1 - \beta b_3 & b_3 - b_2\alpha \end{bmatrix},$$

one has the following system of four linear algebraic equations:

$$\begin{bmatrix} 0 & \alpha & -1 & 0 \\ -\alpha & 0 & -\beta & \alpha \\ 1 & \beta & 0 & -1 \\ 0 & -\alpha & 1 & 0 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since the rank of the coefficient matrix of the above system is equal to 2 for all $\alpha, \beta \in \mathbb{R}$, and matrices $M_0 = E$ and $M_1 = A$ are linearly independent over $\mathbb{R}$, one concludes that $\mathcal{L}_c(A) = \text{span}_{\mathbb{R}}\{E, A\}$. Consider now the case $\alpha = 0, \beta = 0$. The condition $[A, B] = \mu A$ leads to

$$b_1 = c_2 + c_3, \quad b_2 = c_1, \quad b_3 = 0, \quad b_4 = c_2, \quad \mu = c_3,$$

with $c_1, c_2, c_3 \in \mathbb{R}$ being arbitrary constants, which shows that $\mathcal{L}_n(A)$ has dimension 3. As basis of $\mathcal{L}_n(A)$, one can take the basis of $\mathcal{L}_c(A)$ (corresponding to the choice $c_3 = 0$) completed with the matrix B found by letting $c_1 = 0, c_2 = 0$ and $c_3 = 1$.

2.2 Darboux Polynomials and First Integrals

The following classical theorem, which is also used in other chapters, is stated and proven here because its statement and proof have strong similarities with the subsequent results. The theorem itself can be proven directly in the discrete-time case, and by an alternative simpler proof based on the Jordan form of A in the continuous-time case.

Theorem 2.10 *Let $\varpi(t) = \det(e^{At})$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\varpi(t) = \det(A^t)$ if $\mathbb{T} = \mathbb{Z}$); then $\frac{d\varpi(t)}{dt} = \text{trace}(A)\varpi(t)$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\varpi(t+1) = \det(A)\varpi(t)$ if $\mathbb{T} = \mathbb{Z}$) and $\varpi(0) = 1$.*

Proof Clearly, $\varpi(0) = \det(E) = 1$ (also when $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$). Let $v_i(t) \in \mathbb{R}^n$ be the i th column of matrix e^{At} (respectively, A^t); then, $\Delta v_i = Av_i$ and $v_i(0) = e_i$, with e_i being the i th column of the $n \times n$ identity matrix E . The proof follows from the multi-linearity of the determinant in the continuous-time case:

$$\begin{aligned} \Delta \varpi &= \Delta \det([v_1 \ v_2 \ \dots \ v_n]) \\ &= \det([\Delta v_1 \ v_2 \ \dots \ v_n]) + \det([v_1 \ \Delta v_2 \ \dots \ v_n]) + \dots + \det([v_1 \ v_2 \ \dots \ \Delta v_n]) \\ &= \det([Av_1 \ v_2 \ \dots \ v_n]) + \det([v_1 \ Av_2 \ \dots \ v_n]) + \dots + \det([v_1 \ v_2 \ \dots \ Av_n]) \\ &= \text{trace}(A) \det([v_1 \ v_2 \ \dots \ v_n]) = \text{trace}(A)\varpi, \quad \text{if } \mathbb{T} = \mathbb{R}, \end{aligned}$$

and by the following relationships in the discrete-time case:

$$\begin{aligned} \Delta \varpi &= \Delta \det([v_1 \ v_2 \ \dots \ v_n]) \\ &= \det([\Delta v_1 \ \Delta v_2 \ \dots \ \Delta v_n]) = \det([Av_1 \ Av_2 \ \dots \ Av_n]) \\ &= \det(A) \det([v_1 \ v_2 \ \dots \ v_n]) = \det(A)\varpi, \quad \text{if } \mathbb{T} = \mathbb{Z}. \quad \square \end{aligned}$$

The concept of the Darboux polynomial extends the concept of polynomial first integral and the concept of left eigenfunction for linear systems. The definition of the

Darboux polynomial is given in the literature (see, e.g., [56, 88, 96]) for nonlinear systems but, in view of the importance of this concept, the special properties of Darboux polynomials for linear systems are studied in this section.

Definition 2.6 A scalar polynomial $\omega(x)$ is a *Darboux polynomial* (a *polynomial semi-invariant*) of systems (2.1a), (2.1b) if there exists a $\lambda \in \mathbb{R}$ such that the following relation holds for all $x \in \mathbb{R}^n$:

$$\Delta\omega = \lambda\omega, \quad (2.17)$$

with $\Delta\omega = L_{Ax}\omega$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\Delta\omega = \omega \circ Ax$ if $\mathbb{T} = \mathbb{Z}$). The number λ is called *characteristic value*.

In the subsequent Chaps. 3 and 4, the definition of the Darboux polynomial is given for nonlinear systems, by allowing λ to be a polynomial function of x . Here, a simple reasoning on the degree of polynomials and the linearity of systems (2.1a), (2.1b) imply that λ is necessarily constant. Therefore, there is no loss of generality in assuming λ constant as is done in Definition 2.6.

From Definition 2.6, a polynomial first integral is a Darboux polynomial with $\lambda = 0$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\lambda = 1$ if $\mathbb{T} = \mathbb{Z}$). It is worth pointing out that, if not empty, the set $\mathcal{J}_\omega$ of all $x \in \mathbb{R}^n$ such that $\omega(x) = 0$, with $\omega(x)$ being a Darboux polynomial of systems (2.1a), (2.1b), is an invariant subspace of systems (2.1a), (2.1b), i.e., if $x(0) \in \mathbb{R}^n$ is such that $\omega(x(0)) = 0$, then $\omega(x(t)) = 0$, $\forall t \in \mathbb{T}$ ($t \geq 0$ if $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$), along the solutions of systems (2.1a), (2.1b). To be more precise, by (2.17), $\omega(t) = e^{\lambda t}\omega(0)$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\omega(t) = \lambda^t\omega(0)$ if $\mathbb{T} = \mathbb{Z}$), whence $\omega(t) = 0$, $\forall t \in \mathbb{T}$ ($t \geq 0$ if $\mathbb{T} = \mathbb{Z}$ and $\det(A) = 0$), if and only if $\omega(0) = 0$. Clearly, the same is true for a polynomial first integral.

The following four theorems characterize the Darboux polynomials of systems (2.1a), (2.1b).

Theorem 2.11 Let $\mathbb{T} = \mathbb{R}$ and $M_i \in \mathcal{L}_n(A)$, $i = 1, \dots, n-1$. Assume that the function $\omega(x)$ defined as follows:

$$\omega(x) := \det(\Omega(x)), \quad \Omega(x) = [Ax \ M_1x \ \dots \ M_{n-1}x], \quad (2.18)$$

is not identically equal to zero. Then, the following properties hold:

- (2.11.1) relation (2.17) holds with $\lambda = \text{trace}(A)$, i.e., ω is a Darboux polynomial of system (2.1a), with characteristic value $\lambda = \text{trace}(A)$;
- (2.11.2) if the polynomial ω can be factorized as $\omega = \prod_i \omega_i^{v_i}$, with ω_i 's being co-prime real polynomials and $v_i \in \mathbb{Z}^+$, then there exist constants $\lambda_i \in \mathbb{R}$ such that $\sum_i v_i \lambda_i = \text{trace}(A)$ and such that (2.17) holds with $\omega = \omega_i$ and $\lambda = \lambda_i$.

Proof Consider, first, Statement (2.11.1) of the theorem. By linear algebra, applying Δ to ω , one finds that

$$\Delta\omega = \det([A\Delta x \ M_1x \ \dots \ M_{n-1}x]) + \det([Ax \ M_1\Delta x \ \dots \ M_{n-1}x])$$

$$+ \cdots + \det([Ax \ M_1x \ \dots \ M_{n-1} \Delta x]);$$

since $\Delta x = Ax$ along the solutions of (2.1a), it follows that

$$\begin{aligned} \Delta\omega &= \det([AAx \ M_1x \ \dots \ M_{n-1}x]) + \det([Ax \ M_1Ax \ \dots \ M_{n-1}x]) \\ &\quad + \cdots + \det([Ax \ M_1x \ \dots \ M_{n-1}Ax]); \end{aligned}$$

$M_iA = AM_i + \mu_iA$, because $M_i \in \mathcal{L}_n(A)$, which implies

$$\begin{aligned} \Delta\omega &= \det([AAx \ M_1x \ \dots \ M_{n-1}x]) \\ &\quad + \det([Ax \ AM_1x + \mu_1Ax \ \dots \ M_{n-1}x]) \\ &\quad + \cdots + \det([Ax \ M_1x \ \dots \ AM_{n-1}x + \mu_{n-1}Ax]), \end{aligned}$$

and, therefore, by linear algebra, one concludes that

$$\Delta\omega = \text{trace}(A) \det([Ax \ M_1x \ \dots \ M_{n-1}x]) = \text{trace}(A)\omega,$$

thus proving Statement (2.11.1) of the theorem. As for Statement (2.11.2) of the theorem, from $\omega = \prod_i \omega_i^{v_i}$, one finds that

$$\frac{L_{Ax}\omega}{\omega} = \sum_i v_i \frac{L_{Ax}\omega_i}{\omega_i}. \quad (2.19)$$

Since $L_{Ax}\omega_i = \frac{\partial \omega_i}{\partial x} Ax$, it follows that $\frac{L_{Ax}\omega_i}{\omega_i}$ is a proper rational function. Since all denominators of the rational functions $\frac{L_{Ax}\omega_i}{\omega_i}$ are co-prime, such functions do not have common poles (common roots of the denominators). Let x^o be a root of ω_j : then, $\omega_j(x^o) = 0$ and $\omega_i(x^o) \neq 0$, $\forall i \neq j$; since $\frac{L_{Ax}\omega}{\omega}$ is constant, $\sum_i v_i \frac{L_{Ax}\omega_i}{\omega_i}$ is constant only if $L_{Ax}\omega_j(x^o) = 0$; then, iterating through all roots of the polynomials ω_i , and taking into account that each $\frac{L_{Ax}\omega_i}{\omega_i}$ is proper, one concludes that each $\frac{L_{Ax}\omega_i}{\omega_i}$ is equal to a certain constant λ_i . Finally, equation (2.19) shows that $\sum_i v_i \lambda_i = \text{trace}(A)$, taking into account that $\frac{L_{Ax}\omega}{\omega} = \text{trace}(A)$ and $\frac{L_{Ax}\omega_i}{\omega_i} = \lambda_i$, thus proving Statement (2.11.2) of the theorem. $\square$

Note that Theorem 2.11 extends Theorem 2.10 (in the case $\mathbb{T} = \mathbb{R}$), because if $x(0)$ is chosen so that $\omega(x(0)) = 1$, then $\omega(x(t))$ is just the function $\varpi(t)$ of Theorem 2.10.

The next theorem is the analogous of Theorem 2.11 for discrete-time systems, but two important differences have to be stressed: in the discrete-time case, matrices M_i are required to commute with A and a statement analogous to Statement (2.11.2) of Theorem 2.11 does not hold.

Theorem 2.12 *Let $\mathbb{T} = \mathbb{Z}$ and $M_i \in \mathcal{L}_c(A)$, $i = 1, \dots, n-1$. Assume that the function $\omega(x)$ defined as follows:*

$$\omega(x) := \det(\Omega(x)), \quad \Omega(x) = [Ax \ M_1x \ \dots \ M_{n-1}x],$$

is not identically equal to zero. Then, relation (2.17) holds with $\lambda = \det(A)$, i.e., ω is a Darboux polynomial of system (2.1b), with characteristic value $\lambda = \det(A)$.

Proof Taking into account that $M_i A = A M_i$, one has

$$\begin{aligned}\Delta\omega &= \det(\Omega(\Delta x)) = \det(\Omega(Ax)) = \det([AAx \ M_1 Ax \ \dots \ M_{n-1} Ax]) \\ &= \det([AAx \ AM_1 x \ \dots \ AM_{n-1} x]) = \det(A) \det(\Omega(x)) = \det(A)\omega(x),\end{aligned}$$

as to be proven. $\square$

By Theorem 2.6, if matrix A is semi-simple and has distinct eigenvalues, then $\mathcal{L}_n(A) = \mathcal{L}_c(A)$; for this reason, the continuous-time and discrete-time cases are considered jointly in the next theorem.

Theorem 2.13 *Consider jointly both cases $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. Let A be semi-simple with distinct eigenvalues, let n_r be the number of real eigenvalues of A , let n_c be the number of pairs of complex conjugate eigenvalues of A (in particular, $n_r + 2n_c = n$), and let the eigenvalues of A be ordered as*

$$\{\lambda_1, \dots, \lambda_{n_r}, \lambda_{n_r+1}, \lambda_{n_r+1}^*, \dots, \lambda_{n_r+n_c}, \lambda_{n_r+n_c}^*\}.$$

Then, the polynomial $\omega(x)$ defined in Theorems 2.11 and 2.12 can be factorized as follows:

$$\omega(x) = k \hat{\omega}_1(x) \cdots \hat{\omega}_{n_r+n_c}(x), \quad (2.20)$$

where $k \in \mathbb{R}$ is a constant, and

$$\hat{\omega}_i(x) = \begin{cases} u_i^\top x, & i = 1, \dots, n_r, \\ (u_i^\top x)(u_i^\top x)^*, & i = n_r+1, \dots, n_r+n_c, \end{cases}$$

with $u_i^\top$ being the left eigenvector relative to eigenvalue λ_i , $i = 1, \dots, n_r + n_c$.

Proof Let $\Omega(x)$ be defined as in Theorems 2.11 and 2.12. Since one of the bases of $\mathcal{L}_c(A)$ is given by $\{E, A, \dots, A^{n-1}\}$, one has $\Omega(x) = \bar{\Omega}(x)T$ for some invertible matrix $T \in \mathbb{R}^{n \times n}$, where

$$\bar{\Omega}(x) = [Ex \ Ax \ \dots \ A^{n-1}x]. \quad (2.21)$$

For $i = 1, \dots, n_r + n_c$, one has

$$\begin{aligned}u_i^\top \bar{\Omega}(x) &= [u_i^\top Ex \ u_i^\top Ax \ \dots \ u_i^\top A^{n-1}x] \\ &= [(u_i^\top x) \ \lambda_i (u_i^\top x) \ \dots \ \lambda_i^{n-1} (u_i^\top x)] \\ &= (u_i^\top x) [1 \ \lambda_i \ \dots \ \lambda_i^{n-1}];\end{aligned}$$

hence, defining the (possibly, complex) matrices

$$U := \begin{bmatrix} u_1^\top \\ \vdots \\ u_{n_r}^\top \\ u_{n_r+1}^\top \\ \vdots \\ u_{n_r+n_c}^\top \\ u_{n_r+1}^{*\top} \\ \vdots \\ u_{n_r+n_c}^{*\top} \end{bmatrix}, \quad V := \begin{bmatrix} 1 & \lambda_1 & \dots & \lambda_1^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r} & \dots & \lambda_{n_r}^{n-1} \\ 1 & \lambda_{n_r+1} & \dots & \lambda_{n_r+1}^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r+n_c} & \dots & \lambda_{n_r+n_c}^{n-1} \\ 1 & \lambda_{n_r+1}^* & \dots & (\lambda_{n_r+1}^*)^{n-1} \\ \vdots & \vdots & \dots & \vdots \\ 1 & \lambda_{n_r+n_c}^* & \dots & (\lambda_{n_r+n_c}^*)^{n-1} \end{bmatrix},$$

one obtains

$$\begin{aligned} U \Omega(x) &= U \bar{\Omega}(x) T \\ &= \text{diag}\{u_1^\top x, \dots, u_{n_r}^\top x, u_{n_r+1}^\top x, \dots, u_{n_r+n_c}^\top x, u_{n_r+1}^{*\top} x, \dots, u_{n_r+n_c}^{*\top} x\} V T; \end{aligned}$$

taking the determinant of both sides of the above equation, one proves the theorem with $k = \frac{\det(V) \det(T)}{\det(U)}$, where $\det(V) \neq 0$ because the eigenvalues of A are distinct. $\square$

The following theorem shows, also in the case that matrix A has repeated eigenvalues, that a Darboux polynomial can be computed by starting from every left eigenvector of matrix A .

Theorem 2.14 *Let $u^\top$ be a left eigenvector of A , i.e., $u^\top A = \lambda u^\top$. Then,*

$$\omega(x) = \begin{cases} u^\top x, & \text{if } \lambda \in \mathbb{R}, \\ (u^\top x)(u^\top x)^*, & \text{if } \lambda \notin \mathbb{R}, \end{cases}$$

is a Darboux polynomial of systems (2.1a), (2.1b).

Proof If $\lambda \in \mathbb{R}$, then

$$\begin{aligned} \Delta \omega(x) &= \Delta u^\top x = u^\top \Delta x = u^\top A x = \lambda u^\top x \\ &= \lambda \omega(x), \end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value λ . Consider now the case $\lambda \notin \mathbb{R}$. When $\mathbb{T} = \mathbb{Z}$,

$$\begin{aligned} \Delta \omega(x) &= \Delta(u^\top x) \Delta(u^{*\top} x) = (u^\top \Delta x)(u^{*\top} \Delta x) = (u^\top A x)(u^{*\top} A x) \\ &= (\lambda u^\top x)(\lambda^* u^{*\top} x) = \lambda \lambda^* (u^\top x)(u^{*\top} x) \\ &= |\lambda|^2 \omega(x), \end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value $|\lambda|^2$. When $\mathbb{T} = \mathbb{R}$,

$$\begin{aligned}\Delta\omega(x) &= (u^{*\top}x)\Delta(u^\top x) + (u^\top x)\Delta(u^{*\top}x) \\ &= (u^{*\top}x)(u^\top \Delta x) + (u^\top x)(u^{*\top} \Delta x) \\ &= (u^{*\top}x)(u^\top Ax) + (u^\top x)(u^{*\top} Ax) \\ &= \lambda(u^{*\top}x)(u^\top x) + \lambda^*(u^\top x)(u^{*\top}x) \\ &= (\lambda + \lambda^*)\omega(x) = 2\operatorname{Re}(\lambda)\omega(x),\end{aligned}$$

i.e., $\omega(x)$ is a Darboux polynomial with characteristic value $2\operatorname{Re}(\lambda)$. $\square$

Remark 2.6 Let ω_1 and ω_2 be two Darboux polynomials of systems (2.1a), (2.1b) with characteristic values λ_1 and λ_2 , respectively: $\Delta\omega_i = \lambda_i\omega_i$, $i = 1, 2$. Then, $\omega = \omega_1^{v_1}\omega_2^{v_2}$, with $v_i \in \mathbb{Z}^{\geq}$, is still a Darboux polynomial of systems (2.1a), (2.1b), with characteristic value $\lambda = v_1\lambda_1 + v_2\lambda_2$ if $\mathbb{T} = \mathbb{R}$ ($\lambda = \lambda_1^{v_1}\lambda_2^{v_2}$ if $\mathbb{T} = \mathbb{Z}$). To be more precise, if $\mathbb{T} = \mathbb{R}$, then

$$\begin{aligned}\Delta\omega &= v_1\omega_1^{v_1-1}\omega_2^{v_2}\Delta\omega_1 + v_2\omega_1^{v_1}\omega_2^{v_2-1}\Delta\omega_2 = (v_1\lambda_1 + v_2\lambda_2)\omega_1^{v_1}\omega_2^{v_2} \\ &= (v_1\lambda_1 + v_2\lambda_2)\omega,\end{aligned}$$

whereas, if $\mathbb{T} = \mathbb{Z}$, then

$$\Delta\omega = (\Delta\omega_1)^{v_1}(\Delta\omega_2)^{v_2} = \lambda_1^{v_1}\lambda_2^{v_2}\omega_1^{v_1}\omega_2^{v_2} = \lambda_1^{v_1}\lambda_2^{v_2}\omega.$$

The second part of the next remark suggests a practical way for the computation of first integrals for linear systems.

Remark 2.7 Following the same reasoning of the proof of Statement (2.11.2) of Theorem 2.11, one can demonstrate the following claims. If $\mathbb{T} = \mathbb{R}$ and I is a rational first integral of system (2.1a), then I can be factorized as $I = \prod_i \omega_i^{v_i}$, with ω_i being Darboux polynomials of system (2.1a) and v_i being (positive or negative) integers. Conversely, both in the continuous-time and discrete-time cases, if $\omega_1, \omega_2, \dots$ are Darboux polynomials of systems (2.1a), (2.1b), $\Delta\omega_i = \lambda_i\omega_i$, such that $\sum_i v_i\lambda_i = 0$ (respectively, $\prod_i \lambda_i^{v_i} = 1$), with v_i being either positive or negative integers, then $I = \prod_i \omega_i^{v_i}$ is a rational first integral of systems (2.1a), (2.1b).

Example 2.10 Constants $\lambda_i \in \mathbb{R}$ appearing in Statement (2.11.2) of Theorem 2.11 need not be eigenvalues of matrix A . For instance, if $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, then

$$\omega(x) = \det([Ax \ x]) = \det\left(\begin{bmatrix} x_1 + x_2 & x_1 \\ -x_1 + x_2 & x_2 \end{bmatrix}\right) = x_1^2 + x_2^2$$

satisfies

$$\Delta\omega = \begin{cases} [2x_1 \ 2x_2] \begin{bmatrix} x_1 + x_2 \\ -x_1 + x_2 \end{bmatrix} = 2(x_1^2 + x_2^2), & \text{if } \mathbb{T} = \mathbb{R}, \\ (F_1^2 + F_2^2)|_{F_1=x_1+x_2, F_2=x_2-x_1} = 2(x_1^2 + x_2^2), & \text{if } \mathbb{T} = \mathbb{Z}, \end{cases}$$

with 2 that is not eigenvalue of A (where $\det(A) = \text{trace}(A) = 2$).

In the next example, the concept of irreducible polynomial is needed. A polynomial with real coefficients is said to be *irreducible over $\mathbb{R}$* if it is not constant and cannot be rewritten as the product of two non-constant polynomials with real coefficients.

Example 2.11 Consider matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda_1 \lambda_2 \lambda_3 & -\lambda_1 \lambda_3 - \lambda_1 \lambda_2 - \lambda_2 \lambda_3 & \lambda_1 + \lambda_3 + \lambda_2 \end{bmatrix},$$

with $\lambda_1, \lambda_2, \lambda_3$ being scalars. If $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$, then such a matrix has three real eigenvalues λ_1, λ_2 and λ_3 with respective real left eigenvectors:

$$u_1 = \begin{bmatrix} \lambda_2 \lambda_3 \\ -\lambda_3 - \lambda_2 \\ 1 \end{bmatrix}, \quad u_2 = \begin{bmatrix} \lambda_1 \lambda_3 \\ -\lambda_1 - \lambda_3 \\ 1 \end{bmatrix}, \quad u_3 = \begin{bmatrix} \lambda_1 \lambda_2 \\ -\lambda_1 - \lambda_2 \\ 1 \end{bmatrix},$$

which are linearly independent over $\mathbb{R}$ when $\lambda_i \neq \lambda_j, i \neq j$. Two linear symmetries of Ax , such that $\det(\Omega) \neq 0$, are given by Ex and A^2x , thus yielding

$$\begin{aligned} \omega(x) &= \det(\Omega(x)) = \det([Ax \ Ex \ A^2x]) \\ &= (\lambda_2 \lambda_3 x_1 - (\lambda_3 + \lambda_2)x_2 + x_3)(\lambda_1 \lambda_3 x_1 - (\lambda_1 + \lambda_3)x_2 + x_3) \\ &\quad \times (\lambda_1 \lambda_2 x_1 - (\lambda_1 + \lambda_2)x_2 + x_3) \\ &= (u_1^\top x)(u_2^\top x)(u_3^\top x). \end{aligned}$$

Note that if $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$, for some constants $\alpha, \beta \in \mathbb{R}, \beta \neq 0$, then the two complex factors $(u_1^\top x)(u_2^\top x)$ lead to a real Darboux polynomial associated with Ax , irreducible over $\mathbb{R}$,

$$\begin{aligned} \omega_4 &= (u_1^\top x)(u_2^\top x) \\ &= (2\lambda_3\alpha + \lambda_3^2 + \alpha^2 + \beta^2)x_2^2 + (-2\lambda_3^2\alpha - 2\lambda_3\beta^2 - 2\lambda_3\alpha^2)x_1x_2 \\ &\quad + (-2\lambda_3 - 2\alpha)x_2x_3 + (\lambda_3^2\alpha^2 + \lambda_3^2\beta^2)x_1^2 + 2\lambda_3\alpha x_1x_3 + x_3^2, \end{aligned}$$

such that $\Delta\omega_4 = \lambda\omega_4$, where $\lambda = \lambda_1 + \lambda_2 = 2\alpha$ if $\mathbb{T} = \mathbb{R}$ (respectively, $\lambda = \lambda_1\lambda_2 = \alpha^2 + \beta^2$ if $\mathbb{T} = \mathbb{Z}$). Moreover, note that the same factor $(u_i^\top x)$ appears more than once as a factor of $\omega(x)$ in case of multiple eigenvalues.

Example 2.12 Consider again matrix A of Example 2.11 with $\lambda_1 = \lambda_2 = \lambda_3 = 0$. Let

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Clearly, Ex and Mx are two linear symmetries of Ax . Let

$$\omega(x) = \det([Ax \ Ex \ Mx]) = \det\left(\begin{bmatrix} x_2 & x_1 & x_1 \\ x_3 & x_2 & 3x_2 \\ 0 & x_3 & 5x_3 \end{bmatrix}\right) = -2x_3(2x_1x_3 - x_2^2);$$

let $\omega_1(x) = x_3$ and $\omega_2(x) = 2x_1x_3 - x_2^2$. Clearly, $\omega_1(x) = [0 \ 0 \ 1]x$, where $[0 \ 0 \ 1]$ is a left eigenvector of A with eigenvalue $\lambda = 0$, is a Darboux polynomial associated with Ax , both in the continuous-time and discrete-time cases. Furthermore, since

$$L_{Ax}\omega_2(x) = [2x_3 \ -2x_2 \ 2x_1] \begin{bmatrix} x_2 \\ x_3 \\ 0 \end{bmatrix} = 0,$$

ω_2 is another Darboux polynomial of the continuous-time system (not corresponding to any left eigenvector), whereas since

$$\omega_2 \circ Ax = (2F_1F_3 - F_2^2)|_{F_1=x_2, F_2=x_3, F_3=0} = -x_3^2,$$

the factor ω_2 of ω is not a Darboux polynomial of the discrete-time system.

Remark 2.8 Example 2.12 shows that, although there is a strong relationship between left eigenvectors and Darboux polynomials associated with Ax , there may exist Darboux polynomials of (2.1a), (2.1b) that are not generated by left eigenvectors.

Theorem 2.15 Assume that $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$. Let $\Omega(x) = [A^0x \ A^1x \ \dots \ A^{n-1}x]$ and $\omega = \det(\Omega)$. Then, all linear systems $\Delta x = Bx$, with $B \in \mathcal{L}_c(A)$, $B \neq 0$, have ω as a Darboux polynomial.

Proof If $B \in \mathcal{L}_c(A)$, $B \neq 0$, then $B = \sum_{i=0}^{n-1} \mu_i A^i$, with $\mu_j \neq 0$ for at least one index j . Let $\hat{\Omega}(x) = [Bx \ A^1x \ \dots \ A^{n-1}x]$ and $\hat{\omega} = \det(\hat{\Omega})$; assume that $\hat{\omega} \neq 0$, otherwise define $\hat{\Omega}$ as the matrix obtained from Ω by replacing one of the last $n-1$ columns with Bx . By construction, $\hat{\omega}$ is a Darboux polynomial of $\Delta x = Bx$, and in addition

$$\begin{aligned} \hat{\omega}(x) &= \det([Bx \ A^1x \ \dots \ A^{n-1}x]) = \det\left(\left[\sum_{i=0}^{n-1} \mu_i A^i x \ A^1x \ \dots \ A^{n-1}x\right]\right) \\ &= \det([\mu_0 A^0x \ A^1x \ \dots \ A^{n-1}x]) = \mu_0 \det([A^0x \ A^1x \ \dots \ A^{n-1}x]) \\ &= \mu_0 \omega(x), \end{aligned}$$

where $\mu_0 \neq 0$ by $\hat{\omega} \neq 0$. □

By Corollary 2.1, recall that if either A is semi-simple with distinct eigenvalues or if A is a Jordan block, then $\{A^0, A^1, \dots, A^{n-1}\}$ is a basis of $\mathcal{L}_c(A)$.

Example 2.13 Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$; a basis of $\mathcal{L}_c(A)$ is $\{A^0, A^1\}$; hence, any $B \in \mathcal{L}_c(A)$ can be written as $B = \mu_0 A^0 + \mu_1 A^1 = \begin{bmatrix} \mu_0 & \mu_1 \\ -\mu_1 & \mu_0 \end{bmatrix}$, for constant $\mu_0, \mu_1 \in \mathbb{R}$. Then, letting $\Omega(x) = [A^0 x \ A^1 x] = \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}$, one computes $\omega(x) = \det(\Omega(x)) = -(x_1^2 + x_2^2)$. Under the assumption that $\mu_0 \neq 0$, letting

$$\hat{\Omega}(x) = [Bx \ A^1 x] = \begin{bmatrix} \mu_0 x_1 + \mu_1 x_2 & x_2 \\ -\mu_1 x_1 + \mu_0 x_2 & -x_1 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_0(x_1^2 + x_2^2) \neq 0$; similarly, if $\mu_1 \neq 0$, letting

$$\hat{\Omega}(x) = [A^0 x \ Bx] = \begin{bmatrix} x_1 & \mu_0 x_1 + \mu_1 x_2 \\ x_2 & -\mu_1 x_1 + \mu_0 x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_1(x_1^2 + x_2^2) \neq 0$. This shows that all systems having a dynamic matrix in $\mathcal{L}_c(A)$ share the same Darboux polynomial $x_1^2 + x_2^2$.

Example 2.14 Let $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$; A is not semi-simple. The linear centralizer $\mathcal{L}_c(A)$ has dimension two and $\mathcal{L}_c(A) = \text{span}_{\mathbb{R}}\{A^0, A^1\}$. Any $B \in \mathcal{L}_c(A)$ can be expressed as $B = \mu_0 A^0 + \mu_1 A^1 = \begin{bmatrix} \mu_0 + \mu_1 & \mu_1 \\ -\mu_1 & \mu_0 - \mu_1 \end{bmatrix}$, for constant $\mu_0, \mu_1 \in \mathbb{R}$. Then, by $\Omega(x) = [A^0 x \ A^1 x] = \begin{bmatrix} x_1 & x_1 + x_2 \\ x_2 & -x_1 - x_2 \end{bmatrix}$, one computes $\omega(x) = \det(\Omega(x)) = -(x_1 + x_2)^2$. Under the assumption that $\mu_0 \neq 0$, letting

$$\hat{\Omega}(x) = [Bx \ A^1 x] = \begin{bmatrix} (\mu_0 + \mu_1)x_1 + \mu_1 x_2 & x_1 + x_2 \\ -\mu_1 x_1 + (\mu_0 - \mu_1)x_2 & -x_1 - x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_0(x_1 + x_2)^2 \neq 0$; similarly, under the assumption that $\mu_1 \neq 0$, letting

$$\hat{\Omega}(x) = [A^0 x \ Bx] = \begin{bmatrix} x_1 & (\mu_0 + \mu_1)x_1 + \mu_1 x_2 \\ x_2 & -\mu_1 x_1 + (\mu_0 - \mu_1)x_2 \end{bmatrix},$$

one computes $\hat{\omega}(x) = \det(\hat{\Omega}(x)) = -\mu_1(x_1 + x_2)^2 \neq 0$. This shows how all systems having a dynamic matrix belonging to $\mathcal{L}_c(A)$ have $(x_1 + x_2)^2$ as Darboux polynomial ($x_1 + x_2$ in the continuous-time case).

Remark 2.9 Assume that the Jordan form of A has not two Jordan blocks corresponding to the same eigenvalue, i.e., that $A^0, A^1, \dots, A^{n-1}$ are linearly independent over $\mathbb{R}$; since $[A^i, A^j] = 0$, by the analysis carried out in Sect. 1.6, one concludes that the rows of $\Omega^{-1}(x) = [A^0 x \ A^1 x \ \dots \ A^{n-1} x]^{-1}$ are exact one-forms. Then, the diffeomorphism $y = \varphi(x)$ such that $\frac{\partial \varphi}{\partial x} = \Omega^{-1}$ satisfies $L_{A^i x} \varphi = e_{i+1}$, with e_i being the i th column of the $n \times n$ identity matrix E . This can be useful to compute $n - 1$ independent first integrals of $\frac{dx}{dt} = Ax$, when A is not semi-simple, as illustrated in the following example.

Example 2.15 Let

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix};$$

then,

$$\Omega(x) = [A^0 x \ A^1 x \ A^2 x] = \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_3 & 0 \\ x_3 & 0 & 0 \end{bmatrix}.$$

The rows of Ω^{-1} are exact one-forms and yield, by integration, the diffeomorphism $y = \varphi(x)$, with

$$\varphi(x) = \begin{bmatrix} \ln(|x_3|) \\ \frac{x_2}{x_3} \\ \frac{x_1}{x_3} - \frac{1}{2} \frac{x_2^2}{x_3^2} \end{bmatrix};$$

it is not difficult to see that the first and last entry of $\varphi(x)$ are functionally independent first integrals of $\frac{dx}{dt} = Ax$.

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