Chapter 2
Dynamic Surface Control

2.1 Motivation

As mentioned earlier, dynamic surface control (DSC) has been developed to overcome an "explosion of terms", which is generally problematic in integrator backstepping control, through the use of dynamic filters [28]. Its design procedure can applied to both a Lipschitz and a non-Lipschitz nonlinear system, and its existence for semi-global stability was shown in [94]. However, a systematic method to choose appropriate gains and filter time constants for a dynamic surface controller has not been fully addressed yet in the literature. Typically, their values may be determined through somewhat heuristic methods, e.g. high gain assignment, and these heuristic methods do not always guarantee stability. Moreover, there is no analytic way to check the stability for the given gains and filter time constants without numerous simulations.

In order to illustrate the DSC design procedure, as well as the choice of surface gains and filter time constants, consider the following example:

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^2 := x_2 + f_1(x_1), \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u. \end{aligned} \tag{2.1}$$

The goal is to make x_1 track the desired value, $x_{1d}(t) = \sin t$. Let

$$S_1 = x_1 - x_{1d}(t).$$

Then, after taking a derivative and using (2.1),

$$\dot{S}_1 = \dot{x}_1 - \dot{x}_{1d} = x_2 + f_1 - \dot{x}_{1d}.$$

Using the idea of a low-pass filter as in the DSC design procedure [37], consider the following choice of $\bar{x}_2$ and introduce a first-order low-pass filter as follows:

$$\bar{x}_2 := \dot{x}_{1d}(t) - f_1 - K_1 S_1,$$
$$\tau_2 \dot{x}_{2d} + x_{2d} = \bar{x}_2, \quad x_{2d}(0) := \bar{x}_2(0).$$

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Then, after defining the second sliding surface as

$$S_2 = x_2 - x_{2d},$$

similarly we can choose $\bar{x}_3$ and the second low-pass filter as follows:

$$\begin{aligned} \bar{x}_3 &:= \dot{x}_{2d} - K_2 S_2, \\ \tau_3 \dot{x}_{3d} + x_{3d} &= \bar{x}_3, \quad x_{3d}(0) = \bar{x}_3(0). \end{aligned} \tag{2.2}$$

Finally, the control input u is calculated with the definition of the third sliding surface

$$\begin{aligned} S_3 &= x_3 - x_{3d}, \\ \dot{S}_3 &= u - \dot{x}_{3d}, \\ u &= \dot{x}_{3d} - K_3 S_3 = \frac{\bar{x}_3 - x_{3d}}{\tau_3} - K_3 S_3 \end{aligned}$$

where the last equality comes from (2.2).

Next, we need to assign the set of controller gains, K_i and τ_i, to guarantee stability and/or ultimate error boundedness. As discussed for the second-order nonlinear system in Sect. 1.5, similarly it is guaranteed that there exists a set of controller gains to make the errors of the closed-loop system ultimately bounded. However, we need to find the specific controller gains to guarantee stability. Suppose $K_i = 50$ for $i = 1, 2, 3$, and $\tau_i = 0.021$ for $i = 2, 3$ for simplicity. Then, the time responses of x_1, S_1, and u are shown in Fig. 2.1. As noticed in the plot of S_1, the first surface error is diverging, thus the system becomes unstable. However, when either $\tau_i = 0.019$ or $K_i = 47$, which is slightly smaller than the original gain set, the system becomes stable and the error is bounded on a small boundary, as shown in Fig. 2.2. These results show that design of the controller gains itself is an important issue in the framework of dynamic surface control even though the design procedure is well structured.

In general, instability of the closed-loop system with DSC may happen by two main causes: one is due to inappropriate gain assignment and the other is when the given controller cannot compensate for model uncertainty or disturbances. In this chapter, we will focus on the development of a systematic procedure of stability and performance analysis for design of the set of controller gains without considering the uncertainty. Model uncertainty and disturbances are considered in Chap. 3. One of challenging problems regarding design of the controller gains is to determine the filter time constants (τ_i). If the gain assignment scheme (1.34) introduced in the example of Sect. 1.5 is used, we need to know M, which is an upper bound of η, and η is in general highly nonlinear with respect to error subspaces, S_i and ξ_i and depends on K_i and τ_i as well. One ad hoc approach to design τ_i is to choose the value as small as possible. However, the filter time constant cannot be reduced arbitrarily in most applications due to hardware performance limitations, such as the control sampling frequency in a real-time implementation. Furthermore, it is desired in some cases to increase the surface gains, K_i, to compensate for the uncertainty. Thus it results in an increase of M and thus requires smaller filter time constants.

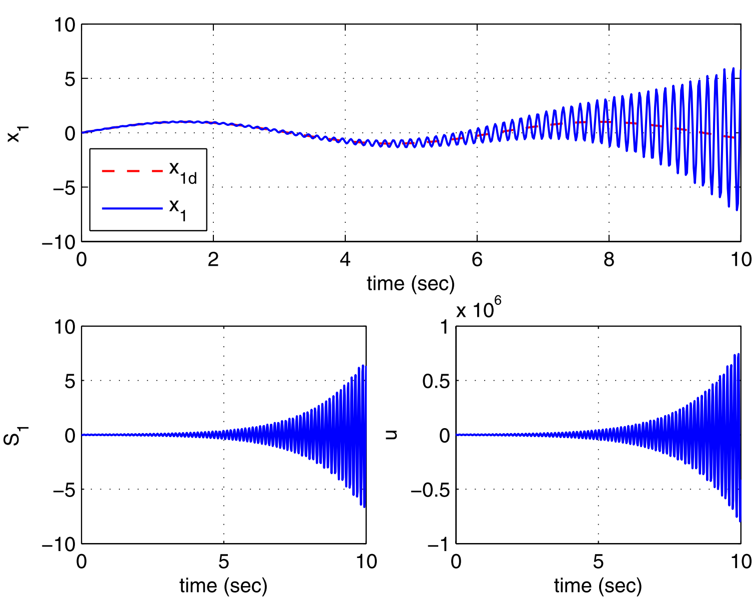

Fig. 2.1 Tracking performance and control input of DSC with $K_i = 50$ and $\tau_i = 0.021$

Therefore, we need a compromise for choosing the surface gains (K_i) and filter time constants (τ_i) to balance the goals of stability and robustness.

The remainder of this chapter is divided as follows: Sect. 2.3 outlines the development of the DSC controller structure for the stabilization problem to the system given in Sect. 2.2. Then, it is shown in Section 2.4 that the closed-loop error dynamics is derived and regarded as a linear system subject to a perturbation. Once the error dynamics is derived with the given bound assumption of the perturbation, the analysis and design method for the stabilization problem is proposed in the framework of an LMI approach in Sect. 2.5. Finally, Sect. 2.6 will discuss the performance analysis of the closed-loop nonlinear system for the tracking problem in the term of quadratic ultimate boundedness.

2.2 Problem Statement

Consider the class of nonlinear systems

$$
\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1),\\
\dot{x}_2 &= x_3 + f_2(x_1, x_2),\\
&\vdots\\
\dot{x}_{n-1} &= x_n + f_{n-1}(x_1, \ldots, x_{n-1}),\\
\dot{x}_n &= u + f_n(x_1, \ldots, x_n)
\end{aligned}
\tag{2.3}
$$

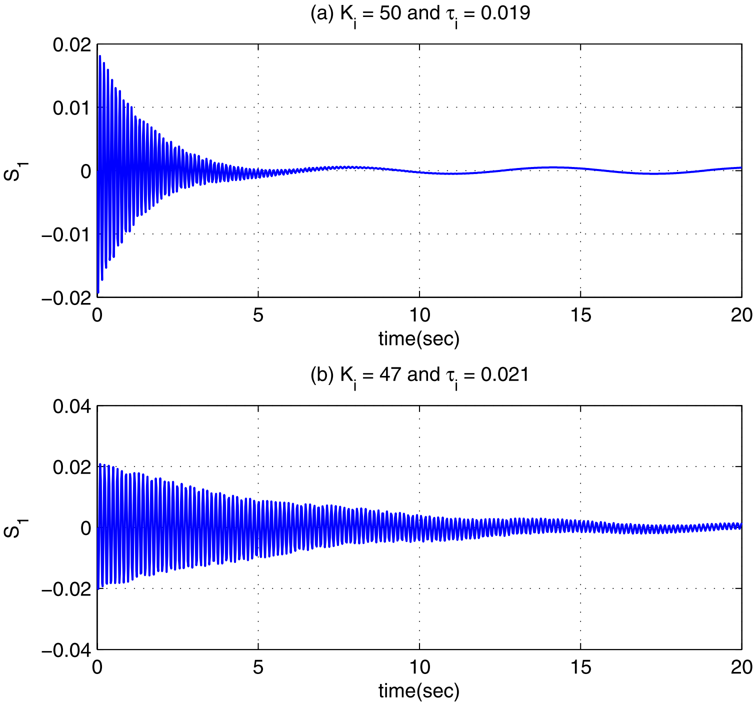

Fig. 2.2 Tracking performance of DSC with different gain sets

where $f(x)$ and $[\partial f(x)/\partial x]$ are continuous on $\mathscr{D} \in \Re^n$, and $f_i : \mathscr{D} \to \Re$ is in *strict-feedback* form in the sense that the f_i depend only on $x_1, \ldots, x_i$ [55]. Then, it is implied that f is locally Lipschitz in x on $\mathscr{D}$ and the existence and uniqueness of the solution of (2.3) are guaranteed [51]. Furthermore, by continuity, $[\partial f(x)/\partial x]$ is bounded on $\mathscr{D}_i$ which is convex and compact, and contained in $\mathscr{D}$. Therefore, there exists a constant $\gamma > 0$ such that

$$\left\| \frac{\partial f(x)}{\partial x} \right\| = \left\| J(x) \right\| \leq \gamma \tag{2.4}$$

for all x in a convex subset $\mathscr{D}_i \subset \mathscr{D}$ where $J(x)$ is a Jacobian matrix of f in the form of a lower triangular matrix, i.e.,

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \frac{\partial f_n}{\partial x_3} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}.$$

Without loss of generality, we will assume that $f_i(0, \ldots, 0) = 0$. If $f_i(0, \ldots, 0) = f_{i0} \neq 0$, any equilibrium point can be shifted to the origin via a change of variables [51]. That is, by defining new variables $y_i = x_i - f_{i0}$, the derivatives of y_i is given by

$$\dot{y}_i = x_{i+1} + f_i(x_1 + f_{10}, \ldots, x_i + f_{i0}) \triangleq y_{i+1} + g_i(y_1, \ldots, y_i)$$

where $g_i(0, \ldots, 0) = 0$. Furthermore, it is assumed that f_i is a nonlinear function. If the f_i includes linear combinations of x_i, the linear part can be isolated from f_i and the class of systems can be then rewritten as

$$\dot{x} = Ax + Bu + \bar{f}(x) \tag{2.5}$$

where the matrix $A = U + \bar{A}$ is a *lower-Hessenberg* matrix, i.e., a matrix whose elements above the first super-diagonal are all zero, the matrix $U = \mathrm{diag}([1, \ldots, 1], 1)$ is a square matrix whose first super-diagonal elements are one and elsewhere zero, the matrix $\bar{A} = f(x) - \bar{f}(x)$ is the linear part of $f(x)$ in the strict-feedback form, and $B = [0 \cdots 0\ 1]^T \in \Re^n$.

2.3 Design Procedure

The standard design procedure for the DSC which stabilizes the Lipschitz nonlinear system is described in [94]. An outline of this procedure is as follows: Define the first error surface as $S_1 := x_1 - x_{1d}$ where x_{1d} is the desired value as the control objective, e.g., $x_{1d} = \dot{x}_{1d} = 0$ for the stabilization problem. After taking the time derivative of S_1 and using (2.3),

$$\dot{S}_1 = x_2 + f_1(x_1) - \dot{x}_{1d}. \tag{2.6}$$

The surface error S_1 will converge to zero if $S_1 \dot{S}_1 < 0$, however there is no direct control over the surface dynamics. But if x_2 is considered as the forcing term for the surface dynamics, then the sliding condition outside some boundary layer is satisfied if $x_2 = \bar{x}_2$ where

$$\bar{x}_2 = \dot{x}_{1d} - f_1(x_1) - K_1 S_1. \tag{2.7}$$

Consequently, the next step is to force $x_2 \to \bar{x}_2$, so define $S_2 := x_2 - x_{2d}$ where x_{2d} equals $\bar{x}_2$ passed through a first-order low-pass filter, i.e.,

$$\tau_2 \dot{x}_{2d} + x_{2d} = \bar{x}_2, \quad x_{2d}(0) := \bar{x}_2(0). \tag{2.8}$$

Similarly, if we choose $\bar{x}_3$ as

$$\bar{x}_3 = \dot{x}_{2d} - f_2(x_1, x_2) - K_2 S_2 \tag{2.9}$$

and force $x_3 \to \bar{x}_3$. Continuing this process for each consecutive state, define the $(i-1)$th error surface as $S_{i-1} = x_{i-1} - x_{(i-1)d}$ and $\bar{x}_i$ is

$$\bar{x}_i = \dot{x}_{(i-1)d} - f_{i-1}(x_1, \ldots, x_{i-1}) - K_{i-1} S_{i-1}. \tag{2.10}$$

x_{id} is obtained by filtering $\bar{x}_i$, i.e.,

$$\tau_i \dot{x}_{id} + x_{id} = \bar{x}_i, \quad x_{id}(0) := \bar{x}_i(0). \tag{2.11}$$

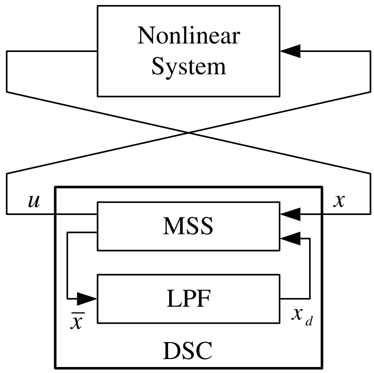

Fig. 2.3 Schematic structure of DSC systems

After continuing this procedure for $2 \le i \le n-1$, define $S_n := x_n - x_{nd}$. Finally, the control input is chosen as

$$\begin{aligned} u &= \dot{x}_{nd} - f_n(x_1, \ldots, x_n) - K_n S_n \\ &= \frac{\bar{x}_n - x_{nd}}{\tau_n} - f_n(x_1, \ldots, x_n) - K_n S_n. \end{aligned} \tag{2.12}$$

It is noted that $\dot{x}_{nd}$ is replaced by $(\bar{x}_n - x_{nd})/\tau_n$ based on (2.11).

The structural design procedure of DSC can be summarized using the graphical representation as shown in Fig. 2.3. In words, the DSC is composed of two blocks: multiple sliding surface (MSS) and low-pass filter (LPF) blocks. As introduced in Sect. 1.4, the MSS block defines multiple sliding surfaces (S_i) and calculates the forcing state values ($\bar{x}_i$), based on the state information as well as the filtered signal (x_{id}). Then, all the forcing state values pass through the LPF block and the corresponding filtered values go back into the MSS. After n iterations, the control input(u) is calculated and fed into the system.

Remark 2.1 The design procedure of DSC can be applied to a more general class of nonlinear system such as [29]

$$\begin{aligned} \dot{x}_1 &= g_1(x_1)x_2 + f_1(x_1), \\ \dot{x}_2 &= g_2(x_1, x_2)x_3 + f_2(x_1, x_2), \\ &\vdots \\ \dot{x}_n &= g_n(x_1, \ldots, x_n)u + f_n(x_1, \ldots, x_n) \end{aligned} \tag{2.13}$$

where $f_i : \mathscr{D}_i \to \Re$, $g_i : \mathscr{D}_i \to \Re$. Then, (2.10) is rewritten as

$$\bar{x}_i = [\dot{x}_{(i-1)d} - f_{i-1} - K_{i-1}S_{i-1}]/g_{i-1}.$$

Then the control input is rewritten as

$$u = [\dot{x}_{nd} - f_n - K_n S_n]/g_n = \left[\frac{\bar{x}_n - x_{nd}}{\tau_n} - f_n - K_n S_n\right] \Big/ g_n.$$

2.4 Augmented Error Dynamics

Once the design procedure is applied, the closed-loop error dynamics needs to be derived for stability analysis. By subtracting and adding $\bar{x}_{i+1}$ and $x_{(i+1)d}$ in x_{i+1}, and $\dot{x}_{nd}$ in u, respectively, they can be rewritten as $x_{i+1} = \bar{x}_{i+1} + [x_{i+1} - x_{(i+1)d}] + [x_{(i+1)d} - \bar{x}_{i+1}]$ and $u = \dot{x}_{nd} + (u - \dot{x}_{nd})$. Then, the system equation in the right hand side terms of (2.3) can be written as

$$\begin{aligned} \dot{x}_i &= \bar{x}_{i+1} + [x_{i+1} - x_{(i+1)d}] + [x_{(i+1)d} - \bar{x}_{i+1}] + f_i \quad \text{for } i = 1, \ldots, n-1, \\ \dot{x}_n &= \dot{x}_{nd} + (u - \dot{x}_{nd}) + f_n. \end{aligned}$$

By use of (2.10) and the definition of error surfaces, the above equations can be described as the error equations of DSC as follows:

$$\begin{aligned} \dot{S}_1 &= -K_1 S_1 + S_2 + (x_{2d} - \bar{x}_2), \\ &\vdots \\ \dot{S}_{n-1} &= -K_{n-1} S_{n-1} + S_n + (x_{nd} - \bar{x}_n), \\ \dot{S}_n &= -K_n S_n. \end{aligned} \tag{2.14}$$

In addition, we need to consider the augmented error dynamics due to inclusion of a set of the first-order low-pass filters. Let us define the filter error as $\xi_i := x_{id} - \bar{x}_i$ for $2 \le i \le n$. Then, the filter dynamics is

$$\dot{\xi}_i = \dot{x}_{id} - \dot{\bar{x}}_i = -\xi_i/\tau_i - \dot{\bar{x}}_i \tag{2.15}$$

where the last equality comes from (2.11). By differentiating (2.10), we can write $\dot{\bar{x}}_i$ as

$$\begin{aligned} \dot{\bar{x}}_2 &= -\dot{f}_1 + \ddot{x}_{1d} - K_1 \dot{S}_1, \\ \dot{\bar{x}}_i &= -\dot{f}_{i-1} + \ddot{x}_{(i-1)d} - K_{i-1} \dot{S}_{i-1} \\ &= -\dot{f}_{i-1} - \dot{\xi}_{i-1}/\tau_{i-1} - K_{i-1} \dot{S}_{i-1} \end{aligned}$$

for $i = 3, \ldots, n$. Combining (2.15) with (2.16), we have the filter error dynamics,

$$\begin{aligned} \dot{\xi}_2 - K_1 \dot{S}_1 &= -\frac{\xi_2}{\tau_2} + \dot{f}_1 - \ddot{x}_{1d}, \\ \dot{\xi}_3 - \frac{\dot{\xi}_2}{\tau_2} - K_2 \dot{S}_2 &= -\frac{\xi_3}{\tau_3} + \dot{f}_2, \\ &\vdots \\ \dot{\xi}_n - \frac{\dot{\xi}_{n-1}}{\tau_{n-1}} - K_{n-1} \dot{S}_{n-1} &= -\frac{\xi_n}{\tau_n} + \dot{f}_{n-1}. \end{aligned} \tag{2.16}$$

Therefore, the overall error dynamics including both the nth-order closed-loop nonlinear system in (2.14) and $(n-1)$th-order low-pass filter error equations in (2.16) can be given as

$$T\dot{z} = A_z z + \bar{B}_w \dot{f} + \bar{B}_e \ddot{x}_{1d} \tag{2.17}$$

where the error state $z \in \Re^{n_z}$ and $w \in \Re^{n_w}$ are

$$\begin{aligned} z &= [S\ \xi]^T = [S_1\ \cdots\ S_n\ \xi_2\ \cdots\ \xi_n]^T \in \Re^{2n-1}, \\ S &= [S_1\ \cdots\ S_n], \qquad \xi = [\xi_2\ \cdots\ \xi_n], \\ \dot{f} &= [\dot{f}_1\ \cdots\ \dot{f}_{n-1}]^T \in \Re^{n-1}, \end{aligned}$$

the system matrices in the above equation are

$$T = \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -K & T_\xi \end{bmatrix}, \qquad A_z = \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix},$$
$$\bar{B}_w = \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_{n_w} \end{bmatrix}, \qquad \bar{B}_e = \begin{bmatrix} 0_n \\ -b_r \end{bmatrix},$$

and the corresponding sub-block matrices are the following:

$$T_\xi = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\frac{1}{\tau_2} & 1 & \cdots & 0 & 0 \\ 0 & -\frac{1}{\tau_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -\frac{1}{\tau_{n-1}} & 1 \end{bmatrix} = \mathbf{I}_{n-1} + \mathrm{diag}\left(\left[-\frac{1}{\tau_2}, \ldots, -\frac{1}{\tau_{n-1}}\right], -1\right),$$

$$K = \left[\mathrm{diag}(K_1, \ldots, K_{n-1})\ 0_{n-1}\right] \in \Re^{(n-1)\times n},$$

$$A_{11} = \begin{bmatrix} -K_1 & 1 & \cdots & 0 \\ 0 & -K_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & -K_n \end{bmatrix} \in \Re^{n\times n}, \qquad A_{12} = \begin{bmatrix} \mathbf{I}_{n-1} \\ 0_{n-1}^T \end{bmatrix} \in \Re^{n\times(n-1)},$$

$$A_{22} = -\,\mathrm{diag}\left(\frac{1}{\tau_2}, \ldots, \frac{1}{\tau_n}\right) \in \Re^{(n-1)\times(n-1)}, \qquad b_r = [1\ 0\ \cdots\ 0]^T \in \Re^{n_w}.$$

It is noted that $\dot{f}$ is defined as $\dot{f} = [\dot{f}_1\ \cdots\ \dot{f}_{n-1}]^T \in \Re^{n-1}$, not $\dot{f} = [\dot{f}_1\ \cdots\ \dot{f}_n]^T \in \Re^n$ since $\dot{f}_n$ does not affect the filter error dynamics as seen in (2.16). Therefore, the reduced order of Jacobian matrix J is redefined as

$$J(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & 0 & 0 & \cdots & 0 \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_{n-1}}{\partial x_1} & \frac{\partial f_{n-1}}{\partial x_2} & \frac{\partial f_{n-1}}{\partial x_3} & \cdots & \frac{\partial f_{n-1}}{\partial x_{n-1}} \end{bmatrix} \in \Re^{(n-1)\times(n-1)}$$

and without loss of generality, there exists a constant $\gamma > 0$ satisfying the inequality (2.4) for all x in a convex subset $\mathscr{D}_i \subset \mathscr{D}$.

Furthermore, since T_ξ is full rank of $n-1$, both T and T_ξ are invertible with inverses given by

$$T_\xi^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ \frac{1}{\tau_2} & 1 & \cdots & 0 & 0 \\ \frac{1}{\tau_2\tau_3} & \frac{1}{\tau_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\tau_2\cdots\tau_{n-1}} & \frac{1}{\tau_3\cdots\tau_{n-1}} & \cdots & \frac{1}{\tau_{n-1}} & 1 \end{bmatrix},$$

we can compute the inverse matrix of T using the following property in linear algebra: if the matrices X and Z are square and invertible,

$$\begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}^{-1} = \begin{bmatrix} X^{-1} & 0 \\ -Z^{-1}YX^{-1} & Z^{-1} \end{bmatrix}$$

so that

$$T^{-1} = \begin{bmatrix} I_n & 0 \\ T_\xi^{-1}K & T_\xi^{-1} \end{bmatrix}.$$

Then, after multiplying T^{-1} to both sides in (2.17), the augmented closed-loop error dynamics is rewritten as

$$\dot{z} = A_{cl}z + B_w\dot{f} + B_e\ddot{x}_{1d} \tag{2.18}$$

where

$$A_{cl} = T^{-1}A_z = \begin{bmatrix} A_{11} & A_{12} \\ T_\xi^{-1}KA_{11} & T_\xi^{-1}(KA_{12}+A_{22}) \end{bmatrix} \in \Re^{n_z\times n_z},$$

$$B_w = T^{-1}\bar{B}_w = \begin{bmatrix} \mathbf{0}_{n\times n_w} \\ T_\xi^{-1} \end{bmatrix} \in \Re^{n_z\times n_w}, \qquad B_e = T^{-1}\bar{B}_e = \begin{bmatrix} 0_n \\ -T_\xi^{-1}b_r \end{bmatrix} \in \Re^{n_z}.$$

Finally, after decoupling $\dot{f}$ in (2.18) into a vanishing and a nonvanishing term, i.e., the nonvanishing term may not be zero for $z=0$, while the vanishing term becomes zero when $z=0$, the augmented error dynamics is summarized as follows:

Lemma 2.1 *For the given class of nonlinear system* (2.3), *the augmented closed-loop error dynamics with DSC is*

$$\begin{aligned} \dot{z} &= A_{cl}z + B_w w + B_r r, \\ \|w\| &\le \gamma\|C_z z\| \end{aligned} \tag{2.19}$$

where

$$\dot{f} = [\dot{f}_1 \ \cdots \ \dot{f}_{n-1}]^T = \frac{\partial f}{\partial x}\dot{x} = J(x)C_z z + J_1\dot{x}_{1d} = w + J_1\dot{x}_{1d} \in \Re^{n_w},$$

$$w = J(x)C_z z, \quad J = \frac{\partial f}{\partial x}, \ J_1 \text{ is the first column of } J,$$

$$r = \begin{bmatrix} J_1 \dot{x}_{1d} \\ \ddot{x}_{1d} \end{bmatrix} \in \Re^{n_w+1} := \Re^{n_r},$$

$$B_r = \begin{bmatrix} \mathbf{0}_{n \times n_w} & 0_n \\ T_\xi^{-1} & -T_\xi^{-1} b_r \end{bmatrix} = [B_w \ B_e] \in \Re^{n_z \times n_r},$$

and $C_z = [\underline{A}_{11} \ T_\xi] \in \Re^{n_w \times n_z}$ *where the notation* $\underline{A}_{11}$ *is the matrix where the last row of the matrix* A_{11} *is eliminated.*

The proof is given in Appendix A.1. It is worth noting that the perturbation term, w, only affects filter error dynamics since the first block matrix in the matrix B_w in (2.18) is a zero matrix. That is, the augmented error dynamics is also rewritten as

$$\begin{aligned} \dot{z} &= T^{-1}(A_z + \bar{B}_w J C_z) z + T^{-1} \bar{B}_r r \\ &= A(x) z + B_r r \end{aligned} \tag{2.20}$$

where

$$A(x) = T^{-1}(A_z + \bar{B}_w J C_z) = \begin{bmatrix} \mathbf{I}_n & \mathbf{0}_{n(n-1)} \\ T_\xi^{-1} K & T_\xi^{-1} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ J(x)\underline{A}_{11} & A_{22} + J(x) T_\xi \end{bmatrix}$$
$$= \begin{bmatrix} A_{11} & A_{12} \\ T_\xi^{-1}(K A_{11} + J \underline{A}_{11}) & T_\xi^{-1}(K A_{12} + A_{22} + J T_\xi) \end{bmatrix}.$$

Example 2.1 (Derivation of closed-loop error dynamics) Let us consider the third-order nonlinear system as follows:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_1^2 := x_2 + f_1(x_1), \\ \dot{x}_2 &= x_3 - x_1 x_2^2 := x_3 + f_2(x_1, x_2), \\ \dot{x}_3 &= u \end{aligned} \tag{2.21}$$

where f_i are locally Lipschitz on $\Re^3$ since it is continuously differentiable on $\Re^3$. However, it is not globally Lipschitz, since $[\partial f / \partial x]$ is not bounded on $\Re^3$. On any compact subset of $\Re^3$, f_i is Lipschitz. The Jacobian matrix is given by

$$J(x) := \left[\frac{\partial f}{\partial x} \right] = \begin{bmatrix} 2x_1 & 0 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}.$$

Suppose that the control objective is $x_1 \to x_{1d}(t)$. If the design procedure of DSC in Sect. 2.3 is applied to the system, the first error surface, synthetic input $\bar{x}_2$, and first-order low-pass filter are

$$\begin{aligned} S_1 &:= x_1 - x_{1d}, \\ \dot{S}_1 &= \dot{x}_1 - \dot{x}_{1d} = x_2 + f_1 - \dot{x}_{1d}, \\ \bar{x}_2 &:= \dot{x}_{1d} - f_1 - K_1 S_1, \\ \tau_2 \dot{x}_{2d} + x_{2d} &= \bar{x}_2, \quad x_{2d}(0) = \bar{x}_2(0). \end{aligned}$$

Similarly, the synthetic input $\bar{x}_3$ and control input u are

$$
\begin{aligned}
S_2 &:= x_2 - x_{2d}, \\
\dot{S}_2 &= \dot{x}_2 - \dot{x}_{2d} = x_3 + f_2 - \dot{x}_{2d}, \\
\bar{x}_3 &:= \dot{x}_{2d} - f_2 - K_2 S_2, \\
\tau_3 \dot{x}_{3d} + x_{3d} &= \bar{x}_3, \quad x_{3d}(0) = \bar{x}_3(0), \\
S_3 &:= x_3 - x_{3d}, \\
\dot{S}_3 &= u - \dot{x}_{3d}, \\
u &:= \dot{x}_{3d} - K_2 S_3.
\end{aligned}
$$

Then, the augmented error dynamics is

$$
\dot{z} = A_{cl} z + B_w \dot{f} + B_d \ddot{x}_{1d} \tag{2.22}
$$

where $z = [S_1\ S_2\ S_3\ \xi_2\ \xi_3]^T \in \Re^5$, $\dot{f} = [\dot{f}_1\ \dot{f}_2]^T \in \Re^2$, and the matrices are

$$
A_{cl} = \left[\begin{array}{ccc|cc}
-K_1 & 1 & 0 & 1 & 0 \\
0 & -K_2 & 1 & 0 & 1 \\
0 & 0 & -K_3 & 0 & 0 \\
\hline
-K_1^2 & K_1 & 0 & K_1 - \frac{1}{\tau_2} & 0 \\
-\frac{K_1^2}{\tau_2} & \frac{K_1}{\tau_2} - K_2^2 & K_2 & \frac{K_1}{\tau_2} - \frac{1}{\tau_2^2} & K_2 - \frac{1}{\tau_3}
\end{array}\right],
$$

$$
B_w = \left[\begin{array}{cc}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\hline
1 & 0 \\
\frac{1}{\tau_2} & 1
\end{array}\right], \qquad
B_d = \left[\begin{array}{c}
0 \\
0 \\
0 \\
\hline
-1 \\
-\frac{1}{\tau_2}
\end{array}\right].
$$

Furthermore, $\dot{f}$ is

$$
\begin{aligned}
\dot{f}_1 &= J_{11}\{S_2 + \xi_2 - K_1 S_1\} + J_{11}\dot{x}_{1d} = J_{11} c_{z1} z + J_{11}\dot{x}_{1d} = \bar{J}_1 C_z z + J_{11}\dot{x}_{1d}, \\
\dot{f}_2 &= J_{21}(c_{z1} z + \dot{x}_{1d}) + J_{22} c_{z2} z = [J_{21}\ J_{22}] \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} z + J_{21}\dot{x}_{1d} = \bar{J}_2 C_z z + J_{21}\dot{x}_{1d}
\end{aligned}
$$

where

$$
C_z := \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} = \begin{bmatrix} -K_1 & 1 & 0 & 1 & 0 \\ 0 & -K_2 & 1 & -\frac{1}{\tau_2} & 1 \end{bmatrix}.
$$

Therefore,

$$
\dot{f} = J C_z z + J_1 \dot{x}_{1d}
$$

where $J_1 = [J_{11}\ J_{21}]^T \in \Re^2$.

Finally, by use of Lemma 2.1, the closed-loop error dynamics can be written as

$$
\begin{aligned}
\dot{z} &= A_{cl} z + B_w w + B_r r, \\
w &= J C_z z
\end{aligned}
$$

where $r = [J_{11}\dot{x}_{1d}\ J_{21}\dot{x}_{1d}\ \ddot{x}_{1d}]^T \in \Re^3$ and

$$B_r = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 1 & 0 & -1 \\ \frac{1}{\tau_2} & 1 & -\frac{1}{\tau_2} \end{bmatrix}.$$

If $\|\cdot\|_\infty$ for vectors in $\Re^3$ and the induced matrix norm for matrices are used, we have

$$\|w\|_\infty = \|JC_z z\|_\infty \le \left\|J(x)\right\|_\infty \|C_z z\|_\infty$$

where

$$\left\|J(x)\right\|_\infty = \max\left\{2|x_1|, \left|x_2^2\right| + 2|x_1 x_2|\right\}.$$

If we are interested in the tracking problem over the convex set $\mathscr{D}_\delta = \{x \in \Re^3 | |x_1| \le \delta_1, |x_2| \le \delta_2\}$, all points in $\mathscr{D}_\delta$ satisfy

$$2|x_1| \le 2\delta_1, \qquad \left|x_2^2\right| + 2|x_1 x_2| \le \delta_2^2 + 2\delta_1\delta_2.$$

Hence,

$$\|w\|_\infty \le \gamma \|C_z z\|_\infty$$

where $\gamma := 2\delta_1 + \delta_2^2 + 2\delta_1\delta_2$.

2.5 Quadratic Stabilization

If either stabilization (i.e., $x_{1d} = 0$) or regulation (i.e., $x_{1d} = c \ne 0$) problem is considered, the r in (2.19) is zero since $\dot{x}_{1d} = \ddot{x}_{1d} = 0$. Based on Lemma 2.1, the augmented closed-loop error dynamics becomes

$$\dot{z} = A_{cl} z + B_w w, \quad w = JC_z z. \tag{2.23}$$

Moreover, $J(x)$ can be written as a nonlinear function of z, i.e., $J(x) = G(z)$ because x can be expressed as a function of z as follows [94]:

$$\begin{aligned} x_1 &= S_1 := \eta_1(S_1), \\ x_2 &= S_2 + \xi_2 + \bar{x}_2 = S_2 + \xi_2 - K_1 S_1 - f_1\left\{\eta_1(S_1)\right\} := \eta_2(S_1, S_2, \xi_2). \end{aligned}$$

By induction, for $3 \le i \le n$,

$$\begin{aligned} x_i &= S_i + \xi_i + \bar{x}_i = S_i + \xi_i - \frac{\xi_{i-1}}{\tau_{i-1}} - K_{i-1} S_{i-1} - f_{i-1}(\eta_1, \dots, \eta_{i-1}) \\ &:= \eta_i(S_1, \dots, S_i, \xi_2, \dots, \xi_i) \end{aligned}$$

where η_i are continuous nonlinear functions and $\eta_i(0, \dots, 0) = 0$. Therefore, the error dynamics for the stabilization or regulation problem can be described graphically as shown in Fig. 2.4.

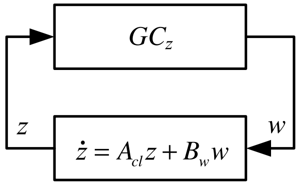

Fig. 2.4 Graphical interpretation for augmented closed-loop error dynamics

Definition 2.1 Let $z = 0$ be an exponentially stable equilibrium point of the nominal system (2.25) when A_{cl} is Hurwitz for the given set of controller gains, $\Theta = \{K_1, \ldots, K_n, \tau_2, \ldots, \tau_n\}$. Then, a nonlinear system (2.3) is *quadratically stabilizable* via DSC if there exists a positive definite matrix P such that

$$\frac{d}{dt}V(z) = \frac{d}{dt}\left(z^T P z\right) = (A_{cl}z + B_w w)^T P z + z^T P (A_{cl}z + B_w w) < 0. \tag{2.24}$$

It is important to note that we are interested in finding a quadratic Lyapunov function which will be calculated in the framework of linear matrix inequality (LMI) and convex optimization although there may exist a different type of Lyapunov function to guarantee the stability. Therefore, we will investigate how the quadratic Lyapunov function satisfying LMI (2.24) can be found for different types of bounds of w, which is the vanishing perturbation. If a quadratic Lyapunov function exists for this system, then the system is said to be *quadratically stable*. The concept of quadratic stabilization was introduced for the robust stabilization of uncertain linear systems by linear feedback [6, 70]. These results can be applied to the error dynamics above as long as the perturbation term w is bounded by a linear function of the augmented error z.

2.5.1 Nominal Error Dynamics

For the stabilization problem, the augmented error dynamics (2.23) can be regarded as linear nominal error dynamics,

$$\dot{z} = A_{cl}z \tag{2.25}$$

subject to a nonlinear perturbation function w, which is the vanishing perturbation in the sense that $w = 0$ for $z = 0$ from $w = JC_z z$ for all x [51]. If the matrix A_{cl} is Hurwitz; that is, $\mathrm{Re}\{\lambda_i(A_{cl})\} < 0$ for all eigenvalues of A_{cl}, the nominal system (2.25) is exponentially stable at $z = 0$. This is well known in linear system theory.

The assignment of $\lambda(A_{cl})$ in (2.25) can be considered as a generalized eigenvalue problem as follows:

$$\begin{aligned} \lambda v = A_{cl}v \quad &\Longrightarrow \quad \lambda T v = A_z v \quad \Longrightarrow \quad (\lambda T - A_z)v = 0 \\ &\Longrightarrow \quad \det(\lambda T - A_z) = 0. \end{aligned}$$

Moreover, using the definitions of T and A_z in (2.17), it can be written as

$$\lambda T - A_z = \lambda \begin{bmatrix} \mathbf{I}_n & \mathbf{0} \\ -K & T_\xi \end{bmatrix} - \begin{bmatrix} A_{11} & A_{12} \\ \mathbf{0} & A_{22} \end{bmatrix} = \begin{bmatrix} \lambda \mathbf{I}_n - A_{11} & -A_{12} \\ -\lambda K & \lambda T_\xi - A_{22} \end{bmatrix},$$
$$\det(\lambda T - A_z) = \det\left(\begin{bmatrix} \lambda \mathbf{I}_n - A_{11} & -A_{12} \\ -\lambda K & \lambda T_\xi - A_{22} \end{bmatrix}\right) = 0. \tag{2.26}$$

Since the nth row of $\lambda T - A_z$ has only one nonzero element such as $\lambda + K_n$ in the nth column, $-K_n$ is always one of eigenvalues of A_{cl}. After using this fact and eliminating the nth row and column of $\lambda T - A_z$, (2.26) is equivalent to

$$\det\left(\begin{bmatrix} \lambda \mathbf{I}_{n-1} - \tilde{A}_{11} & -\mathbf{I}_{n-1} \\ -\lambda \tilde{K} & \lambda T_\xi - A_{22} \end{bmatrix}\right) = 0 \tag{2.27}$$

where

$$\tilde{K} = \operatorname{diag}(K_1, \ldots, K_{n-1}) \in \Re^{(n-1)\times(n-1)},$$
$$\tilde{A}_{11} = \begin{bmatrix} -K_1 & 1 & \cdots & 0 \\ 0 & -K_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \cdots & -K_{n-1} \end{bmatrix} \in \Re^{(n-1)\times(n-1)}.$$

Furthermore, using linear algebra, (2.27) can be converted into computation of a lower dimension matrix. The basic idea is as follows: if the matrices A and D are square, and D is invertible,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & BD^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ C & D \end{bmatrix},$$
$$\det\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\right) = \det(A)\det(C)$$

so that

$$\det\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) = \det\left(A - BD^{-1}C\right)\det(D).$$

Therefore, if $(\lambda T_\xi - A_{22})$ is invertible, i.e.

$$\det(\lambda T_\xi - A_{22}) \neq 0 \quad \text{or} \quad \det\left(\lambda I - T_\xi^{-1} A_{22}\right) \neq 0,$$

(2.27) is equivalent to

$$\det\left[\lambda \mathbf{I}_{n-1} - \tilde{A}_{11} - \lambda(\lambda T_\xi - A_{22})^{-1}\tilde{K}\right] = 0 \quad \text{or}$$
$$\det\left[\lambda \mathbf{I}_{n-1} - \tilde{A}_{11} - \lambda\left(\lambda I - T_\xi^{-1} A_{22}\right)^{-1} T_\xi^{-1}\tilde{K}\right] = 0. \tag{2.28}$$

It is noted that $T_\xi^{-1}A_{22}$ is a lower triangular matrix such that

$$T_\xi^{-1}A_{22} = -\begin{bmatrix} \frac{1}{\tau_2} & 0 & \cdots & 0 & 0 \\ \frac{1}{\tau_2^2} & \frac{1}{\tau_3} & \cdots & 0 & 0 \\ \frac{1}{\tau_2^2\tau_3} & \frac{1}{\tau_3^2} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{1}{\tau_2^2\cdots\tau_{n-1}} & \frac{1}{\tau_3^2\cdots\tau_{n-1}} & \cdots & \frac{1}{\tau_{n-1}^2} & \frac{1}{\tau_n} \end{bmatrix}$$

so the eigenvalues of $T_\xi^{-1}A_{22}$ are diagonal elements such as

$$\lambda_i\left(T_\xi^{-1}A_{22}\right) = -\frac{1}{\tau_i}.$$

Therefore, either $(\lambda T_\xi - A_{22})$ or $(\lambda I - T_\xi^{-1}A_{22})$ is invertible if $\lambda \neq -1/\tau_i$ for $i = 2, \ldots, n$. Thus, (2.28) can be used as long as $\lambda \neq -1/\tau_i$.

Remark 2.2 If we consider a special class of the systems (2.5), i.e., linear systems, they becomes

$$\dot{x} = Ax + Bu \tag{2.29}$$

where the matrix A is a *lower-Hessenberg* matrix described in (2.5). Then, the closed-loop error dynamics is described as

$$\dot{z} = \tilde{A}_{cl}z \tag{2.30}$$

where the matrix $\tilde{A}_{cl}$ is

$$A_{cl} = \begin{bmatrix} A_{11} & A_{12} \\ T_\xi^{-1}\tilde{K}A_{11} & T_\xi^{-1}(\tilde{K}A_{12} + A_{22}) \end{bmatrix} \in \Re^{n_z \times n_z}$$

and

$$\tilde{K} = K + (\underline{A} - U)$$

where $\underline{A}$ is the matrix the last row of the matrix $A - U$ is eliminated, i.e., $(A - U)_{1:n-1,1:n}$. Therefore, the linear system in (2.29) is *quadratically stabilizable* via DSC if there is a set of controller gains, Θ, such that $\tilde{A}_{cl}$ in (2.30) is Hurwitz.

Example 2.2 (Eigenvalues of nominal error dynamics) If the second-order nonlinear system is controlled by DSC with a set of gains $\Theta = \{K_1, K_2, \tau_2\}$ as was done in Sect. 1.5, A_{cl} in (1.36) for different gains K_1 and K_2 is rewritten as

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -K_1 & 0 & 1 \end{bmatrix}, \qquad A_z = \begin{bmatrix} -K_1 & 1 & 1 \\ 0 & -K_2 & 0 \\ 0 & 0 & -1/\tau_2 \end{bmatrix}.$$

Following (2.26), the characteristic equation is

$$\det(\lambda T - A_z) = \det\left(\begin{bmatrix} \lambda + K_1 & -1 & -1 \\ 0 & \lambda + K_2 & 0 \\ -\lambda K_1 & 0 & \lambda + 1/\tau_2 \end{bmatrix}\right) = 0.$$

Since $-K_2$ is the eigenvalue of A_{cl}, the characteristic equation for the reduced-order nominal error dynamics can be written as follows: using (2.27),

$$\det\left(\begin{bmatrix} \lambda + K_1 & -1 \\ -\lambda K_1 & \lambda + \frac{1}{\tau_2} \end{bmatrix}\right) = 0,$$
$$(\lambda + K_1)(\lambda + 1/\tau_2) - K_1\lambda = 0$$

or using (2.28), if $\lambda \neq -1/\tau_2$,

$$(\lambda + K_1) - (\lambda + 1/\tau_2)^{-1}\lambda K_1 = 0,$$
$$\lambda^2 + \frac{1}{\tau_2}\lambda + \frac{K_1}{\tau_2} = 0.$$

Therefore, A_{cl} is Hurwitz as long as $\tau_2 > 0$.

Example 2.3 (Stabilization of nominal error dynamics) Consider the nonlinear system in (2.1) whose the control objective is $x_1 \to 0$. Then, if DSC is applied, the augmented error dynamics is described as follows:

$$\dot{z} = A_{cl} z$$

subject to a nonlinear perturbation w where $z = [S_1\ S_2\ S_3\ \xi_2\ \xi_3]^T \in \Re^5$ and the matrix A_{cl} is derived in (2.22). If $\lambda \neq -1/\tau_i$ for $i = 2, 3$, $\lambda = -K_2$ is one of eigenvalues of A_{cl}. Then, using (2.28), the characteristic equation for the reduced-order nominal error dynamics is

$$\det\left[\begin{pmatrix} \lambda + K_1 & -1 \\ 0 & \lambda + K_2 \end{pmatrix} - \begin{pmatrix} \lambda + \frac{1}{\tau_2} & 0 \\ \frac{1}{\tau_2^2} & \lambda + \frac{1}{\tau_3} \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ \frac{1}{\tau_2} & 1 \end{pmatrix} \begin{pmatrix} \lambda K_1 & 0 \\ 0 & \lambda K_2 \end{pmatrix}\right]$$
$$= 0.$$

Although one may derive the characteristic equation algebraically, it can be also calculated using Symbolic Math Toolbox™ and MATLAB® [98] and used for higher-order systems. As shown in the MATLAB Program 2-1 below, the computed equation is

$$\frac{\tau_2\tau_3\lambda^4 + (\tau_2 + \tau_3)\lambda^3 + (K_2\tau_2 + 1)\lambda^2 + (K_1 + K_2)\lambda + K_1K_2}{(\tau_2\lambda + 1)(\tau_3\lambda + 1)} = 0.$$

Therefore, inequality conditions of Θ for A_{cl} to be Hurwitz may be derived analytically if the Routh stability criterion, which is introduced in most classical feedback control textbooks, is used.

Suppose that the given set of controller gains is $K = K_1 = K_2 = K_3$ and $\tau = \tau_2 = \tau_3$ for simplicity. Then, the above characteristic equation is rewritten as

$$\frac{\tau^2\lambda^4 + 2\tau\lambda^3 + (K\tau + 1)\lambda^2 + 2K\lambda + K^2}{(\tau\lambda + 1)^2} = 0.$$

Using the Routh stability criterion, the inequality condition for A_{cl} to be Hurwitz is derived to be

$$\tau K < 1 \tag{2.31}$$

for $\lambda \neq -1/\tau$. As used in Sect. 2.1, let $K = 50$ and $\tau = 0.021$, not satisfying (2.31). Then, $\lambda(A_{cl})$ have $\lambda(A_{cl}) = \{-50.0, -48.207 \pm i7.5522, 0.588 \pm i48.7915\}$, thus A_{cl} is not Hurwitz. Therefore, the error vector, z, is diverging due to the eigenvalues with positive real part. Consequently, the control input u is also diverging (see Fig. 2.1). However, when either the surface gains are changed to $K = 49$ or filter time constants to $\tau = 0.019$, the set of controller gains satisfy the inequality condition (2.31) and the corresponding eigenvalues are

$$\lambda(A_{cl}) = \{-51.2998, -43.6278, -47.0, -0.1553 \pm i47.3083\}$$

for $K = 47$ and

$$\lambda(A_{cl}) = \{-60.2603, -43.6702, -50.0, -0.6663 \pm i51.2946\}$$

for $\tau = 0.019$, respectively. Therefore, A_{cl} is Hurwitz for both cases and it is expected that the error vector converges to the origin for the nominal error dynamics.

MATLAB Program 2-1

```
%***** Generation of the symbolic objects *****
x = sym('x');
K1 = sym('K1');  K2 = sym('K2');
tau2 = sym('tau2');  tau3 = sym('tau3');

%***** Symbolic math computation *****
K = [x+K1  -1; 0  x+K2 ];
P = [x+1/tau2  0; 1/tau2^ 2  x+1/tau3];
Q = [1 0; 1/tau2 1];
R = [x*K1 0; 0 x*K2];
f = det(K - inv(P)*Q*R)
```

2.5.2 Norm-Bounded Error Dynamics

According to [37], it is said that semi-global stability of the resulting closed-loop system under the DSC is guaranteed by showing the existence of a set of controller gains for exponential regulation of DSC for Lipschitz nonlinear systems, and the corresponding proof is given in [110]. However, as motivated in Sect. 1.5, determination of the controller gains is not straightforward since it requires the upper bound of a highly nonlinear function of S_i, ξ_i, K_i, and τ_i, e.g., see the function η in (1.30). Although higher surface gains and lower filter time constants are preferable ideally, the filter time constant cannot be arbitrarily small in most applications due to hardware limitation and the surface gains cannot be arbitrarily large because they may result in input saturation to relatively small surface errors due to uncertainties and/or disturbances.

In this section, the LMI approach will be proposed in the framework of the augmented error dynamics which will allow us to calculate the Lyapunov function candidate $V(z) = z^T P z$ where $P > 0$ while the existence of the Lyapunov function candidate $V(z) = \frac{1}{2}z^T z$ is shown for stability in [37]. Using Lemma 2.1, the closed-loop error dynamics for the stabilization or regulation problem is written as

$$\begin{cases} \dot{z} = A_{cl}z + B_w w, \\ w = J(x)C_z z, \\ \|w\| \le \gamma \|C_z z\| := \|\tilde{C}_z z\| \end{cases} \tag{2.32}$$

where γ is a Lipschitz constant on the convex set $\mathscr{D}_i \in \mathscr{D}$ and $\tilde{C}_z = \gamma C_z$. It is noted that the calculation of γ is straightforward and simpler than the upper bound of the nonlinear function since $J(x)$ is a function of x in a convex domain $\mathscr{D}_i \subset \mathscr{D}$.

Quadratic stability of the augmented error dynamics under the DSC is guaranteed by the following theorem.

Theorem 2.1 *Suppose that the closed-loop error dynamics (2.32) is given for the given set of controller gains, $\Theta = \{K_1, \dots, K_n, \tau_2, \dots, \tau_n\}$, for all x in a domain $\mathscr{D}$. If there exist $P > 0$ and $\sigma \ge 0$ such that*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + \sigma \tilde{C}_z^T \tilde{C}_z & P B_w \\ B_w^T P & -\sigma I \end{bmatrix} < 0, \tag{2.33}$$

the origin in (2.32) is then exponentially stable in $\mathscr{D}$. Thus the nonlinear system (2.3) is quadratically stabilizable via DSC with the given Θ on $\mathscr{D}$.

Proof Suppose the quadratic Lyapunov function $V(z) = z^T P z$ satisfies

$$\dot{V}(z) = z^T \left(A_{cl}^T P + P A_{cl}\right) z + w^T B_w^T P z + z^T P B_w w < 0 \tag{2.34}$$

for all nonzero z. Then it is claimed that this is equivalent to

$$\dot{V}(z) = \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_w \\ B_w^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0 \tag{2.35}$$

for all nonzero z and w satisfying $w^T w \le z^T \tilde{C}_z^T \tilde{C}_z z$. To show the equivalence, we need to show that the set

$$\mathscr{A} = \left\{(z, w) | z \ne 0\right\}$$

equals the set

$$\mathscr{B} = \left\{(z, w) | (z, w) \ne 0, \|w\| \le \left\|\tilde{C}_z z\right\|\right\}.$$

It suffices to show that $\{(z, w) | z = 0, w \ne 0, \|w\| \le \|\tilde{C}_z z\|\} = \emptyset$, which is an empty set. Therefore, the inequality condition that $\dot{V}(z) < 0$ for all nonzero z is equivalent to (2.35) for any nonzero (z, w) satisfying

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} -\tilde{C}_z^T \tilde{C}_z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0. \tag{2.36}$$

The S-procedure can then be used to give a sufficient condition for a quadratic constraint to be satisfied given that some other quadratic constraints are also satisfied. The S-procedure can be summarized as follows [12]: Given matrices P_i for $i = 0, \dots, n$, then the condition

$$x^T P_0 x < 0 \quad \text{for all } x \ne 0 \quad \text{such that} \quad x^T P_i x < 0, \quad i = 1, \dots, n$$

holds if $\exists \sigma_i \geq 0$ for $i = 1, \ldots, n$ such that

$$x^T P_0 x - \sum_{i=1}^{n} \sigma_i x^T P_i x < 0, \quad \forall x \neq 0.$$

Using the S-procedure, the inequality condition (2.34) for the quadratic stability is equivalent to the existence of $P > 0$ and $\sigma \geq 0$ satisfying

$$z^T \left(A_{cl}^T P + P A_{cl}\right) z + 2 z^T P B_w w - \sigma \left\{ w^T w - (\tilde{C}_z z)^T (\tilde{C}_z z) \right\} < 0 \tag{2.37}$$

or

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_w \\ B_w^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} - \sigma \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} -\tilde{C}_z^T \tilde{C}_z & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0$$

which is equivalent to LMI (2.33). □

For a specific class of nonlinear functions f, quadratic stability is guaranteed globally as follows:

Corollary 2.1 *Suppose f is continuous and globally Lipschitz in $\Re^n$. If there exist $P > 0$ and $\sigma \geq 0$ satisfying LMI* (2.33) *for the given set of controller gains Θ, the origin in* (2.23) *is globally exponentially stable. Thus the nonlinear system* (2.3) *is globally quadratically stabilizable via DSC with the given Θ.*

Since f_i are known functions, $\dot{f}_i$ can be derived mathematically and the upper bound of $\|\dot{f}\|$ can be computed for the given convex set as done in Example 2.1. That is, we first need to define the convex and compact set $\mathscr{D}_i \subset \mathscr{D}$ to calculate γ. Then, the quadratic stability analysis can be performed using Theorem 2.1 for the given γ. However, it may be more interesting to estimate the maximum value of γ to guarantee the quadratic stability for the given set of controller gains. This allows us to estimate the region of attraction for the stabilization problem. Since the error dynamics is regarded as a norm-bounded linear differential inclusion (LDI) [12], the region of attraction can be computed in the framework of convex optimization and the computed region is defined as the convex and compact set $\mathscr{D}_i \subset \mathscr{D}$. Then we do not need to compute γ, but are able to compute a region of attraction.

Theorem 2.2 *Suppose that the closed-loop error dynamics* (2.32) *is given for the given set of controller gains, Θ. If A_{cl} is Hurwitz, i.e., there exist $P > 0$ and $Q = Q^T > 0$ such that*

$$P A_{cl} + A_{cl}^T P = -Q, \tag{2.38}$$

and $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)\|B_w C_z\|_2}$ for $\mathscr{D}_i = \{x \in \Re^n \,|\, \|J\| \leq \gamma\} \subset \mathscr{D}$, the origin in (2.32) *is exponentially stable on $\mathscr{D}_i$. Thus a nonlinear system* (2.3) *is quadratically stabilizable via DSC with the given Θ on $\mathscr{D}_i$. Furthermore, $\mathscr{D}_i$ is the region of attraction.*

Proof Suppose the quadratic Lyapunov candidate, $V(z) = z^T Pz$, satisfies (refer to [51, §9])

$$\begin{aligned}\lambda_{\min}(P)\|z\|_2^2 \le V(z) \le \lambda_{\max}(P)\|z\|_2^2,\\ \frac{\partial V}{\partial z}A_{cl}z = -z^T Qz \le -\lambda_{\min}(Q)\|z\|_2^2,\\ \left\|\frac{\partial V}{\partial z}\right\|_2 = \left\|2z^T P\right\|_2 \le 2\|P\|_2\|z\|_2 = 2\lambda_{\max}(P)\|z\|_2.\end{aligned}$$

The derivative of $V(z)$ along the trajectories of the perturbed error dynamics satisfies

$$\begin{aligned}\dot{V}(z) &= z^T\left(A_{cl}^T P + PA_{cl}\right)z + w^T B_w^T Pz + z^T PB_w w\\ &= -z^T Qz + (B_w JC_z z)^T Pz + z^T P(B_w JC_z z)\\ &\le -\lambda_{\min}(Q)\|z\|_2^2 + 2\gamma\lambda_{\max}(P)\|B_w C_z\|_2\|z\|_2^2\end{aligned}$$

for all x in the convex set $\mathscr{D}_i = \{x \in \Re^n \mid \|J\| \le \gamma\}$. Hence, $\dot{V}(z) < 0$ if $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)\|B_w C_z\|_2}$. Therefore, the origin is exponentially stable for all x in $\mathscr{D}_i \subset \mathscr{D}$. □

Remark 2.3 It is known that the maximum of γ is calculated when $Q = I$ [51, §9], that is, $\gamma < \frac{1}{2\lambda_{\max}(P)\|B_w\|\|C_z\|}$.

There is another way to calculate the maximum value of γ which is computed in the framework of convex optimization. First, consider *quadratic stability margin* which is the largest nonnegative α for which the origin in (2.32) satisfying $w^T w \le \alpha^2 z^T C_z^T C_z z$ is exponentially stable. That is, if the inequality condition (2.37) is satisfied for all nonzero z and w on $\mathscr{D}_\alpha = \{x \in \Re^n \mid \|w\| \le \alpha\|C_z z\|\}$. Then, by defining $\tilde{P} = P/\sigma$, the inequality condition (2.37) can rewritten as

$$z^T\left(A_{cl}^T\tilde{P} + \tilde{P}A_{cl}\right)z + 2z^T\tilde{P}B_w w - \left\{w^T w - \beta(C_z z)^T(C_z z)\right\} < 0, \tag{2.39}$$

where $\beta = \alpha^2$. This inequality is equivalent to LMI (2.40). Then, the quadratic stability margin can be computed as follows:

Algorithm 2.1 Quadratic stability margin of closed-loop error dynamics

$$\begin{aligned}&\text{maximize} && \beta\\ &\text{subject to} && \tilde{P} > 0, \qquad \beta \ge 0,\\ &&& \begin{bmatrix} A_{cl}^T\tilde{P} + \tilde{P}A_{cl} + \beta C_z^T C_z & \tilde{P}B_w\\ B_w^T\tilde{P} & -I\end{bmatrix} < 0.\end{aligned} \tag{2.40}$$

To solve the above algorithm, the convex optimization programming method called `CVX` can be used [32]. It allows us to solve standard problems such as linear programs (LPs), quadratic programs (QPs), second-order cone programs (SOCPs), and semidefinite programs (SDP) and simplifies the task of specifying the problem. Furthermore, it supports two core solvers, SeDuMi [93] and SDPT3 [100]. MATLAB Program 2-2 below shows an example to program the Algorithm 2.1 using `CVX` in the framework of MATLAB.

MATLAB Program 2-2

```
%***** Define the dimension of matrices *****
n = size(Acl, 1);
m = size(Bw, 2);

%***** cvx version *****
cvx_begin
   variable P(n,n) symmetric;
   variable beta;
   maximize(beta);
   beta >= 0;
   P == semidefinite(n);
   -[Acl'*P + P*Acl + beta*Cz'*Cz P*Bw; Bw'*P -eye(m)] == semidefinite(n+m);
cvx_end
```

Example 2.4 (Region of attraction for the second-order nonlinear system) Consider the second-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 - ax_1^2 := x_2 + f_1(x_1), \\ \dot{x}_2 &= u - x_2 \end{aligned} \tag{2.41}$$

where a is a known positive constant. The control objective is $x_1(t) \to 0$. Then, after applying DSC to the system, the augmented error dynamics is

$$\begin{aligned} \dot{z} &= A_{cl} z + B_w w, \\ w &= J(x) C_z z \end{aligned} \tag{2.42}$$

where $z = [S_1 \;\; S_2 \;\; \xi_2]^T \in \Re^3$, w is

$$w = -2ax_1\dot{x}_1 = -2ax_1(S_2 + \xi_2 - K_1 S_1) := J(x)C_z z \tag{2.43}$$

where $J(x) = \partial f_1/\partial x_1 = -2ax_1$, $C_z = [-K_1 \;\; 1 \;\; 1] \in \Re^{1\times 3}$, and the matrices A_{cl}, B_w are

$$A_{cl} = \left[\begin{array}{cc|c} -K_1 & 1 & 1 \\ 0 & -K_2 & 0 \\ \hline -K_1^2 & K_1 & K_1 - \frac{1}{\tau_2} \end{array}\right], \qquad B_w = \left[\begin{array}{c} 0 \\ 0 \\ \hline 1 \end{array}\right]. \tag{2.44}$$

Moreover,

$$|w| = 2a|x_1||C_z z| \le \gamma |C_z z| \tag{2.45}$$

for the convex set $\mathscr{D}_i = \{x \in \Re^2 | |x_1| \le \frac{\gamma}{2a}\}$.

Suppose the set of controller gains is $\{K_1, K_2, \tau_2\} = \{1, 2, 0.1\}$. Then $\lambda(A_{cl}) = \{-1.127, -2, -8.873\}$ is Hurwitz and the solution of (2.38) for $Q = I$ is

$$P = \begin{bmatrix} 0.4600 & 0.1588 & 0.0400 \\ 0.1588 & 0.3412 & 0.0235 \\ 0.0400 & 0.0235 & 0.0600 \end{bmatrix}.$$

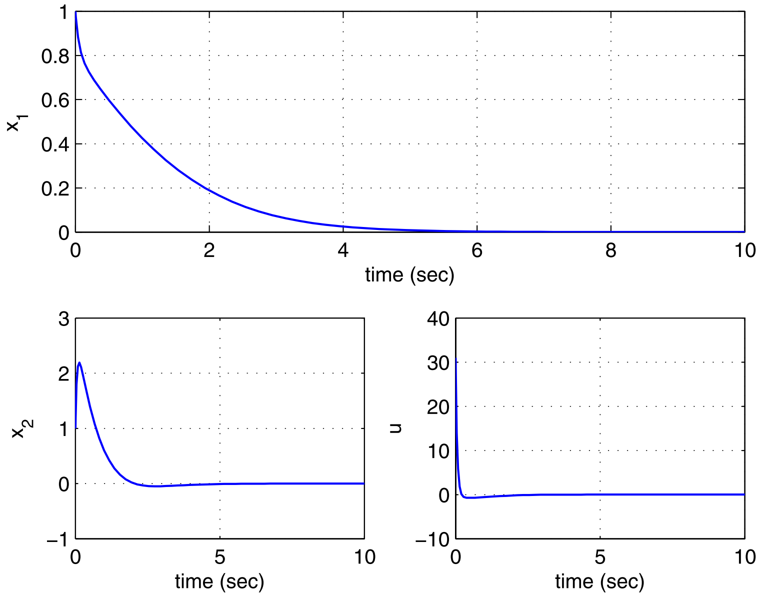

Fig. 2.5 Time responses of x_1, x_2, and u of DSC

Based on Corollary 2.2, the origin in (2.42) is exponentially stable if

$$\gamma < \frac{1}{2\lambda_{\max}(P)\|B_w C_z\|_2} = 0.5026.$$

That is, it is exponentially stable if $2a|x_1| < 0.5026$ from (2.45). Finally, we can define a region of attraction as the domain $\mathscr{D}_1 = \{x \in \Re^2 | |x_1| < 0.2513/a \text{ for } a > 0\}$ where the origin is exponentially stable.

Next, we consider the quadratic stability margin to maximize the domain in the sense that γ is maximized in the framework of convex optimization. That is, if the quadratic stability margin is calculated using Algorithm 2.1 (MATLAB Program 2-2), the solutions $\tilde{P}$ and $\alpha = \sqrt{\beta}$ are

$$\alpha = 10, \qquad \tilde{P} = \begin{bmatrix} 109.9980 & -78.5688 & -9.9998 \\ -78.5688 & 144.1719 & 21.4281 \\ -9.9998 & 21.4281 & 9.9998 \end{bmatrix}.$$

Therefore, the origin is exponentially stable if $\gamma < 10$, which is about 20 times larger than the previous result. Then we can define the larger domain such as $\mathscr{D}_2 = \{x \in \Re^2 | |x_1| < 5/a \text{ for } a > 0\}$.

Let $a = 5$ and $\{x_1(0),\ x_2(0)\} = \{1,\ 1\}$, thus x is on the boundary of $\mathscr{D}_2$. Figure 2.5 shows the time responses of x and the control input u for the given set of controller gains and initial conditions. It is shown that $x \to 0$, and $\mathscr{D}_2$ is a region of attraction.

2.5.3 Diagonal Norm-Bounded Error Dynamics

Although the perturbation terms $w = J(x)C_z z$ is norm-bounded, $\|w\| \le \gamma \|C_z z\|$, for some cases we can compute a tighter upper bound of w by calculating the componentwise upper bound of w. Suppose there are nonnegative constants γ_i such that

$$\|\bar{J}_i\| \le \gamma_i \quad \text{for all } x \in \mathscr{D}_i \subset \mathscr{D} \tag{2.46}$$

where $\bar{J}_i$ is the ith row of the Jacobian matrix J. Then, the componentwise upper bound of w is

$$\begin{aligned}
|w_i| &= \left|\sum_{j=1}^{i} J_{ij}(x) c_{zj} z\right| = \left| [J_{i1}\ J_{i2}\ \cdots\ J_{ii}] \begin{bmatrix} c_{z1} \\ c_{z2} \\ \vdots \\ c_{zi} \end{bmatrix} z \right| \\
&\le \left\| [J_{i1}\ J_{i2}\ \cdots\ J_{ii}] \right\| \left\| \begin{bmatrix} c_{z1} \\ c_{z2} \\ \vdots \\ c_{zi} \end{bmatrix} z \right\|
\end{aligned}$$

where the inequality comes from Cauchy–Schwartz inequality. Then, using (2.46),

$$|w_i| \le \gamma_i \|C_{zi} z\| := \|\tilde{C}_{zi} z\| \tag{2.47}$$

where

$$C_{zi} = \left[c_{z1}^T\ c_{z2}^T\ \cdots\ c_{zi}^T\right]^T \in \Re^{n_i \times n_z}, \qquad \tilde{C}_{zi} = \gamma_i C_{zi}.$$

Therefore, since w is bounded componentwise by the function of z, the closed-loop error dynamics is called the diagonal norm-bounded error dynamics and written as

$$\begin{cases} \dot{z} = A_{cl} z + B_w w, \\ w = J(x) C_z z, \\ \|w_i\| \le \|\tilde{C}_{zi} z\|. \end{cases} \tag{2.48}$$

Example 2.5 (Diagonal norm-bound of w) Consider the third-order nonlinear system in Example 2.1 and the control objective is $x(t) \to 0$. Then, the error dynamics is given as

$$\dot{z} = A_{cl} z + B_w w, \quad w = J C_z z$$

where

$$J(x) := \left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} 2x_1 & 0 \\ x_2^2 & 2x_1 x_2 \end{bmatrix}.$$

Then, there are nonnegative constants γ_1 and γ_2 such that

$$\begin{aligned}
|\bar{J}_1| &= |J_{11}| = |2x_1| \le 2\delta_1 := \gamma_1, \\
|\bar{J}_2| &= \left|[J_{21}\ J_{22}]\right|_2 = \sqrt{x_2^4 + 4x_1^2 x_2^2} \le 2x_1^2 + x_2^2 \le 2\delta_1^2 + \delta_2^2 := \gamma_2
\end{aligned}$$

for all $x \in \mathscr{D}_\delta = \{x \in \Re^3 | |x_1| \le \delta_1, |x_2| \le \delta_2\}$. Then,

$$|w_1| = |J_{11} c_{z1} z| \le \gamma_1 |c_{z1} z| = 2\delta_1 |c_{z1} z| := \left\| \tilde{C}_{z1} z \right\|_2,$$

$$|w_2| = |J_{21} c_{z1} z + J_{22} c_{z2} z| \le \gamma_2 \left\| \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} z \right\| := \left\| \tilde{C}_{z2} z \right\|_2$$

where $\tilde{C}_{z1} = 2\delta_1 c_{z1} \in \Re^{1\times 5}$, $\tilde{C}_{z2} = (2\delta_1^2 + \delta_2^2)\left[\begin{smallmatrix} c_{z1} \\ c_{z2} \end{smallmatrix}\right] \in \Re^{2\times 5}$, and c_{zi} are defined in Example 2.1.

Theorem 2.3 *Suppose that the diagonal norm-bounded error dynamics* (2.48) *is given for the given set of controller gains,* Θ*. The nonlinear system* (2.3) *is quadratically stabilizable via DSC for the given* Θ *on* $\mathscr{D}$ *if there exist* $P > 0$ *and* $\Sigma = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_{n_w}) \ge 0$ *such that*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + \tilde{C}_z^T \Sigma_B \tilde{C}_z & P B_w \\ B_w^T P & -\Sigma \end{bmatrix} < 0 \tag{2.49}$$

where $\tilde{C}_z = [\tilde{C}_{z1}^T, \ldots, \tilde{C}_{zn_w}^T]^T$, $\tilde{C}_{zi} \in \Re^{n_i \times n_z}$, *and* $\Sigma_B = \text{diag}(\sigma_1,\ \sigma_2 \mathbf{I}_2 \ldots,\ \sigma_{n_w} \mathbf{I}_{n_w})$ *is the diagonal block matrix.*

Proof We need to show the existence of a quadratic function that decreases along every nonzero trajectory of (2.23). Let a quadratic Lyapunov function be $V_z(t) = z(t)^T P z(t)$ where $P > 0$. The derivative of the function satisfies

$$\frac{d}{dt} V_z(t) = (A_{cl} z + B_w w)^T P z + z^T P (A_{cl} z + B_w w) < 0 \tag{2.50}$$

for all nonzero z. This is equivalent to

$$\dot{V}(z) = \begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_w \\ B_w^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} < 0$$

for all nonzero (z, w) satisfying (2.53), which is

$$\begin{bmatrix} z \\ w_i \end{bmatrix}^T \begin{bmatrix} -\tilde{C}_{zi}^T \tilde{C}_{zi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ w_i \end{bmatrix} < 0 \quad \text{for } i = 1, \ldots, n_w. \tag{2.51}$$

Using the S-procedure [12], the inequality condition (2.51) holds if there exist nonnegative constants $\sigma_1, \ldots, \sigma_{n_w}$ such that

$$z^T \left(A_{cl}^T P + P A_{cl}\right) z + 2 z^T P B_w w - \sum_{i=1}^{n_w} \sigma_i \left\{ w_i^T w_i - (\tilde{C}_{zi} z)^T (\tilde{C}_{zi} z) \right\} < 0 \tag{2.52}$$

or

$$\begin{bmatrix} z \\ w \end{bmatrix}^T \begin{bmatrix} A_{cl}^T P + P A_{cl} & P B_w \\ B_w^T P & 0 \end{bmatrix} \begin{bmatrix} z \\ w \end{bmatrix} - \sum_{i=1}^{n_w} \sigma_i \begin{bmatrix} z \\ w_i \end{bmatrix}^T \begin{bmatrix} -\tilde{C}_{zi}^T \tilde{C}_{zi} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z \\ w_i \end{bmatrix} < 0.$$

With $\Sigma = \text{diag}(\sigma_1, \sigma_2 \mathbf{I}_2, \ldots, \sigma_{n_w} \mathbf{I}_{n_w})$, this inequality is equivalent to the LMI (2.49). □

Example 2.6 (Global stabilization of a Lipschitz system) Consider the third-order globally Lipschitz nonlinear system:

$$\begin{aligned}\dot{x}_1 &= x_2 + \sin(\omega x_1) := x_2 + f_1,\\ \dot{x}_2 &= x_3 + x_1 \sin(\omega x_2) := x_3 + f_2,\\ \dot{x}_3 &= u\end{aligned} \tag{2.53}$$

where l and ω_i are the known constants. The control objective is $x_1 \to 0$. Then, after applying DSC to the system, the augmented error dynamics is

$$\dot{z} = A_{cl} z + B_w w, \quad w = J C_z z$$

where z, A_{cl}, and B_w are derived in Example 2.1, and w is

$$\begin{aligned} w &= \begin{bmatrix} \dot{f}_1 \\ \dot{f}_2 \end{bmatrix} = \begin{bmatrix} \omega \dot{x}_1 \cos(\omega x_1) \\ \dot{x}_1 \sin(\omega x_2) + \omega \dot{x}_2 \cos(\omega x_2) \end{bmatrix} = J \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} z \\ &= J C_z z \end{aligned}$$

where

$$J = \begin{bmatrix} \omega \cos(\omega x_1) & 0 \\ \sin(\omega x_2) & \omega \cos(\omega x_2) \end{bmatrix},$$

$$C_z = \begin{bmatrix} -K_1 & 1 & 0 & 1 & 0 \\ 0 & -K_2 & 1 & -\frac{1}{\tau_2} & 1 \end{bmatrix} \in \Re^{2\times 5}.$$

If the Euclidean norm is used, the norm-bound of w is

$$\|w\|_2 \le \|J\|_2 \|C_z z\|_2 \le \gamma \|C_z z\|_2.$$

Since $J(x)$ is not a constant matrix, it is not straightforward to calculate $\|J\|_2$. However, the upper bound of $\|J\|_\infty$ can be calculated using $\|\cdot\|_\infty$ of the vector norm and the induced matrix norm:

$$\|J\|_\infty = \max\left\{\left|\omega \cos(\omega x_1)\right|, \left|\sin(\omega x_2)\right| + \left|\omega \cos(\omega_2 x_2)\right|\right\} \le 1 + \omega.$$

Then using norm equivalence, γ can be calculated as

$$\|J\|_2 \le \sqrt{2}\|J\|_\infty \le \sqrt{2}(1+\omega) := \gamma.$$

If the diagonal norm-bound of w for all $x \in \Re^3$ is considered

$$\begin{aligned} |w_1| &= |J_{11} c_{z1} z| \le \omega |c_{z1} z| := \left|\tilde{C}_{z1} z\right|,\\ |w_2| &= |J_{21} c_{z1} z + J_{22} c_{z2} z| \le \left\|[J_{21}\ J_{22}]\right\|_2 \left\| \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} z \right\|_2 \\ &\le \sqrt{1+\omega^2} \left\| \begin{bmatrix} c_{z1} \\ c_{z2} \end{bmatrix} z \right\|_2 := \left\|\tilde{C}_{z2} z\right\|_2 \end{aligned}$$

where $\tilde{C}_{z1} = \omega c_{z1} \in \Re^{1\times 5}$ and $\tilde{C}_{z2} = \sqrt{1+\omega^2}\left[\begin{smallmatrix} c_{z1} \\ c_{z2} \end{smallmatrix}\right] \in \Re^{2\times 5}$.

Suppose that $\omega = 1$ for simulation. We consider the following sets of controller gains: $\Theta = \{K_1, K_2, K_3, \tau_2, \tau_3\}$ where $\{K_1, K_2, K_3\} = \{4, 6, 6\}$. Then, as derived in Example 2.3, the characteristic equation for the nominal error dynamics is

$$\tau_2 \tau_3 \lambda^4 + (\tau_2 + \tau_3)\lambda^3 + (6\tau_2 + 1)\lambda^2 + 10\lambda + 24 = 0.$$

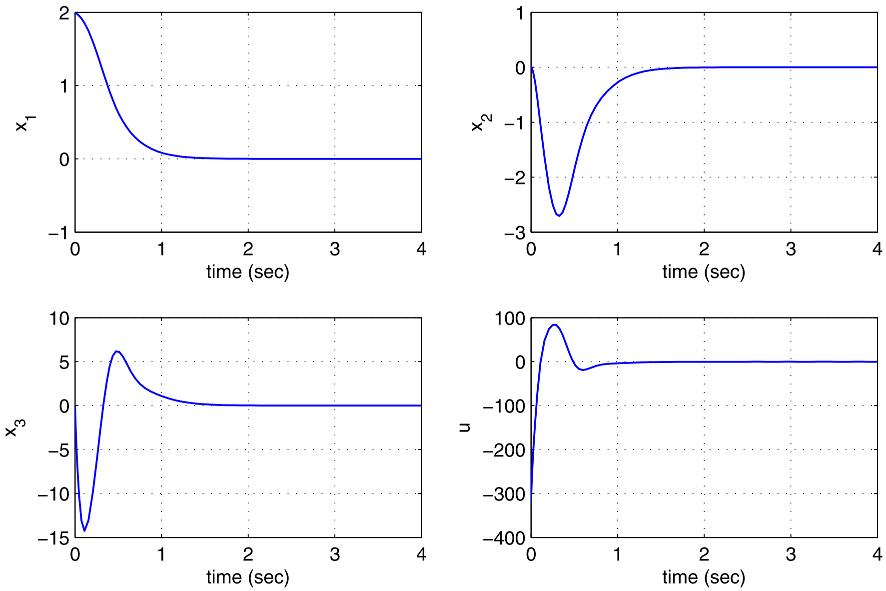

Fig. 2.6 Time responses of x and u for the given initial condition

The inequality condition with respect to τ_2 and τ_3 can be derived for A_{cl} to be Hurwitz using the Routh stability criterion. If $\tau = \tau_2 = \tau_3$, the inequality condition is $\tau \leq 0.2632$. It is interesting to note that the inequality condition is $\tau \leq 0.0263$ for A_{cl} to be Hurwitz when 10 times greater values of K_i are used in Example 2.3. It implies that higher gains K_i may allow smaller τ_i to make A_{cl} Hurwitz. It is intuitive that if higher gains are assigned, larger change of ξ_i is expected and the smaller τ_i makes ξ_i smaller.

When τ is chosen as 0.028, the eigenvalues of A_{cl} are

$$\lambda(A_{cl}) = \{-3.7872, -6, -12.0213 \pm i6.3943, -43.5988\}$$

and A_{cl} is thus Hurwitz. With respect to two different upper bounds of w, either LMI (2.33) or LMI (2.49) can be computed numerically and there exist solutions P for both cases for the given Θ_1. Therefore, it is expected that the origin of the augmented error dynamics is globally exponentially stable. For the initial conditions given as $x_1(0) = 2$ and $x_2(0) = x_3(0) = 0$, the time responses of x and u are plotted in Fig. 2.6. It is shown that x in (2.53) is stabilized as expected.

2.6 Ultimate and Quadratic Boundedness

If we consider a tracking problem such that the control objective is $x_1(t) \to x_{1d}(t)$, the augmented error dynamics from Lemma 2.1 then is

$$\dot{z} = A_{cl}z + B_w w + B_r r, \quad w = JC_z z.$$

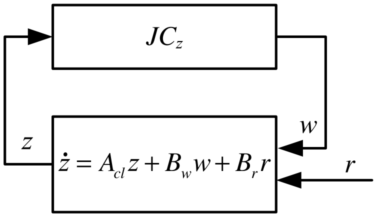

Fig. 2.7 Graphical interpretation for augmented closed-loop error dynamics

Moreover, for some compact and convex set $\mathscr{D}_i \subset \mathscr{D}$,

$$\|J\| \le \gamma$$

for all $x \in \mathscr{D}_i$, and x_{1d} is the feasible output trajectory as follows:

Definition 2.2 $x_{1d}(t)$ is a feasible output trajectory in $\mathscr{D}$ if $x_{1d}(t)$ is a C^2 function and $[x_{1d}\ \dot{x}_{1d}\ \ddot{x}_{1d}]^T$ is uniformly bounded on the convex and compact set $\mathscr{D}_i \subset \mathscr{D}$.

Suppose the convex and compact set $\mathscr{D}_i := \{x | \|x\| \le c,\ c > 0\}$ is defined in $\mathscr{D}$. Then, r is bounded as

$$\|r\|_2^2 = \left\| \begin{bmatrix} J_1 & 0_{n_w} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_{1d} \\ \ddot{x}_{1d} \end{bmatrix} \right\|_2^2 = J_1^T J_1 \dot{x}_{1d}^2 + \ddot{x}_{1d}^2 \le \left(\gamma^2 + 1\right) c^2 := r_0^2. \tag{2.54}$$

Therefore, the augmented error dynamics for the tracking problem is

$$\begin{cases} \dot{z} = A_{cl} z + B_w w + B_r r, & w = J C_z z, \\ \|w\| \le \gamma \|C_z z\|, & \|r\| \le r_0. \end{cases} \tag{2.55}$$

As shown in Fig. 2.7, r is not the vanishing perturbation in the sense that $r \ne 0$ for $z = 0$ while w is regarded as the vanishing perturbation. If the error dynamics includes the nonvanishing perturbation, we can no longer study stability of the origin in the error dynamics and expect that $z(t) \to 0$ as $t \to \infty$. We may hope that $z(t)$ will be ultimately bounded by a small bound if the nonvanishing perturbation, r, is small in some sense. This concept of ultimate boundedness in [51] is applied to the augmented error dynamics in (2.55) and its ultimate bound will be approximated in the form of a quadratic function, $z^T P z$, using the concept of *quadratic boundedness* in [14, 15].

Definition 2.3 Suppose $V(z)$ is a continuously differentiable, positive definite function and a set $\Omega_c = \{z \in \Re^{n_z} | V(z) \le c\}$ is compact for some $c > 0$. Let $\Delta = \{z | \varepsilon \le V(z) \le c\}$ for some positive constant $\varepsilon < c$. If the derivative of V along the trajectories of the error dynamics (2.19) satisfies

$$\dot{V}(z) \le -W(z), \quad \forall z \in \Delta$$

where $W(z)$ is a continuous positive definite function. A set $\Omega_\varepsilon = \{z | V(z) \le \varepsilon\}$ is an uniform ultimate error bound if there exists a positive constant c, and for every $\delta \in (0, c)$ there is a positive constant $T = T(\delta)$ such that

$$z(t_0) \in \Omega_\delta \implies z(t) \in \Omega_\varepsilon, \quad \forall t \ge t_0 + T$$

where $\Omega_\delta = \{z \in \Re^{n_z} | V(z) \le \delta\}$.

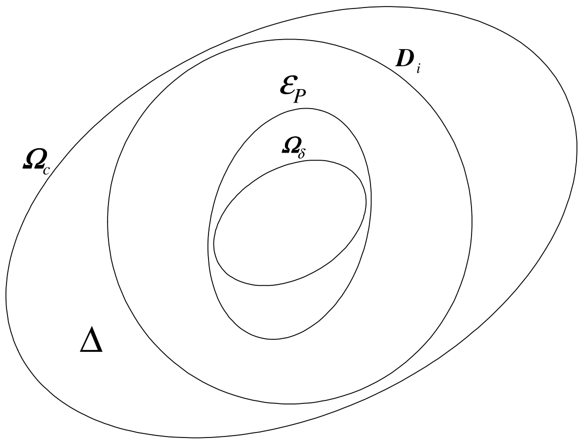

Fig. 2.8 Graphical interpretation of domain

Definition 2.4 Suppose a set of controller gains, Θ, is given, $\|r\| \neq 0$, and x_{id} in r is the feasible output trajectory. The nonlinear system (2.3) is *quadratically trackable* via DSC for a feasible output trajectory if the error dynamics (2.19) is *quadratically bounded with Lyapunov matrix* P, i.e., if there exists $P > 0$ such that

$$z^T Pz > 1 \quad \text{implies } (A_{cl}z + B_w w + B_r r)^T Pz + z^T P(A_{cl}z + B_w w + B_r r) < 0$$

for all nonzero $z \in \mathscr{E}_P = \{z \in \Re^{n_z} | z^T Pz \leq 1\}$. Then, it has the following properties for the set $\mathscr{E}_P$:

(i) The set $\mathscr{E}_P$ is *controlled invariant* via DSC, i.e. if for all $z(0) \in \mathscr{E}_P$ the solution $z(t) \in \mathscr{E}_P$ for all $t > 0$.
(ii) The set $\mathscr{E}_P$ contains the *reachable* set from the origin, i.e., if $z(t)$ is any solution with $z(0) = 0$, then $z(t) \in \mathscr{E}_P$ for all $t > 0$.
(iii) The set $\mathscr{E}_P$ is the uniform ultimate error bound.

All domains defined in Definitions 2.2 to 2.4 are described graphically in Fig. 2.8. In this section, we will compute the smallest quadratic error bound ($\mathscr{E}_P$) in some sense which contains the ultimate error bound (Ω_δ) when the x_{1d} is the feasible trajectory, i.e., $[x_{1d}\ \dot{x}_{1d}\ \dot{x}_{1d}]^T \in \mathscr{D}_i$.

Example 2.7 (Boundedness of the closed-loop error dynamics) Consider the system (2.53) in Example 2.6 with no uncertainty. Suppose the control objective is $x_1 \to x_{1d}(t) = \sin t$. If DSC is applied, we obtain the error dynamics as follows:

$$\begin{aligned} \dot{z} &= A_{cl}z + B_w w + B_r r, \\ w &= JC_z z \end{aligned}$$

where z, A_{cl}, B_w are defined in Example 2.1, and w, J, C_z are derived in Example 2.6. Moreover,

$$r = \begin{bmatrix} J_1 \dot{x}_{1d} \\ \ddot{x}_{1d} \end{bmatrix} = \begin{bmatrix} \omega\cos(\omega x_1)\dot{x}_{1d} \\ \sin(\omega x_2)\dot{x}_{1d} \\ \ddot{x}_{1d} \end{bmatrix}, \qquad B_r = \begin{bmatrix} 0 & 0 & 0 & 1 & 1/\tau_2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1/\tau_2 \end{bmatrix}^T.$$

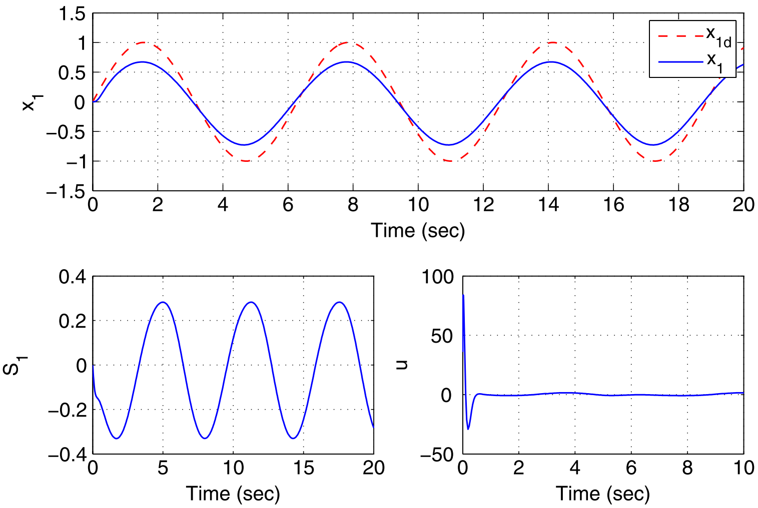

Fig. 2.9 Tracking performance and control input of DSC for Θ_1

Since f is globally Lipschitz and $\dot{x}_{1d}^2 + \ddot{x}_{1d}^2 = 1$,

$$\|r\|_2^2 = \left[\omega^2 \cos(\omega x_1)^2 + \sin(\omega x_2)^2\right]\dot{x}_{1d}^2 + \ddot{x}_{1d}^2 \le \omega^2 + 1 := r_0^2$$

for all $x \in \Re^3$. Then, the diagonal norm-bounded error dynamics is

$$\begin{aligned} \dot{z} &= A_{cl}z + B_w w + B_r r, \quad w = JC_z z, \\ |w_i| &\le \|\tilde{C}_{zi}z\|_2 \quad \text{for } i = 1, 2, \qquad \|r\| \le r_0 \end{aligned} \tag{2.56}$$

where $\tilde{C}_{zi}$ are defined in Example 2.6 and $r_0 = \sqrt{\omega^2 + 1}$.

First consider the same controller gain set $\Theta_1 = \{4, 6, 6, 0.028, 0.028\}$ which is used for the stabilization problem in Example 2.6. Figure 2.9 shows that the maximum error of S_1 reaches up to about 0.3 although it does not diverge. If the surface gains are increased to $\{K_1, K_2, K_3\} = \{40, 60, 60\}$ to reduce the maximum error of S_1, A_{cl} in (2.56) is not Hurwitz, i.e., $\lambda(A_{cl}) = \{0.5623 \pm \mathrm{i}42.2411, -36.2766 \pm \mathrm{i}19.9836, -60\}$. Therefore, the error vector, z, is diverging due to the eigenvalues with positive real part. Consequently, the control input u also diverges (refer to Fig. 2.1). However, when $\Theta_2 = \{40, 60, 60, 0.026, 0.026\}$, A_{cl} becomes Hurwitz, i.e., $\lambda(A_{cl}) = \{-0.1157 \pm \mathrm{i}43.8554, -38.3459 \pm \mathrm{i}19.3785, -60\}$. Figure 2.10 shows that the error S_1 converges to about ± 0.05 around the origin with a high frequency oscillation. Therefore, this example motivates the question of how to estimate the ultimate error bound as well as to ensure the quadratic stability outside the error bound for the given controller gains.

If $\tilde{r} := r/r_0$ and $\tilde{B}_r := r_0 B_r$ in (2.55), the error dynamics can rewritten as

$$\begin{aligned} \dot{z} &= A_{cl}z + B_w w + \tilde{B}_r \tilde{r}, \quad w = JC_z z, \\ \|w\| &\le \gamma \|C_z z\|, \qquad \|\tilde{r}\| \le 1. \end{aligned} \tag{2.57}$$

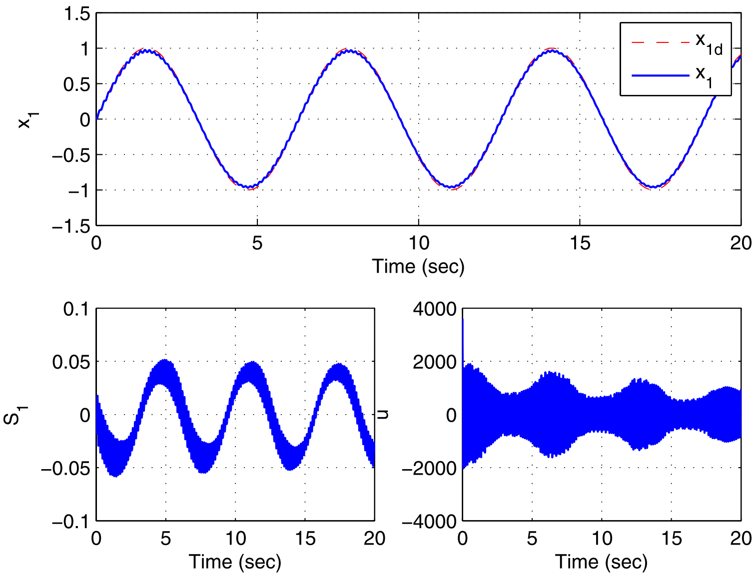

Fig. 2.10 Tracking performance and control input of DSC for Θ_2

Without loss of generality, it can be considered that $\tilde{r}$ is a unit-peak input. Then, the following theorem describes the condition for guaranteeing quadratic tracking as well as the computation of the matrix P for a given set of controller gains.

Theorem 2.4 *For the given set of controller gains, Θ, suppose that the closed-loop error dynamics (2.57) is given on the domain $\mathscr{D}_i \subset \mathscr{D}$ and x_{1d} is a feasible output trajectory. The nonlinear system (2.3) is quadratically trackable via DSC if there exist $P > 0$, $\sigma \geq 0$, and $\alpha \geq 0$ such that*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + \alpha P + \sigma \tilde{C}_z^T \tilde{C}_z & P B_w & P \tilde{B}_r \\ B_w^T P & -\sigma I & 0 \\ \tilde{B}_r^T P & 0 & -\alpha I \end{bmatrix} < 0 \tag{2.58}$$

where $\tilde{C}_z = \gamma C_z$ and $\tilde{B}_r = r_0 B_r$.

Proof Suppose that there exist a function $V_z = z^T P z$, with $P > 0$, such that

$$\frac{d}{dt} V_z(t) = \left(A_{cl} z + B_w w + \tilde{B}_r \tilde{r}\right)^T P z + z^T P \left(A_{cl} z + B_w w + \tilde{B}_r \tilde{r}\right) < 0 \tag{2.59}$$

for every nonzero $(z, w, \tilde{r})$ satisfying $\tilde{r}^T \tilde{r} \leq 1$ and $z^T P z \geq 1$. Using the S-procedure, we see that a sufficient condition for the inequality conditions (2.59) to hold is the existence of $P > 0$, $K \geq 0$, and $\alpha \geq 0$ such that

$$\begin{aligned} & z^T\left(A_{cl}^T P + P A_{cl}\right)z + 2z^T(P B_w w + P\tilde{B}_r \tilde{r}) - \alpha\left(\tilde{r}^T\tilde{r} - z^T P z\right) \\ & \quad - \sigma\left\{w^T w - (\tilde{C}_z z)^T(\tilde{C}_z z)\right\} < 0. \end{aligned} \tag{2.60}$$

The above inequality is equivalent to LMI (2.58). □

As done for the stabilization problem in Sect. 2.5.3, if a componentwise upper bound of w is obtained

$$|w_i| \leq \|\tilde{C}_{zi} z\|_2,$$

the quadratic boundedness can be stated as follows:

Corollary 2.2 *For the given set of controller gains, Θ, suppose that the closed-loop error dynamics (2.57) is given on the domain $\mathscr{D}_i \subset \mathscr{D}$, w is diagonally bounded, $|w_i| \leq \|\tilde{C}_z z\|_2$, and x_{1d} is the feasible output trajectory. The nonlinear system (2.3) is quadratically trackable via DSC if there exist $P > 0$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_{n_w}) \geq 0$ and $\alpha \geq 0$ such that*

$$\begin{bmatrix} A_{cl}^T P + P A_{cl} + \alpha P + \tilde{C}_z^T \Sigma_B \tilde{C}_z & P B_w & P\tilde{B}_r \\ B_w^T P & -\Sigma & 0 \\ \tilde{B}_r^T P & 0 & -\alpha I \end{bmatrix} < 0 \tag{2.61}$$

where $\tilde{C}_z = [\tilde{C}_{z1}^T \ \cdots \ \tilde{C}_{zn_w}^T]^T$, $\tilde{C}_{zi} \in \Re^{n_i \times n_z}$, and $\Sigma_B = \mathrm{diag}(\sigma_1, \sigma_2\mathbf{I}_2, \ldots, \sigma_{n_w}\mathbf{I}_{n_w})$ is the diagonal block matrix.

While the given inequality (2.58) or (2.61) gives an approximation on the ultimate error bound, we would like to find the *smallest* upper bound to accurately estimate the reachable set. Therefore, we need to determine an appropriate measure of size for the ellipsoid, such as the volume or the largest semi-axis. For the purposes of this chapter, the largest semi-axis (or diameter) of the ellipsoid will be used to measure its size. Since w and r in the error dynamics (2.57) enter only the ξ subspace in the sense that the upper block matrices of B_w and B_r are the zero matrix, it is natural that the largest diameter of the ellipsoid may be defined in the ξ subspace and it might be overestimated due to a conservative upper bound of w and r. Thus if the largest diameter is minimized, a smaller quadratic error bound can be computed. Although the inequality (2.58) or (2.61) is not an LMI as stated, it can be posed as a convex optimization problem for the fixed α, so we have to fix the gain α and obtain the smallest ellipsoid by maximizing the smallest eigenvalue of P. Therefore, the LMI relaxation problem is written as

Algorithm 2.2 For a fixed $\alpha \in [a,\ b]$,

$$\begin{array}{ll} \text{maximize} & \lambda_{\min}(P) \\ \text{subject to} & P > 0, \qquad \sigma \geq 0, \quad \text{LMI (2.58)} \\ & \text{or} \\ & P > 0, \qquad \Sigma \geq 0, \quad \text{LMI (2.61)}. \end{array} \tag{2.62}$$

As shown in MATLAB Program 2-3, a specific α which gives the smallest ellipsoid can be determined by iterating over a range of values. It should be noted that the objective function can be changed depending on what we want to minimize, e.g., a volume or the largest semi-axis of the ellipsoidal set [12]. It is also remarked that other types of input such as unit energy, componentwise unit energy and componentwise peak inputs can be also considered in the proposed framework for quadratic tracking or quadratic boundedness [12].

MATLAB Program 2-3

```
%***** Define the dimension of matrices *****
n = size(Acl, 1);
m = size(Bw, 2);
nr = size(Br, 2);

%***** Define the interval of alpha *****
alpha = logspace(-2,2,40);

for k = 1:length(alpha),
%***** cvx version *****
  cvx_begin sdp
    variable P(n,n) symmetric;
    variable beta;
    maximize( lambda_min(P) );
    beta >= 0;
    P == semidefinite(n);
    -[Acl'*P + P*Acl + alpha(K)*P + beta*Cz'*Cz P*Bw P*Br;
    Bw'*P -beta*eye(m) zeros(m,nr);
    Br'*P zeros(nr,m) -alpha(k)*eye(nr)] == semidefinite(n+m+nr);
  cvx_end
end
```

Example 2.8 (Quadratic ultimate bound for a second-order nonlinear system) Consider the second-order nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 - x_1^2, \\ \dot{x}_2 &= u - x_2. \end{aligned} \tag{2.63}$$

The control objective is $x_1 \to x_{1d} = \sin t$.

Let $\mathscr{D}_i = \{x \in \Re^2 || x_i| \le c \text{ for } c \ge 1\}$. Then, the x_{1d} is a feasible trajectory in the sense that $[x_{1d}\ \dot{x}_{1d}\ \ddot{x}_{1d}]^T = [\sin t\ \ \cos t\ -\sin t]^T$ is uniformly bounded on $\mathscr{D}_0$. If DSC is applied to the system, the augmented error dynamics is

$$\begin{aligned} \dot{z} &= A_{cl} z + B_w w + B_r r, \\ w &= -2x_1 C_z z := J_{11} C_z z, \\ |w| &= 2|x_1||C_z z| \le 2c|C_z z| := \gamma |C_z z| \quad \text{for } \forall x \in \mathscr{D}_0 \end{aligned} \tag{2.64}$$

where z, A_{cl}, B_w, and C_z are given in (2.42) of Example 2.4,

$$r = [J_{11}\dot{x}_{1d}\ \ddot{x}_{1d}]^T = [J_{11}\cos t\ -\sin t]^T \in \Re^2,$$

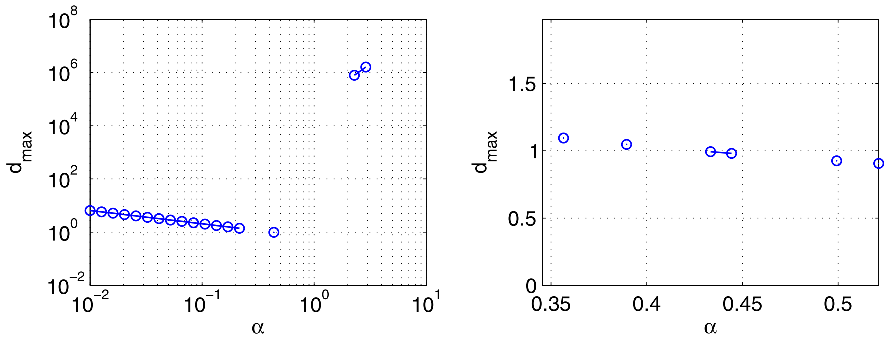

Fig. 2.11 Maximum radius of ellipsoid along line search of α

$$B_r = \begin{bmatrix} 0_2 & 0_2 \\ 1 & -1 \end{bmatrix} \in \Re^{3\times 2}.$$

Furthermore,

$$\|r\|_2^2 = J_{11}^2 \cos^2 t + \sin^2 t \le 4c^2 + 1 := r_0^2$$

for all $x \in \mathscr{D}_i$. Then, the augmented error dynamics (2.64) can be rewritten as

$$\begin{aligned} \dot{z} &= A_{cl}z + B_w w + \tilde{B}_r \tilde{r}, \\ |w| \le \gamma |C_z z| &= |\tilde{C}_z z|, \qquad \|\tilde{r}\| < 1 \end{aligned} \tag{2.65}$$

where $\tilde{B}_r = r_0 B_r = \sqrt{4c^2+1} B_r$, and $\tilde{r} = r/r_0 = r/\sqrt{4c^2+1}$.

Suppose that $c = 1$ for computation of algorithm (2.62) and a set of controller gains is $\Theta_1 = \{K_1, K_2, \tau_2\} = \{1, 2, 0.02\}$ which makes $\lambda(A_{cl}) = \{-1.0208, -2, -48.9792\}$ Hurwitz. To compute the ultimate bound of the closed-loop error dynamics, the LMI (2.62) is solved iteratively for fixed α. That is, after the 40 logarithmically equally spaced points between 10^{-2} and 10^2 are generated for α's, the minimum of the maximum diameter, which is $d_{\max} = 2/\sqrt{\lambda_{\min}(P)}$, is obtained when $\alpha = 0.4375$ (see in the left plot of Fig. 2.11). Then the 20 linearly equally spaced points between 0.3455 and 0.5541 are generated and the iterative computation of LMI (2.62) is performed for each α. Finally, for $\alpha = 0.5212$, the corresponding solution P is

$$P = \begin{bmatrix} 0.1655 & -0.1578 & -0.0032 \\ -0.1578 & 1.1185 & 0.0030 \\ -0.0032 & 0.0030 & 0.0049 \end{bmatrix} \times 10^3,$$

and the corresponding maximum diameter of the ellipsoid, $d_{\max}$, is 0.4528 which is the semi-axis in the ξ_2 axis.

To validate the computed quadratic ultimate error bound, Fig. 2.12 shows the time responses of x and u for the given Θ_1 after simulation. The upper plot in Fig. 2.13 shows that the error trajectory stays in the quadratic error bound which is calculated above in the sense that $z(t)^T P z(t) \le 1$ after 2.1846 seconds and the

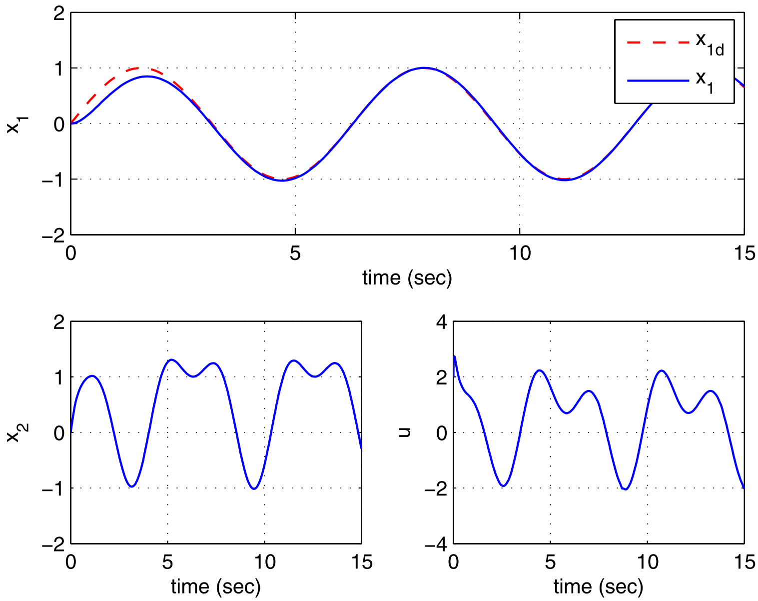

Fig. 2.12 Time responses of x_1, x_2, and u for Θ_1

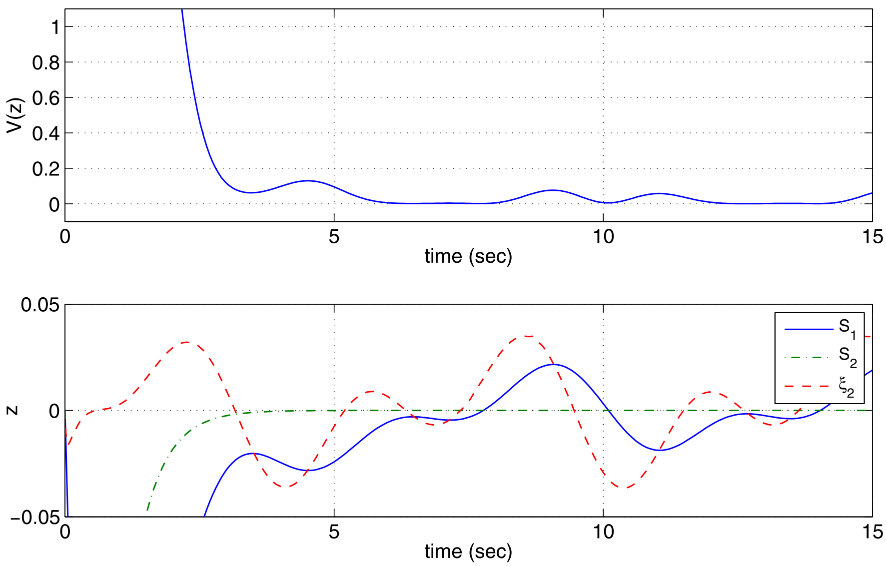

Fig. 2.13 Quadratic function level $V(z) = z^T P z$ and time responses of z

bottom plot shows that all errors are less than $d_{\max}/2$ after a certain time. It is interesting to note that the semi-axes of the quadratic error bound are $\{d_{S_1}, d_{S_2}, d_{\xi_2}\} =$

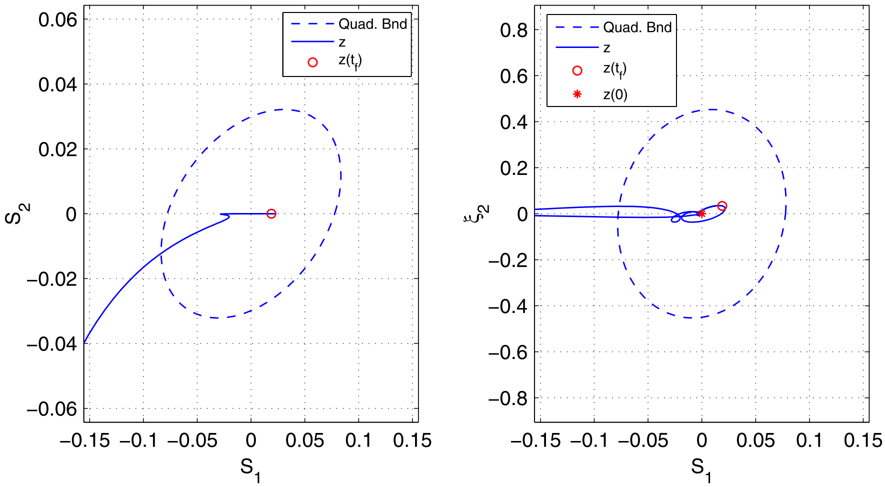

Fig. 2.14 Quadratic bound and z in S_1–S_2 and S_1–ξ_2 planes

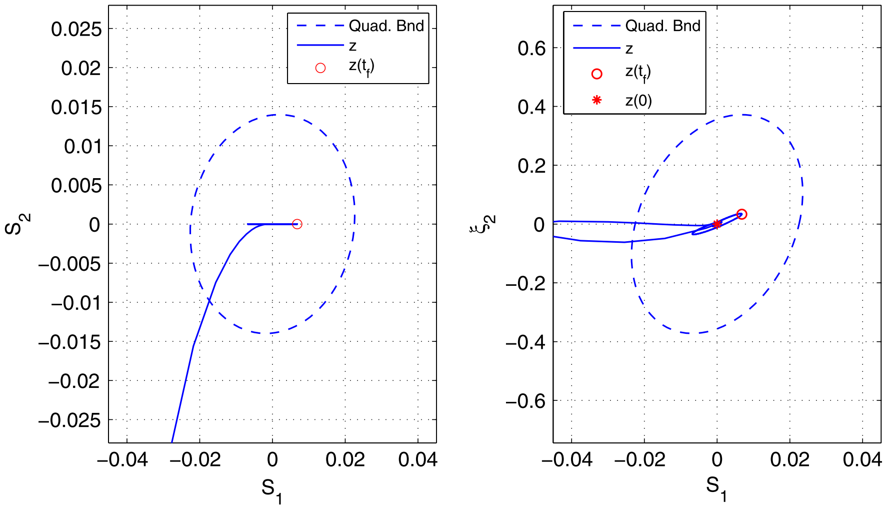

Fig. 2.15 Quadratic bound and z in S_1–S_2 and S_1–ξ_2 planes

$\{0.1555, 0.0643, 0.9056\}$ and d_{ξ_2} is the most overestimated. As expected, both w and r enter through the ξ subspace in the sense that the upper block matrices of B_w and B_r are a zero matrix. So d_{ξ_2} may be estimated in a conservative way, thus this is the reason to minimize the maximum diameter of the quadratic error bound in the framework of convex optimization. If the quadratic error bound is projected onto S_1–S_2 or S_1–ξ_2 plane, it is shown in Fig. 2.14 that z stays in the quadratic bound once it reaches the bound.

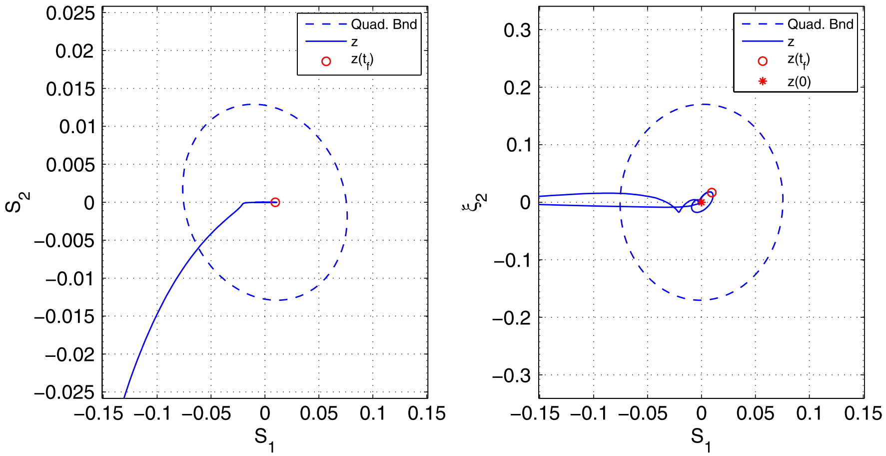

Fig. 2.16 Quadratic bound and z in S_1–S_2 and S_1–ξ_2 planes

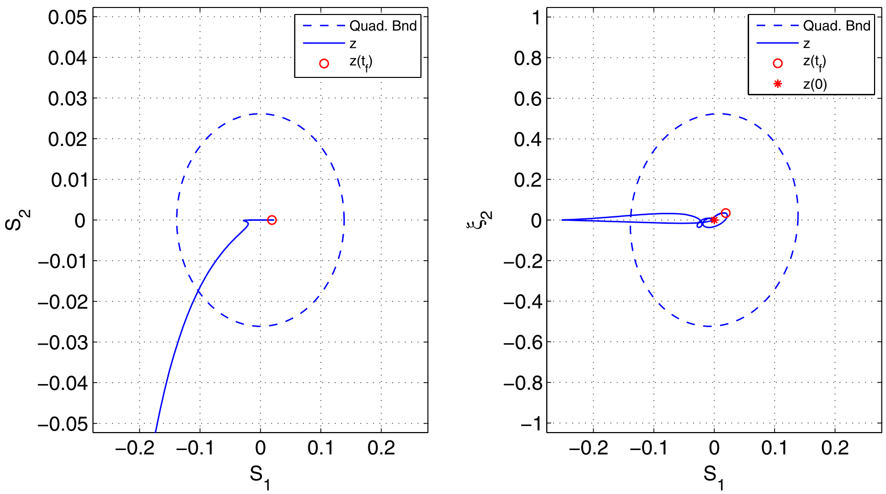

Fig. 2.17 Quadratic bound and z in S_1–S_2 and S_1–ξ_2 planes

Let us consider that the gains K_i are larger. Suppose the second set of controller gains are $\Theta_2 = \{5, 10, 0.02\}$ for which $\lambda(A_{cl}) = \{-5.6351, -10, -44.3649\}$ is Hurwitz. Similarly the LMI (2.62) can be solved iteratively for a fixed α. Then, the minimum of a maximum diameter of a quadratic error bound is obtained as 0.7442 for $\alpha = 0.8358$ and the corresponding semi-axis diameters are $\{d_{S_1}, d_{S_2}, d_{\xi_2}\} = \{0.0451, 0.028, 0.7441\}$ as shown in Fig. 2.15. As expected, a smaller ultimate bound is estimated for Θ_2 and especially the semi-axes in S_1 and S_2 axis are much reduced such that $\frac{d_{S_1}(\Theta_1)}{d_{S_1}(\Theta_2)} = \frac{0.1555}{0.0451} = 3.4479$, $\frac{d_{S_2}(\Theta_1)}{d_{S_2}(\Theta_2)} = \frac{0.0643}{0.028} = 2.2963$. If the time constant τ_2 becomes 0.01, i.e. $\Theta_3 = \{1, 2, 0.01\}$, the semi-axis diameters are calculated as $\{d_{S_1}, d_{S_2}, d_{\xi_2}\} = \{0.151, 0.0258, 0.3408\}$ for $\alpha = 1.8326$ as shown in Fig. 2.16. The semi-axes in ξ_2 and S_2 axis are much reduced such that $\frac{d_{\xi_2}(\Theta_1)}{d_{\xi_2}(\Theta_3)} = \frac{0.9056}{0.3408} = 2.6573$, $\frac{d_{S_2}(\Theta_1)}{d_{S_2}(\Theta_3)} = \frac{0.0643}{0.0258} = 2.4922$. Therefore, the results imply that higher surface gains (K_i) and smaller time constants (τ_i) improve the tracking performance. However, as seen in Example 2.3, higher surface gains may result in instability of the closed-loop system. Consequently, the tradeoff between stability and tracking performance should be considered to determine the control gains.

It is also important to note that the upper bounds of w and r affect the performance of the error bound estimation. In this example, $c = 1$ is used under the assumption that x_1 tracks x_{1d} with very small error after a certain time. If $c = 2$ is assigned in a more conservative way, a larger error bound will be estimated as shown in Fig. 2.17. If semi-axis diameters in the S_1 axis are compared, it is shown that $\frac{d_{S_1}(c=1)}{d_{S_1}(c=2)} = \frac{0.1555}{0.2771} = 0.5612$. Therefore, it is necessary to have tighter upper bounds of w and r for better estimation of an error bound.

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