

Chapter 2

Illustrative Example

Here the procedure for the construction of Lyapunov functionals described in Chap. 1 is used for the investigation of p -stability of the scalar difference equation with constant coefficients and $p = 2k, k = 1, 2, \dots$ via different representations of the considered equation in the form (1.7) different conditions for asymptotic p -stability are obtained.

2.1 First Way of Construction of Lyapunov Functional

Consider the equation

$$x_{i+1} = ax_i + bx_{i-1} + \sigma x_{i-1} \xi_{i+1}, \quad i \in Z, \quad (2.1)$$

where a, b, σ are known constants, and $\xi_i, i \in Z$, is a sequence of $\mathfrak{F}_i$ -adapted mutually independent random variables with $\mathbf{E}\xi_i = 0, \mathbf{E}\xi_i^2 = 1$.

To begin with, asymptotic mean square stability is investigated, i.e., the case $p = 2$.

Using the four steps of the procedure for the construction of Lyapunov functionals, we obtain the following.

1. The right-hand side of (2.1) is already represented in the form (1.7), where $\tau = 0$,

$$\begin{aligned} F_1(i, x_i) &= ax_i, & F_2(i, x_{-1}, \dots, x_i) &= bx_{i-1}, \\ F_3(i, x_{-1}, \dots, x_i) &= 0, & G_1(i, j, x_j) &= 0, \quad j = 0, \dots, i, \\ G_2(i, j, x_{-1}, \dots, x_j) &= 0, & j &= 0, \dots, i-1, \\ G_2(i, i, x_{-1}, \dots, x_i) &= \sigma x_{i-1}. \end{aligned}$$

2. The auxiliary equation (1.8) in this case has the form $y_{i+1} = ay_i$. The function $v_i = y_i^2$ is a Lyapunov function for this equation if $|a| < 1$, since $\Delta v_i = (a^2 - 1)y_i^2$.

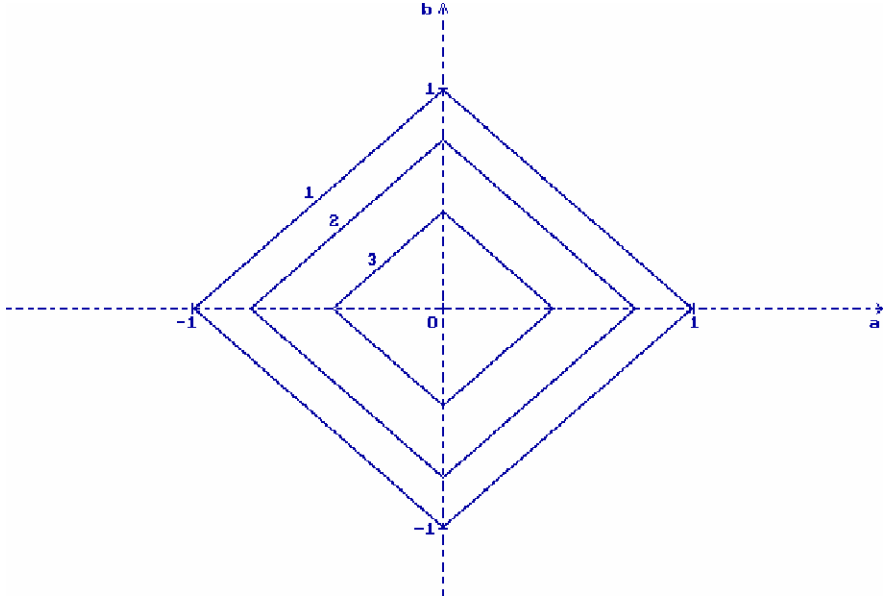


Fig. 2.1 Stability regions for (2.1) given by condition (2.2) for different values of σ^2 : (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$

3. The main part V_{1i} of the Lyapunov functional $V_i = V_{1i} + V_{2i}$ has to be chosen in the form $V_{1i} = x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ for (2.1) we get

$$\begin{aligned}\mathbf{E}\Delta V_{1i} &= \mathbf{E}(x_{i+1}^2 - x_i^2) = \mathbf{E}[(ax_i + bx_{i-1} + \sigma x_{i-1}\xi_{i+1})^2 - x_i^2] \\ &\leq (a^2 - 1 + |ab|)\mathbf{E}x_i^2 + A\mathbf{E}x_{i-1}^2, \quad A = b^2 + |ab| + \sigma^2.\end{aligned}$$

4. Choosing the additional functional V_{2i} in the form $V_{2i} = A\mathbf{E}x_{i-1}^2$ we find that the functional $V_i = V_{1i} + V_{2i}$ satisfy the conditions of Theorem 1.1, provided that

$$|a| + |b| < \sqrt{1 - \sigma^2}. \quad (2.2)$$

So, inequality (2.2) is a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1).

Note, however, that on making the assumption (2.2) the functional V_{1i} satisfies the conditions of Theorem 1.2. So it is an option to use the additional functional V_{2i} .

The stability regions for (2.1) given by inequality (2.2) are shown on Fig. 2.1 for different values of σ^2 : (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

2.2 Second Way of Construction of Lyapunov Functional

Let us use now another representation of (2.1).

1. Represent the right-hand side of (2.1) in the form (1.7), where $\tau = 0$,

$$\begin{aligned} F_1(i, x_i) &= (a + b)x_i, & F_2(i, x_{-1}, \dots, x_i) &= 0, \\ F_3(i, x_{-1}, \dots, x_i) &= -bx_{i-1}, & G_1(i, j, x_j) &= 0, \quad j = 0, \dots, i, \\ G_2(i, j, x_{-1}, \dots, x_j) &= 0, & j &= 0, \dots, i - 1, \\ G_2(i, i, x_{-1}, \dots, x_i) &= \sigma x_{i-1}. \end{aligned}$$

2. Auxiliary equation (1.8) in this case is $y_{i+1} = (a + b)y_i$. The function $v_i = y_i^2$ is the Lyapunov function for this equation if $|a + b| < 1$, since $\Delta v_i = ((a + b)^2 - 1)y_i^2$.
3. The functional V_{li} has be chosen in the form

$$V_{li} = (x_i + bx_{i-1})^2. \quad (2.3)$$

Calculating $\mathbf{E}\Delta V_{li}$ via (2.1), (2.3), we get

$$\begin{aligned} \mathbf{E}\Delta V_{li} &= \mathbf{E}[(x_{i+1} + bx_i)^2 - (x_i + bx_{i-1})^2] \\ &= \mathbf{E}[(a + b - 1)x_i + \sigma x_{i-1}\xi_{i+1}][a + b + 1)x_i + 2bx_{i-1} + \sigma x_{i-1}\xi_{i+1}] \\ &\leq [(a + b)^2 - 1 + |b(a + b - 1)|]\mathbf{E}x_i^2 + B\mathbf{E}x_{i-1}^2, \\ B &= \sigma^2 + |b(a + b - 1)|. \end{aligned}$$

Via Theorem 1.2 we find that the inequality

$$(a + b)^2 + 2|b(a + b - 1)| + \sigma^2 < 1 \quad (2.4)$$

is a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1).

Note that condition (2.4) can also be written in the form

$$|a + b| < \sqrt{1 - \sigma^2}, \quad \sigma^2 < (1 - a - b)(1 + a + b - 2|b|). \quad (2.5)$$

The stability regions defined by conditions (2.5) are shown on Fig. 2.2 for different values of σ^2 : (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

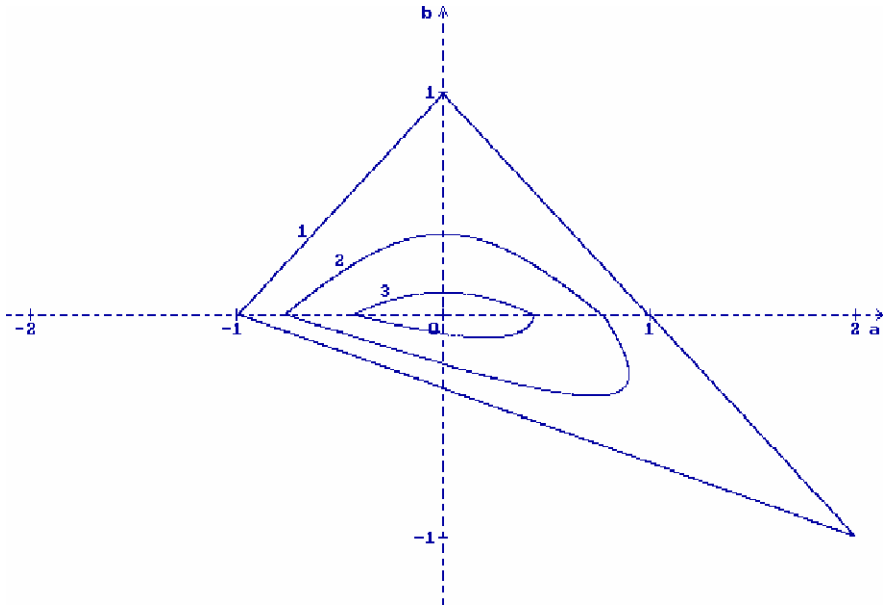


Fig. 2.2 Stability regions for (2.1) given by condition (2.5) for different values of σ^2 : (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$

2.3 Third Way of Construction of Lyapunov Functional

In some cases new stability conditions can be obtained by iterating the right-hand side of the equation under consideration. For example, from (2.1) we have

$$\begin{aligned} x_{i+1} &= a(ax_{i-1} + bx_{i-2} + \sigma x_{i-2}\xi_i) + bx_{i-1} + \sigma x_{i-1}\xi_{i+1} \\ &= (a^2 + b)x_{i-1} + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}. \end{aligned} \quad (2.6)$$

1. Representing the right-hand side of (2.6) in the form (1.7), put $\tau = 0$,

$$F_1(i, x_i) = F_3(i, x_{-1}, \dots, x_i) = 0,$$

$$F_2(i, x_{-1}, \dots, x_i) = (a^2 + b)x_{i-1} + abx_{i-2},$$

$$G_1(i, j, x_j) = 0, \quad j = 0, \dots, i,$$

$$G_2(i, j, x_{-1}, \dots, x_j) = 0, \quad j = 0, \dots, i-2,$$

$$G_2(i, i-1, x_{-1}, \dots, x_{i-1}) = a\sigma x_{i-2}, \quad G_2(i, i, x_{-1}, \dots, x_i) = \sigma x_{i-1}.$$

2. The auxiliary equation is $y_{i+1} = 0, i \in \mathbb{Z}$. The function $v_i = y_i^2$ is the Lyapunov function for this equation, since $\Delta v_i = y_{i+1}^2 - y_i^2 = -y_i^2$.

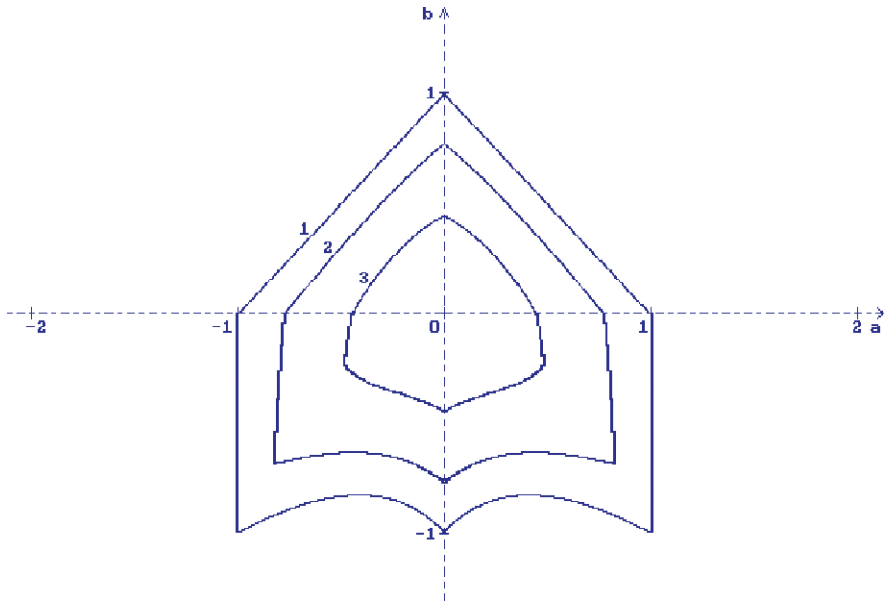


Fig. 2.3 Stability regions for (2.1) given by condition (2.7) for different values of σ^2 : (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$

3. The functional V_{1i} has to be chosen in the form $V_{1i} = x_i^2$. Calculating $\mathbf{E}\Delta V_{1i}$ via (2.6), we have

$$\begin{aligned}
 \mathbf{E}\Delta V_{1i} &= \mathbf{E}(x_{i+1}^2 - x_i^2) \\
 &= \mathbf{E}\left[\left((a^2 + b)x_{i-1} + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}\right)^2 - x_i^2\right] \\
 &\leq -\mathbf{E}x_i^2 + A_1\mathbf{E}x_{i-1}^2 + A_2\mathbf{E}x_{i-2}^2, \\
 A_1 &= |ab||a^2 + b| + (a^2 + b)^2 + \sigma^2, \\
 A_2 &= |ab||a^2 + b| + a^2b^2 + \sigma^2a^2.
 \end{aligned}$$

It follows from Theorem 1.2 that a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) has the form $A_1 + A_2 < 1$ or

$$|a^2 + b| + |ab| < \sqrt{1 - \sigma^2(1 + a^2)}. \quad (2.7)$$

Stability regions defined by condition (2.7) are shown on Fig. 2.3 for different values of σ^2 : (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

Using one more iteration, via (2.6) and (2.1) we obtain

$$\begin{aligned}
 x_{i+1} &= (a^2 + b)(ax_{i-2} + bx_{i-3} + \sigma x_{i-3}\xi_{i-1}) \\
 &\quad + abx_{i-2} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}
 \end{aligned}$$

$$\begin{aligned}
&= a(a^2 + 2b)x_{i-2} + b(a^2 + b)x_{i-3} \\
&\quad + \sigma(a^2 + b)x_{i-3}\xi_{i-1} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1}.
\end{aligned}$$

For $V_{1i} = x_i^2$ we have

$$\begin{aligned}
\mathbf{E}\Delta V_{1i} &= \mathbf{E}(x_{i+1}^2 - x_i^2) \\
&= [(a(a^2 + 2b)x_{i-2} + b(a^2 + b)x_{i-3} \\
&\quad + \sigma(a^2 + b)x_{i-3}\xi_{i-1} + a\sigma x_{i-2}\xi_i + \sigma x_{i-1}\xi_{i+1})^2 - x_i^2] \\
&\leq -\mathbf{E}x_i^2 + \sigma^2 \mathbf{E}x_{i-1}^2 + B_2 \mathbf{E}x_{i-2}^2 + B_3 \mathbf{E}x_{i-3}^2, \\
B_2 &= a^2(a^2 + 2b)^2 + |ab(a^2 + b)(a^2 + 2b)| + \sigma^2 a^2, \\
B_3 &= b^2(a^2 + b)^2 + |ab(a^2 + b)(a^2 + 2b)| + \sigma^2(a^2 + b)^2.
\end{aligned}$$

Via Theorem 1.2 a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) takes the form $\sigma^2 + B_2 + B_3 < 1$ or

$$|a(a^2 + 2b)| + |b(a^2 + b)| < \sqrt{1 - \sigma^2(1 + a^2 + (a^2 + b)^2)}. \quad (2.8)$$

After the next iteration one can obtain a sufficient condition for asymptotic mean square stability of the trivial solution of (2.1) in the form

$$\begin{aligned}
&|a^2(a^2 + 2b) + b(a^2 + b)| + |ab(a^2 + 2b)| \\
&< \sqrt{1 - \sigma^2(1 + a^2 + (a^2 + b)^2 + a^2(a^2 + 2b)^2)}. \quad (2.9)
\end{aligned}$$

Stability regions defined by conditions (2.7) (the bound number 1), (2.8) (the bound number 2) and (2.9) (the bound number 3) are shown on Fig. 2.4 for $\sigma^2 = 0$. One can see that each next iteration increases the stability region.

2.4 Fourth Way of Construction of Lyapunov Functional

Consider now the case $\tau = 1$.

1. Represent (2.1) in the form (1.7) using

$$\begin{aligned}
F_1(i, x_{i-1}, x_i) &= ax_i + bx_{i-1}, & G_2(i, i, x_{-1}, \dots, x_i) &= \sigma x_{i-1}, \\
F_2(i, x_{-1}, \dots, x_i) &= F_3(i, x_{-1}, \dots, x_i) = 0, \\
G_1(i, j, x_j) &= 0, & j &= 0, \dots, i, \\
G_2(i, j, x_{-1}, \dots, x_j) &= 0, & j &= 0, \dots, i-1.
\end{aligned}$$

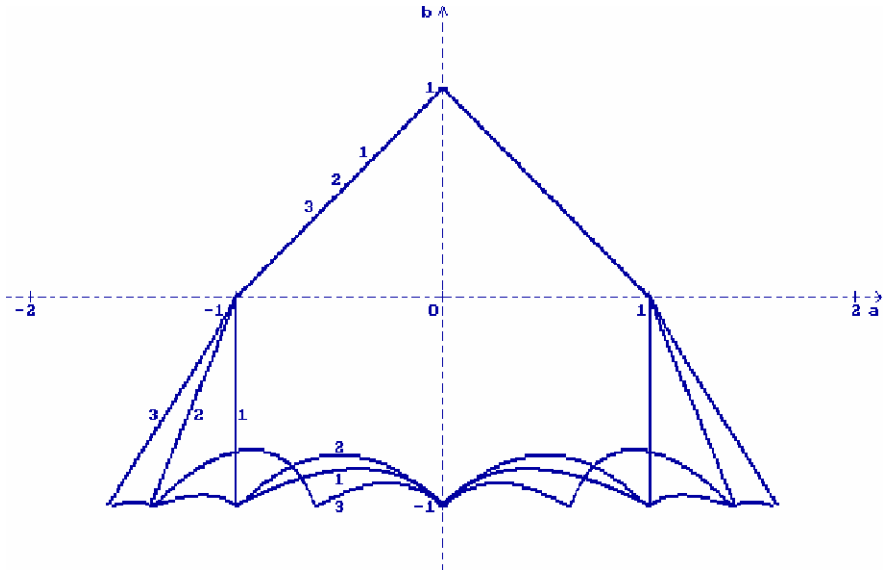


Fig. 2.4 Stability regions for (2.1) given for $\sigma^2 = 0$ by the conditions: (1) (2.7), (2) (2.8), (3) (2.9)

Put

$$x(i) = \begin{pmatrix} x_{i-1} \\ x_i \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \sigma & 0 \end{pmatrix}.$$

Then (2.1) can be written in the matrix form

$$x(i+1) = Ax(i) + Bx(i)\xi_{i+1}. \quad (2.10)$$

2. Introduce the vector $y(i)$ such that $y'(i) = (y_{i-1}, y_i)$. Then the auxiliary equation has the form

$$y(i+1) = Ay(i). \quad (2.11)$$

Let C be an arbitrary positive semidefinite symmetric matrix. Suppose that the solution D of the matrix equation

$$A'DA - D = -C \quad (2.12)$$

is a positive semidefinite matrix with $d_{22} > 0$. In this case the function $v_i = y'(i)Dy(i)$ is the Lyapunov function for (2.11). In particular, if

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}, \quad c_1 \geq 0, \quad c_2 > 0, \quad (2.13)$$

then the elements of the matrix D are

$$\begin{aligned} d_{11} &= c_1 + b^2 d_{22}, & d_{12} &= \frac{ab}{1-b} d_{22}, \\ d_{22} &= \frac{(c_1 + c_2)(1-b)}{(1+b)[(1-b)^2 - a^2]}. \end{aligned} \quad (2.14)$$

The matrix D is positive semidefinite with $d_{22} > 0$ if and only if

$$|b| < 1, \quad |a| < 1 - b. \quad (2.15)$$

3. The functional V_{1i} has been chosen in the form $V_{1i} = x'(i)Dx(i)$. Calculating $\mathbf{E}\Delta V_{1i}$, via (2.10), (2.11) and (2.13) we get

$$\begin{aligned} \mathbf{E}\Delta V_{1i} &= \mathbf{E}[x'(i+1)Dx(i+1) - x'(i)Dx(i)] \\ &= \mathbf{E}[(Ax(i) + Bx(i)\xi_{i+1})' D (Ax(i) + Bx(i)\xi_{i+1}) - x'(i)Dx(i)] \\ &= -\mathbf{E}x'(i)Cx(i) + \sigma^2 d_{22} \mathbf{E}x_{i-1}^2 \\ &= -c_2 \mathbf{E}x_i^2 + (\sigma^2 d_{22} - c_1) \mathbf{E}x_{i-1}^2. \end{aligned}$$

Let us suppose that

$$\frac{\sigma^2(1-b)}{(1+b)[(1-b)^2 - a^2]} < 1. \quad (2.16)$$

Then $\sigma^2 d_{22} < c_1 + c_2$. Choosing $c_1 = \sigma^2 d_{22}$ via Corollary 1.2 we find that the inequalities (2.15), (2.16) are necessary and sufficient conditions for asymptotic mean square stability of the trivial solution of (2.1).

In particular, if $a = b = 0$, then the inequality $\sigma^2 < 1$ is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of the equation $x_{i+1} = \sigma x_{i-1} \xi_{i+1}$. In fact, in this case we have $\mathbf{E}x_{2i-k}^2 = \sigma^{2i} \mathbf{E}x_{-k}^2$, $k = 0, 1$; $i = 0, 1, \dots$

Remark 2.1 It is easy to see that the stability conditions (2.15) and (2.16) do not depend on c_1, c_2 . So, for simplicity one can put $c_1 = 0, c_2 = 1$.

Stability regions defined by conditions (2.15), (2.16) are shown on Fig. 2.5 for different values of σ^2 : (1) $\sigma^2 = 0$; (2) $\sigma^2 = 0.4$; (3) $\sigma^2 = 0.8$.

On Fig. 2.6 one can see the stability regions corresponding to conditions (2.2), (2.5), (2.7) and (2.15), (2.16) for (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.8$.

2.5 One Generalization

Let us obtain a condition for asymptotic p -stability of the trivial solution of (2.1) in the case $p = 2k, k = 1, 2, \dots$

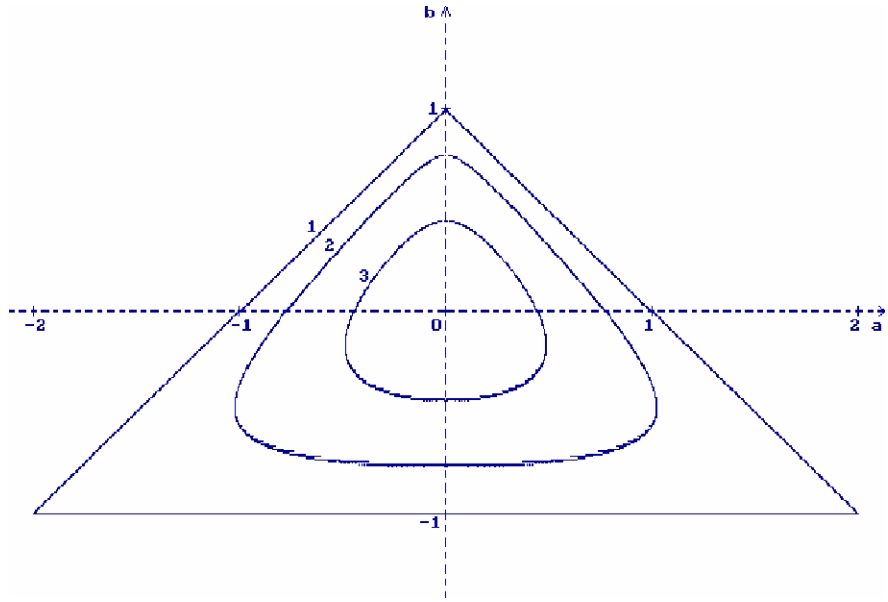


Fig. 2.5 Stability regions for (2.1) given by the conditions (2.15) and (2.16), for different values of σ^2 : (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.4$, (3) $\sigma^2 = 0.8$

Put $\lambda_r = \mathbf{E}\xi_i^r$, $r = 1, \dots, 2k$, and suppose that ξ_i is a sequence of $\mathfrak{F}_i$ -adapted mutually independent random variables with $\lambda_{2m-1} = 0$, $\lambda_{2m} < \infty$, $m = 1, \dots, k$.

Choose $V_{1i} = x_i^{2k}$. Then for (2.1) we have

$$\begin{aligned} \mathbf{E}\Delta V_{1i} &= \mathbf{E}[(ax_i + bx_{i-1} + \sigma x_{i-1}\xi_{i+1})^{2k} - x_i^{2k}] \\ &= \mathbf{E}\left[\sum_{j=0}^{2k} C_{2k}^j (ax_i + bx_{i-1})^j (\sigma x_{i-1}\xi_{i+1})^{2k-j} - x_i^{2k}\right]. \end{aligned}$$

Using $\lambda_{2m-1} = 0$, $\lambda_0 = 1$ and putting $j = 2l$, we obtain

$$\begin{aligned} \mathbf{E}\Delta V_{1i} &= \sum_{l=0}^k C_{2k}^{2l} \mathbf{E}(ax_i + bx_{i-1})^{2l} (\sigma x_{i-1})^{2(k-l)} \lambda_{2(k-l)} - \mathbf{E}x_i^{2k} \\ &= \sum_{l=0}^k C_{2k}^{2l} \mathbf{E}\left(\sum_{j=0}^{2l} C_{2l}^j (ax_i)^j (bx_{i-1})^{2l-j}\right) (\sigma x_{i-1})^{2(k-l)} \lambda_{2(k-l)} - \mathbf{E}x_i^{2k} \\ &\leq \sum_{l=0}^k C_{2k}^{2l} \sum_{j=0}^{2l} C_{2l}^j |a|^j |b|^{2l-j} \sigma^{2(k-l)} \lambda_{2(k-l)} \mathbf{E}|x_i|^j |x_{i-1}|^{2k-j} - \mathbf{E}|x_i|^{2k}. \end{aligned}$$

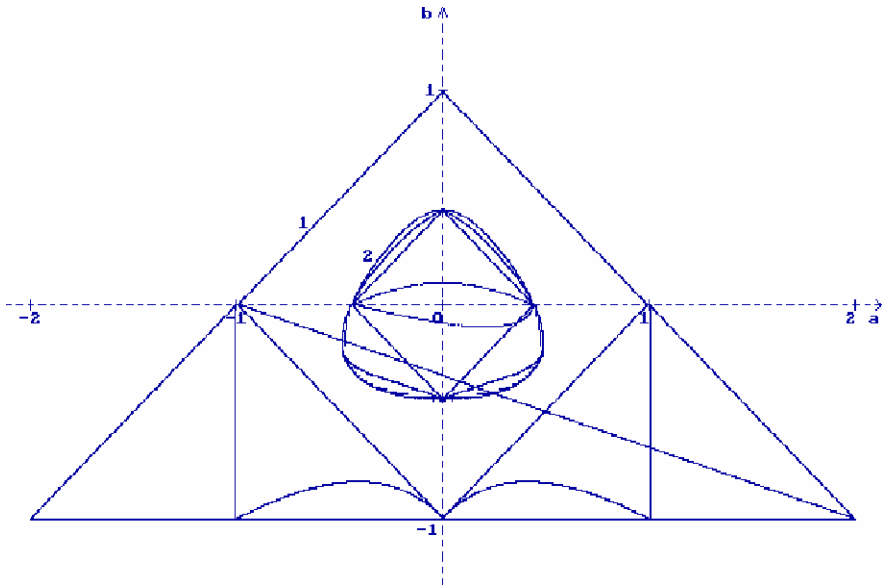


Fig. 2.6 Stability regions for (2.1) given by conditions (2.2), (2.5) and (2.7), and (2.15) and (2.16) for two values of σ^2 : (1) $\sigma^2 = 0$, (2) $\sigma^2 = 0.8$

Via Lemma 1.1

$$|x_i|^j |x_{i-1}|^{2k-j} \leq \frac{j}{2k} |x_i|^{2k} + \frac{2k-j}{2k} |x_{i-1}|^{2k}.$$

So,

$$\begin{aligned} \mathbf{E} \Delta V_{1i} &\leq \sum_{l=0}^k C_{2k}^{2l} \sum_{j=0}^{2l} C_{2l}^j |a|^j |b|^{2l-j} \sigma^{2(k-l)} \lambda_{2(k-l)} \\ &\quad \times \left(\frac{j}{2k} \mathbf{E} |x_i|^{2k} \frac{2k-j}{2k} \mathbf{E} |x_{i-1}|^{2k} \right) - \mathbf{E} |x_i|^{2k}. \end{aligned}$$

Using Theorem 1.2 we obtain the following sufficient condition for asymptotic $2k$ -stability of the trivial solution of (2.1):

$$\sum_{l=0}^k C_{2k}^{2l} (|a| + |b|)^{2l} \sigma^{2(k-l)} \lambda_{2(k-l)} < 1. \quad (2.17)$$

Note that for $k = 1$ and $\lambda_2 = 1$ condition (2.17) coincides with (2.2). If $k = 2$, then condition (2.17) gives

$$(|a| + |b|)^4 + 6\lambda_2 \sigma^2 (|a| + |b|)^2 + \lambda_4 \sigma^4 < 1$$

or

$$|a| + |b| < \sqrt{\sqrt{1 + (9\lambda_2^2 - \lambda_4)\sigma^4} - 3\lambda_2\sigma^2}, \quad \lambda_4\sigma^4 < 1. \quad (2.18)$$

In particular, if ξ_i is Gaussian, then $\lambda_4 = 3\lambda_2^2$ and the stability condition (2.18) takes the form

$$|a| + |b| < \sqrt{\sqrt{1 + 6\lambda_2^2\sigma^4} - 3\lambda_2\sigma^2}, \quad 3\lambda_2^2\sigma^4 < 1.$$



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