# Chapter 2
# Kazhdan–Lusztig Cells and Cellular Bases

The aim of this chapter is to develop a general framework for studying the representation theory of Iwahori–Hecke algebras associated with finite Coxeter groups.

The motivating example is the representation theory of the symmetric group $\mathfrak{S}_n$. Frobenius showed around 1900 that the irreducible representations of $\mathfrak{S}_n$ over a field of characteristic 0 are naturally parametrised by the partitions of $n$. In the 1970s, James [181] developed a "characteristic-free" approach to the representation theory of $\mathfrak{S}_n$, where Specht modules and certain bilinear forms on them play a crucial role. Dipper and James [62] extended this theory to Iwahori–Hecke algebras associated with $\mathfrak{S}_n$. A considerable simplification was then achieved through the powerful new ideas introduced by Murphy [256], [257]. In fact, what we nowadays call the "Murphy basis" is an example of a "cellular basis" in the formal sense defined later by Graham and Lehrer [144].

Here, we shall construct such a "cellular basis" in the sense of Graham and Lehrer, for the generic algebra $\mathbf{H}$ associated with an arbitrary (finite) Coxeter group $W$. For this purpose, we need two basic ingredients:

(1) a basis of $\mathbf{H}$ with certain specific multiplicative properties and
(2) a suitable partial ordering on $\mathrm{Irr}(W)$.

Already Graham and Lehrer identified the Kazhdan–Lusztig basis $\{C_w \mid w \in W\}$ (see Section 2.1) of $\mathbf{H}$ as a natural candidate for (1). However, it is only in some very special examples (in type $A$ or $B$) that $\{C_w\}$ itself has the required multiplicative properties. But in any case, this new basis of $\mathbf{H}$ provides the necessary tools to define a partial ordering on $\mathrm{Irr}(W)$; see Section 2.2.

In order to proceed, we have to rely on certain deep properties of the basis $\{C_w\}$ for which no elementary proofs are known. Sections 2.3–2.5 are devoted to a discussion of these properties, which appear as conjectures **P1**–**P15** in Lusztig's book [231]. We can then put all the pieces together and construct, following [111], [112], a "cellular basis" for $\mathbf{H}$; see Sections 2.6 and 2.7. In the final section, we present an elementary treatment of the case where $W \cong \mathfrak{S}_n$.

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## 2.1 The Kazhdan–Lusztig Basis

Let $W$ be a finite Coxeter group and $L\colon W \to \Gamma$ a weight function, where $\Gamma$ is an abelian group admitting a monomial order $\leqslant$ such that $L(s) \geqslant 0$ for all $s \in S$ (as in Chapter 1). Let $\mathbf{H} = \mathbf{H}_A(W,S,L)$ be the corresponding generic Iwahori–Hecke algebra over $A = R[\Gamma]$, where $R \subseteq \mathbb{C}$ is a subring as in 1.2.1. The main purpose of this section is to introduce the *Kazhdan–Lusztig basis* $\{C_w \mid w \in W\}$ of $\mathbf{H}$. This basis first appeared in [195], in the equal-parameter case. Then Lusztig [219] showed that the construction also works in the general multiparameter case. These results are now readily accessible in Lusztig's book [231], so we will outline the main constructions and formulate the main results but refer to [231] for further details.

**2.1.1.** Given elements $y,w \in W$, we write $y \leqslant w$ if $y$ can be obtained by omitting some terms in a reduced expression for $w$. This defines a partial order relation on $W$, called the *Bruhat–Chevalley order*. Here are some properties (see [231, Chap. 2]):

(a) Let $w \in W$ and $s \in S$. Then $sw < w$ if and only if $l(sw) = l(w) - 1$.
(b) Let $y,w \in W$ and $s \in S$ be such that $sw < w$. Then

$$y \leqslant w \qquad \Leftrightarrow \qquad \begin{cases} sy \leqslant sw & \text{if } sy < y, \\ y \leqslant sw & \text{if } sy > y. \end{cases}$$

Note that (b) provides a recursive description of $\leqslant$.

**2.1.2.** Let $w_0 \in W$ be the longest element. For any $w \in W$, we can write uniquely

$$T_w T_{w_0} = \sum_{y \in W} R^*_{y,w}\, T_{yw_0}, \qquad \text{where } R^*_{y,w} \in \mathbb{Z}[\Gamma].$$

If $w = 1$, then $R^*_{1,1} = 0$ and $R^*_{y,1} = 0$ for all $y \neq 1$. Now assume that $w \neq 1$ and let $s \in S$ be such that $sw < w$. Then one easily checks the following relation:

$$R^*_{y,w} = \begin{cases} R^*_{sy,sw} & \text{if } sy < y, \\ R^*_{sy,sw} + (v_s - v_s^{-1})R^*_{y,sw} & \text{if } sy > y. \end{cases}$$

By using 2.1.1 and the above formulae, we obtain (see also [231, 4.5 and 4.7])

(a) $\qquad R^*_{w,w} = 1 \qquad \text{and} \qquad R^*_{y,w} = 0 \quad \text{unless } y \leqslant w,$

(b) $\qquad \bar{R}^*_{y,w} = (-1)^{l(y)+l(w)} R^*_{y,w}.$

(Here, $\bar{a}$ for any $a \in A$ is defined in Example 1.2.6.) The above recursion formulae are the same as those for the elements $r_{y,w}$ in [231, 4.4]. Consequently, we have

(c) $$T^{-1}_{w^{-1}} = \sum_{y \in W} \bar{R}^*_{y,w}\, T_y \qquad \text{for any } w \in W.$$

(The relation between the expressions for $T_w T_{w_0}$ and $T^{-1}_{w^{-1}}$ already appeared in the remarks following [195, Lemma A.4].)

**2.1.3.** We set $\Gamma_{\geqslant 0} = \{g \in \Gamma \mid g \geqslant 0\}$ and denote by $\mathbb{Z}[\Gamma_{\geqslant 0}]$ the set of all integral linear combinations of terms $\varepsilon^g$, where $g \geqslant 0$. The notations $\mathbb{Z}[\Gamma_{>0}]$, $\mathbb{Z}[\Gamma_{\leqslant 0}]$, $\mathbb{Z}[\Gamma_{<0}]$ have a similar meaning. Then, by the proof of [231, Theorem 5.2] (see also [228, 7.10]), there exists a unique collection of elements $\{P^*_{y,w} \mid y, w \in W\} \subseteq \mathbb{Z}[\Gamma]$ satisfying the following conditions:

(a) $P^*_{w,w} = 1$ and $P^*_{y,w} = 0$ unless $y \leqslant w$; furthermore, $P^*_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$ if $y < w$.
(b) For any $y, w \in W$, we have

$$\bar{P}^*_{y,w} = \sum_{z \in W:\, y \leqslant z \leqslant w} R^*_{y,z} P^*_{z,w}.$$

Note that $P^*_{y,w}$ can be constructed recursively using (a) and (b); see Example 2.1.5. (Here, the notation is as in [219]; $R^*_{y,w}$, $P^*_{y,w}$ are denoted by $r_{y,w}$, $p_{y,w}$ in [231].)

**Definition 2.1.4 (Kazhdan and Lusztig [195], Lusztig [219]).** For $w \in W$, we set

$$C_w := \sum_{y \in W} (-1)^{l(w)+l(y)} \bar{P}^*_{y,w} T_y \ \in \mathbf{H},$$

with $P^*_{y,w}$ as in 2.1.3. The elements $\{C_w \mid w \in W\}$ form an $A$-basis of $\mathbf{H}$; to see this just note that, by 2.1.3(a), we have $T_w \in C_w + \sum_{y \in W:\, y<w} \mathbb{Z}[\Gamma_{>0}] T_y$ for any $w \in W$.

For $x, y \in W$, let us write

$$C_x C_y = \sum_{z \in W} h_{x,y,z} C_z, \qquad \text{where} \qquad h_{x,y,z} \in A.$$

**Example 2.1.5.** The formulae in 2.1.2 yield a straighforward algorithm for computing the polynomials $R^*_{y,w}$. As already mentioned above, the formulae in 2.1.3 can then be used to construct $P^*_{y,w}$ recursively. Indeed, given $y < w$, note that

$$\bar{P}^*_{y,w} - P^*_{y,w} = \sum_{z \in W:\, z < y \leqslant w} R^*_{y,z} P^*_{z,w}.$$

Proceeding by induction on $l(w) - l(y)$, all terms on the right-hand side are known. Then $P^*_{y,w}$ itself is determined by the additional condition that $P^*_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$.

For example, it is clear that $C_1 = T_1$. Now let $s \in S$. Then $R^*_{1,s} = v_s - v_s^{-1}$ and so

$$\bar{P}^*_{1,s} - P^*_{1,s} = \sum_{z \in W:\, 1 < z \leqslant s} R^*_{1,z} P^*_{z,s} = R^*_{1,s} P^*_{s,s} = v_s - v_s^{-1}.$$

Since $P^*_{1,s} \in \mathbb{Z}[\Gamma_{<0}]$, it follows that $P^*_{1,s} = 0$ (if $L(s) = 0$) and $P^*_{1,s} = v_s^{-1}$ (if $L(s) > 0$). Thus, we obtain

$$C_s = \begin{cases} T_s & \text{if } L(s) = 0, \\ T_s - v_s T_1 & \text{if } L(s) > 0. \end{cases}$$

In order to see some more complicated polynomials $P^*_{y,w}$, let $s, t \in S$ be such that $m_{st} \geqslant 3$ and assume that $L(t) \geqslant L(s) > 0$. Then the above precedure yields

$$C_{st} = T_{st} - v_t T_s - v_s T_t + v_s v_t T_1$$

and

$$C_{tst} = T_{tst} - v_t T_{st} - v_t T_{ts} + v_t^2 T_s$$
$$+\begin{cases} v_s v_t T_t - v_s v_t^2 T_1 & \text{if } L(s) = L(t), \\ (v_s v_t - v_s^{-1} v_t) T_t - (v_s v_t^2 - v_s^{-1} v_t^2) T_1 & \text{if } L(t) > L(s). \end{cases}$$

With some more effort, it is possible to write down explicit formulae for all basis elements $C'_w$, where $W$ is of type $I_2(m)$; see [231, Chap. 7], [132, Exc. 11.4].

We can now state the following characterisation of the element $C_w$. This version of the characterisation (which works for finite $W$) is due to Lusztig [232]; the proof is very similar to (but the statement as such is different from) the original one in [195], [219] (which relied on the "bar involution" on $\mathbf{H}$).

**Theorem 2.1.6 (Kazhdan and Lusztig [195], Lusztig [219], [231], [232]).** *For any $w \in W$, the element $C_w$ is uniquely determined by the following two conditions:*

$$C_w \in T_w + \sum_{y \in W} \mathbb{Z}[\Gamma_{>0}] T_y \qquad \textit{and} \qquad C_w T_{w_0} \in \sum_{y \in W} \mathbb{Z}[\Gamma_{\leqslant 0}] T_y.$$

*Proof.* Let us verify that $C_w$ satisfies the above two conditions. The first one is clear by 2.1.3(a); furthermore, using the relations (a), (b) in 2.1.2, we obtain:

$$\begin{aligned} C_w T_{w_0} &= \sum_{y \in W} (-1)^{l(w)+l(y)} \Big( \sum_{z \in W : y \leqslant z \leqslant w} (-1)^{l(y)+l(z)} R^*_{y,z} \bar{P}^*_{z,w} \Big) T_{yw_0} \\ &= \sum_{y \in W} (-1)^{l(w)+l(y)} \Big( \sum_{z \in W : y \leqslant z \leqslant w} \overline{R}^*_{y,z} \overline{P}^*_{z,w} \Big) T_{yw_0} \\ &= \sum_{y \in W} (-1)^{l(w)+l(y)} P^*_{y,w} T_{yw_0}, \end{aligned}$$

where the last equality holds by 2.1.3(b). Thus, we have in fact

$$C_w T_{w_0} \in T_{ww_0} + \sum_{y \in W : y < w} \mathbb{Z}[\Gamma_{<0}] T_{yw_0} \quad \subseteq \quad \sum_{y \in W} \mathbb{Z}[\Gamma_{\leqslant 0}] T_y,$$

as required. Using this expression for $C_w T_{w_0}$, one easily deduces the following statement. Let $h = \sum_{x \in W} a_x C_x \in \mathbf{H}$, where $a_x \in \mathbb{Z}[\Gamma]$ for all $x \in W$. Then we have

$$(*) \qquad hT_{w_0} \in \sum_{y \in W} \mathbb{Z}[\Gamma_{\leqslant 0}] T_y \quad \Rightarrow \quad a_x \in \mathbb{Z}[\Gamma_{\leqslant 0}] \quad \text{for all } x \in W.$$

This immediately implies the uniqueness of $C_w$. Indeed, assume that $\tilde{C}_w \in \mathbf{H}$ also satisfies the desired conditions. Let $h := C_w - \tilde{C}_w$. Then we have

$$h \in \sum_{y \in W} \mathbb{Z}[\Gamma_{>0}] T_y \subseteq \sum_{y \in W} \mathbb{Z}[\Gamma_{>0}] C_y \qquad \text{and} \qquad hT_{w_0} \in \sum_{y \in W} \mathbb{Z}[\Gamma_{\leqslant 0}] T_y.$$

Hence, using $(*)$, we conclude that, in an expression of $h$ as a linear combination of basis elements $\{C_x\}$, all coefficients must be zero and so $\tilde{C}_w = C_w$. □

*Remark 2.1.7.* As in [195], [219], we set $C'_w := (-1)^{l(w)} C_w^\dagger$ for all $w \in W$, where $\dagger$ is defined in Example 1.2.6. (The element $C'_w$ is denoted by $c_w$ in [231].) Using the formula $T_w^\dagger = (-1)^{l(w)} T_{w^{-1}}^{-1}$ and the relations in 2.1.2, 2.1.3, we obtain

$$C'_w = \sum_{z\in W} (-1)^{l(z)} \bar{P}^*_{z,w} T_z^\dagger = \sum_{z\in W} \bar{P}^*_{z,w} T_{z^{-1}}^{-1} = \sum_{y\in W} \big(\bar{P}^*_{z,w} \bar{R}^*_{y,z}\big) T_y = \sum_{y\in W} P^*_{y,w} T_y.$$

Furthermore, applying $\dagger$ to the relation $C_x C_y = \sum_{z\in W} h_{x,y,z} C_z$, we obtain

$$C'_x C'_y = \sum_{z\in W} (-1)^{l(x)+l(y)+l(z)} h_{x,y,z} C'_z \qquad \text{for any } x,y \in W.$$

We shall write $h'_{x,y,z} := (-1)^{l(x)+l(y)+l(z)} h_{x,y,z}$ for any $x,y,z \in W$.

Thus, any statement about $C_w$ has an equivalent version for $C'_w$ (where typically some signs need to be arranged). For applications to representation theory, it is more convenient to work with $C_w$; see, for example, Remark 2.1.12. In this book, we will systematically work with $C_w$.

**Theorem 2.1.8 (Kazhdan and Lusztig [195], Lusztig [219], [231, Chap. 6]).** *For any $x,y,z \in W$, we have $h_{x,y,z} = \bar{h}_{x,y,z}$. Furthermore, for $s \in S$ and $w \in W$, we have*

$$C_s C_w = \begin{cases} C_{sw} & \text{if } L(s) = 0, \\ -(v_s + v_s^{-1}) C_w & \text{if } L(s) > 0 \text{ and } sw < w, \\ C_{sw} - \displaystyle\sum_{y\in W: sy<y<w} (-1)^{l(w)+l(y)} \mu^s_{y,w} C_y & \text{if } L(s) > 0 \text{ and } sw > w, \end{cases}$$

*where $\mu^s_{y,w} \in \mathbb{Z}[\Gamma]$ is such that $\bar{\mu}^s_{y,w} = \mu^s_{y,w}$.*

(The analogous formulae for the elements $\{C'_w\}$ are proved in [231, Chap. 6]; then it remains to use the conversion formulae in 2.1.7.)

**2.1.9.** There is a direct recursive algorithm for simultaneously computing

$$\{P^*_{y,w} \mid y,w \in W\} \quad \text{and}$$
$$\{\mu^s_{y,w} \mid s \in S, y,w \in W \text{ such that } L(s) > 0 \text{ and } sy < y < w < sw\},$$

without reference to the polynomials $\{R^*_{y,w}\}$. Recall that, first of all, we have

(a) $\qquad P^*_{w,w} = 1$ for all $w \in W \qquad$ and $\qquad P^*_{y,w} = 0$ unless $y < w$;

see 2.1.3. We shall now list some further properties of these elements. Let $y,w \in W$ be such that $y < w$. Let $t \in S$ be such that $tw < w$. Then we have

(b1) $\qquad P^*_{y,w} = P^*_{ty,tw} \qquad$ if $L(t) = 0$,

(b2) $\qquad P^*_{y,w} = v_t^{-1} P^*_{ty,w} \qquad$ if $L(t) > 0$, $ty > y$,

(b3) $\qquad P^*_{y,w} = v_t P^*_{y,tw} + P^*_{ty,tw} - \displaystyle\sum_{z\in W: y\leqslant z<tw, tz<z} P^*_{y,z} \mu^t_{z,tw} \qquad$ if $L(t) > 0$, $ty < y$.

Furthermore, for any $s \in S$ such that $L(s) > 0$ and $sy < y < w < sw$, we have

(c1) $$\mu^s_{y,w} - v_s P^*_{y,w} + \sum_{z \in W: y<z<w, sz<z} P^*_{y,z}\, \mu^s_{z,w} \in \mathbb{Z}[\Gamma_{<0}],$$

(c2) $$\overline{\mu}^s_{y,w} = \mu^s_{y,w}.$$

See [231, Chap. 6] and [132, §11.1]. In order to describe a recursion based on the above properties, we need to define an ordering on all pairs of elements $(y,w)$, where $y,w \in W$ and $y \leqslant w$. This is done as follows:

$$(y',w') \sqsubseteq (y,w) \quad \stackrel{\text{def}}{\Leftrightarrow} \quad w' < w \quad \text{or} \quad w' = w \text{ and } y \leqslant y'.$$

The recursion starts with the pair $(y,w) = (1,1)$. We have $P^*_{1,1} = 1$ and there are no $\mu$-polynomials to determine for this pair. Now let $(y,w)$ be such that $w \neq 1$ and $y \leqslant w$. Assume that $P^*_{y',w'}$ and the relevant $\mu$-polynomials are already known for all pairs $(y',w') \sqsubset (y,w)$. Then we proceed as follows.

(1) First we determine $P^*_{y,w}$. If $y = w$, then $P^*_{w,w} = 1$. If $y < w$, then choose some $t \in S$ such that $tw < w$. There are three cases to distinguish:

 (i) If $L(t) = 0$, then $(ty,tw) \sqsubset (y,w)$ and so the right-hand side of (b1) is known by induction.
 (ii) If $L(t) > 0$ and $ty > y$, then $(ty,w) \sqsubset (y,w)$ and so the right-hand side of (b2) is known by induction.
 (iii) If $L(t) > 0$ and $ty < y$, then all terms on the right-hand side of (b3) involve pairs $(y',w')$, where $w' < w$. In particular, $(y',w') \sqsubset (y,w)$ for all such pairs and so, by induction, the right-hand side of (b3) is known.

(2) Now assume that $y < w$. Then we have to determine $\mu^s_{y,w}$ for any $s \in S$ such that $L(s) > 0$ and $sy < y < w < sw$. For this purpose, we set

$$\alpha := v_s P^*_{y,w} - \sum_{z \in W: y<z<w, sz<z} P^*_{y,z}\, \mu^s_{z,w}.$$

 (i) For all $z$ appearing in the above sum, we have $(y,z) \sqsubset (y,w)$ and $(z,w) \sqsubset (y,w)$ and, hence, the corresponding terms are known by induction. By (1), we also know $P^*_{y,w}$. Thus, $\alpha$ is determined.
 (ii) Write $\alpha = \alpha_+ + \alpha_0 + \alpha_-$, where $\alpha_+ \in \mathbb{Z}[\Gamma_{>0}]$, $\alpha_- \in \mathbb{Z}[\Gamma_{<0}]$ and $\alpha_0 \in \mathbb{Z}$ are uniquely determined. By (c1) and (c2), we have $\mu^s_{y,w} = \alpha_+ + \alpha_0 + \overline{\alpha}_+$. Thus, $\mu^s_{y,w}$ is determined.

For readers with an interest in "computer algebra" we just mention that it is an excellent programming exercise to implement the above recursion on a computer. For further details see, for example, DuCloux [75] and his COXETER system, CHEVIE [105], [118], and the references in these articles.

The above recursion formulae can actually be used to establish some further properties of $P^*_{y,w}$ and $\mu^s_{y,w}$. We illustrate this with a few examples.

**Example 2.1.10.** Let $y,w \in W$ and $s \in S$. Then we claim that

(a) $$v_s \mu^s_{y,w} \in \mathbb{Z}[\Gamma_{>0}], \quad \text{where} \quad L(s) > 0 \text{ and } sy < y < w < sw.$$

Indeed, by 2.1.9(c2), this is equivalent to showing that $v_s^{-1}\mu^s_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$. Multiplying 2.1.9(c1) by $v_s^{-1}$, we obtain

$$v_s^{-1}\mu^s_{y,w} - P^*_{y,w} + \sum_{z\in W: y<z<w, sz<z} P^*_{y,z}\left(v_s^{-1}\mu^s_{z,w}\right) \in \mathbb{Z}[\Gamma_{<0}].$$

By an inductive argument, we can assume that we already know that $v_s^{-1}\mu^s_{z,w} \in \mathbb{Z}[\Gamma_{<0}]$ for all $z$ in the above sum. Hence, we also have $v_s^{-1}\mu^s_{y,w} \in \mathbb{Z}[\Gamma_{<0}]$, as required.

Assume, furthermore, that we are in the equal-parameter case where $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Then $\mathbb{Z}[\Gamma]$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. Let $y,w \in W$ and $s \in S$ be such that $sy < y < w < sw$. We have just seen that $v\mu^s_{y,w} \in v\mathbb{Z}[v]$. Hence, we have $\mu^s_{y,w} \in \mathbb{Z}[v]$. Since $\overline{\mu}^s_{y,w} = \mu^s_{y,w}$, we conclude that $\mu^s_{y,w} \in \mathbb{Z}$. In fact, we have

(b) $$\mu^s_{y,w} = \text{coefficient of } v^{-1} \text{ in } P^*_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}].$$

Indeed, since $\mu^s_{y,w} \in \mathbb{Z}$, the relation in 2.1.9(c1) reduces to the condition that $\mu^s_{y,w} - vP^*_{y,w} \in v^{-1}\mathbb{Z}[v^{-1}]$, which immediately yields the above statement.

**Example 2.1.11.** Let $y,w \in W$ be such that $y \leqslant w$ and set $P_{y,w} := v_w v_y^{-1} P^*_{y,w}$. Then the following holds:

(a) If $L(s) > 0$ for all $s \in S$, then $P_{y,w} \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ is non-zero, with constant term 1.

This is proved as follows (see also [231, Prop. 5.4]). If $y = w$, then $P_{w,w} = 1$ and so (a) holds. Now assume that $y < w$ and choose some $t \in S$ such that $tw < w$. If $ty > y$, then 2.1.9(b2) yields $P_{y,w} = P_{ty,w}$ and so (a) holds by induction. (Note that $y \leqslant tw$ by 2.1.1(b) and, hence, $ty \leqslant t(tw) = w$.) If $ty < y$, then 2.1.9(b3) yields

$$P_{y,w} = v_t^2 P_{y,tw} + P_{ty,tw} - \sum_{z\in W: y\leqslant z<tw, tz<z} P_{y,z}\, v_{tw} v_z^{-1}\left(v_t \mu^t_{z,tw}\right).$$

By Example 2.1.10 and induction, we have $P_{y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ and $v_t\mu^t_{z,tw} \in \mathbb{Z}[\Gamma_{>0}]$ for all $z$ in the above sum. Hence, we conclude that $P_{y,w} \equiv P_{ty,tw} \bmod \mathbb{Z}[\Gamma_{>0}]$. Since $ty \leqslant tw$ by 2.1.1(b), this yields (a) by induction.

Note that if $L(s) = 0$ for some $s \in S$, then the conclusion in (a) no longer holds. For example, if $L(s) = 0$, then $C'_s = T_s$ and so $P_{1,s} = 0$.

*Remark 2.1.12.* Assume that $L(s) > 0$ for all $s \in S$. Then the basis $\{C_w\}$ gives rise to a $W$-graph structure on $W$. Indeed, let us set $I(w) := \{s \in S \mid sw < w\}$ for $w \in W$. Furthermore, if $y,w \in W$ and $s \in S$ are such that $s \in I(y)$ and $s \notin I(w)$, we set

$$m^s_{y,w} := \begin{cases} 1 & \text{if } y = sw, \\ -(-1)^{l(w)+l(y)}\mu^s_{y,w} & \text{if } y < w, \\ 0 & \text{otherwise.} \end{cases}$$

Then we see that the data $\{I(w)\}$, $\{m_{y,w}^s\}$ give rise to a $W$-graph structure on the set $W$, in the sense of Definition 1.4.11. Note that $v_s m_{y,w}^s \in \mathbb{Z}[\Gamma_{>0}]$ by Example 2.1.10.

**2.1.13.** Recall that $\mathbf{H}$ is a symmetric algebra, where $\{T_w \mid w \in W\}$ and $\{T_{w^{-1}} \mid w \in W\}$ form a pair of dual bases of $\mathbf{H}$. Since each $C_w$ equals $T_w$ plus a $\mathbb{Z}[\Gamma_{>0}]$-linear combination of basis elements $T_z$ ($z \in W$), it easily follows that

(a) $$\tau(C_{x^{-1}}C_y) \in \delta_{xy} + \mathbb{Z}[\Gamma_{>0}] \qquad \text{for all } x,y \in W;$$

(see [220, 5.3.3] where this appeared in the equal-parameter case). Now set

$$D_w := T_w + \sum_{y \in W\,:\, w<y} \overline{P}^*_{yw_0,ww_0}\, T_y \qquad (w \in W),$$

where $w_0 \in W$ is the longest element. Then, $\{C_w \mid w \in W\}$ and $\{D_{w^{-1}} \mid w \in W\}$ form a pair of dual bases; that is, we have

(b) $$\tau(C_x D_{y^{-1}}) = \delta_{xy} \qquad \text{for all } x,y \in W.$$

In particular, $h_{x,y,z} = \tau(C_x C_y D_{z^{-1}})$ for all $x,y,z \in W$, a relation which will be used repeatedly in what follows. The relation (b) follows from the following identity:

$$\sum_{z \in W\,:\, y \leqslant z \leqslant w} (-1)^{l(w)+l(y)} P^*_{y,z} P^*_{ww_0,zw_0} = \delta_{yw} \qquad \text{for all } y \leqslant w \text{ in } W,$$

which appeared as [195, Theorem 3.1] in the equal-parameter case; see [231, 10.7 and 11.4] or [103, §2] for the general case. Once the above identity is proved, one also obtains the following relation (see [231, 11.6] or [103, 2.6]):

(c) $$\mu^s_{ww_0,yw_0} = -(-1)^{l(w)+l(y)}\, \mu^s_{y,w}$$

for any $s \in S$ and $y,w \in W$ such that $sy < y < w < sw$.

**2.1.14.** The $A$-linear map $\mathbf{H} \to \mathbf{H}$, $h \mapsto h^\flat$, defined by $T_w^\flat = T_{w^{-1}}$ ($w \in W$) is an anti-involution of $\mathbf{H}$; see Example 1.2.5. Applying $\flat$ to the relation $T^{-1}_{w^{-1}} = \sum_{y \in W} \bar{R}^*_{y,w} T_y$, we find that $R^*_{y^{-1},w^{-1}} = R^*_{y,w}$ for all $y,w \in W$. Then, using 2.1.3, it also follows that

$$C_w^\flat = C_{w^{-1}} \qquad \text{and} \qquad P^*_{y,w} = P^*_{y^{-1},w^{-1}} \qquad \text{for all } y,w \in W.$$

We can now apply the general definitions concerning "cells" in Section 1.6 to the algebra $\underline{H} = \mathbf{H}$ with its basis $\{C_w \mid w \in W\}$. Thus, we obtain pre-order relations $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{LR}}$ on $W$. Recall, for example, that $\leqslant_{\mathscr{L}}$ is defined as the transitive closure of the relation $\leftarrow_{\mathscr{L}}$; by the multiplicition formulae in Theorem 2.1.6, we have

$$y \leftarrow_{\mathscr{L}} w \quad \Leftrightarrow \quad \begin{cases} \text{either } y = sw, & \text{where } s \in S \text{ is such that } L(s) = 0 \text{ or } sw > w, \\ \text{or } \mu^s_{y,w} \neq 0, & \text{where } s \in S,\ L(s) > 0 \text{ and } sy < y < w < sw. \end{cases}$$

Furthermore, we have $y \leqslant_{\mathscr{R}} w$ if and only if $y^{-1} \leqslant_{\mathscr{L}} w^{-1}$. And, finally, $\leqslant_{\mathscr{LR}}$ is the union of $\leqslant_{\mathscr{L}}$ and $\leqslant_{\mathscr{R}}$.

**Definition 2.1.15.** The left, right or two-sided cells defined, in the sense of 1.6.1, by taking $\underline{H} = \mathbf{H}$ with its basis $\{C_w \mid w \in W\}$, are called the left, right or two-sided *Kazhdan–Lusztig cells* of $W$ respectively.

From now on, unless explicitly stated otherwise, the symbols $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{LR}}$, $\sim_{\mathscr{L}}$, $\sim_{\mathscr{R}}$, $\sim_{\mathscr{LR}}$ will always refer to the pre-order relations defined using the Kazhdan–Lusztig basis $\{C_w\}$ of $\mathbf{H}$.

**Lemma 2.1.16 (Lusztig [231, 8.6]).** *Given $w \in W$, define $\mathscr{L}(w) := \{s \in S \mid sw < w$ and $L(s) > 0\}$ and $\mathscr{R}(w) := \mathscr{L}(w^{-1})$. Then the following hold:*

*(a) If $z, y \in W$ are such that $z \leqslant_{\mathscr{L}} y$, then $\mathscr{R}(y) \subseteq \mathscr{R}(z)$.*
*(b) If $z, y \in W$ are such that $z \leqslant_{\mathscr{R}} y$, then $\mathscr{L}(y) \subseteq \mathscr{L}(z)$.*

*In particular, the function $w \mapsto \mathscr{R}(w)$ is constant on left cells and the function $w \mapsto \mathscr{L}(w)$ is constant on right cells.*

*Proof.* Since the formulation in [231, 8.6] does not include the possibility that $L(s) = 0$ for some $s \in S$, let us briefly sketch the argument. To prove (a), we may assume that $z, y$ are related by an elementary step in the definition of $\leqslant_{\mathscr{L}}$; that is, there is some $s \in S$ such that $h'_{s,y,z} \neq 0$. If $L(s) > 0$, then the argument is exactly the same as in [231, 8.6], using the fact that ${}^t\mathbf{H} := \langle C_w \mid w \in W, wt < w\rangle_A \subseteq \mathbf{H}$ is a left ideal for any $t \in S$ such that $L(t) > 0$; see [231, 8.4].

Now assume that $L(s) = 0$. Then $z = sy$ by the multiplication formulae in Theorem 2.1.6. Let $t \in \mathscr{R}(y)$. If $sy > y$, then $\mathscr{R}(y) \subseteq \mathscr{R}(sy)$ and so $t \in \mathscr{R}(z)$, as required. Finally, assume that $sy < y$; then $l(sy) = l(yt)$. If we had $zt > z$, then $l(syt) = l(y)$ and so $syt = y$; see [132, 1.2.6]. Hence, $s, t$ would be conjugate in this case and so $L(s) = L(t)$, which is a contradiction. Thus, we must have $zt < z$, as required.

The proof of (b) is analogous. □

**Example 2.1.17.** Assume that $L(s) > 0$ for all $s \in S$. Let $w \in W$ be such that $w \sim_{\mathscr{L}} 1$. Then Lemma 2.1.16 implies that $\mathscr{R}(w) = \mathscr{R}(1) = \varnothing$ and so $w = 1$. Hence, $\{1\}$ is a left Kazhdan–Lusztig cell.

Similarly, let $w \in W$ be such that $w \sim_{\mathscr{L}} w_0$, where $w_0 \in W$ is the longest element. Then Lemma 2.1.16 implies that $\mathscr{R}(w) = \mathscr{R}(w_0) = S$ and so $w = w_0$. Hence, $\{w_0\}$ is a left Kazhdan–Lusztig cell. We have the following explicit formula:

$$C_{w_0} = \sum_{w \in W} (-1)^{l(w_0)+l(w)} \varepsilon^{L(w_0)-L(w)} T_w.$$

Indeed, if $w \in W$ is such that $w \neq w_0$, then there exists some $s \in S$ such that $sw > w$. Hence, the formula in 2.1.9(b2) yields that $P^*_{w,w_0} = v_s^{-1} P^*_{sw,w_0}$. By a simple downward induction on $l(w)$, we conclude that $P^*_{w,w_0} = \varepsilon^{L(w)-L(w_0)}$, as required.

**Example 2.1.18.** Let $W$ be of type $I_2(m)$ ($m \geqslant 3$); that is, we have $W = \langle s_1, s_2 \rangle$, where $s_1^2 = s_2^2 = (s_1 s_2)^m = 1$. Let $L$ be a weight function where $b := L(s_1) \geqslant 0$ and $a := L(s_2) \geqslant 0$; here, $a = b$ if $m$ is odd.

The relations $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$ and $\leqslant_{\mathscr{LR}}$ are determined in [231, 8.8]. (See also [132, Exc. 11.4] for the case $a \neq b$.) For any $k \geqslant 0$, write $1_k = s_1 s_2 s_1 \cdots$ ($k$ factors) and $2_k = s_2 s_1 s_2 \cdots$ ($k$ factors); note that $1_m = 2_m$. With this notation, we have:

- If $m$ is odd and $a = b > 0$, then the left cells are
$$\{1_0\}, \quad \{1_m\}, \quad \{2_1, 1_2, 2_3, \ldots, 1_{m-1}\}, \quad \{1_1, 2_2, 1_3, \ldots, 2_{m-1}\}.$$
- If $m$ is even and $a = b > 0$, then the left cells are
$$\{1_0\}, \quad \{1_m\}, \quad \{2_1, 1_2, 2_3, \ldots, 2_{m-1}\}, \quad \{1_1, 2_2, 1_3, \ldots, 1_{m-1}\}.$$
- If $m$ is even and $b > a > 0$, then the left cells are
$$\{1_0\}, \ \{2_1\}, \ \{1_{m-1}\}, \ \{1_m\}, \ \{1_1, 2_2, 1_3, \ldots, 2_{m-2}\}, \ \{1_2, 2_3, 1_4, \ldots, 2_{m-1}\}.$$
- If $m$ is even and $b > a = 0$, then the left cells are
$$\{1_0, 2_1\}, \ \{1_m, 1_{m-1}\}, \ \{1_1, 2_2, 1_3, \ldots, 2_{m-2}\}, \ \{1_2, 2_3, 1_4, \ldots, 2_{m-1}\}.$$

The two-sided cells and the partial order induced on them are given by

$$\begin{gathered}
\{1_m\} \leqslant_{\mathscr{LR}} W \setminus \{1_0, 1_m\} \leqslant_{\mathscr{LR}} \{1_0\} \quad (a = b > 0), \\
\{1_m\} \leqslant_{\mathscr{LR}} \{1_{m-1}\} \leqslant_{\mathscr{LR}} W \setminus \{1_0, 2_1, 1_{m-1}, 1_m\} \leqslant_{\mathscr{LR}} \{2_1\} \leqslant_{\mathscr{LR}} \{1_0\} \quad (b > a > 0), \\
\{1_m, 1_{m-1}\} \leqslant_{\mathscr{LR}} W \setminus \{1_0, 2_1, 1_{m-1}, 1_m\} \leqslant_{\mathscr{LR}} \{1_0, 2_1\} \quad (b > a = 0).
\end{gathered}$$

Recall that, in Definition 1.6.4, we have introduced the left, right and two-sided $\tilde{\mathbf{J}}$-cells of $W$, using the algebra $\underline{H} = \tilde{\mathbf{J}}$ with its basis $\{t_w \mid w \in W\}$. In the above example where $W$ is of type $I_2(m)$, the two-sided Kazhdan–Lustzig cells are precisely the two-sided $\tilde{\mathbf{J}}$-cells determined in Example 1.7.3. If this was known to be true in general, then our task in this book would be considerably simpler! (We will discuss this in more detail in Section 2.5.) To close this section, we will show by a general argument that, at least, the Kazhdan–Lusztig cells are always unions of $\tilde{\mathbf{J}}$-cells.

**2.1.19.** We will now bring back into the picture the balanced matrix representations $\{\rho^\lambda \mid \lambda \in \Lambda\}$ and the corresponding leading matrix coefficients $c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}}$; see Section 1.4. Recall that, for any $w \in W$, we have

$$\varepsilon^{\mathbf{a}_\lambda} \rho^\lambda(T_w) \in M_{d_\lambda}(\mathscr{O}_0) \qquad \text{and} \qquad c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}} \equiv \varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(T_w) \bmod \mathfrak{m}$$

for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$. Now consider the expressions for $C_w$ and $D_w$. Since $\overline{P}_{y,w}^* \in \mathbb{Z}[\Gamma_{>0}]$ for all $y \neq w$, we deduce that

$$\begin{gathered}
\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(C_w) \in \mathscr{O}_0 \qquad \text{and} \qquad \varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(D_w) \in \mathscr{O}_0, \\
\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(T_w) \equiv \varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(C_w) \equiv \varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{s}\mathfrak{t}}^\lambda(D_w) \equiv c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}} \mod \mathfrak{m},
\end{gathered}$$

for all $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$. Thus, the leading matrix coefficients can be taken with respect to any of the bases $\{T_w\}$, $\{C_w\}$ or $\{D_w\}$.

**Proposition 2.1.20.** *Every left Kazhdan–Lusztig cell of $W$ is a union of left $\tilde{\mathbf{J}}$-cells (see Definition 1.6.4). Analogous statements hold for right and two-sided cells. In particular, if $x, y, z \in W$ are such that $\tilde{\gamma}_{x,y,z} \neq 0$, then the elements $x^{\pm 1}$, $y^{\pm 1}$, $z^{\pm 1}$ all lie in the same two-sided Kazhdan–Lusztig cell.*

*Proof.* Let $y, z \in W$ belong to the same left $\tilde{\mathbf{J}}$-cell. It will be sufficient to consider an elementary step in the definition of this relation; that is, we can assume that

$$\tilde{\gamma}_{x,y,z^{-1}} = \Big(\sum_{\lambda\in\Lambda}\sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u}\in M(\lambda)} f_\lambda^{-1} c_{x,\lambda}^{\mathfrak{st}} c_{y,\lambda}^{\mathfrak{tu}} c_{z,\lambda}^{\mathfrak{us}}\Big) \neq 0 \quad \text{for some } x\in W.$$

We deduce that

$$\sum_{\mathfrak{u}\in M(\lambda)} c_{y,\lambda}^{\mathfrak{tu}} c_{z^{-1},\lambda}^{\mathfrak{us}} \neq 0 \qquad \text{for some } \lambda\in\Lambda \text{ and } \mathfrak{s},\mathfrak{t}\in M(\lambda).$$

Using the relations in 2.1.19 we obtain that

$$\varepsilon^{2\mathbf{a}_\lambda}\rho_{\mathfrak{ts}}^{\lambda}(C_yD_{z^{-1}}) \equiv \sum_{\mathfrak{u}\in M(\lambda)} \big(\varepsilon^{\mathbf{a}_\lambda}\rho_{\mathfrak{tu}}^{\lambda}(C_y)\big)\big(\varepsilon^{\mathbf{a}_\lambda}\rho_{\mathfrak{us}}^{\lambda}(D_{z^{-1}})\big) \equiv \sum_{\mathfrak{u}\in M(\lambda)} c_{y,\lambda}^{\mathfrak{tu}} c_{z^{-1},\lambda}^{\mathfrak{us}}$$

modulo $\mathfrak{m}$. Since the expression on the right-hand side is non-zero modulo $\mathfrak{m}$, we conclude that $\rho^\lambda(C_yD_{z^{-1}}) \neq 0$ and so $C_yD_{z^{-1}} \neq 0$. Since $\tau$ is non-degenerate, we have $\tau(C_wC_yD_{z^{-1}}) \neq 0$ for some $w\in W$. This yields $h_{w,y,z} = \tau(C_wC_yD_{z^{-1}}) \neq 0$ and so $z \leqslant_{\mathscr{L}} y$ (in the Kazhdan–Lusztig pre-order). Similarly, we find that

$$\varepsilon^{2\mathbf{a}_\lambda}\rho_{\mathfrak{ts}}^{\lambda}(D_yC_{z^{-1}}) \equiv \sum_{\mathfrak{u}\in M(\lambda)} \big(\varepsilon^{\mathbf{a}_\lambda}\rho_{\mathfrak{tu}}^{\lambda}(D_y)\big)\big(\varepsilon^{\mathbf{a}_\lambda}\rho_{\mathfrak{us}}^{\lambda}(C_{z^{-1}})\big) \equiv \sum_{\mathfrak{u}\in M(\lambda)} c_{y,\lambda}^{\mathfrak{tu}} c_{z^{-1},\lambda}^{\mathfrak{us}}$$

modulo $\mathfrak{m}$ and, hence, $D_yC_{z^{-1}} \neq 0$. Again, we can find some $w\in W$ such that $\tau(C_wD_yC_{z^{-1}}) \neq 0$. It follows that $h_{z^{-1},w,y^{-1}} = \tau(C_{z^{-1}}C_wD_y) = \tau(C_wD_yC_{z^{-1}}) \neq 0$. Hence, we have $y^{-1} \leqslant_{\mathscr{R}} z^{-1}$ and so $y \leqslant_{\mathscr{L}} z$. Thus, we have shown that $y,z$ belong to the same left Kazhdan–Lusztig cell. The arguments for right and two-sided cells are analogous. The last statement (involving $\tilde{\gamma}_{x,y,z}$) follows from Corollary 1.6.7. $\square$

## 2.2 A Pre-order Relation on Irr$(W)$

We have just seen that the weight function $L\colon W\to\Gamma$ and the monomial order $\leqslant$ on $\Gamma$ give rise to the Kazhdan–Lusztig pre-order relations $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{LR}}$ on $W$. We will now use the two-sided relation $\leqslant_{\mathscr{LR}}$ to define a pre-order relation on the set $\mathrm{Irr}_{\mathbb{K}}(W) = \{E^\lambda \mid \lambda\in\Lambda\}$. Recall that, in Proposition 1.6.11, we constructed a natural surjective map

$$\mathrm{Irr}_{\mathbb{K}}(W) \to \{\text{set of two-sided } \tilde{\mathbf{J}}\text{-cells of } W\}, \qquad E^\lambda \mapsto \mathscr{F}_\lambda.$$

By Proposition 2.1.20, we also know that each $\mathscr{F}_\lambda$ is contained in a two-sided Kazhdan–Lusztig cell. This leads us to the following definition.

**Definition 2.2.1 (Cf. Lusztig [220, 5.15]).** Let $\lambda,\mu\in\Lambda$. Then we define

$$E^\lambda \preceq_L E^\mu \quad \overset{\text{def}}{\Leftrightarrow} \quad x\leqslant_{\mathscr{LR}} y \quad \text{for all } x\in\mathscr{F}_\lambda \text{ and } y\in\mathscr{F}_\mu,$$

where $\leqslant_{\mathscr{LR}}$ is the Kazhdan–Lusztig pre-order relation on $W$; see 2.1.14. Since each two-sided $\tilde{\mathbf{J}}$-cell is contained in a two-sided Kazhdan–Lusztig cell, we have

$$E^\lambda \preceq_L E^\mu \qquad \Leftrightarrow \qquad x \leqslant_{\mathscr{LR}} y \quad \text{for } \textit{some } x \in \mathscr{F}_\lambda \text{ and } \textit{some } y \in \mathscr{F}_\mu.$$

Furthermore, we write $E^\lambda \sim_L E^\mu$ if $E^\lambda \preceq_L E^\mu$ and $E^\mu \preceq_L E^\lambda$. Thus, $E^\lambda \sim_L E^\mu$ if and only if $\mathscr{F}_\lambda$, $\mathscr{F}_\mu$ are contained in the same two-sided Kazhdan–Lusztig cell.

We wish to mention right away that the relation $\sim_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ is not yet completely understood nor explicitly known in all cases (and even less so the relation $\preceq_L$). In this section we will, therefore, content ourselves with giving some examples and explaining the open questions. Of particular interest for us will be the relation between $\preceq_L$ and the function $E^\lambda \mapsto \mathbf{a}_\lambda$. We will see that even the first example that one might think of, namely the case where $W \cong \mathfrak{S}_n$, requires a considerable amount of work; see Example 2.2.13 and Section 2.8. We begin by showing that $\preceq_L$ and $\sim_L$ can be expressed without reference to the map $E^\lambda \mapsto \mathscr{F}_\lambda$.

**2.2.2.** By the general method described in 1.6.1, each left Kazhdan–Lusztig cell $\mathfrak{C}$ gives rise to a representation of $\mathbf{H}$. This is constructed as follows. Let $[\mathfrak{C}]_A$ be an $A$-module with a free $A$-basis $\{e_w \mid w \in \mathfrak{C}\}$. Then the action of $\mathbf{H}$ on $[\mathfrak{C}]_A$ is given by

(a) $$C_w.e_x := \sum_{y\in\mathfrak{C}} h_{w,x,y}\, e_y, \qquad \text{where } w \in W \text{ and } x \in \mathfrak{C}.$$

(Similarly, right cells give rise to right $\mathbf{H}$-modules and two-sided cells give rise to $\mathbf{H}$-bimodules.) Now let $\theta\colon A \to k$ be a ring homomorphism into a field $k$. Then $[\mathfrak{C}]_k := k \otimes_A [\mathfrak{C}]_A$ is a left module for the specialised algebra $\mathbf{H}_k$. In particular, let $\theta_1\colon A \to \mathbb{K}$ be the ring homomorphism such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$, as in 1.2.1. Then we obtain a module $[\mathfrak{C}]_1 := \mathbb{K} \otimes_A [\mathfrak{C}]_A$ for $\mathbb{K}W = \mathbb{K} \otimes_A \mathbf{H}$. For any $\lambda \in \Lambda$, denote by $m(\mathfrak{C},\lambda)$ the multiplicity of $E^\lambda$ as an irreducible constituent of $[\mathfrak{C}]_1$. Then the "specialisation argument" in Example 1.2.4 immediately shows that

(b) $$m(\mathfrak{C},\lambda) = \text{ multiplicity of } E_\varepsilon^\lambda \text{ as an irreducible constituent of } [\mathfrak{C}]_K,$$

where $[\mathfrak{C}]_K$ is the $\mathbf{H}_K$-module obtained from $[\mathfrak{C}]_A$ by extending scalars from $A$ to $K$.

*Remark 2.2.3.* Assume that $L(s) > 0$ for all $s \in S$. Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell and consider the left cell module $[\mathfrak{C}]_A$. As in Remark 2.1.12, we see that the action of $\mathbf{H}$ on $[\mathfrak{C}]_A$ is given by a $W$-graph, where $X = \mathfrak{C}$, $I(x) = \mathscr{L}(x)$ $(x \in \mathfrak{C})$ and

$$m_{x,y}^s = \begin{cases} 1 & \text{if } x = sy > y, \\ -(-1)^{l(y)+l(x)}\mu_{x,y}^s & \text{if } sx < x < y < sy, \\ 0 & \text{otherwise.} \end{cases}$$

**Lemma 2.2.4.** *Let $\lambda \in \Lambda$ and $\mathfrak{C}$ be a left Kazhdan–Lusztig cell such that $m(\mathfrak{C},\lambda) > 0$. Then we have $E^\lambda \leftrightsquigarrow_L w$ for some $w \in \mathfrak{C}$, that is, $w \in \mathfrak{C} \cap \mathscr{F}_\lambda$.*

*Proof.* (Compare with the proof of Theorem 1.8.1.) Consider the identity

$$\sum_{w\in\mathfrak{C}} C_w D_{w^{-1}} = \sum_{x\in W}\sum_{y\in\mathfrak{C}} h_{x,y,y}\, D_{x^{-1}}.$$

(This is proved by multiplying both sides by $C_z$ for some $z \in W$ and applying the trace form $\tau$.) Now note that $\mathrm{trace}(C_x, [\mathfrak{C}]_A) = \sum_{y\in\mathfrak{C}} h_{x,y,y}$. Taking also into account the formula $[\mathfrak{C}]_K \cong \bigoplus_{\mu\in\Lambda} m(\mathfrak{C},\mu)\, E_\varepsilon^\mu$, we obtain

$$\sum_{y\in\mathfrak{C}} h_{x,y,y} = \sum_{\mu\in\Lambda} m(\mathfrak{C},\mu)\,\chi^\mu(C_x) \qquad \text{for all } x\in W.$$

Then the orthogonality relations in Proposition 1.2.12 yield that

$$\chi^\lambda\Big(\sum_{w\in\mathfrak{C}} C_w D_{w^{-1}}\Big) = \sum_{\mu\in\Lambda} m(\mathfrak{C},\mu)\Big(\sum_{x\in W}\chi^\mu(C_x)\,\chi^\lambda(D_{x^{-1}})\Big) = m(\mathfrak{C},\lambda)\, d_\lambda\, \mathbf{c}_\lambda.$$

Multiplying this relation by $\varepsilon^{2\mathbf{a}_\lambda}$ and taking constant terms, we deduce that

$$\sum_{\mathfrak{s},\mathfrak{t}\in M(\lambda)}\sum_{w\in\mathfrak{C}} c_{w,\lambda}^{\mathfrak{st}}\, c_{w^{-1},\lambda}^{\mathfrak{ts}} = m(\mathfrak{C},\lambda)\, d_\lambda\, f_\lambda.$$

Since the right-hand side is non-zero by assumption, we conclude that $c_{w,\lambda}^{\mathfrak{st}} \neq 0$ for some $w\in\mathfrak{C}$ and some $\mathfrak{s},\mathfrak{t}\in M(\lambda)$, as required. □

**Corollary 2.2.5.** *Let $\lambda,\mu\in\Lambda$ and $\mathfrak{C},\mathfrak{C}'$ be left Kazhdan–Lusztig cells such that $m(\mathfrak{C},\lambda)>0$ and $m(\mathfrak{C}',\mu)>0$. Then $E^\lambda \preceq_L E^\mu$ if and only if $w \leqslant_{\mathscr{LR}} w'$ for some $w\in\mathfrak{C}$ and some $w'\in\mathfrak{C}'$ (where $\leqslant_{\mathscr{LR}}$ denotes the Kazhdan–Lusztig pre-order relation).*

*In particular, $E^\lambda \sim_L E^\mu$ if and only if $\mathfrak{C},\mathfrak{C}'$ are contained in the same two-sided Kazhdan–Lusztig cell.*

*Proof.* First assume that $E^\lambda \preceq_L E^\mu$. By definition, this means that $x \leqslant_{\mathscr{LR}} y$ for all $x\in\mathscr{F}_\lambda$ and $y\in\mathscr{F}_\mu$. Now, by Lemma 2.2.4, there exist elements $w\in\mathfrak{C}\cap\mathscr{F}_\lambda$ and $w'\in\mathfrak{C}'\cap\mathscr{F}_\mu$. Hence, we have $w\leqslant_{\mathscr{LR}} w'$, as required.

Conversely, assume that $w\leqslant_{\mathscr{LR}} w'$ for some $w\in\mathfrak{C}$ and some $w'\in\mathfrak{C}'$. Since $m(\mathfrak{C},\lambda)>0$, there exists some $w_1\in\mathfrak{C}\cap\mathscr{F}_\lambda$; see Lemma 2.2.4. Similarly, there exists some $w_1'\in\mathfrak{C}'\cap\mathscr{F}_\mu$. Hence, we have $w_1\sim_{\mathscr{L}} w\leqslant_{\mathscr{LR}} w'\sim_{\mathscr{L}} w_1'$ and so $w_1\leqslant_{\mathscr{LR}} w_1'$. As pointed out in Definition 2.2.1, this already implies that $E^\lambda\preceq_L E^\mu$. □

*Remark 2.2.6.* Let $W = W_1\times\cdots\times W_d$ be the decomposition into irreducible components. Correspondingly, we have

$$\mathrm{Irr}_{\mathbb{K}}(W) = \{E^{\lambda_1}\boxtimes\cdots\boxtimes E^{\lambda_d} \mid \lambda_i\in\Lambda_i\}, \quad \text{where} \quad \mathrm{Irr}_{\mathbb{K}}(W_i) = \{E^{\lambda_i}\mid\lambda_i\in\Lambda_i\}.$$

Thus, as in Remark 1.3.5, we identify $\Lambda = \Lambda_1\times\cdots\times\Lambda_d$. Furthermore, we have $\mathbf{H}\cong\mathbf{H}_1\otimes_A\cdots\otimes_A\mathbf{H}_d$, where $\mathbf{H}_i$ is the generic algebra associated with $W_i$, $L|_{W_i}$. The Kazhdan–Lusztig basis of $\mathbf{H}$ behaves well with respect to this decomposition, that is, if $w = w_1\cdot\ldots\cdot w_d$, where $w_i\in W_i$, then $C_w = C_{w_1}\cdot\ldots\cdot C_{w_d}$. It follows that

$$E^\lambda\preceq_L E^\mu \qquad\Leftrightarrow\qquad E^{\lambda_i}\preceq_{L_i} E^{\mu_i} \quad \text{for } i=1,\ldots,d,$$

where $\lambda=(\lambda_1,\ldots,\lambda_d)$ and $\mu=(\mu_1,\ldots,\mu_d)$. Thus, the determination of $\preceq_L$ can be reduced to the case where $(W,S)$ is irreducible.

**2.2.7.** Assume that $W, L$ are such that the following data are explicitly available:

- The matrices $\{\rho^\lambda(T_s) \mid s \in S\}$ for all $\lambda \in \Lambda$. (Recall the algorithm in 1.4.9 for turning any given representation into a balanced representation.)
- All the polynomials $\{P^*_{y,w}\}$ and $\{\mu^s_{y,w}\}$. (See the recursive description in 2.1.9.)

Since the invariants $\mathbf{a}_\lambda$ are also known (see Section 1.3), we can then work out all leading matrix coefficients $c^{\mathfrak{st}}_{w,\lambda}$ and the Kazhdan–Lusztig pre-order relations $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{LR}}$. This, in turn, enables us to explicitly determine the pre-order relation $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$, via the characterisation in Corollary 2.2.5. Now, the above data are available for $W$ of type $I_2(m)$ (any $m \geqslant 3$), $H_3$, $H_4$, $F_4$. We will now go through these examples one by one and describe the relation $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ in each case.

**Example 2.2.8.** Let $W$ be of type $I_2(m)$ $(m \geqslant 3)$; that is, we have $W = \langle s_1, s_2 \rangle$, where $s_1^2 = s_2^2 = (s_1s_2)^m = 1$. Recall from Example 1.3.7 the description of $\mathrm{Irr}_{\mathbb{K}}(W)$. By Example 2.1.18, we know the left and two-sided cells. It it also not difficult to determine the cell modules $[\mathfrak{C}]_1$ and decompose them into irreducibles. Let $\psi$ denote the sum of all the two-dimensional representations.

- If $m$ is odd and $L(s_1) = L(s_2) > 0$, then the left cell $\{1_0\}$ affords $1_W$, $\{2_m\}$ affords sgn, and the two remaining left cells afford $\psi$.
- If $m$ is even and $L(s_1) = L(s_2) > 0$, then $\{1_0\}$ affords $1_W$, $\{2_m\}$ affords sgn, the first of the two remaining left cells affords $\psi \oplus \mathrm{sgn}_2$, and the second affords $\psi \oplus \mathrm{sgn}_1$.
- If $m$ is even and $L(s_1) > L(s_2) > 0$, then $\{1_0\}$ affords $1_W$, $\{2_1\}$ affords $\mathrm{sgn}_1$, $\{1_{m-1}\}$ affords $\mathrm{sgn}_2$, $\{1_m\}$ affords sgn, and the two remaining left cells afford $\psi$.
- If $m$ is even and $L(s_1) > L(s_2) = 0$, then $\{1_0, 2_1\}$ affords $1_W \oplus \mathrm{sgn}_1$, $\{1_m, 1_{m-1}\}$ affords $\mathrm{sgn} \oplus \mathrm{sgn}_2$, and the two remaining left cells afford $\psi$.

Using this information together with the knowledge of $\leqslant_{\mathscr{LR}}$ (see Example 2.1.18) and of $\mathbf{a}_\lambda$ (see Example 1.3.7), we find that the pre-order $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ is "linear" in the sense that, for any $\lambda, \mu \in \Lambda$, we have

$$E^\lambda \preceq_L E^\mu \qquad \Leftrightarrow \qquad \mathbf{a}_\mu \leqslant \mathbf{a}_\lambda.$$

In particular, $E^\lambda \sim_L E^\mu$ if and only if $\mathbf{a}_\lambda = \mathbf{a}_\mu$.

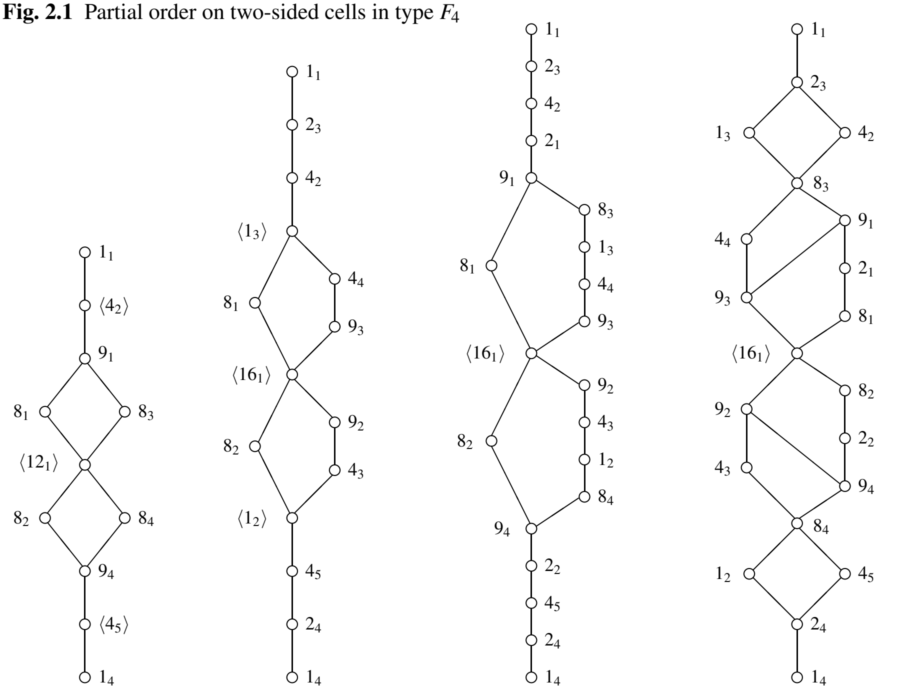
**Example 2.2.9.** Let $W$ be of type $H_3$ or $H_4$. Then all generators are conjugate, so we are automatically in the equal-parameter case. Assume that $L(s) > 0$ for $s \in S$. Then Alvis [2] has computed all polynomials $P^*_{y,w}$ and $\mu^s_{y,w}$. In this way, he explicitly determined the relations $\leqslant_{\mathscr{L}}$ and $\leqslant_{\mathscr{LR}}$; he also found the decomposition of the left cell representations into irreducibles (see [216, §5] for type $H_3$). The partial order induced by $\leqslant_{\mathscr{LR}}$ on the set of two-sided cells is, in fact, a total order.[1] The equivalence classes of $\mathrm{Irr}_{\mathbb{K}}(W)$ under $\sim_L$ are explicitly known by [218, §13] and [2, 3.5]. This information, together with the invariants $\mathbf{a}_\lambda$, is listed in [132, Tables C.1 and C.2]. It turns out that, again, the pre-order $\preceq_L$ is "linear" such that

[1] This statement is not contained in [2]; we thank Alvis (personal communication, 2008) for having verified this using his programs for producing the data in [2].

$$E^\lambda \preceq_L E^\mu \qquad \Leftrightarrow \qquad \mathbf{a}_\mu \leqslant \mathbf{a}_\lambda$$

for any $\lambda, \mu \in \Lambda$. In particular, $E^\lambda \sim_L E^\mu$ if and only if $\mathbf{a}_\lambda = \mathbf{a}_\mu$.

**Fig. 2.1** Partial order on two-sided cells in type $F_4$




$a = b$ $\qquad$ $b = 2a$ $\qquad$ $2a > b > a$ $\qquad$ $b > 2a$

Brackets $\langle\ \rangle$ indicate a two-sided cell with several irreducible components, given as follows:

$\langle 4_2 \rangle = \{2_1, 2_3, 4_2\}$, $\langle 4_5 \rangle = \{2_2, 2_4, 4_5\}$, $\langle 1_3 \rangle = \{1_3, 2_1, 8_3, 9_1\}$, $\langle 1_2 \rangle = \{1_2, 2_2, 8_4, 9_4\}$,

$\langle 12_1 \rangle = \{1_2, 1_3, 4_1, 4_3, 4_4, 6_1, 6_2, 9_2, 9_3, 12_1, 16_1\}$, $\quad \langle 16_1 \rangle = \{4_1, 6_1, 6_2, 12_1, 16_1\}$.

Otherwise, the two-sided cell contains just one irreducible respresentation.

**Example 2.2.10.** Let $W$ be of type $F_4$ with generators labelled as in Table 1.1 (p. 2). Assume that $a := L(s_1) = L(s_2) > 0$ and $b := L(s_3) = L(s_4) > 0$. (The case where $L(s_i) = 0$ for some $i$ will be considered in Remark 2.4.13.) We may also assume without loss of generality that $b \geqslant a$. The pre-order relations $\leqslant_{\mathscr{L}}$, $\leqslant_{\mathscr{R}}$, $\leqslant_{\mathscr{LR}}$ have been determined in [105], based on an explicit computation of all the polynomials $P^*_{y,w}$ and $\mu^s_{y,w}$ using CHEVIE [118]. The resulting pre-order relations $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ are given in Figure 2.1. The notation for the irreducible representations is compatible with that in Table 1.2 (p. 16). For example, $1_1$ is the trivial representation, $1_4$ is the sign representation and $4_2$ is the reflection representation. The pre-order $\preceq_L$ is not "linear" in these cases, but by inspection of the tables we notice that, at least, the following property holds:

$$E^\lambda \preceq_L E^\mu \qquad \Rightarrow \qquad \mathbf{a}_\mu \leqslant \mathbf{a}_\lambda \quad \text{(with equality only if } E^\lambda \sim_L E^\mu\text{)}.$$

In particular, if $E^\lambda \sim_L E^\mu$, then $\mathbf{a}_\lambda = \mathbf{a}_\mu$.

*Remark 2.2.11.* The diagrams in Figure 2.1 have a striking symmetry. This is a general phenomenon. Indeed, recall the definition of the bijection $\lambda \mapsto \lambda^\dagger$ on $\Lambda$ from Example 1.2.6. By Corollary 1.6.16, we have $\mathrm{Irr}_{\mathbb{K}}(W \mid \mathscr{F} w_0) = \mathrm{Irr}_{\mathbb{K}}(W \mid \mathscr{F})^\dagger$ for every two-sided $\tilde{\mathbf{J}}$-cell $\mathscr{F}$ of $W$. Now, 2.1.13(c) implies that if $x, y \in W$ are such that $x \leqslant_{\mathscr{LR}} y$, then $yw_0 \leqslant_{\mathscr{LR}} xw_0$. It follows that, for any $\lambda, \mu \in \Lambda$, we have

$$E^\lambda \preceq_L E^\mu \qquad \Leftrightarrow \qquad E^{\mu^\dagger} \preceq_L E^{\lambda^\dagger}.$$

Thus, the pre-order $\preceq_L$ admits a natural symmetry with respect to $\lambda \mapsto \lambda^\dagger$.

**2.2.12.** Assume that $W$ is of crystallographic type and that we are in the equal-parameter case where $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. It has recently been shown [113] that then $\preceq_L$ admits a geometric interpretation, and this actually yields an explicit description of $\preceq_L$. It would be beyond the scope of this book to explain this in detail, but we can at least outline the general idea, assuming some familiarity with the theory of algebraic groups and Lusztig's work [220] (see also Section 4.4).

So let $G$ be a connected reductive algebraic group (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$, where $p$ is a large prime), with Weyl group $W$. Then, by the *Springer correspondence* (see [197], [221], [282]), we can naturally associate with every $E^\lambda \in \mathrm{Irr}_{\mathbb{K}}(W)$ a pair consisting of a unipotent class of $G$, which we denote by $O_\lambda$, and a $G$-equivariant irreducible local system on $O_\lambda$. Thus, we obtain a map

$$\mathrm{Irr}_{\mathbb{K}}(W) \to \{\text{set of unipotent classes in } G\}, \quad E^\lambda \mapsto O_\lambda.$$

(The local system on $O_\lambda$ will not play a role for our purposes here.) We now need the concept of a "special" unipotent class. This is defined as follows. Given $\lambda \in \Lambda$, let $\mathbf{b}_\lambda$ be the smallest $i \geqslant 0$ such that $E^\lambda$ is a constituent of the $i$th symmetric power of the natural reflection representation of $W$. Lusztig [215] observed that we always have $\mathbf{a}_\lambda \leqslant \mathbf{b}_\lambda$. We say that $E^\lambda$ is a *special representation* if $\mathbf{a}_\lambda = \mathbf{b}_\lambda$; let

$$\mathscr{S}(W) := \{\lambda \in \Lambda \mid \mathbf{a}_\lambda = \mathbf{b}_\lambda\}.$$

Following Lusztig [215], the classes $\{O_\lambda \mid \lambda \in \mathscr{S}(W)\}$ are called the *special unipotent classes* of $G$ (although the word "special" only appeared in later references; see also 4.3.13). By [220, 13.1.1], we have

$$\mathbf{a}_\lambda = \dim \mathscr{B}_u \qquad \text{for any } \lambda \in \mathscr{S}(W),$$

where $u \in O_\lambda$ and $\mathscr{B}_u$ is the variety of Borel subgroups in $G$ containing $u$. Now [113, Cor. 5.6] shows that, for any $\lambda, \mu \in \mathscr{S}(W)$, we have

$$(*)\qquad E^\lambda \preceq_L E^\mu \qquad\Leftrightarrow\qquad O_\lambda \subseteq \overline{O}_\mu := \text{Zariski closure of } O_\mu.$$

The map $E^\lambda \mapsto O_\lambda$ is explicitly known in all cases; see Carter [45, §13.3] and the references there. Also, the Zariski closure relation among the special unipotent classes of $G$ is explicitly known in all cases; see Carter [45, §13.4] and Spaltenstein [280]. Hence, $(*)$ provides an explicit description of the pre-order $\preceq_L$ for special representations. On the other hand, given any $\lambda \in \Lambda$, we have

$$E^\lambda \sim_L E^{\lambda_0} \quad \text{for a unique } \lambda_0 \in \mathscr{S}(W);$$

see [220, 4.14 and 5.25]. Hence, since the equivalence relation $\sim_L$ is explicitly known by Lusztig [220, 4.4–4.13 and 5.25], the relation $\preceq_L$ is determined once we know it for special representations. Finally, by [220, 5.27], the function $\lambda \mapsto \mathbf{a}_\lambda$ is known to be constant on the equivalence classes under $\sim_L$. Hence, by the above characterisation of $\mathbf{a}_\lambda$ for $\lambda \in \mathscr{S}(W)$, we also find that, for any $\lambda, \mu \in \Lambda$, we have

$$E^\lambda \preceq_L E^\mu \quad\Rightarrow\quad \mathbf{a}_\mu \leqslant \mathbf{a}_\lambda \quad (\text{with equality only if } E^\lambda \sim_L E^\mu).$$

(In the next two sections, we will say more about the proof of this implication.)

**Example 2.2.13.** In the setting of 2.2.12, let $W$ be of type $A_{n-1}$. Then $W \cong \mathfrak{S}_n$ and $\Lambda$ is the set of all partitions of $n$; see Example 1.3.8. By [220, 4.4], all irreducible representations of $W$ are special. Now $W$ is the Weyl group of $G = \mathrm{GL}_n$ (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$, where $p$ is a large prime). Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0) \in \Lambda$. Then the Springer correspondence associates with $E^\lambda$ the unipotent class $O_\lambda$ consisting of all unipotent matrices in $G$ whose Jordan normal form has blocks of size $\lambda_1, \lambda_2, \ldots$; see Springer [282, p. 293], [45, §13.3]. By [104, §2.6] and 2.2.12$(*)$, we have

$$E^\lambda \preceq_L E^\mu \qquad\Leftrightarrow\qquad O_\lambda \subseteq \overline{O}_\mu \qquad\Leftrightarrow\qquad \lambda \trianglelefteq \mu,$$

where $\trianglelefteq$ denotes the *dominance order*, which is defined as follows. Write $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0)$ and $\mu = (\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant 0)$. Then

$$\lambda \trianglelefteq \mu \quad\stackrel{\text{def}}{\Leftrightarrow}\quad \sum_{1\leqslant i\leqslant d} \lambda_i \leqslant \sum_{1\leqslant i\leqslant d} \mu_i \quad (\text{for all } d \geqslant 1).$$

It then follows by a completely elementary argument that we have the implication

$$\lambda \trianglelefteq \mu \qquad\Rightarrow\qquad \lambda = \mu \quad \text{or} \quad \mathbf{a}_\lambda > \mathbf{a}_\mu;$$

see, for example, [132, Exc. 5.6]. See Corollary 2.8.14 for a much more direct and elementary proof of the above characterisation of $\preceq_L$ (following [107]).

**Example 2.2.14.** In the setting of 2.2.12, let $W$ be of type $B_n$. Then $\Lambda$ is the set of all pairs of partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$; see Example 1.3.9. For any $(\lambda, \mu) \in \Lambda$ and $(\lambda', \mu') \in \Lambda$, let us define

$$(\lambda,\mu) \preceq (\lambda',\mu') \quad \overset{\text{def}}{\Leftrightarrow} \quad \begin{cases} \displaystyle\sum_{1\leqslant i\leqslant d} (\lambda_i+\mu_i) \leqslant \sum_{1\leqslant i\leqslant d} (\lambda_i'+\mu_i') \\ \displaystyle\lambda_d + \sum_{1\leqslant i<d} (\lambda_i+\mu_i) \leqslant \lambda_d' + \sum_{1\leqslant i<d} (\lambda_i'+\mu_i'), \\ \text{(for all } d \geqslant 1) \end{cases}$$

where $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0)$, $\lambda' = (\lambda_1' \geqslant \lambda_2' \geqslant \ldots \geqslant 0)$, $\mu = (\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0)$ and $\mu' = (\mu_1' \geqslant \mu_2' \geqslant \ldots \geqslant 0)$. By [220, 4.5], we have:

$$(\lambda,\mu) \in \mathscr{S}(W) \qquad \Leftrightarrow \qquad \lambda_i + 1 \geqslant \mu_i \geqslant \lambda_{i+1} \quad \text{(for all } i \geqslant 1).$$

Now $W$ is the Weyl group of $G = \mathrm{SO}_{2n+1}$ (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$, where $p$ is a large prime). Then, by Spaltenstein [281, §4] and 2.2.12($*$), we have

$$E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \quad \Leftrightarrow \quad O_{(\lambda,\mu)} \subseteq \overline{O}_{(\lambda',\mu')} \quad \Leftrightarrow \quad (\lambda,\mu) \preceq (\lambda',\mu')$$

for $(\lambda,\mu) \in \mathscr{S}(W)$ and $(\lambda',\mu') \in \mathscr{S}(W)$. See [122, §5] for further details.

**Example 2.2.15.** Let again $W$ be of type $B_n$ and consider the reflection subgroup $\tilde{W} \subseteq W$ in Example 1.3.10. Then $\tilde{W}$ is of type $D_n$ and this is the Weyl group of $G = \mathrm{SO}_{2n}$ (over $\mathbb{C}$ or over $\overline{\mathbb{F}}_p$, where $p$ is a large prime). To be consistent with the notation in Example 1.3.10, the equal-parameter weight function on $\tilde{W}$ will be denoted by $\tilde{L}$. Now $\mathrm{Irr}_{\mathbb{K}}(\tilde{W})$ is described in terms of the restrictions of the irreducible representations of $W$ to $\tilde{W}$. Given $\tilde{E} \in \mathrm{Irr}_{\mathbb{K}}(\tilde{W})$ and $(\lambda,\mu) \in \Lambda$, we write

$$\tilde{E} \mid E^{(\lambda,\mu)} \quad \overset{\text{def}}{\Leftrightarrow} \quad \tilde{E} \text{ is a constituent of the restriction of } E^{(\lambda,\mu)} \text{ to } \tilde{W}.$$

To characterise the special representations in $\mathrm{Irr}_{\mathbb{K}}(\tilde{W})$, it is convenient to define

$$\tilde{\mathscr{S}}(W) := \{(\lambda,\mu) \in \Lambda) \mid \lambda_i \geqslant \mu_i \geqslant \lambda_{i+1} - 1 \text{ for all } i \geqslant 1\},$$

where we write $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0)$ and $\mu = (\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0)$. Now let $\tilde{E} \in \mathrm{Irr}_{\mathbb{K}}(\tilde{W})$. Then, by [220, 4.6], we have

$$\tilde{E} \text{ is special} \qquad \Leftrightarrow \qquad \tilde{E} \mid E^{(\lambda,\mu)} \text{ for some } (\lambda,\mu) \in \tilde{\mathscr{S}}(W).$$

Note that if $\tilde{E}$ is special, then $(\lambda,\mu)$ on the right-hand side is uniquely determined: just observe that if both $(\lambda,\mu)$ and $(\mu,\lambda)$ belong to $\tilde{\mathscr{S}}(W)$, then $\lambda = \mu$.

Now, by Spaltenstein [281, §4] and 2.2.12($*$), we obtain the following result. Let $\tilde{E}, \tilde{E}' \in \mathrm{Irr}_{\mathbb{K}}(\tilde{W})$ be special representations. Let $(\lambda,\mu) \in \tilde{S}(W)$ and $(\lambda',\mu') \in \tilde{\mathscr{S}}(W)$ be such that $\tilde{E} \mid E^{(\lambda,\mu)}$ and $\tilde{E}' \mid E^{(\lambda',\mu')}$. Then

$$\tilde{E} \preceq_{\tilde{L}} \tilde{E}' \qquad \Leftrightarrow \qquad \begin{cases} \tilde{E} = \tilde{E}' & \text{if } \lambda = \lambda' = \mu = \mu', \\ (\lambda,\mu) \preceq (\lambda',\mu') & \text{otherwise,} \end{cases}$$

where $(\lambda,\mu) \preceq (\lambda',\mu')$ is defined in Example 2.2.15. See [122, §5] for further details.

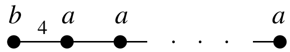
**Example 2.2.16.** Assume that $W$ is of type $E_6$, $E_7$ or $E_8$. The equivalence classes of $\mathrm{Irr}_{\mathbb{K}}(W)$ under $\sim_L$ are listed in Lusztig [220, 4.11–4.13]; see also [132, App. C]. The Springer correspondence is explicitly described in the tables in [45, 13.3].

($E_6$) We have $|\mathrm{Irr}_{\mathbb{K}}(W)| = 25$ and there are 17 equivalence classes under $\sim_L$. The partially ordered set of special unipotent classes is printed in [45, p. 441].

($E_7$) We have $|\mathrm{Irr}_{\mathbb{K}}(W)| = 60$ and there are 35 equivalence classes under $\sim_L$. The partially ordered set of special unipotent classes is printed in [45, p. 443].

($E_8$) We have $|\mathrm{Irr}_{\mathbb{K}}(W)| = 112$ and there are 46 equivalence classes under $\sim_L$. The partially ordered set of special unipotent classes is printed in [45, p. 445].

**Example 2.2.17.** Let $W$ be of type $B_n$ and $L\colon W \to \Gamma$ be a weight function given by

$B_n$ 

where $b > (n-1)a > 0$.

Recall that $\Lambda$ is the set of all pairs of partitions $(\lambda, \mu)$ such that $|\lambda| + |\mu| = n$; see Example 1.3.9. By [122, Example 5.1] (which relies on the series of papers by Bonnafé, Geck, Iancu [21], [26], [108], [114], [121]), we have

(a) $$E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \quad \Leftrightarrow \quad (\lambda,\mu) \trianglelefteq (\lambda',\mu').$$

Here, $\trianglelefteq$ denotes the *dominance order* on pairs of partitions, which is defined by

(b) $$(\lambda,\mu) \trianglelefteq (\lambda',\mu') \quad \overset{\text{def}}{\Leftrightarrow} \quad \begin{cases} \displaystyle\sum_{1\leqslant i\leqslant d} \lambda_i \leqslant \sum_{1\leqslant i\leqslant d} \lambda_i' \\ \displaystyle|\lambda| + \sum_{1\leqslant i\leqslant d} \mu_i \leqslant |\lambda'| + \sum_{1\leqslant i\leqslant d} \mu_i', \\ \text{(for all } d \geqslant 1) \end{cases}$$

where $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0)$, $\lambda' = (\lambda_1' \geqslant \lambda_2' \geqslant \ldots \geqslant 0)$, $\mu = (\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant 0)$ and $\mu' = (\mu_1' \geqslant \mu_2' \geqslant \ldots \geqslant 0)$. Furthermore, by [121, Cor. 5.5], we have

(c) $$E^{(\lambda,\mu)} \preceq_L E^{(\lambda',\mu')} \quad \Rightarrow \quad \mathbf{a}_{(\lambda',\mu')} \leqslant \mathbf{a}_{(\lambda,\mu)},$$

with equality only if $(\lambda,\mu) = (\lambda',\mu')$.

For the (infinitely many) remaining open cases in type $B_n$, at least a conjecture is formulated in [122, 4.11].

*Remark 2.2.18.* Lusztig's definition [220, 5.15] of a pre-order relation on $\mathrm{Irr}_{\mathbb{K}}(W)$ looks somewhat different from that in Definition 2.2.1, but it is really the same. Let us briefly indicate why this is the case. By the Artin–Wedderburn theorem, the split semisimple algebra $\mathbf{H}_K$ decomposes as a direct sum of simple two-sided ideals $\mathbf{H}_K = \bigoplus_{\lambda\in\Lambda} \mathbf{H}_K(\lambda)$, where $\mathbf{H}_K(\lambda)$ is the sum of all left ideals in $\mathbf{H}_K$ which are isomorphic to $E_\varepsilon^\lambda$ (as left $\mathbf{H}_K$-modules). On the other hand, for any $y \in W$, we have the two-sided ideals $\mathfrak{I}_y^{\mathscr{LR}}$ and $\hat{\mathfrak{I}}_y^{\mathscr{LR}}$ defined by the general procedure in 1.6.2, with respect to the basis $\{C_w \mid w \in W\}$ of $\mathbf{H}_K$. Now let $\mathfrak{T}$ be a two-sided Kazhdan–Lusztig cell and $\lambda \in \Lambda$. Then we claim that the following two statements are equivalent:

(a) $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ for some $w \in \mathfrak{T}$.
(b) $\mathbf{H}_K(\lambda) \subseteq \mathfrak{I}_y^{\mathscr{LR}}$ and $\mathbf{H}_K(\lambda) \not\subseteq \hat{\mathfrak{I}}_y^{\mathscr{LR}}$ for some $y \in \mathfrak{T}$.

Indeed, if (a) holds, then $c_{w,\lambda}^{\mathfrak{st}} \neq 0$ and $c_{w^{-1},\lambda}^{\mathfrak{uv}} \neq 0$ for some $\mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v} \in M(\lambda)$; see 1.6.10. Hence, by 2.1.19, we have $\rho^\lambda(C_w) \neq 0$ and $\rho^\lambda(D_{w^{-1}}) \neq 0$. This yields $C_w.\mathbf{H}_K(\lambda) \neq \{0\}$ and $D_{w^{-1}}.\mathbf{H}_K(\lambda) \neq \{0\}$. Then the argument in the proof of [220, Lemma 5.2] shows that (b) holds. Conversely, assume that (b) holds. Then the inclusion $\mathbf{H}_K(\lambda) \subseteq \mathfrak{I}_y^{\mathscr{LR}}$ induces an $(\mathbf{H}_K,\mathbf{H}_K)$-bimodule homomorphism $\varphi\colon \mathbf{H}_K(\lambda) \rightarrow \mathfrak{I}_y^{\mathscr{LR}}/\hat{\mathfrak{I}}_y^{\mathscr{LR}}$, which is non-zero since $\mathbf{H}_K(\lambda) \not\subseteq \hat{\mathfrak{I}}_y^{\mathscr{LR}}$. Now let us just consider the left $\mathbf{H}_K$-module structure. Since $\mathfrak{T}$ is a union of left cells, the left $\mathbf{H}_K$-module $\mathfrak{I}_y^{\mathscr{LR}}/\hat{\mathfrak{I}}_y^{\mathscr{LR}}$ has a filtration by left cell modules $[\mathfrak{C}_i]_K$, where each $\mathfrak{C}_i$ is a left cell contained in $\mathfrak{T}$. Hence, there exists a non-zero $\mathbf{H}_K$-module homomorphism $E_\varepsilon^\lambda \rightarrow [\mathfrak{C}_i]_K$ for some $i$. Then $m(\mathfrak{C}_i,\lambda) > 0$ and so there exists some $w \in \mathfrak{C}_i$ such that $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$; see Lemma 2.2.4. Thus, the equivalence of (a) and (b) is proved. Once this is established, we can conclude that

$$\text{(c)} \quad E^\lambda \preceq_L E^\mu \quad \Leftrightarrow \quad \begin{cases} \text{there exists some } w \in W \text{ such that} \\ \mathbf{H}_K(\lambda) \subseteq \mathfrak{I}_w^{\mathscr{LR}}, \ \mathbf{H}_K(\mu) \subseteq \mathfrak{I}_w^{\mathscr{LR}}, \ \mathbf{H}_K(\mu) \not\subseteq \hat{\mathfrak{I}}_w^{\mathscr{LR}}. \end{cases}$$

The condition on the right-hand side is the one used by Lusztig [220, 5.15].

## 2.3 On Lusztig's Conjectures, I

In the previous section, we have defined a pre-order relation $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ and we have seen that, in many examples, the following implication holds for any $\lambda,\mu \in \Lambda$:

$$(\clubsuit) \qquad E^\lambda \preceq_L E^\mu \quad \Rightarrow \quad \mathbf{a}_\mu \leqslant \mathbf{a}_\lambda \quad (\text{with equality only if } E^\lambda \sim_L E^\mu).$$

This property will turn out to be *the* key to our main results on representations of Hecke algebras at roots of unity. The somewhat weaker implication

$$(\clubsuit') \qquad E^\lambda \sim_L E^\mu \qquad \Rightarrow \qquad \mathbf{a}_\mu = \mathbf{a}_\lambda$$

was a key ingredient in Lusztig's work [220] on characters of reductive groups over finite fields. Now, a general proof of these apparently simple statements is not yet known. And in those situations where ($\clubsuit$) and ($\clubsuit'$) are known to hold, the proofs rely on deep results from algebraic geometry, or explicit computations. It is the purpose of this and the following two sections to discuss this in some more detail.

Assume first that $W$ is a Weyl group and that we are in the equal-parameter case. Then the proof of ($\clubsuit'$) in [220, Chap. 5] relies on the theory of primitive ideals in enveloping algebras. Subsequently, Lusztig [225] found a new proof which uses a geometric interpretation of $\{C_w\}$ and the results in [222], [223]. This interpretation implies, for example, that all coefficients of the polynomials $P_{y,w}^*$ are non-negative integers. In the general multiparameter case, such a geometric interpretation is not known – and the coefficients of $P^*_{y,w}$ may be strictly negative; see Example 2.1.5!

In his book [231, Chap. 14], Lusztig has extended the known situation in the equal-parameter case and stated 15 conjectural properties **P1–P15** of the basis $\{C_w\}$ which should hold for any Coxeter group (finite or infinite) and in the general multiparameter case. In [231, Chap. 20], Lusztig shows that (♣) and (♣′) are formal consequences of **P1–P15**. Thus, **P1–P15** appear to provide the appropriate framework for establishing substantial results concerning the representation theory of **H**.

(See 2.4.1 for a summary of the cases where **P1–P15** are known to hold.)

**2.3.1.** For the convenience of the reader, we state here Lusztig's conjectures **P1–P15** in [231, Chap. 14] in the general framework involving a totally ordered abelian group $\Gamma$, and taking into account the possibility that $L(s)=0$ for some $s \in S$. Also note that these properties are formulated in [231] with respect to the basis $\{C'_w\}$, but, using the formulae in Remark 2.1.7, it is a straightforward matter to switch back and forth between $C_w$ and $C'_w$. The following definitions originally appeared in [222], in the equal-parameter case. For a fixed $z \in W$, we set

$$\mathbf{a}(z) := \min\{g \in \Gamma_{\geqslant 0} \mid \varepsilon^g h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}] \text{ for all } x,y \in W\}.$$

Given $x,y,z \in W$, we define $c_{x,y,z^{-1}} \in \mathbb{Z}$ by

$$c_{x,y,z^{-1}} := \text{ constant term of } \varepsilon^{\mathbf{a}(z)} h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}].$$

Furthermore, if $z \in W$ is such that $P^*_{1,z} \neq 0$, we define an element $\Delta(z) \in \Gamma_{\geqslant 0}$ and an integer $0 \neq n_z \in \mathbb{Z}$ by the condition

$$\varepsilon^{\Delta(z)} P^*_{1,z} \equiv n_z \mod \mathbb{Z}[\Gamma_{<0}]; \qquad \text{see [231, 14.1]}.$$

(Note that we can only have $P^*_{1,z}=0$ if $L(s)=0$ for some $s \in S$; see Example 2.1.11; this is the only place where we explicitly have to mention if $L(s)$ equals zero or not.) Finally, we set $\mathscr{D} := \{z \in W \mid P^*_{1,z} \neq 0 \text{ and } \mathbf{a}(z)=\Delta(z)\}$.

**Conjecture 2.3.2 (Lusztig [231, 14.2]).** *The following properties hold.*

**P1.** *For any $z \in W$ such that $P^*_{1,z} \neq 0$, we have $\mathbf{a}(z) \leqslant \Delta(z)$.*
**P2.** *If $d \in \mathscr{D}$ and $x,y \in W$ satisfy $c_{x,y,d} \neq 0$, then $x=y^{-1}$.*
**P3.** *If $y \in W$, there exists a unique $d \in \mathscr{D}$ such that $c_{y^{-1},y,d} \neq 0$.*
**P4.** *If $x,y \in W$ are such that $x \leqslant_{\mathscr{LR}} y$, then $\mathbf{a}(y) \geqslant \mathbf{a}(x)$. In particular, if $x \sim_{\mathscr{LR}} y$, then $\mathbf{a}(x)=\mathbf{a}(y)$.*
**P5.** *If $d \in \mathscr{D}$, $y \in W$, $c_{y^{-1},y,d} \neq 0$, then $c_{y^{-1},y,d}n_d = (-1)^{l(d)}$.*
**P6.** *If $d \in \mathscr{D}$, then $d^2=1$.*
**P7.** *For any $x,y,z \in W$, we have $c_{x,y,z}=c_{y,z,x}$.*
**P8.** *Let $x,y,z \in W$ be such that $c_{x,y,z} \neq 0$. Then $x \sim_{\mathscr{L}} y^{-1}$, $y \sim_{\mathscr{L}} z^{-1}$, $z \sim_{\mathscr{L}} x^{-1}$.*
**P9.** *If $x \leqslant_{\mathscr{L}} y$ and $\mathbf{a}(x)=\mathbf{a}(y)$, then $x \sim_{\mathscr{L}} y$.*
**P10.** *If $x \leqslant_{\mathscr{R}} y$ and $\mathbf{a}(x)=\mathbf{a}(y)$, then $x \sim_{\mathscr{R}} y$.*
**P11.** *If $x \leqslant_{\mathscr{LR}} y$ and $\mathbf{a}(x)=\mathbf{a}(y)$, then $x \sim_{\mathscr{LR}} y$.*

**P12.** *Let $I \subseteq S$ and $W_I$ be the parabolic subgroup generated by $I$. If $y \in W_I$, then $\mathbf{a}(y)$ computed in terms of $W_I$ is equal to $\mathbf{a}(y)$ computed in terms of $W$.*

**P13.** *Any left cell $\mathfrak{C}$ of $W$ contains a unique element $d \in \mathscr{D}$. We have $c_{x^{-1},x,d} \neq 0$ for all $x \in \mathfrak{C}$.*

**P14.** *For any $z \in W$, we have $z \sim_{\mathscr{L}\mathscr{R}} z^{-1}$.*

**P15.** *If $w, w', x, y \in W$ are such that $\mathbf{a}(x) = \mathbf{a}(y)$, then*

$$\sum_{z\in W} h_{x,w',z} \otimes_{\mathbb{Z}} h_{w,z,y} = \sum_{z\in W} h_{z,w',y} \otimes_{\mathbb{Z}} h_{w,x,z} \quad \textit{in } \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma].$$

We just remark (a) that there are some logical dependences between these properties (for example, "**P1** + **P3** $\Rightarrow$ **P5**" by [231, 14.5]) and (b) that some of these properties seem to be more crucial than others (for example, **P4** will appear almost everywhere while **P6** will not be needed in the whole discussion below).

*Remark 2.3.3.* In 2.1.14, we have seen that $C_w^\flat = C_{w^{-1}}$ for all $w \in W$, where $h \mapsto h^\flat$ is the anti-automorphism of $\mathbf{H}$ defined by $T_w^\flat = T_{w^{-1}}$. This immediately implies that

$$h_{x,y,z} = h_{y^{-1},x^{-1},z^{-1}}, \qquad \mathbf{a}(z) = \mathbf{a}(z^{-1}) \qquad \text{and} \qquad c_{x,y,z} = c_{y^{-1},x^{-1},z^{-1}}$$

for all $x, y, z \in W$. Furthermore, we have $n_z = n_{z^{-1}}$, $\Delta(z) = \Delta(z^{-1})$, $\mathscr{D} = \mathscr{D}^{-1}$.

*Remark 2.3.4.* **P14** holds for finite $W$ by Lemma 1.6.6 and Proposition 2.1.20. (See [220, 5.2] for the equal-parameter case.) The reason why it appears in the above list is that Conjecture 2.3.2 is stated in [231] for arbitrary (possibly infinite) Coxeter groups satisfying a certain boundedness condition.

*Remark 2.3.5.* Assume that we are in the equal-parameter case where $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Now $A$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. One easily checks that there is a well-defined ring homomorphism $\alpha \colon \mathbf{H} \to \mathbf{H}$ such that $\alpha(v) = -v$, $\alpha(r) = r$ for all $r \in R$ and $\alpha(T_w) = (-1)^{l(w)} T_w$ for all $w \in W$. Hence, by the characterisation in Theorem 2.1.6, we must have $\alpha(C_w) = (-1)^{l(w)} C_w$. Applying $\alpha$ to the relation $C_x C_y = \sum_{z\in W} h_{x,y,z} C_z$, we deduce that

(a) $$h_{x,y,z}(-v) = (-1)^{l(x)+l(y)+l(z)}\, h_{x,y,z}(v) \qquad \text{for all } x, y, z \in W.$$

This also implies that

(b) $$(-1)^{l(x)+l(y)+l(z)} c_{x,y,z} = (-1)^{\mathbf{a}(z)} c_{x,y,z} \qquad \text{for all } x, y, z \in W.$$

(These observations are due to Lusztig [222, 3.2].)

*Remark 2.3.6.* Let $\mathbf{J}$ be the free $\mathbb{Z}$-module with basis $\{t_w \mid w \in W\}$. We define an element of $\mathbf{J}$ by $1_{\mathbf{J}} := \sum_{d\in\mathscr{D}} n_d t_d$. We define a bilinear product on $\mathbf{J}$ by

$$t_x t_y = \sum_{z\in W} \gamma_{x,y,z} t_{z^{-1}}, \qquad \text{where} \qquad \gamma_{x,y,z} := (-1)^{l(x)+l(y)+l(z)} c_{x,y,z^{-1}}.$$

(Note that this agrees with the notation in [231, 13.6].) Using **P1**, **P4**, one deduces that $\mathbf{J}$ is an associative ring, where $1_{\mathbf{J}}$ is the identity. Since we will not need this construction here, we refer to Lusztig [231, Chap. 18] for further details. For the identification with our algebra $\tilde{\mathbf{J}}$, see Proposition 2.3.16 below.

*Remark 2.3.7.* We note that **P15** really is a statement about a certain bimodule structure (which appeared in [216], [222, 9.2]). To see this, consider the ring $\mathscr{A} = R[\Gamma] \otimes_R R[\Gamma]$ and let $\mathscr{E}$ be a free $\mathscr{A}$-module with basis $\{e_z \mid z \in W\}$. Let

$$\begin{aligned} \mathbf{H}_1 &= \mathscr{A} \otimes_A \mathbf{H}, && \text{where } A \text{ is embedded into } \mathscr{A} \text{ via } a \mapsto 1 \otimes a, \\ \mathbf{H}_2 &= \mathscr{A} \otimes_A \mathbf{H}, && \text{where } A \text{ is embedded into } \mathscr{A} \text{ via } a \mapsto a \otimes 1. \end{aligned}$$

By **P4** and the definition of the Kazhdan–Lusztig pre-order relation $\leqslant_{\mathscr{LR}}$, there is a left action of $\mathbf{H}_1$ on $\mathscr{E}$ via

$$C_w.e_x = \sum_{z \in W : \mathbf{a}(z) = \mathbf{a}(x)} (1 \otimes h_{w,x,z})\, e_z \qquad \text{for } x, w \in W.$$

Similarly, there is a right action of $\mathbf{H}_2$ on $\mathscr{E}$ via

$$e_x.C_w = \sum_{z \in W : \mathbf{a}(z) = \mathbf{a}(x)} (h_{x,w,z} \otimes 1)\, e_z \qquad \text{for } x, w \in W.$$

Now let $x, w, w' \in W$. Then

$$C_w.(e_x.C_{w'}) = \sum_{\substack{z \in W \\ \mathbf{a}(z) = \mathbf{a}(x)}} (h_{x,w',z} \otimes 1)\, C_w.e_z = \sum_{\substack{y,z \in W \\ \mathbf{a}(y) = \mathbf{a}(z) = \mathbf{a}(x)}} (h_{x,w',z} \otimes h_{w,z,y})\, e_y.$$

Here, we recognise the terms appearing on the left-hand side of **P15**. Similarly, when we expand $(C_w.e_x).C_{w'}$, we will recognise the terms appearing on the right-hand side of **P15**. Thus, we conclude that **P15** holds if and only if $\mathscr{E}$ is an $(\mathbf{H}_1, \mathbf{H}_2)$-bimodule.

As already mentioned in the introduction to this section, Lusztig [231, Chap. 20] has shown that ($\clubsuit'$) formally follows from **P1**–**P15**. We will now give a somewhat streamlined exposition of this deduction which, eventually, only requires **P1**, **P4**. (The stronger property ($\clubsuit$) will be considered in the next section.) For this purpose, we need to relate the functions $\mathbf{a}(z)$ and $\mathbf{a}_\lambda$. The following result (which first appeared in [114]) seems to be the only known connection between these two functions which can be proved without assuming any of the properties **P1**–**P15**.

**Lemma 2.3.8.** *Let $\lambda \in \Lambda$ and $w \in W$ be such that $E^\lambda \leftrightsquigarrow_L w$. Then $\mathbf{a}(w) \geqslant \mathbf{a}_\lambda$.*

*Proof.* By assumption, there exist some $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ such that $c_{w,\lambda}^{\mathfrak{st}} \neq 0$. Furthermore, by 2.1.19, we have $\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{st}}^\lambda(D_w) \equiv c_{w,\lambda}^{\mathfrak{st}} \mod \mathfrak{m}$. Now we claim that

(a) $$\rho_{\mathfrak{st}}^\lambda(D_w) = \sum_{x,y \in W} \mathbf{c}_\lambda^{-1}\, \rho_{\mathfrak{st}}^\lambda(D_{x^{-1}})\, \rho_{\mathfrak{ss}}^\lambda(D_{y^{-1}})\, h_{x,y,w^{-1}}.$$

This is seen as follows. Let $x,y \in W$. Then $h_{x,y,w^{-1}} = \tau(C_xC_yD_w)$; see 2.1.13. Furthermore, $\tau = \sum_{\mu\in\Lambda} \mathbf{c}_\mu^{-1}\chi^\mu$ and so $h_{x,y,w^{-1}} = \sum_{\mu\in\Lambda} \mathbf{c}_\mu^{-1}\chi^\mu(C_xC_yD_w)$. This yields

$$h_{x,y,w^{-1}} = \sum_{\mu\in\Lambda}\sum_{\mathfrak{u},\mathfrak{u}',\mathfrak{v}\in M(\mu)} \mathbf{c}_\mu^{-1}\,\rho^\mu_{\mathfrak{u}\mathfrak{u}'}(C_x)\,\rho^\mu_{\mathfrak{u}'\mathfrak{v}}(C_y)\,\rho^\mu_{\mathfrak{v}\mathfrak{u}}(D_w).$$

Now multiply on both sides by $\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(D_{x^{-1}})\,\rho^\lambda_{\mathfrak{s}\mathfrak{s}}(D_{y^{-1}})$ and sum over all $x,y\in W$. Using the Schur relations in Proposition 1.2.12, a straightforward computation yields (a).

Now note that $\mathbf{c}_\lambda^{-1} = f_\lambda^{-1}\,\varepsilon^{2\mathbf{a}_\lambda}/(1+g_\lambda)$, where $g_\lambda \in F[\Gamma_{>0}]$. Hence, we obtain

$$\varepsilon^{\mathbf{a}(w)}\,\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(D_w) = \sum_{x,y\in W}\frac{f_\lambda^{-1}}{1+g_\lambda}\big(\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(D_{x^{-1}})\big)\big(\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{s}\mathfrak{s}}(D_{y^{-1}})\big)\big(\varepsilon^{\mathbf{a}(w)}h_{x,y,w^{-1}}\big).$$

All terms in the above sum lie in $\mathscr{O}_0$; see 2.1.19 and also note that $\mathbf{a}(w)=\mathbf{a}(w^{-1})$ by Remark 2.3.3. Hence the whole sum will lie in $\mathscr{O}_0$ and so $\varepsilon^{\mathbf{a}(w)}\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(\mathbf{D}_w)\in\mathscr{O}_0$. Since $\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(D_w)\not\equiv 0 \bmod \mathfrak{m}$, we conclude that $\mathbf{a}(w)\geqslant\mathbf{a}_\lambda$, as claimed. ☐

**Lemma 2.3.9.** *Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell and $\lambda\in\Lambda$ be such that $m(\mathfrak{C},\lambda)>0$.*

*(a) If $y\in W$ is such that $\rho^\lambda(C_y)\neq 0$, then $y'\leqslant_{\mathscr{R}} y$ for some $y'\in\mathfrak{C}$.*
*(b) If $z\in W$ is such that $\rho^\lambda(D_{z^{-1}})\neq 0$, then $z\leqslant_{\mathscr{R}} z'$ for some $z'\in\mathfrak{C}$.*

*Proof.* Since $m(\mathfrak{C},\lambda)>0$, we have that $E^\lambda_\varepsilon$ is an irreducible constituent of $[\mathfrak{C}]_K$; see 2.2.2. Now assume that $\rho^\lambda(C_y)\neq 0$; that is, $C_y$ does not act as zero on $E^\lambda_\varepsilon$. Then $C_y$ cannot act as zero on $[\mathfrak{C}]_K$ either. By the definition of this action, there exist some $x,y'\in\mathfrak{C}$ such that $h_{y,x,y'}\neq 0$. In particular, $y'\leqslant_{\mathscr{R}} y$. Thus, (a) is proved. Now assume that $\rho^\lambda(D_{z^{-1}})\neq 0$. Then $D_{z^{-1}}$ cannot act as zero on $[\mathfrak{C}]_K$. So, by the definition of this action, there exists some $z'\in\mathfrak{C}$ such that $D_{z^{-1}}C_{z'}\neq 0$. Since $\tau$ is non-degenerate, there exists some $x\in W$ such that $\tau(C_xD_{z^{-1}}C_{z'})\neq 0$. This yields $h_{z',x,z}=\tau(C_{z'}C_xD_{z^{-1}})=\tau(C_xD_{z^{-1}}C_{z'})\neq 0$ and so $z\leqslant_{\mathscr{R}} z'$, as required. ☐

**Lemma 2.3.10.** *Assume that* **P4** *holds. Let $x,y,z\in W$. Then*

$$c_{x,y,z} = \sum_\lambda \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u}\in M(\lambda)} f_\lambda^{-1}\,c^{\mathfrak{s}\mathfrak{t}}_{x,\lambda}\,c^{\mathfrak{t}\mathfrak{u}}_{y,\lambda}\,c^{\mathfrak{u}\mathfrak{s}}_{z,\lambda},$$

*where the first sum runs over all $\lambda\in\Lambda$ such that $\mathbf{a}_\lambda=\mathbf{a}(z)$.*

*Proof.* We have $h_{x,y,z^{-1}}=\tau(C_xC_yD_z)$ and $\tau=\sum_{\lambda\in\Lambda}\mathbf{c}_\lambda^{-1}\chi^\lambda$. This yields

$$\begin{aligned} h_{x,y,z^{-1}} &= \sum_{\lambda\in\Lambda}\mathbf{c}_\lambda^{-1}\,\mathrm{trace}\big(\rho^\lambda(C_x)\,\rho^\lambda(C_y)\,\rho^\lambda(D_z)\big)\\ &= \sum_{\lambda\in\Lambda}\sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u}\in M(\lambda)}\mathbf{c}_\lambda^{-1}\,\rho^\lambda_{\mathfrak{s}\mathfrak{t}}(C_x)\,\rho^\lambda_{\mathfrak{t}\mathfrak{u}}(C_y)\,\rho^\lambda_{\mathfrak{u}\mathfrak{s}}(D_z).\end{aligned}$$

Now note that $\mathbf{c}_\lambda^{-1}=f_\lambda^{-1}\,\varepsilon^{2\mathbf{a}_\lambda}/(1+g_\lambda)$, where $g_\lambda\in F[\Gamma_{>0}]$. Hence, we obtain

$$\varepsilon^{\mathbf{a}(z)} h_{x,y,z^{-1}} = \sum_{\lambda \in \Lambda} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} \frac{f_\lambda^{-1}}{1+g_\lambda} \left(\varepsilon^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{st}}(C_x)\right)\left(\varepsilon^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{tu}}(C_y)\right)\left(\varepsilon^{\mathbf{a}(z)} \rho^\lambda_{\mathfrak{us}}(D_z)\right).$$

By 2.1.19, the terms $\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{st}}(C_x)$, $\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{tu}}(C_y)$ and $\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{us}}(D_z)$ lie in $\mathscr{O}_0$. Let $\lambda \in \Lambda$ be such that $\rho^\lambda_{\mathfrak{us}}(D_z) \neq 0$. Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell such that $m(\mathfrak{C},\lambda) > 0$. Then, by Lemma 2.3.9(b), there exists some $w \in \mathfrak{C}$ such that $z^{-1} \leqslant_{\mathscr{R}} w$. By **P4**, we must have $\mathbf{a}(w) \leqslant \mathbf{a}(z^{-1}) = \mathbf{a}(z)$. Furthermore, by Lemma 2.2.4, there exists some $w'' \in \mathfrak{C} \cap \mathscr{F}_\lambda$. Hence, by **P4** and Lemma 2.3.8, we have $\mathbf{a}_\lambda \leqslant \mathbf{a}(w'') = \mathbf{a}(w) \leqslant \mathbf{a}(z)$. But if $\mathbf{a}_\lambda < \mathbf{a}(z)$, then $\varepsilon^{\mathbf{a}(z)}\rho^\lambda_{\mathfrak{us}}(D_z) \in \mathfrak{m}$ for all $\mathfrak{s},\mathfrak{u} \in M(\lambda)$ and so these terms do not contribute to $\varepsilon^{\mathbf{a}(z)} h_{x,y,z^{-1}} \bmod \mathbb{Z}[\Gamma_{>0}]$. We conclude that

$$\varepsilon^{\mathbf{a}(z)} h_{x,y,z^{-1}} \equiv \Bigg( \sum_{\substack{\lambda \in \Lambda \\ \mathbf{a}_\lambda = \mathbf{a}(z)}} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} f_\lambda^{-1}\, c^{\mathfrak{st}}_{x,\lambda}\, c^{\mathfrak{tu}}_{y,\lambda}\, c^{\mathfrak{us}}_{z,\lambda} \Bigg) \bmod \mathfrak{m}.$$

This yields the desired formula for $c_{x,y,z}$. ☐

**Corollary 2.3.11.** *Assume that* **P4** *holds. Then* $\{\mathbf{a}(z) \mid z \in W\} \subseteq \{\mathbf{a}_\lambda \mid \lambda \in \Lambda\}$.

*Proof.* Given $z \in W$, let $x, y \in W$ be such that $c_{x,y,z} \neq 0$. Then Lemma 2.3.10 shows that there exists some $\lambda \in \Lambda$ such that $\mathbf{a}_\lambda = \mathbf{a}(z)$. ☐

**Lemma 2.3.12 (Lusztig [222, 6.1]).** *Assume that* **P4** *holds. Then* **P7** *also holds. Furthermore, if* $c_{x,y,z} \neq 0$, *then* $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$.

*Proof.* We first show that, for any $x', y', z \in W$, we have

$$(*) \qquad c_{x',y',z} = \text{ constant term of } \varepsilon^{\mathbf{a}(z)}\tau(T_{x'}T_{y'}D_z) \in \mathbb{Z}[\Gamma_{\geqslant 0}].$$

Indeed, as already noted in Definition 2.1.4, we have

$$T_w = C_w + \sum_{w' \in W:\, w' < w} \alpha_{w,w'}\, C_{w'}, \qquad \text{where} \qquad \alpha_{w,w'} \in \mathbb{Z}[\Gamma_{>0}].$$

Since $\mathbf{a}(z) = \mathbf{a}(z^{-1})$ and $c_{x',y',z} \equiv \varepsilon^{\mathbf{a}(z)}\tau(C_{x'}C_{y'}D_z) \bmod \mathbb{Z}[\Gamma_{>0}]$, this shows that

$$c_{x',y',z} \equiv \varepsilon^{\mathbf{a}(z)}\tau(T_{x'}T_{y'}D_z) + \sum_{x''<x'} \sum_{y''<y'} \alpha_{x',x''}\alpha_{y',y''}\varepsilon^{\mathbf{a}(z)}\tau(C_{x''}C_{y''}D_z) \bmod \mathbb{Z}[\Gamma_{>0}].$$

Since $\varepsilon^{\mathbf{a}(z)}\tau(C_{x''}C_{y''}D_z) \in \mathbb{Z}[\Gamma_{\geqslant 0}]$, we see that $(*)$ holds.

Now we argue as follows. Let $x, y, z \in W$ and set $c := c_{x,y,z}$. Assume first that $c \neq 0$. Hence, by $(*)$, we have that $\varepsilon^{\mathbf{a}(z)}\tau(T_xT_yD_z) \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ has constant term $c$. Now, writing $D_x$ in terms of the $T$-basis (see 2.1.13) and using $(*)$, we see that $\varepsilon^{\mathbf{a}(z)}\tau(D_xT_yD_z) \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ has constant term $c$ and, hence, $\varepsilon^{\mathbf{a}(z)}\tau(T_yD_zD_x) \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ has constant term $c$. Using once more the expression of $D_z$ in 2.1.13, we obtain

$$\varepsilon^{\mathbf{a}(z)}\tau(T_yT_zD_x) = \varepsilon^{\mathbf{a}(z)}\tau(T_yD_zD_x) - \sum_{w \in W:\, z<w} \overline{P}^*_{ww_0,zw_0}\,\varepsilon^{\mathbf{a}(z)}\tau(T_yT_wD_x).$$

Now, since $c = c_{x,y,z} \neq 0$, we have $h_{x,y,z^{-1}} \neq 0$ and, hence, $z^{-1} \leqslant_{\mathscr{R}} x$. So, by **P4**, we have $\mathbf{a}(x) \leqslant \mathbf{a}(z^{-1}) = \mathbf{a}(z)$. Combining this with $(*)$, we deduce that $\varepsilon^{\mathbf{a}(z)}\tau(T_yT_wD_x) \in \mathbb{Z}[\Gamma_{\geqslant 0}]$ for all $y, w \in W$. Consequently,

$$\varepsilon^{\mathbf{a}(z)}\tau(T_yT_zD_x) \equiv \varepsilon^{\mathbf{a}(z)}\tau(T_yD_zD_x) \equiv c \quad \mod \mathbb{Z}[\Gamma_{\geqslant 0}].$$

Using $(*)$, this shows that $\mathbf{a}(x) \geqslant \mathbf{a}(z)$. Since we also have $\mathbf{a}(z) \geqslant \mathbf{a}(x)$, we conclude that $\mathbf{a}(x) = \mathbf{a}(z)$ and, hence, $c_{y,z,x} = c$. Since $c \neq 0$, we can repeat the whole argument with $c_{y,z,x}$ and find that $c_{z,x,y} = c$; furthermore, $\mathbf{a}(z) = \mathbf{a}(y)$.

Thus, if one of the numbers $c_{x,y,z}$, $c_{y,z,x}$, $c_{z,x,y}$ is non-zero, then these three numbers are equal to each other and we have $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. If all three numbers are zero, they are again equal. □

**Lemma 2.3.13 (Lusztig [231, 14.5]).** *Assume that* **P1** *holds. Then*

$$\sum_{d\in\mathscr{D}} (-1)^{l(d)}\, n_d\, c_{x^{-1},y,d} = \delta_{xy} \qquad \textit{for any } x, y \in W.$$

*Proof.* Since $C_{y^{-1}}C_x = \sum_{z\in W} h_{y^{-1},x,z}C_z$ and $\tau(C_z) = (-1)^{l(z)}\overline{P}^*_{1,z}$, we have

$$\begin{aligned}\tau(C_{y^{-1}}C_x) &= \sum_{z\in W} (-1)^{l(z)} h_{y^{-1},x,z}\overline{P}^*_{1,z}\\ &= \sum_{z\in W:\, P^*_{1,z}\neq 0} \big(\varepsilon^{\mathbf{a}(z)}h_{y^{-1},x,z}\big)\big((-1)^{l(z)}\varepsilon^{-\mathbf{a}(z)}\overline{P}^*_{1,z}\big).\end{aligned}$$

By 2.1.13(a), the left-hand side is congruent to $\delta_{xy}$ modulo $\mathbb{Z}[\Gamma_{>0}]$. Now consider the right-hand side. By the definition of $\Delta(z)$, we have $\varepsilon^{-\Delta(z)}\overline{P}^*_{1,z} \equiv n_z \mod \mathbb{Z}[\Gamma_{>0}]$. Since **P1** is assumed to hold, we have $\mathbf{a}(z) \leqslant \Delta(z)$. This yields that

$$\varepsilon^{-\mathbf{a}(z)}\overline{P}^*_{1,z} \equiv \begin{cases} n_z \mod \mathbb{Z}[\Gamma_{>0}] & \text{if } z \in \mathscr{D},\\ 0 \mod \mathbb{Z}[\Gamma_{>0}] & \text{otherwise.}\end{cases}$$

Hence, we obtain $\delta_{xy} \equiv \tau(C_{y^{-1}}C_x) \equiv \sum_{d\in\mathscr{D}}(-1)^{l(d)}c_{y^{-1},x,d^{-1}}n_d \mod \mathbb{Z}[\Gamma_{>0}]$. Finally, by Remark 2.3.3, we have $c_{y^{-1},x,d^{-1}} = c_{x^{-1},y,d}$, which yields the desired formula. □

**Proposition 2.3.14.** *Assume that* **P1**, **P4** *hold. If* $\lambda \in \Lambda$ *and* $w \in W$ *are such that* $E^\lambda \leftrightsquigarrow_L w$, *then* $\mathbf{a}(w) = \mathbf{a}_\lambda$. *In particular, (♣′) holds.*

*Proof.* By Lemma 2.3.8, we already know that $\mathbf{a}(w) \geqslant \mathbf{a}_\lambda$. So it will now be sufficient to prove that $\mathbf{a}(w) \leqslant \mathbf{a}_\lambda$. For this purpose, we consider the identity

$$\sum_{d\in\mathscr{D}} (-1)^{l(d)} n_d C_w C_d = \sum_{d\in\mathscr{D}, y\in W} (-1)^{l(d)} n_d h_{w,d,y} C_y.$$

Applying $\rho^\lambda$ and multiplying by $\varepsilon^{\mathbf{a}_\lambda + \mathbf{a}(w)}$, we obtain

$$\sum_{d\in\mathscr{D}} (-1)^{l(d)} n_d \varepsilon^{\mathbf{a}_\lambda + \mathbf{a}(w)} \rho^\lambda_{\mathfrak{st}}(C_wC_d) = \sum_{d\in\mathscr{D}, y\in W} (-1)^{l(d)} n_d \big(\varepsilon^{\mathbf{a}(w)}h_{w,d,y}\big)\big(\varepsilon^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{st}}(C_y)\big).$$

Assume that the terms corresponding to $d \in \mathscr{D}$, $y \in W$ give a non-zero contribution to the sum on the right-hand side; that is, $h_{w,d,y} \neq 0$ and $\rho^{\lambda}_{\mathfrak{st}}(C_y) \neq 0$. Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell such that $m(\mathfrak{C},\lambda) > 0$. By Lemma 2.3.9(a), there exists some $z' \in \mathfrak{C}$ such that $z' \leqslant_{\mathscr{R}} y$. On the other hand, since $h_{w,d,y} \neq 0$, we have $y \leqslant_{\mathscr{R}} w$. Furthermore, by Lemma 2.2.4, there exists some $w' \in \mathfrak{C} \cap \mathscr{F}_\lambda$. Thus, we obtain

$$w, w' \in \mathscr{F}_\lambda, \qquad w', z' \in \mathfrak{C}, \qquad z' \leqslant_{\mathscr{R}} y \leqslant_{\mathscr{R}} w.$$

Since every two-sided Kazhdan–Lusztig cell is a union of two-sided $\tilde{\mathbf{J}}$-cells (see Proposition 2.1.20) and also a union of left Kazhdan–Lusztig cells, we conclude that $w, w', y, z'$ all lie in the same two-sided Kazhdan–Lusztig cell. In particular, since **P4** holds, we have $\mathbf{a}(y) = \mathbf{a}(w)$. Hence, using 2.1.19, the right-hand side of the above identity can be rewritten as

$$\begin{aligned}\sum_{d\in\mathscr{D}, y\in W} &(-1)^{l(d)} n_d \big(\varepsilon^{\mathbf{a}(y)}\, h_{w,d,y}\big)\big(\varepsilon^{\mathbf{a}_\lambda}\, \rho^{\lambda}_{\mathfrak{st}}(C_y)\big)\\ &\equiv \sum_{d\in\mathscr{D}, y\in W} (-1)^{l(d)} n_d\, \gamma_{w,d,y^{-1}}\, c^{\mathfrak{st}}_{y,\lambda} \quad \mod \mathfrak{m}.\end{aligned}$$

By Lemma 2.3.12, we have $\gamma_{w,d,y^{-1}} = \gamma_{y^{-1},w,d}$. Hence, Lemma 2.3.13 yields that

$$\sum_{d\in\mathscr{D}, y\in W} (-1)^{l(d)} n_d\, \gamma_{w,d,y^{-1}}\, c^{\mathfrak{st}}_{y,\lambda} \equiv c^{\mathfrak{st}}_{w,\lambda} \quad \mod \mathfrak{m}.$$

Since $c^{\mathfrak{st}}_{w,\lambda} \neq 0$, we can go back to the left-hand side of the original identity above and conclude that $\sum_{d\in\mathscr{D}}(-1)^{l(d)} n_d \varepsilon^{\mathbf{a}_\lambda + \mathbf{a}(w)} \rho^{\lambda}_{\mathfrak{st}}(C_w C_d) \not\equiv 0 \bmod \mathfrak{m}$. Thus, we have

$$\sum_{d\in\mathscr{D}} \sum_{\mathfrak{u}\in M(\lambda)} (-1)^{l(d)} n_d \big(\varepsilon^{\mathbf{a}(w)}\, \rho^{\lambda}_{\mathfrak{su}}(C_w)\big)\big(\varepsilon^{\mathbf{a}_\lambda}\, \rho^{\lambda}_{\mathfrak{ut}}(C_d)\big) \not\equiv 0 \bmod \mathfrak{m}.$$

So there must be some $d \in \mathscr{D}$ and some $\mathfrak{u} \in M(\lambda)$ such that $\varepsilon^{\mathbf{a}(w)} \rho^{\lambda}_{\mathfrak{ut}}(C_d) \notin \mathfrak{m}$. Consequently, by Proposition 1.4.10(c) and 2.1.19, we have $\mathbf{a}_\lambda \geqslant \mathbf{a}(w)$. Thus, we have shown that $\mathbf{a}(w) = \mathbf{a}_\lambda$ if $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$.

Now assume that $\lambda, \mu \in \Lambda$ are such that $E^\lambda \sim_L E^\mu$. By definition, this means that $w \sim_{\mathscr{LR}} w'$, where $w, w' \in W$ are such that $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ and $E^\mu \leftsquigarrow\!\!\rightsquigarrow_L w'$. Using **P4**, we obtain $\mathbf{a}_\lambda = \mathbf{a}(w) = \mathbf{a}(w') = \mathbf{a}_\mu$; that is, (♣′) holds. □

*Remark 2.3.15.* In the above discussion, we have not found it necessary to use any of the properties **P2**, **P3**, **P5**, **P6**, **P13** in Lusztig's list. All of these express properties of the elements in $\mathscr{D}$. It seems that these are logically independent of **P1**, **P4**, **P15**.

Finally, we show that our algebra $\tilde{\mathbf{J}}$ constructed in Section 1.5 really is an incarnation of Lusztig's asymptotic ring $\mathbf{J}$ (see Remark 2.3.6).

**Proposition 2.3.16 (Cf. [114, §3]).** *Assume that* **P1**, **P4** *hold. Then*

$$\tilde{\gamma}_{x,y,z} = c_{x,y,z} \qquad \text{and} \qquad \tilde{n}_w = \begin{cases} (-1)^{l(w)} n_w & \text{if } w \in \mathscr{D}, \\ 0 & \text{otherwise}, \end{cases}$$

*for all $x,y,z,w \in W$. In particular, Conjecture 1.5.12(a) holds; that is, $\tilde{\gamma}_{x,y,z}$ and $\tilde{n}_w$ are integers. Furthermore, we have $\tilde{\mathscr{D}} = \mathscr{D}$.*

Thus, the map $t_w \mapsto (-1)^{l(w)} t_w$ defines an algebra isomorphism $\tilde{\mathbf{J}} \xrightarrow{\sim} \mathbb{K} \otimes_{\mathbb{Z}} \mathbf{J}$, where $\mathbf{J}$ is Lusztig's asymptotic ring; see Remark 2.3.6.

*Proof.* Let $x,y,z \in W$. By **P4** and Proposition 2.1.20, we have $\tilde{\gamma}_{x,y,z} = 0$ unless $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. The analogous statement also holds for $c_{x,y,z}$ by Lemma 2.3.12. Thus, in order to show that $\tilde{\gamma}_{x,y,z} = c_{x,y,z}$, we can assume without loss of generality that $\mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. But, in this case, we have

$$c_{x,y,z} = \sum_{\lambda} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} f_\lambda^{-1} c_{x,\lambda}^{\mathfrak{s}\mathfrak{t}} c_{y,\lambda}^{\mathfrak{t}\mathfrak{u}} c_{z,\lambda}^{\mathfrak{u}\mathfrak{s}}$$

by Lemma 2.3.10, where the first sum runs over all $\lambda \in \Lambda$ such that $\mathbf{a}_\lambda = \mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. On the other hand, we have

$$\tilde{\gamma}_{x,y,z} = \sum_{\lambda \in \Lambda} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} f_\lambda^{-1} c_{x,\lambda}^{\mathfrak{s}\mathfrak{t}} c_{y,\lambda}^{\mathfrak{t}\mathfrak{u}} c_{z,\lambda}^{\mathfrak{u}\mathfrak{s}}.$$

But, by Proposition 2.3.14, the leading matrix coefficients appearing in the above expression are zero unless $\mathbf{a}_\lambda = \mathbf{a}(x) = \mathbf{a}(y) = \mathbf{a}(z)$. Thus, the desired identity between $\tilde{\gamma}_{x,y,z}$ and $c_{x,y,z}$ is proved.

The identity in Lemma 2.3.13 and the fact that **P7** holds (see Lemma 2.3.12) now imply that $\sum_{d \in \mathscr{D}} (-1)^{l(d)} n_d t_d \in \tilde{\mathbf{J}}$ is an identity element in $\tilde{\mathbf{J}}$; see the analogous argument in the proof of Proposition 1.5.5. Since the identity element of $\tilde{\mathbf{J}}$ is uniquely determined, we obtain the desired statement about $\tilde{n}_w$. $\square$

## 2.4 On Lusztig's Conjectures, II

The conjectural properties **P1–P15** are known to hold in a number of situations (including the equal-parameter case), but a general proof is still missing. There does not even seem to be a general idea of how to prove one of the crucial properties **P1**, **P4**, **P15** for an arbitrary weight function $L$. In this section, we first give a summary about the present state of knowledge concerning the validity of **P1–P15**. This will be followed by a detailed discussion of the case where $L(s) = 0$ for some $s \in S$.

**2.4.1.** Here is a summary of the cases where **P1–P15** are known to hold.

(a) **P1–P15** hold for any finite $W$, assuming that we are in the equal-parameter case where $\Gamma = \mathbb{Z}$ and there is some $a > 0$ such that $L(s) = a$ for all $s \in S$. (Here, $A$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$.) Indeed, as already mentioned, Lusztig [231, Chap. 15] deduces **P1–P15** from the following "positivity" properties:

$$P_{y,w}^* \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}] \quad \text{and} \quad h'_{x,y,z} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}] \quad \text{for all } x,y,z,w \in W.$$

(Recall that $h'_{x,y,z} = (-1)^{l(x)+l(y)+l(z)} h_{x,y,z}$; see also Remark 2.3.5.) If $W$ is a Weyl group, then these "positivity" properties follow from a geometric interpretation; see Kazhdan and Lusztig [196], Lusztig and Vogan [233] and Springer [283]. If $W$ is of type $I_2(m)$ (any $m \geqslant 2$), $H_3$ or $H_4$, they follow by explicit computations; see Alvis [2] and DuCloux [76].

(b) **P1–P15** have been checked by explicit computations for $W$ of type $I_2(m)$ (any $m \geqslant 3$) and any weight function such that $L(s) > 0$ for $s \in S$; see [76], [114, §5].

(c) **P1–P15** have been checked by explicit computations (with the help of a computer and CHEVIE [118]) for $W$ of type $F_4$ and any weight function such that $L(s) > 0$ for $s \in S$; see [105], [114, §5].

(d) **P1–P15** hold for $W$ of type $B_n$ and any weight function $L\colon W \to \Gamma$ given by

$$B_n \qquad \overset{b}{\bullet} \overset{4}{\text{——}} \overset{a}{\bullet} \text{——} \overset{a}{\bullet} \text{——} \;\cdots\; \text{——} \overset{a}{\bullet} \qquad \text{where } b > (n-1)a > 0.$$

See the series of papers by Bonnafé, Geck, Iancu [21], [26], [108], [114], [121].

(e) **P1–P15** hold if $(W,S)$ is irreducible, $\Gamma = \mathbb{Z}$ and $L(s) = 0$ for some $s \in S$; see 2.4.8 below. (Here, we are essentially reduced to the equal-parameter case; see also Lusztig [224], [225], where a more general setting is considered.)

It is beyond the scope of this book to discuss the proofs of (a)–(d) in any more detail. An elementary proof of **P1–P15** for $W \cong \mathfrak{S}_n$ is given in [107]; see also Section 2.8. The geometric arguments used in (a) can be extended to the so-called *quasi-split case*, in which some choices of unequal parameters occur; see Table 4.1 (p. 227). (The proofs are sketched in [219] and [231, Chap. 16].) Of course, it would be highly desirable to find general proofs (at least for **P1**, **P4**, **P15**) which uniformly work for any $W, L$. The above results imply the following general statement:

**Corollary 2.4.2.** *Let $W$ be any finite Coxeter group and $L_0\colon W \to \Gamma_0$ be the universal weight function in Example 1.1.9. Let $\leqslant$ be a monomial order on $\Gamma_0$ such that we are in the "asymptotic case" as in Example 1.1.11(c). Then* **P1–P15** *hold for $W, L_0$.*

For the remainder of this section, we address in some more detail the question of what happens when $W$ is a finite Coxeter group and $L(s) = 0$ for some $s \in S$.

**2.4.3.** Let $\Omega \subseteq W$ be the parabolic subgroup generated by all $t \in S$ such that $L(t) = 0$. Then we can break down the structure of $W$ as follows. Let $W_1 \subseteq W$ be the subgroup generated by $S_1 := \{\omega s \omega^{-1} \mid \omega \in \Omega, s \in S \text{ where } L(s) > 0\}$. Then, by Bonnafé and Dyer [24, Theorem 1.1], $W_1$ is a normal subgroup of $W$ such that $W_1 \cap \Omega = \{1\}$; furthermore, we have a semidirect product decomposition

(a) $\quad W = \Omega \ltimes W_1$ and $(W_1, S_1)$ is a Coxeter system.

Given $w \in W$, let $w = s_1 \cdots s_p$ ($s_i \in S$) be a reduced expression. We denote by $l_\Omega(w)$ the number of $i \in \{1, \ldots, p\}$ such that $L(s_i) = 0$, and by $l_1(w)$ the number of $i \in \{1, \ldots, p\}$ such that $L(s_i) > 0$. (Note that these two numbers do not depend on the choice of the reduced expression.) By [24, Cor. 1.3], we have

(b) $\quad l(w) = l_\Omega(w) + l_1(w)$ and $l_1|_{W_1}$ is the length function for $(W_1, S_1)$.

(c) $l_1(\omega w \omega^{-1}) = l_1(w)$ for all $w \in W_1$ and $\omega \in \Omega$.

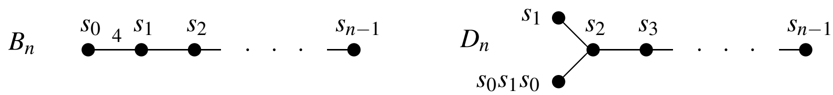
Now let $\tilde{s} \in S_1$ and write $\tilde{s} = \omega s \omega^{-1}$, where $\omega \in \Omega$ and $s \in S$ is such that $L(s) > 0$. Then one readily checks that $L(\tilde{s}) = L(s)$; moreover, we have

(d) The restriction $L|_{W_1} : W_1 \to \Gamma$ is a weight function, which we denote by $L_1$.

Indeed, let $w \in W_1$ and let $w = \tilde{s}_1 \cdots \tilde{s}_p$ ($\tilde{s}_i \in S_1$) be a reduced expression for $w$ with respect to $S_1$. For each $i$, let $s_i \in S$ and $\omega_i \in \Omega$ be such that $\tilde{s}_i = \omega_i s_i \omega_i^{-1}$ and $L(s_i) > 0$. Writing each $\omega_i$ as a product of generators $t \in S$ such that $L(t) = 0$, we obtain an expression for $w$ in terms of $S$ which is not necessarily reduced. But we can extract a reduced expression from it, and this reduced expression will contain the factors $s_1, \ldots, s_p$ and various generators $t \in S$ such that $L(t) = 0$. (By (b), all the factors $s_1, \ldots, s_p$ must occur since $l_1(w) = p$.) Thus, we have $L(w) = L(s_1) + \cdots + L(s_p)$. The argument also shows that $L(\tilde{s}_i) = L(s_i)$ for all $i$ and so $L(w) = L(\tilde{s}_1) + \cdots + L(\tilde{s}_p)$. Hence, $L|_{W_1} : W_1 \to \Gamma$ is a weight function, as required.

**Example 2.4.4.** Assume that $(W,S)$ is irreducible and that $\{1\} \neq \Omega \neq W$. According to the classification in Table 1.1, we are in one of the following cases.

(a) Let $(W,S)$ be of type $B_n$, where the generators are labelled as in the diagram below. Let $L(s_0) = 0$ and $L(s_1) = \cdots = L(s_{n-1}) > 0$. Then $\Omega = \{1, s_0\}$ and $S_1 = \{s_0 s_1 s_0, s_1, s_2, \ldots, s_{n-1}\}$. The Coxeter system $(W_1, S_1)$ is of type $D_n$:




(b) Let $(W,S)$ be of type $B_n$ (with generators labelled as above), but now let $L(s_0) > 0$ and $L(s_1) = \cdots = L(s_{n-1}) = 0$. Then $\Omega = \langle s_1, s_2, \ldots, s_{n-1}\rangle \cong \mathfrak{S}_n$ and $S_1 = \{t_1, t_2, \ldots, t_n\}$, where $t_1 = s_0$ and $t_i = s_{i-1} t_{i-1} s_{i-1}$ for $2 \leqslant i \leqslant n$. One easily checks that all the $t_i$ commute with each other and so $(W_1, S_1)$ is of type $A_1 \times \cdots \times A_1$ ($n$ factors).

(c) Let $(W,S)$ be of type $F_4$, where the generators are labelled as in Table 1.1. Let $L(s_1) = L(s_2) = 0$ and $L(s_3) = L(s_4) > 0$. Then $\Omega = \langle s_1, s_2 \rangle \cong \mathfrak{S}_3$ and $S_1 = \{s_3, s_4, s_2 s_3 s_2, s_1 s_2 s_3 s_2 s_1\}$. One easily checks that $(W_1, S_1)$ is of type $D_4$, where $s_3$, $s_2 s_3 s_2$ and $s_1 s_2 s_3 s_2 s_1$ commute with each other,

(d) Let $(W,S)$ be of type $I_2(m)$, where $m \geqslant 4$ is even. Write $S = \{s_1, s_2\}$ and let $L(s_1) > 0$ and $L(s_2) = 0$. Then $\Omega = \{1, s_2\rangle$ and $S_1 = \{s_1, s_2 s_1 s_2\}$. One easily checks that $(W_1, S_1)$ is of type $I_2(m/2)$.

Note that, in all of the above cases, $L_1$ is a multiple of the length function of $W_1$.

On the level of $\mathbf{H}$, we have the following result.

**Proposition 2.4.5 (Cf. Bonnafé [22, §2.E]).** *Assume we are in the setting of 2.4.3.*

*(a) For all $\omega, \omega' \in \Omega$ and $w \in W$, we have*

$$T_\omega T_w = T_{\omega w}, \qquad T_w T_\omega = T_{w\omega}, \qquad T_\omega T_{\omega'} = T_{\omega\omega'}, \qquad T_{\omega^{-1}} = T_\omega^{-1}.$$

*(b) For any $\tilde{s} \in S_1$ and $w \in W_1$, we have*

$$T_{\tilde{s}}T_w = \begin{cases} T_{\tilde{s}w} & \text{if } l_1(\tilde{s}w) > l_1(w), \\ T_{\tilde{s}w} + (\varepsilon^{L(\tilde{s})} - \varepsilon^{-L(\tilde{s})})T_w & \text{if } l_1(\tilde{s}w) < l_1(w). \end{cases}$$

*In particular, $\mathbf{H}_1 := \langle T_w \mid w \in W_1 \rangle_A \subseteq \mathbf{H}$ is a subalgebra, and this is the generic Iwahori–Hecke algebra associated with $(W_1, S_1)$ and $L_1 \colon W_1 \to \Gamma$.*

*(c) Let $\{C_w \mid w \in W\}$ be the Kazhdan–Lusztig basis of $\mathbf{H}$. Then*

$$C_\omega C_w = C_{\omega w}, \qquad C_w C_\omega = C_{w\omega}, \qquad C_\omega = T_\omega$$

*for all $\omega \in \Omega$ and $w \in W$. Furthermore, if $w \in W_1$, then $C_w \in \mathbf{H}_1$ and this is the Kazhdan–Lusztig basis element constructed within $\mathbf{H}_1$.*

*Proof.* (a) Let $t \in S$ be such that $L(t) = 0$. Then $T_tT_w = T_{tw}$ and $T_wT_t = T_{wt}$ for all $w \in W$ (independently of whether $l(tw) > l(w)$ or $l(tw) < l(w)$). This yields that $T_\omega T_w = T_{\omega w}$ and $T_wT_\omega = T_{w\omega}$ for all $w \in W$ and $\omega \in \Omega$. Hence, (a) follows.

(b) Let $\tilde{s} \in S_1$ and write $\tilde{s} = \omega s \omega^{-1}$ where $\omega \in \Omega$ and $s \in S$ is such that $L(s) > 0$. Let $w \in W_1$ and set $w' = \omega^{-1} w \omega$. Using 2.4.3(b), we obtain

$$l_1(\tilde{s}w) = l_1(\omega^{-1}\tilde{s}w\omega) = l_1(s\omega^{-1}w\omega) = l_1(sw') = l(sw') - l_\Omega(sw').$$

We certainly have $l_\Omega(sw') = l_\Omega(w')$. Using 2.4.3(b), this yields that

$$l_1(\tilde{s}w) - l_1(w) = l(sw') - l(w').$$

Now assume that $l_1(\tilde{s}w) > l_1(w)$. Then the above relation implies that $l(sw') > l(w')$ and so $T_sT_{w'} = T_{sw'}$. By (a), we have $T_{\tilde{s}} = T_\omega T_s T_\omega^{-1}$ and $T_w = T_\omega T_{w'} T_\omega^{-1}$. This yields

$$T_{\tilde{s}}T_w = T_\omega\big(T_sT_{w'}\big)T_\omega^{-1} = T_\omega T_{sw'} T_\omega^{-1} = T_{\omega s w' \omega^{-1}} = T_{\tilde{s}w},$$

as required. Similarly, if $l_1(\tilde{s}w) < l_1(w)$, then $l(sw') < l(w')$ and so $T_sT_{w'} = T_{sw} + (\varepsilon^{L(s)} - \varepsilon^{-L(s)})T_{w'}$. Using (a), we deduce that

$$\begin{aligned} T_{\tilde{s}}T_w &= T_\omega\big(T_sT_{w'}\big)T_\omega^{-1} = T_\omega\big(T_{sw'} + (\varepsilon^{L(s)} - \varepsilon^{-L(s)})T_{w'}\big)T_\omega^{-1} \\ &= \big(T_{\omega s w' \omega^{-1}} + (\varepsilon^{L(s)} - \varepsilon^{-L(s)})T_{\omega w' \omega^{-1}}\big) = T_{\tilde{s}w} + (\varepsilon^{L(s)} - \varepsilon^{-L(s)})T_w, \end{aligned}$$

as required; note that $L(\tilde{s}) = L(s)$. Once these relations are established, we see that $T_{\tilde{s}}T_w \in \mathbf{H}_1$ for all $\tilde{s} \in S_1$ and $w \in W_1$. It follows that $\mathbf{H}_1 \subseteq \mathbf{H}$ is a subalgebra. The relations then show that $\mathbf{H}_1 \cong \mathbf{H}_A(W_1, S_1, L_1)$, as required.

(c) By the formulae in Theorem 2.1.8, Example 2.1.5 and Remark 2.3.3, we have $C_t = T_t$, $C_tC_w = C_{tw}$ and $C_wC_t = C_{wt}$ for $w \in W$ and $t \in S$ such that $L(t) = 0$. This immediately yields the formulae for $C_\omega$, $C_\omega C_w$ and $C_w C_\omega$, where $\omega \in \Omega$. Now let $w \in W_1$ and denote by $\tilde{C}_w$ the Kazhdan–Lusztig basis element constructed inside $\mathbf{H}_1$. In order to show that $\tilde{C}_w = C_w$, we verify that $\tilde{C}_w$ satisfies the two conditions in Theorem 2.1.6 (with respect to $W$). We have $\tilde{C}_w \in T_w + \sum_{y \in W_1} \mathbb{Z}[\Gamma_{>0}]T_y$ and so the first condition is satisfied. Now let $\tilde{w}_0 \in W_1$ be the longest element (with respect to

$S_1$). Then $\tilde{w}_0^{-1}w_0 \in \Omega$ and so we can write $w_0 = \tilde{w}_0\omega_0$, where $\omega_0 \in \Omega$. By Theorem 2.1.6 (applied to $W_1$), we have $\tilde{C}_w T_{\tilde{w}_0} \in \sum_{y\in W_1} \mathbb{Z}[\Gamma_{\leqslant 0}]T_y$. Since $T_y T_{\omega_0} = T_{y\omega_0}$ for all $y \in W_1$, we deduce that $\tilde{C}_w T_{w_0} = \tilde{C}_w T_{\tilde{w}_0} T_{\omega_0} \in \sum_{y\in W_1} \mathbb{Z}[\Gamma_{\leqslant 0}]T_{y\omega_0}$ and so the second condition also holds. Hence, we must have $\tilde{C}_w = C_w$, as required. □

*Remark 2.4.6.* Define $\mathbf{H}_\omega := T_\omega.\mathbf{H}_1 = \mathbf{H}_1.T_\omega$ for $\omega \in \Omega$. Then Proposition 2.4.5 shows that

$$\mathbf{H} = \bigoplus_{\omega\in\Omega} \mathbf{H}_\omega \qquad \text{and} \qquad \mathbf{H}_\omega.\mathbf{H}_{\omega'} = \mathbf{H}_{\omega\omega'} \quad \text{for all } \omega,\omega' \in \Omega.$$

Thus, $\mathbf{H}$ is an *extended Iwahori–Hecke algebra* and the subspaces $\{\mathbf{H}_\omega \mid \omega \in \Omega\}$ form an $\Omega$-graded *Clifford system* in $\mathbf{H}$, in the sense of [53, Def. 11.12].

**2.4.7.** Let $w \in W$ and write $w = \omega w_1$, where $\omega \in \Omega$ and $w_1 \in W_1$. By Proposition 2.4.5(c), we have $T_w = T_\omega T_{w_1}$, $C_\omega = T_\omega$ and $C_w = C_\omega C_{w_1}$, where $C_{w_1}$ is the Kazhdan–Lusztig basis element defined within $\mathbf{H}_1$. Hence, we obtain

$$\sum_{y\in W} (-1)^{l(\omega w_1)+l(y)} \bar{P}^*_{y,\omega w_1} T_y = C_{\omega w_1} = C_\omega C_{w_1} = \sum_{y_1\in W_1} (-1)^{l(w_1)+l(y_1)} \bar{P}^*_{y_1,w_1} T_{\omega y_1},$$

where $P^*_{y_1,w_1}$ is defined by the element $C'_{w_1} \in \mathbf{H}_1$. Thus, given any $y \in W$ and writing $y = \omega' y_1$, where $\omega' \in \Omega$, $y_1 \in W_1$, we have

(a) $$P^*_{y,w} = \begin{cases} P^*_{y_1,w_1} & \text{if } \omega = \omega', \\ 0 & \text{otherwise.} \end{cases}$$

A similar relation can be established for the structure constants $h_{x,y,z}$. By definition, given $x_1, x_2 \in W_1$ and $\omega_1, \omega_2 \in \Omega$, we have

$$C_{\omega_1 x_1} C_{\omega_2 x_2} = \sum_{\omega_3\in\Omega, x_3\in W_1} h_{\omega_1 x_1, \omega_2 x_2, \omega_3 x_3} C_{\omega_3 x_3}.$$

Note that, if $\omega_1 = \omega_2 = \omega_3 = 1$, then $h_{x_1,x_2,x_3}$ is a structure constant with respect to the Kazhdan–Lusztig basis in $\mathbf{H}_1$. By the relations in Proposition 2.4.5(c), we have

(b) $$C_\omega^{-1} = C_{\omega^{-1}} \qquad \text{and} \qquad C_{\omega^{-1}} C_{w_1} C_\omega = C_{\omega^{-1} w_1 \omega}$$

for any $\omega \in \Omega$ and $w_1 \in W_1$. Using these relations, we obtain

$$\begin{aligned} C_{\omega_1 x_1} C_{\omega_2 x_2} &= C_{\omega_1} C_{x_1} C_{\omega_2} C_{x_2} = C_{\omega_1\omega_2} C_{\omega_2^{-1} x_1 \omega_2} C_{x_2} \\ &= \sum_{x_3\in W_1} h_{\omega_2^{-1} x_1 \omega_2, x_2, x_3} C_{\omega_1\omega_2 x_3}. \end{aligned}$$

Thus, for any $x_3 \in W_1$ and $\omega_3 \in \Omega$, we have

(c) $$h_{\omega_1 x_1, \omega_2 x_2, \omega_3 x_3} = \begin{cases} h_{\omega_2^{-1} x_1 \omega_2, x_2, x_3} & \text{if } \omega_3 = \omega_1\omega_2, \\ 0 & \text{otherwise.} \end{cases}$$

We see that the structure constants for the Kazhdan–Lusztig basis in $\mathbf{H}$ are completely determined by the structure constants inside $\mathbf{H}_1$.

**2.4.8.** Let $\mathbf{a}(z)$, $\Delta(z)$ $(z \in W)$ and $\mathscr{D}$ be defined as in 2.3.1, with respect to the weight function $L\colon W \to \Gamma$. Define $\mathbf{a}_1(z_1)$, $\Delta_1(z_1)$ $(z_1 \in W_1)$ and $\mathscr{D}_1$ analogously, with respect to the weight function $L_1\colon W_1 \to \Gamma$. Then 2.4.7(c) shows that

(a) $$\mathbf{a}(\omega z_1) = \mathbf{a}_1(z_1) \qquad \text{for all } \omega \in \Omega \text{ and } z_1 \in W_1.$$

Furthermore, since $L_1(\tilde{s}) > 0$ for all $\tilde{s} \in S_1$, we have $P^*_{1,z_1} \neq 0$ for all $z_1 \in W_1$. Then 2.4.7(a) shows that

(b) $$\Delta(\omega z_1) = \Delta_1(z_1) \quad (\text{if } \omega = 1) \qquad \text{and} \qquad \mathscr{D} = \mathscr{D}_1.$$

Assume now that $\Gamma = \mathbb{Z}$ and that $L_1$ is a multiple of the length function of $W_1$. In particular, $A$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. Then the "positivity" properties in 2.4.1(a) hold for $W_1, L_1$. Using Remark 2.1.7 and the formulae in 2.4.7, we conclude that these "positivity" properties also hold for $W, L$:

$$P^*_{y,w} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}] \quad \text{and} \quad h'_{x,y,z} \in \mathbb{Z}_{\geqslant 0}[v, v^{-1}] \quad \text{for all } x, y, z, w \in W.$$

Taking into account (a) and (b), we can now follow Lusztig's arguments in [231, Chap. 15] to conclude that **P1–P15** hold for $W, L$. In particular, we see that **P1–P15** hold for $W, L$ in all situations described in Example 2.4.4 (where $\Gamma = \mathbb{Z}$).

**Proposition 2.4.9 (Bonnafé [22, §2.E]).**

*(a) Let $x_1, x_2 \in W_1$ and $\omega_1, \omega_2 \in \Omega$. Then $\omega_1 x_1 \leqslant_{\mathscr{L}} \omega_2 x_2$ (with respect to $L$) if and only if $x_1 \leqslant_{\mathscr{L}} x_2$ (with respect to $L_1$). Similarly, $x_1\omega_1 \leqslant_{\mathscr{R}} x_2\omega_2$ (with respect to $L$) if and only if $x_1 \leqslant_{\mathscr{R}} x_2$ (with respect to $L_1$).*

*(b) The left cells of $W$ (with respect to $L$) are of the form $\Omega.\mathfrak{C}_1$ where $\mathfrak{C}_1$ is a left cell of $W_1$ (with respect to $L_1$). The left cell module $[\Omega.\mathfrak{C}_1]_A$ is isomorphic to the induced module $\mathrm{Ind}_{\mathbf{H}_1}^{\mathbf{H}}\left([\mathfrak{C}_1]_A\right) := \mathbf{H} \otimes_{\mathbf{H}_1} [\mathfrak{C}_1]_A$.*

*(c) Let $x_1, x_2 \in W_1$ and $\omega_1, \omega_2 \in \Omega$. Then $\omega_1 x_1 \leqslant_{\mathscr{LR}} \omega_2 x_2$ (with respect to $L$) if and only if there exists some $\omega \in \Omega$ such that $x_1 \leqslant_{\mathscr{LR}} \omega x_2 \omega^{-1}$ (with respect to $L_1$).*

*(d) The two-sided cells of $W$ (with respect to $L$) are of the form $\Omega.\mathscr{F}_1.\Omega$, where $\mathscr{F}_1$ is a two-sided cell of $W_1$ (with respect to $L_1$).*

*Proof.* (a) Assume first that $\omega_1 x_1 \leqslant_{\mathscr{L}} \omega_2 x_2$ (with respect to $L$). It is enough to consider the case where $\omega_1 x_1 \leftarrow_{\mathscr{L}} \omega_2 x_2$; that is, there exists some $w \in W$ such that $h_{w,\omega_2 x_2,\omega_1 x_1} \neq 0$. By 2.4.7(c), this structure constant equals $h_{w_1,x_2,x_1}$ (for some $w_1 \in W_1$). Consequently, we have $x_1 \leqslant_{\mathscr{L}} x_2$ (with respect to $L_1$). Conversely, assume that $x_1 \leftarrow_{\mathscr{L}} x_2$ (with respect to $L_1$). Thus, $h_{w_1,x_2,x_1} \neq 0$ for some $w_1 \in W_1$. By 2.4.7(c), $h_{w_1,x_2,x_1}$ also is a structure constant for the Kazhdan–Lusztig basis in $\mathbf{H}$ and so $x_1 \leqslant_{\mathscr{L}} x_2$ (with respect to $L$). Furthermore, $\omega_1 x_1 \sim_{\mathscr{L}} x_1$ and $\omega_2 x_2 \sim_{\mathscr{L}} x_2$. (This immediately follows from the fact that $C_\omega C_w = C_{\omega w}$ for all $\omega \in \Omega$ and $w \in W$; see Proposition 2.4.5(c).) Hence, we also have $\omega_1 x_1 \leqslant_{\mathscr{L}} \omega_2 x_2$ (with respect to $L$). The statement about the relation $\leqslant_{\mathscr{R}}$ is proved using the fact that $x \leqslant_{\mathscr{L}} y \Leftrightarrow x^{-1} \leqslant_{\mathscr{R}} y^{-1}$.

(b) The statement about the left cells is an immediate consequence of (a). Now consider the left cell module $[\Omega.\mathfrak{C}_1]_A$. This module has a basis $\{e_{\omega_1 x_1} \mid \omega_1 \in \Omega, x_1 \in \mathfrak{C}_1\}$, where the action of $C_{\omega w_1}$ ($\omega \in \Omega$, $w_1 \in W_1$) is given by

$$C_{\omega w_1}.e_{\omega_1 x_1} = \sum_{\omega_2 \in \Omega, x_2 \in \mathfrak{C}_1} h_{\omega w_1, \omega_1 x_1, \omega_2 x_2}\, e_{\omega_2 x_2}.$$

Using 2.4.7(c), we obtain that

$$C_{\omega w_1}.e_{\omega_1 x_1} = \sum_{x_2 \in \mathfrak{C}_1} h_{\omega_1^{-1} w_1 \omega_1, x_1, x_2}\, e_{\omega\omega_1 x_2}.$$

On the other hand, by definition, $\mathrm{Ind}_{\mathbf{H}_1}^{\mathbf{H}}([\mathfrak{C}_1]_A)$ has a basis $\{\omega_1 \otimes e_{x_1} \mid \omega_1 \in \Omega, x_1 \in \mathfrak{C}_1\}$, where the action of $C_{\omega w_1}$ ($\omega \in \Omega$, $w_1 \in W_1$) is given by

$$C_{\omega w_1}.(\omega_1 \otimes e_{x_1}) = \sum_{x_2 \in \mathfrak{C}_1} h_{\omega_1^{-1} w_1 \omega_1, x_1, x_2}\, (\omega\omega_1 \otimes e_{x_2}).$$

Hence, $[\Omega.\mathfrak{C}_1]_A \to \mathrm{Ind}_{\mathbf{H}_1}^{\mathbf{H}}([\mathfrak{C}_1]_A)$, $e_{\omega_1 x_1} \mapsto \omega_1 \otimes e_{x_1}$, is an $\mathbf{H}$-module isomorphism

(c) For any $\omega \in \Omega$, the map $w_1 \mapsto \omega w_1 \omega^{-1}$ is a Coxeter group automorphism of $(W_1, S_1)$. Furthermore, by 2.4.7(b), we have $C_\omega^{-1} = C_{\omega^{-1}}$ and $C_{\omega w_1 \omega^{-1}} = C_\omega C_{w_1} C_{\omega^{-1}}$ for all $w_1 \in W_1$. Hence, for any $x_1, x_2 \in W_1$, we have

$$(*) \qquad x_1 \leqslant_{\mathscr{L}} x_2 \quad \Leftrightarrow \quad \omega x_1 \omega^{-1} \leqslant_{\mathscr{L}} \omega x_2 \omega^{-1} \qquad \text{(with respect to } L_1\text{)}.$$

Now let $x_1, x_2 \in W_1$ and $\omega_1, \omega_2 \in \Omega$. Assume first that $\omega_1 x_1 \leqslant_{\mathscr{LR}} \omega_2 x_2$ (with respect to $L$). It is enough to consider the case where $\omega_1 x_1 \leqslant_{\mathscr{L}} \omega_2 x_2$ or $\omega_1 x_1 \leqslant_{\mathscr{R}} \omega_2 x_2$ (with respect to $L$). Note that, in the latter case, we have $(\omega_1 x_1 \omega_1^{-1})\omega_1 \leqslant_{\mathscr{R}} (\omega_2 x_2 \omega_2^{-1})\omega_2$. Hence, using (a), we conclude that $x_1 \leqslant_{\mathscr{L}} x_2$ or $\omega_1 x_1 \omega_1^{-1} \leqslant_{\mathscr{R}} \omega_2 x_2 \omega_2^{-1}$ (with respect to $L_1$). Setting $\omega = 1$ or $\omega = \omega_1^{-1}\omega_2$ according to these two cases, and using $(*)$, we obtain $x_1 \leqslant_{\mathscr{LR}} \omega x_2 \omega^{-1}$ (with respect to $L_1$), as required. Conversely, assume that there is some $\omega \in \Omega$ such that $x_1 \leqslant_{\mathscr{LR}} \omega x_2 \omega^{-1}$ (with respect to $L_1$). But then, by (a), we have $\omega x_2 \omega^{-1} \sim_{\mathscr{L}} x_2 \omega^{-1} \sim_{\mathscr{R}} x_2$ and so $x_1 \leqslant_{\mathscr{LR}} x_2$ (with respect to $L$).

(d) This immediately follows from (c). $\square$

*Remark 2.4.10.* Let $\mathscr{F}$ be a two-sided cell of $W$ (with respect to $L$). By Proposition 2.4.9(d), we have $\mathscr{F} \cap W_1 \neq \varnothing$. We claim that the following implication holds.

(a) If $x_1, x_2 \in \mathscr{F} \cap W_1$ are such that $x_1 \leqslant_{\mathscr{LR}} x_2$ (with respect to $L_1$), then we have $x_1 \sim_{\mathscr{LR}} x_2$ (with respect to $L_1$).

This is seen as follows. By Proposition 2.4.9(c), since $x_2 \leqslant_{\mathscr{LR}} x_1$ (with respect to $L$), there exists some $\omega \in \Omega$ such that $x_2 \leqslant_{\mathscr{LR}} \omega x_1 \omega^{-1}$ (with respect to $L_1$). Hence, since $x_1 \leqslant_{\mathscr{LR}} x_2$ (with respect to $L_1$), we have $x_1 \leqslant_{\mathscr{LR}} \omega x_1 \omega^{-1}$ (with respect to $L_1$). Relation $(*)$ in the proof of Proposition 2.4.9 shows that then we also have $\omega x_1 \omega^{-1} \leqslant_{\mathscr{LR}} \omega^2 x_1 \omega^{-2}$ (with respect to $L_1$). Repeating this argument, we obtain that $\omega^{i-1} x_1 \omega^{-(i-1)} \leqslant_{\mathscr{LR}} \omega^i x_1 \omega^{-i}$ (with respect to $L_1$), for all $i \geqslant 1$. But $\Omega$ has finite order, and so $\omega^i = 1$ for some $i \geqslant 1$. We conclude that $\omega x_1 \omega^{-1} \leqslant_{\mathscr{LR}} x_1$ and, hence, $x_2 \leqslant_{\mathscr{LR}} x_1$ (with respect to $L_1$). Thus, (a) is proved.

**2.4.11.** Assume that $\mathbb{K} \subseteq \mathbb{C}$ is a splitting field for both $W_1$ and $W$. Then we write

$$\mathrm{Irr}_{\mathbb{K}}(W_1) = \{E^{\lambda_1} \mid \lambda_1 \in \Lambda_1\} \qquad \text{and} \qquad \mathrm{Irr}_{\mathbb{K}}(W) = \{E^{\lambda} \mid \lambda \in \Lambda\}.$$

By the argument in Example 1.2.4, we have the following compatibility between specialisation and restriction, where $\lambda \in \Lambda$ and $\lambda_1 \in \Lambda_1$:

$$\begin{aligned} &\text{multiplicity of } E^{\lambda_1} \text{ in the restriction of } E^{\lambda} \text{ to } W_1 \\ =\ &\text{multiplicity of } E_\varepsilon^{\lambda_1} \text{ in the restriction of } E_\varepsilon^{\lambda} \text{ to } \mathbf{H}_{1,K} := K \otimes_A \mathbf{H}_1. \end{aligned}$$

Now, the group $\Omega$ acts on $W_1$ and, hence, on $\mathrm{Irr}_{\mathbb{K}}(W_1)$. Thus, there is an action of $\Omega$ on $\Lambda_1$ (which we write as $\lambda_1 \mapsto \omega.\lambda_1$) such that

(a) $\quad \mathrm{trace}(w_1, E^{\omega.\lambda_1}) = \mathrm{trace}(\omega^{-1} w_1 \omega, E^{\lambda_1}) \qquad$ for all $\lambda_1 \in \Lambda_1$ and $w_1 \in W_1$.

Using this notation, *Clifford's theorem* ([53, 11.1]) states the following: Let $\lambda \in \Lambda$ and $\lambda_1 \in \Lambda_1$ be such that $E^{\lambda_1}$ is a constituent of the restriction of $E^{\lambda}$ to $W_1$. Then this restriction is a direct sum of simple modules of the form $E^{\omega.\lambda_1}$, for various $\omega \in \Omega$. Since we have an $\Omega$-graded Clifford system as in Remark 2.4.6, there is also a version of Clifford's theorem on the level of $\mathbf{H}$ (see [53, (11.16)]):

(b) The restriction of $E_\varepsilon^{\lambda} \in \mathrm{Irr}(\mathbf{H}_K)$ is a direct sum of simple $\mathbf{H}_{1,K}$-modules of the form $E_\varepsilon^{\omega.\lambda_1}$, for various $\omega \in \Omega$.

Now let $\mathfrak{C}_1$ be a left cell of $W_1$ (with respect to $L_1$) and $\omega \in \Omega$. Then, by relation $(*)$ in the proof of Proposition 2.4.9(c), the set $\omega \mathfrak{C}_1 \omega^{-1}$ also is a left cell of $W_1$ (with respect to $L_1$). Now, using the formulae in 2.4.7, one sees that

$$h_{x_1,x_2,x_2} = h_{\omega x_1 \omega^{-1}, \omega x_2 \omega^{-1}, \omega x_3 \omega^{-1}} \qquad \text{for all } x_1, x_2, x_3 \in W_1.$$

Hence, the action of $C_{w_1}$ ($w_1 \in W_1$) on $\omega \mathfrak{C}_1 \omega^{-1}$ is the same as the action of $C_{\omega^{-1} w_1 \omega}$ on $\mathfrak{C}_1$. Combining this with (a), we conclude that

(c) $\qquad m(\mathfrak{C}_1, \mu_1) = m(\omega \mathfrak{C}_1 \omega^{-1}, \omega.\mu_1) \qquad$ for all $\omega \in \Omega$.

With these preparations, we obtain the following corollary.

**Corollary 2.4.12.** *Let $\lambda, \mu \in \Lambda$ and $\lambda_1, \mu_1 \in \Lambda_1$ be such that $E^{\lambda_1}$ appears in the restriction of $E^{\lambda}$ to $W_1$ and $E^{\mu_1}$ appears in the restriction of $E^{\mu}$ to $W_1$. Then $\mathscr{F}_{\lambda_1} \subseteq \mathscr{F}_{\lambda}$ and $\mathscr{F}_{\mu_1} \subseteq \mathscr{F}_{\mu}$. Furthermore, we have*

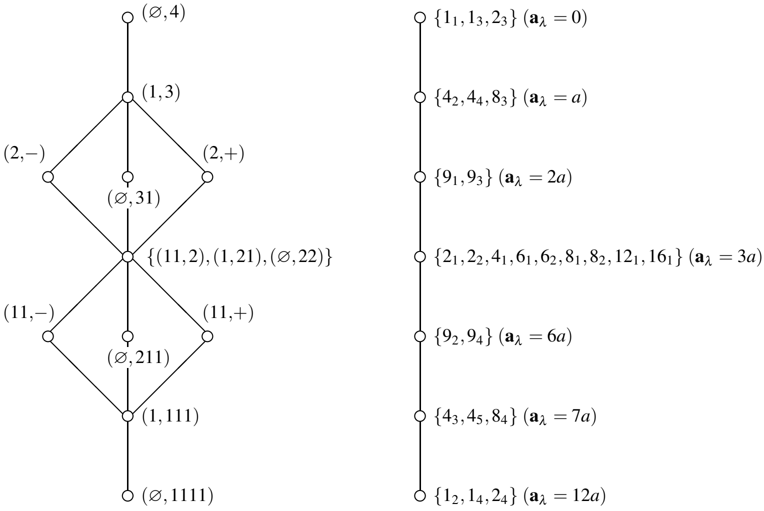
$$E^{\lambda} \preceq_L E^{\mu} \qquad \Leftrightarrow \qquad E^{\lambda_1} \preceq_{L_1} E^{\omega.\mu_1} \quad \textit{for some } \omega \in \Omega.$$

*Proof.* Let $\mathfrak{C}_1$, $\mathfrak{C}_1'$ be left cells of $W_1$ such that $m(\mathfrak{C}_1, \lambda_1) > 0$ and $m(\mathfrak{C}_1', \mu_1) > 0$. By Proposition 2.4.9(b), $\mathfrak{C} := \Omega.\mathfrak{C}_1$ and $\mathfrak{C}' := \Omega.\mathfrak{C}_1'$ are left cells of $W$; furthermore, we have $[\mathfrak{C}]_A \cong \mathrm{Ind}_{\mathbf{H}_1}^{\mathbf{H}}([\mathfrak{C}_1]_A)$ and $[\mathfrak{C}']_A \cong \mathrm{Ind}_{\mathbf{H}_1}^{\mathbf{H}}([\mathfrak{C}'_1]_A)$. Hence, by Frobenius reciprocity, we have $m(\mathfrak{C},\lambda) > 0$ and $m(\mathfrak{C}',\mu) > 0$. Then Corollary 2.2.5 also shows that $\mathscr{F}_{\lambda_1} \subseteq \mathscr{F}_\lambda$ (since $\mathfrak{C}_1 \subseteq \mathfrak{C}$) and $\mathscr{F}_{\mu_1} \subseteq \mathscr{F}_\mu$ (since $\mathfrak{C}'_1 \subseteq \mathfrak{C}'$).

Now assume that $E^\lambda \preceq_L E^\mu$. By Corollary 2.2.5, this implies that $w \leqslant_{\mathscr{LR}} w'$ (with respect to $L$), for some $w \in \mathfrak{C}$ and some $w' \in \mathfrak{C}'$. Let us write $w = \omega_1 w_1$ and $w' = \omega'_1 w'_1$ where $w_1 \in \mathfrak{C}_1$, $w'_1 \in \mathfrak{C}'_1$ and $\omega_1, \omega'_1 \in \Omega$. Then, by Proposition 2.4.9(c), we have $w_1 \leqslant_{\mathscr{LR}} \omega w'_1 \omega^{-1}$ (with respect to $L_1$) for some $\omega \in \Omega$. Using the formula in 2.4.11(c), we conclude that $E^{\lambda_1} \preceq_{L_1} E^{\omega.\mu_1}$, as required.

Conversely, assume that $E^{\lambda_1} \preceq_{L_1} E^{\omega.\mu_1}$, where $\omega \in \Omega$. By 2.4.11(c), we have $m(\omega\mathfrak{C}'_1\omega^{-1}, \omega.\mu_1) = m(\mathfrak{C}'_1, \mu_1) > 0$. So Corollary 2.2.5 implies $w_1 \leqslant_{\mathscr{LR}} \omega w'_1 \omega^{-1}$ (with respect to $L_1$), for some $w_1 \in \mathfrak{C}_1$ and some $w'_1 \in \mathfrak{C}'_1$. Hence, by Proposition 2.4.9(c), we also have $w_1 \leqslant_{\mathscr{LR}} w'_1$ (with respect to $L$). Since $w_1 \in \mathfrak{C}$ and $w'_1 \in \mathfrak{C}'_1$, we can use once more Corollary 2.2.5 and conclude that $E^\lambda \preceq_L E^\mu$. $\square$

**Fig. 2.2** Two-sided cells in type $F_4$ with parameters $0,0,a,a$; see Example 2.4.4(c)




$D_4$ (parameter $a > 0$) $\qquad$ $F_4$ (parameters $0,0,a,a$)

*Remark 2.4.13.* The above result shows that the relation $\preceq_L$ on $\mathrm{Irr}_{\mathbb{K}}(W)$ is completely determined by the relation $\preceq_{L_1}$ on $\mathrm{Irr}_{\mathbb{K}}(W_1)$; an example is given in Figure 2.2. Note that the converse is not true, at least not in any straightforward way. For example, using the above notation, assume that $\lambda = \mu$ and $\omega \in \Omega$ is such that $\lambda_1 = \omega.\mu_1 \neq \mu_1$. Then both sides of the equivalence in Corollary 2.4.3 are trivially true, but we cannot tell whether it is true that $E^{\lambda_1} \preceq_{L_1} E^{\mu_1}$ or not. One can show that, in general, we have:

(a) $$E^{\lambda_1} \preceq_{L_1} E^{\mu_1} \quad \Leftrightarrow \quad \begin{cases} \lambda_1 = \mu_1 & \text{if } \lambda = \mu, \\ E^{\lambda} \preceq_L E^{\mu} & \text{otherwise.} \end{cases}$$

(See Example 2.2.15 for the case where $W_1$ is of type $D_n$ and $W$ is of type $B_n$; see Figure 2.2, where $W_1$ is of type $D_4$ and $W$ is of type $F_4$. The remaining cases are much easier to deal with; we omit further details.)

**Proposition 2.4.14 (Cf. [132, 10.5.6], [101, 4.6]).** *Let $\lambda \in \Lambda$ and $\lambda_1 \in \Lambda_1$ be such that $E^{\lambda_1}$ appears in the restriction of $E^{\lambda}$ to $W_1$. Then*

$$\mathbf{c}_\lambda \dim E^{\lambda} = |\Omega|\,\mathbf{c}_{\lambda_1} \dim E^{\lambda_1}, \quad \mathbf{a}_\lambda = \mathbf{a}_{\lambda_1}, \quad f_\lambda \dim E^{\lambda} = |\Omega|\, f_{\lambda_1} \dim E^{\lambda_1}.$$

*Proof.* Let $d_\lambda = \dim E^{\lambda}$ and denote by $I_{d_\lambda}$ the identity matrix of size $d_\lambda$. Considering a matrix representation $\rho^{\lambda}$ afforded by $E_\varepsilon^{\lambda}$, we have that

$$d_\lambda \mathbf{c}_\lambda I_{d_\lambda} = \sum_{w \in W} \rho^{\lambda}(T_w)\,\rho^{\lambda}(T_{w^{-1}}).$$

(Indeed, the $(\mathfrak{s},\mathfrak{t})$-coefficient of the expression on the right-hand side equals

$$\sum_{w \in W} \sum_{\mathfrak{u} \in M(\lambda)} \rho^{\lambda}_{\mathfrak{s},\mathfrak{u}}(T_w)\,\rho^{\lambda}_{\mathfrak{u},\mathfrak{t}}(T_{w^{-1}}).$$

By the Schur relations in Proposition 1.2.12, this evaluates to $\delta_{\mathfrak{s}\mathfrak{t}} d_\lambda \mathbf{c}_\lambda$, as required.) Now let us write $W = \{w_1\omega \mid w_1 \in W_1, \omega \in \Omega\}$. By Proposition 2.4.5(a), we have $T_{w_1\omega} = T_{w_1}T_\omega$ and $T_\omega^{-1} = T_{\omega^{-1}}$. This yields that

$$\begin{aligned} d_\lambda \mathbf{c}_\lambda I_{d_\lambda} &= \sum_{w_1 \in W_1} \sum_{\omega \in \Omega} \rho^{\lambda}(T_{w_1})\rho^{\lambda}(T_\omega)\,\rho^{\lambda}(T_\omega^{-1})\rho^{\lambda}(T_{w_1^{-1}}) \\ &= |\Omega| \sum_{w_1 \in W_1} \rho^{\lambda}(T_{w_1})\,\rho^{\lambda}(T_{w_1^{-1}}). \end{aligned}$$

Since $E^{\lambda_1}$ appears in the restriction of $E^{\lambda}$ from $W$ to $W_1$, a specialisation argument (see Example 1.2.4) shows that $E_\varepsilon^{\lambda_1}$ appears in the restriction of $E_\varepsilon^{\lambda}$ from $\mathbf{H}_K$ to $\mathbf{H}_{1,K}$. Thus, choosing a suitable basis of $E_\varepsilon^{\lambda}$ we can assume that, for each $w_1 \in W_1$, the matrix $\rho^{\lambda}(T_{w_1})$ has a block diagonal shape, where one of the blocks equals $\rho^{\lambda_1}(T_{w_1})$. Let $d_{\lambda_1} = \dim E^{\lambda_1}$. Considering the corresponding block in the above identity arising from the Schur relations, we obtain

$$d_\lambda \mathbf{c}_\lambda I_{d_{\lambda_1}} = |\Omega| \sum_{w_1 \in W_1} \rho^{\lambda_1}(T_{w_1})\rho^{\lambda_1}(T_{w_1^{-1}}).$$

But then the sum on the right-hand side can be evaluated using the Schur relations for $\mathbf{H}_{1,K}$. This yields the desired identity $d_\lambda \mathbf{c}_\lambda = |\Omega| d_{\lambda_1} \mathbf{c}_{\lambda_1}$. Once this is established, the identities concerning $\mathbf{a}_\lambda$ and $f_\lambda$ are immediate consequences. $\square$

## 2.5 On Lusztig's Conjectures, III

Our aim now is to formulate a version of the properties **P1**–**P15** purely in terms of the invariants $\mathbf{a}_\lambda$ and our algebra $\tilde{\mathbf{J}}$ constructed in Section 1.5.

**Proposition 2.5.1 (Cf. Lusztig [231, 18.9(b)]).** *Assume that* **P1***,* **P4***,* **P15** *hold.*

(a) *Let $w, w', x, y \in W$ and assume that $\mathbf{a}(x) = \mathbf{a}(y)$. Then*

$$\sum_{z \in W} \tilde{\gamma}_{x,w',z^{-1}}\, h_{w,z,y} = \sum_{z \in W} h_{w,x,z}\, \tilde{\gamma}_{z,w',y^{-1}}.$$

(b) *Let $w, w', y \in W$ and assume that $\mathbf{a}(w') = \mathbf{a}(y)$. Then*

$$h_{w,w',y} = \sum_{\substack{z \in W,\, d \in \tilde{\mathscr{D}} \\ \mathbf{a}(z)=\mathbf{a}(d)}} \tilde{n}_d\, h_{w,d,z}\, \tilde{\gamma}_{z,w',y^{-1}}.$$

*Proof.* (a) Consider the identity **P15**; by **P4**, we can assume that on both sides the sum is over all $z \in W$ such that $a := \mathbf{a}(z) = \mathbf{a}(x) = \mathbf{a}(y)$. Now, we can write

$$\varepsilon^a h_{x,w',z} = c_{x,w',z^{-1}} + g_{x,w',z}, \qquad \text{where } g_{x,w',z} \in \mathbb{Z}[\Gamma_{>0}],$$
$$\varepsilon^a h_{z,w',y} = c_{z,w',y^{-1}} + g_{z,w',y}, \qquad \text{where } g_{z,w',y} \in \mathbb{Z}[\Gamma_{>0}].$$

Hence, multiplying both sides of **P15** by $\varepsilon^a \otimes 1$, we obtain

$$1 \otimes \Big( \sum_{z \in W:\, \mathbf{a}(z)=a} c_{x,w',z^{-1}}\, h_{w,z,y} \Big) + \sum_{z \in W:\, \mathbf{a}(z)=a} g_{x,w',z} \otimes h_{w,z,y}$$
$$= 1 \otimes \Big( \sum_{z \in W:\, \mathbf{a}(z)=a} c_{z,w',y^{-1}}\, h_{w,x,z} \Big) + \sum_{z \in W:\, \mathbf{a}(z)=a} g_{z,w',y} \otimes h_{w,x,z}.$$

Finally, $\mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$ is a free $\mathbb{Z}$-module with basis $\{\varepsilon^g \otimes \varepsilon^{g'} \mid g, g' \in \Gamma\}$. Comparing the coefficients of $1 \otimes \varepsilon^{g'}$ on both sides, we obtain the identity

$$\sum_{z \in W} c_{x,w',z^{-1}}\, h_{w,z,y} = \sum_{z \in W} h_{w,x,z}\, c_{z,w',y^{-1}}.$$

The desired identity in (a) now follows from Proposition 2.3.16.

(b) Taking $x = d \in \tilde{\mathscr{D}}$, we multiply both sides of the identity in (a) by $\tilde{n}_d$ and then sum over all $d \in \tilde{\mathscr{D}}$ such that $\mathbf{a}(d) = \mathbf{a}(y)$. This yields

$$\sum_{z \in W} \sum_{\substack{d \in \tilde{\mathscr{D}} \\ \mathbf{a}(y)=\mathbf{a}(d)}} \tilde{n}_d\, \tilde{\gamma}_{d,w',z^{-1}}\, h_{w,z,y} = \sum_{z \in W} \sum_{\substack{d \in \tilde{\mathscr{D}} \\ \mathbf{a}(y)=\mathbf{a}(d)}} \tilde{n}_d\, h_{w,d,z}\, \tilde{\gamma}_{z,w',y^{-1}}.$$

On the right-hand side, we can replace the condition "$\mathbf{a}(y) = \mathbf{a}(d)$" by the condition "$\mathbf{a}(z) = \mathbf{a}(d)$", since $\tilde{\gamma}_{z,w',y^{-1}} \neq 0$ implies $\mathbf{a}(z) = \mathbf{a}(y)$ by **P4** and Proposition 2.1.20. On the other hand, by Lemma 1.5.3(a), we have $\tilde{\gamma}_{d,w',z^{-1}} = \tilde{\gamma}_{w',z^{-1},d}$. Hence, the left-hand side of the above identity equals

$$\sum_{z\in W}\sum_{\substack{d\in\tilde{\mathscr{D}}\\ \mathbf{a}(y)=\mathbf{a}(d)}} \tilde{n}_d\,\tilde{\gamma}_{w',z^{-1},d}\,h_{w,z,y} = \sum_{z\in W}\Big(\sum_{\substack{d\in\tilde{\mathscr{D}}\\ \mathbf{a}(y)=\mathbf{a}(d)}} \tilde{\gamma}_{w',z^{-1},d}\,\tilde{n}_d\Big)h_{w,z,y}.$$

Now, if $\tilde{\gamma}_{w',z^{-1},d}\neq 0$, then $\mathbf{a}(d)=\mathbf{a}(w')$ by **P4** and Proposition 2.1.20. Since $\mathbf{a}(y)=\mathbf{a}(w')$, we can omit the condition "$\mathbf{a}(y)=\mathbf{a}(d)$" in the above sum. So Lemma 1.5.3(b) yields that the above sum evaluates to $h_{w,w',y}$, as required. □

**Corollary 2.5.2.** *Assume that* **P1**, **P4**, **P15** *hold. Then* **P9**, **P10**, **P11** *also hold.*

*Proof.* To prove **P9**, let $y,w\in W$ be such that $y\leqslant_{\mathscr{L}} w$ and $\mathbf{a}(y)=\mathbf{a}(w)$. We must show that $y\sim_{\mathscr{L}} w$. It is enough to consider the case where $y,w$ are related by an elementary step of the relation $\leqslant_{\mathscr{L}}$; that is, we have $h_{x,y,w}\neq 0$ for some $x\in W$. But then Proposition 2.5.1(b) shows that there exist some $z\in W$ and $d\in\tilde{\mathscr{D}}$ such that $\mathbf{a}(z)=\mathbf{a}(d)$, $h_{x,d,z}\neq 0$ and $\tilde{\gamma}_{z,y,w^{-1}}\neq 0$. In particular, by Lemma 1.6.5 and Proposition 2.1.20, $y$ and $w$ belong to the same Kazhdan–Lusztig left cell, as required. Once **P9** is established, **P10** and **P11** easily follow as well; see [231, 14.10, 14.11]. Indeed, to obtain **P10**, just note that $\mathbf{a}(z)=\mathbf{a}(z^{-1})$ and that $y\leqslant_{\mathscr{R}} w$ if and only if $y^{-1}\leqslant_{\mathscr{L}} w^{-1}$. Finally, to prove **P11**, let $y\leqslant_{\mathscr{LR}} w$ be such that $\mathbf{a}(y)=\mathbf{a}(w)$. By definition, there is a sequence $y=y_0,y_1,\ldots,y_m=w$ such that, for each $i\in\{1,\ldots,m\}$, we have $y_{i-1}\leqslant_{\mathscr{L}} y_i$ or $y_{i-1}\leqslant_{\mathscr{R}} y_i$. By **P4**, we have $\mathbf{a}(w)=\mathbf{a}(y_m)\geqslant\mathbf{a}(y_{m-1})\geqslant\ldots\geqslant\mathbf{a}(y_1)\geqslant\mathbf{a}(y_0)=\mathbf{a}(y)$. Since $\mathbf{a}(y)=\mathbf{a}(w)$, we have $\mathbf{a}(y)=\mathbf{a}(y_0)=\mathbf{a}(y_1)=\ldots=\mathbf{a}(y_m)=\mathbf{a}(w)$. Applying **P9** or **P10** to $y_{i-1},y_i$, we obtain $y_{i-1}\sim_{\mathscr{L}} y_i$ or $y_{i-1}\sim_{\mathscr{R}} y_i$. Hence, $y\sim_{\mathscr{LR}} w$. □

**2.5.3.** Let us consider the following three statements (♣), (♠), (♦). These should be regarded as our adaptation of Lusztig's properties **P1**–**P15** in Conjecture 2.3.2 for the purposes of this book. Note that (♣), (♠), (♦) do not refer to the function $\mathbf{a}(z)$ or to $\gamma_{x,y,z}$, as defined in 2.3.1; these have only played an auxiliary role.

(♣) Let $\lambda,\mu\in\Lambda$. If $E^\lambda\preceq_L E^\mu$, then $\mathbf{a}_\mu\leqslant\mathbf{a}_\lambda$. In particular, if $E^\lambda\sim_L E^\mu$, then $\mathbf{a}_\lambda=\mathbf{a}_\mu$. Furthermore, if $E^\lambda\preceq_L E^\mu$ and $\mathbf{a}_\mu=\mathbf{a}_\lambda$, then $E^\lambda\sim_L E^\mu$.

(♠) Let $w,w',x,y\in W$ be such that $x\sim_{\mathscr{LR}} y$. Then

$$\sum_{z\in W}\tilde{\gamma}_{x,w',z^{-1}}\,h_{w,z,y}=\sum_{z\in W}h_{w,x,z}\,\tilde{\gamma}_{z,w',y^{-1}}.$$

(Here, $\sim_{\mathscr{LR}}$ refers to the two-sided Kazhdan–Lusztig relation.)

(♦) Every Kazhdan–Lusztig left cell contains a unique element of $\tilde{\mathscr{D}}$.

Let us briefly recall how the first two statements are deduced from **P1**, **P4**, **P15**. To prove (♣), let $x\in\mathscr{F}_\lambda$ and $y\in\mathscr{F}_\mu$. By Proposition 2.3.14, we have $\mathbf{a}(x)=\mathbf{a}_\lambda$ and $\mathbf{a}(y)=\mathbf{a}_\mu$. So, if $E^\lambda\preceq_L E^\mu$, then $x\leqslant_{\mathscr{LR}} y$ and so $\mathbf{a}_\mu=\mathbf{a}(y)\leqslant\mathbf{a}(x)=\mathbf{a}_\lambda$, using **P4**. If $E^\lambda\preceq_L E^\mu$ and $\mathbf{a}_\lambda=\mathbf{a}_\mu$, then $x\leqslant_{\mathscr{LR}} y$ and $\mathbf{a}(x)=\mathbf{a}(y)$. By Corollary 2.5.2, **P11** holds and so $x\sim_{\mathscr{LR}} y$; hence, $E^\lambda\sim_{\mathscr{LR}} E^\mu$, as required. Finally, if $w,w',x,y\in W$

are as in (♠), then $\mathbf{a}(x) = \mathbf{a}(y)$ by **P4** and so the desired identity holds by Proposition 2.5.1. (One can show that, conversely, (♣) and (♠) imply **P1**, **P4**, **P15**; see [114, 3.8, 4.7].)

Finally, if **P1**, **P4**, **P13** hold, then (♦) holds, since $\tilde{\mathscr{D}} = \mathscr{D}$ by Proposition 2.3.16.

*Remark 2.5.4.* As in Remark 2.3.7, we note that (♠) really is a statement about a certain bimodule structure. Indeed, let $R \subseteq \mathbb{C}$ be an $L$-good subring and consider the algebra $\tilde{\mathbf{J}}$; see Section 1.5. Then $\tilde{\mathbf{J}}_R := \langle t_w \mid w \in W \rangle_R \subseteq \tilde{\mathbf{J}}$ is an $R$-subalgebra of $\tilde{\mathbf{J}}$ and $\tilde{\mathbf{J}} = \mathbb{K} \otimes_R \tilde{\mathbf{J}}_R$. By the identification $C_w \leftrightarrow t_w$, the natural left $\mathbf{H}$-module structure on $\mathbf{H}$ (given by left multiplication) can be transported to a left $\mathbf{H}$-module structure on $\tilde{\mathbf{J}}_A := A \otimes_R \tilde{\mathbf{J}}_R$. Explicitly, the action is given by

$$C_w * t_x = \sum_{z \in W} h_{w,x,z} t_z \qquad \text{for all } x, w \in W.$$

By the definition of the Kazhdan–Lusztig pre-order $\leqslant_{\mathscr{LR}}$, we can define a left $\mathbf{H}$-module structure on $\tilde{\mathbf{J}}_A$ by the formula

$$C_w \diamond t_x = \sum_{z \in W \,:\, z \sim_{\mathscr{LR}} x} h_{w,x,z} t_z \qquad \text{for all } x, w \in W.$$

For any $h \in \mathbf{H}$ and $x \in W$, the difference $h * t_x - h \diamond t_x$ is an $A$-linear combination of terms $t_y$, where $y \leqslant_{\mathscr{LR}} w$ and $y \not\sim_{\mathscr{LR}} w$ (in the Kazhdan–Lusztig pre-order). On the other hand, we have a natural right $\tilde{\mathbf{J}}_A$-module structure on $\tilde{\mathbf{J}}_A$ (given by right multiplication). Then these two actions commute if and only if

$$C_w \diamond (t_x t_{w'}) = (C_w \diamond t_x) t_{w'} \qquad \text{for all } x, w, w' \in W.$$

Writing this out using the defining equations, the above identity is equivalent to

$$\sum_{z \in W \,:\, z \sim_{\mathscr{LR}} y} \tilde{\gamma}_{x,w',z^{-1}} h_{w,z,y} = \sum_{z \in W \,:\, z \sim_{\mathscr{LR}} x} h_{w,x,z} \tilde{\gamma}_{z,w',y^{-1}} \qquad \text{for all } y \in W.$$

Now, by Proposition 2.1.20, we can assume that $z \sim_{\mathscr{LR}} x$ for all $z$ on the left-hand side and $z \sim_{\mathscr{LR}} y$ for all $z$ on the right-hand side. Thus, if $x \not\sim_{\mathscr{LR}} y$, then both sides of the above identity are zero. Hence, the above identity holds if and only if (♠) holds. So we conclude

(a) (♠) holds if and only if $\tilde{\mathbf{J}}_A$ is an $(\mathbf{H}, \tilde{\mathbf{J}}_A)$-bimodule (with the above actions).

Since the algebra $\mathbf{H}$ is generated by $\{C_s \mid s \in S\} \cup \{T_1\}$, we also conclude

(b) in order to verify (♠), it is sufficient to do this assuming that $w = s \in S$.

The following result was proved by Lusztig [223] in the equal-parameter case and in [231, 18.9 and 18.10] in general, assuming that **P1**–**P15** hold. Here, we follow the proof given in [112], which is much less "computational" than that in [223], [231].

**Theorem 2.5.5 (Lusztig [231, 18.9]; see also [112, §5]).** *Assume that property* (♠) *in* 2.5.3 *holds. Then there is a unique unital $A$-algebra homomorphism $\phi\colon \mathbf{H} \to \tilde{\mathbf{J}}_A$*

*such that, for any $h \in \mathbf{H}$ and $w \in W$, the difference $\phi(h)t_w - h * t_w$ is an $A$-linear combination of terms $t_y$, where $y \leqslant_{\mathscr{LR}} w$ and $y \not\sim_{\mathscr{LR}} w$. Explicitly, $\phi$ is given by*

$$\phi(C_w) = \sum_{\substack{z \in W, d \in \tilde{\mathscr{D}} \\ z \sim_{\mathscr{LR}} d}} h_{w,d,z} \tilde{n}_d t_z \qquad (w \in W).$$

*Proof.* In the setting of Remark 2.5.4, the left $\mathbf{H}$-module structure on $\tilde{\mathbf{J}}_A$ gives rise to an $A$-algebra homomorphism

$$\psi \colon \mathbf{H} \to \mathrm{End}_A(\tilde{\mathbf{J}}_A) \quad \text{such that} \quad \psi(h)(t_w) = h \diamond t_w.$$

Since the left action of $\mathbf{H}$ on $\tilde{\mathbf{J}}_A$ commutes with the right action of $\tilde{\mathbf{J}}_A$, the image of $\psi$ lies in $\mathrm{End}_{\tilde{\mathbf{J}}_A}(\tilde{\mathbf{J}}_A^{\mathrm{r}})$, where the superscript "r" indicates that we consider the right action of $\tilde{\mathbf{J}}_A$ on itself. Now, we have a natural $A$-algebra isomorphism

$$\eta \colon \mathrm{End}_{\tilde{\mathbf{J}}_A}(\tilde{\mathbf{J}}_A^{\mathrm{r}}) \to \tilde{\mathbf{J}}_A, \qquad f \mapsto f(1_{\tilde{\mathbf{J}}_A}).$$

We define $\phi = \eta \circ \psi \colon \mathbf{H} \to \tilde{\mathbf{J}}_A$. Then $\phi$ is an $A$-algebra homomorphism such that

$$\phi(h) = \psi(h)(1_{\tilde{\mathbf{J}}_A}) = h \diamond 1_{\tilde{\mathbf{J}}_A} \qquad \text{for all } h \in \mathbf{H}.$$

This yields $\phi(h)t_w = (h \diamond 1_{\tilde{\mathbf{J}}_A})t_w = h \diamond 1_{\tilde{\mathbf{J}}_A} t_w = h \diamond t_w$ or, in other words, the difference $\phi(h)t_w - h * t_w$ is an $A$-linear combination of terms $t_y$, where $y \leqslant_{\mathscr{LR}} w$ and $y \not\sim_{\mathscr{LR}} w$, as required. Furthermore, we immediately obtain the formula

$$\phi(C_w) = C_w \diamond 1_{\tilde{\mathbf{J}}_A} = \sum_{d \in \tilde{\mathscr{D}}} \tilde{n}_d C_w \diamond t_d = \sum_{z \in W, d \in \tilde{\mathscr{D}} : z \sim_{\mathscr{LR}} d} h_{w,d,z} \tilde{n}_d t_z.$$

Since $h_{1,d,z} = \delta_{d,z}$, this yields $\phi(C_1) = 1_{\tilde{\mathbf{J}}_A}$; hence, $\phi$ is unital.

Finally, assume that $\phi' \colon \mathbf{H} \to \tilde{\mathbf{J}}_A$ is another homomorphism satisfying the required conditions. But these imply that $\phi'(h)t_w = h \diamond t_w$ for all $w \in W$ and, hence, $\phi'(h) = \phi'(h)1_{\tilde{\mathbf{J}}_A} = h \diamond 1_{\tilde{\mathbf{J}}_A}$ for all $h \in \mathbf{H}$. So we have $\phi' = \phi$ as required. □

*Remark 2.5.6.* Once Theorem 2.5.5 is established, the further theory of $\tilde{\mathbf{J}}$ and $\mathbf{H}$ can be developed as in [231, Chap. 18–20], with essentially the same proofs. We just single out the following statement; cf. Lusztig [231, 18.11]:

(a) Let $\theta \colon A \to k$ be a specialisation, where $k$ is a commutative ring with 1. Let $\phi_k \colon \mathbf{H}_k \to \tilde{\mathbf{J}}_k$ be the induced map. Then $\ker(\phi_k)$ is a nilpotent ideal of $\mathbf{H}_k$.

*Proof.* Let $\mathscr{F}_1, \ldots, \mathscr{F}_N$ be the two-sided Kazhdan–Lusztig cells of $W$, where the labelling is such that if $x \leqslant_{\mathscr{LR}} y$ for all $x \in \mathscr{F}_i$ and $y \in \mathscr{F}_j$, then $i \leqslant j$. Consequently, each $\mathbf{H}_{k,\leqslant i} := \langle C_w \mid w \in \mathscr{F}_j, 1 \leqslant j \leqslant i \rangle_k$ is a two-sided ideal of $\mathbf{H}_k$. Now let $h \in \ker(\phi_k)$. Then, by Theorem 2.5.5, $h * t_w$ is a $k$-linear combination of terms $t_y$, where $y \leqslant_{\mathscr{LR}} w$ and $y \not\sim_{\mathscr{LR}} w$. Recalling the definition of the $*$-action, we deduce that $h\mathbf{H}_{k,\leqslant i} \subseteq \mathbf{H}_{k,\leqslant i-1}$ for all $i$, where we set $\mathbf{H}_{k,\leqslant 0} = \{0\}$. Hence, given $N$ elements $h_1, \ldots, h_N \in \ker(\phi_k)$, then we have $h_1 \cdots h_N \in \mathbf{H}_{k,\leqslant 0} = \{0\}$. Thus, (a) is proved. □

**Example 2.5.7 (Cf. Lusztig [231, 20.1]).** The inclusion $A \subseteq K$ induces an algebra homomorphism $\phi \colon \mathbf{H}_K \to \tilde{\mathbf{J}}_K$. Since $\mathbf{H}_K$ is semisimple, Remark 2.5.6(a) shows that $\phi$ is an isomorphism. Next, consider the specialisation $\theta_1 \colon A \to \mathbb{K}$ such that $\theta_1(\varepsilon^g) = 1$ for all $g \in \Gamma$. Let $\phi_1 \colon \mathbb{K}W \to \tilde{\mathbf{J}}$ be the induced map. Since $\mathbb{K}W$ is semisimple, Remark 2.5.6(a) shows that $\phi_1$ is an isomorphism. Thus, $\mathbb{K}W \cong \tilde{\mathbf{J}}$ as $\mathbb{K}$-algebras. Finally, the inclusion $\mathbb{K} \subseteq K$ induces an algebra isomorphism $(\phi_1)_K \colon KW \to \tilde{\mathbf{J}}_K$. Hence, the composition

$$\psi = (\phi_1)_K^{-1} \circ \phi_K \colon \mathbf{H}_K \to KW \quad \text{is an algebra isomorphism.}$$

(This first appeared in [216] in the equal-parameter case.) Thus, using $\psi$, one obtains a more natural explanation for the correspondence $\mathrm{Irr}_{\mathbb{K}}(W) \leftrightarrow \mathrm{Irr}(\mathbf{H}_K)$ in 1.2.1. But note that the results in 1.2.1 do not rely on the assumption that (♠) holds!

**Lemma 2.5.8.** *Assume that (♠) holds. Let $x, y, w \in W$ be such that $y \sim_{\mathcal{LR}} w$. Then*

$$h_{x,w,y} = \sum_{\substack{z \in W,\, d \in \tilde{\mathcal{D}} \\ z \sim_{\mathcal{LR}} d}} \tilde{n}_d \, h_{x,d,z} \, \tilde{\gamma}_{z,w,y^{-1}}.$$

*Proof.* The left-hand side of the above identity is the coefficient of $t_y$ in the expansion of $C_x * t_w$, and the right-hand side is the coefficient of $t_y$ in the expansion of $\phi(C_x)t_w$. By Theorem 2.5.5, these two coefficients must be the same. □

**Lemma 2.5.9.** *Assume that (♠) holds. Then the left $\tilde{\mathbf{J}}$-cells are precisely the left Kazhdan–Lusztig cells. (Analogous statements hold for right and two-sided cells.) Furthermore, the following implication holds for any $y, w \in W$:*

$$y \leqslant_{\mathcal{L}} w \quad \text{and} \quad y \sim_{\mathcal{LR}} w \qquad \Rightarrow \qquad y \sim_{\mathcal{L}} w,$$

*where $\leqslant_{\mathcal{L}}$, $\sim_{\mathcal{L}}$ and $\sim_{\mathcal{LR}}$ refer to the Kazhdan–Lusztig relations.*

*Proof.* Recall that, by Proposition 2.1.20, every left (or right or two-sided) $\tilde{\mathbf{J}}$-cell is contained in a left (or right or two-sided respectively) Kazhdan–Lusztig cell. To prove the reverse implications, we begin by showing the following two statements:

(a) Let $y, w \in W$ be such that $y \leqslant_{\mathcal{L}} w$ and $y \sim_{\mathcal{LR}} w$ (with respect to the Kazhdan–Lusztig relations). Then $y, w$ belong to the same left $\tilde{\mathbf{J}}$-cell (and, hence, $y \sim_{\mathcal{L}} w$).
(b) Let $y, w \in W$ be such that $y \leqslant_{\mathcal{R}} w$ and $y \sim_{\mathcal{LR}} w$ (with respect to the Kazhdan–Lusztig relations). Then $y, w$ belong to the same right $\tilde{\mathbf{J}}$-cell.

To prove (a), we may assume that $y, w$ are related by an elementary step in the Kazhdan–Lusztig pre-order relation $\leqslant_{\mathcal{L}}$; that is, we can assume that $h_{x,w,y} \neq 0$ for some $x \in W$. But then Lemma 2.5.8 shows that there exist some $z \in W$ and $d \in \tilde{\mathcal{D}}$ such that $z \sim_{\mathcal{LR}} d$, $h_{x,d,z} \neq 0$ and $\tilde{\gamma}_{z,w,y^{-1}} \neq 0$. In particular, by Lemma 1.6.5, $y$ and $w$ belong to the same left $\tilde{\mathbf{J}}$-cell. Thus, (a) is proved. The proof of (b) is analogous.

Now (a) shows that if $y \sim_{\mathcal{L}} w$, then $y, w$ belong to the same left $\tilde{\mathbf{J}}$-cell. Thus, the left Kazhdan–Lusztig cells coincide with the left $\tilde{\mathbf{J}}$-cells. Using (b), a similar statement holds for right cells. Now consider the two-sided cells. Let $y, w \in W$ be such that $y \sim_{\mathscr{LR}} w$. Then there is a sequence $y = y_0, y_1, \ldots, y_m = w$ in $W$ such that, for each $i \in \{1, \ldots, m\}$, we have $y_{i-1} \leqslant_{\mathscr{L}} y_i$ or $y_{i-1} \leqslant_{\mathscr{R}} y_i$. Since $y \sim_{\mathscr{LR}} w$, all elements $y_i$ belong to the same two-sided Kazhdan–Lusztig cell. Hence, by (a) and (b), all elements $y_i$ belong to the same two-sided $\tilde{\mathbf{J}}$-cell. In particular, $y, w$ belong to the same two-sided $\tilde{\mathbf{J}}$-cell. □

**Example 2.5.10.** Assume that (♠) holds. Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell of $W$. By Lemma 2.5.9, the set $\mathfrak{C}$ also is a left $\tilde{\mathbf{J}}$-cell. Then we claim that

$$\tilde{m}(\mathfrak{C}, \lambda) = m(\mathfrak{C}, \lambda) \qquad \text{for all } \lambda \in \Lambda,$$

where the left-hand side is defined in Theorem 1.8.1 and the right-hand side is defined in 2.2.2. Indeed, by the argument in the proof of Lemma 2.2.4, we have

$$\sum_{\mathfrak{s},\mathfrak{t} \in M(\lambda)} \sum_{w \in \mathfrak{C}} c_{w,\lambda}^{\mathfrak{st}}\, c_{w^{-1},\lambda}^{\mathfrak{ts}} = m(\mathfrak{C}, \lambda)\, d_\lambda\, f_\lambda.$$

By Theorem 1.8.1(b), the left-hand side also equals $d_\lambda \tilde{m}(\mathfrak{C}, \lambda) f_\lambda$, as required. We can now write the relations in Theorem 1.8.1 in the form

$$\sum_{w \in \mathfrak{C}} c_{w,\lambda}\, c_{w^{-1},\mu} = \begin{cases} m(\mathfrak{C}, \lambda)\, f_\lambda & \text{if } \lambda = \mu, \\ 0 & \text{otherwise.} \end{cases}$$

We close this section with some auxiliary results which will be useful at several places below. The proofs of some of these will only require the following weak version of (♣) which we already encountered at the beginning of Section 2.3 (p. 78):

$$(\clubsuit') \qquad E^\lambda \sim_L E^\mu \qquad \Rightarrow \qquad \mathbf{a}_\mu = \mathbf{a}_\lambda$$

**Lemma 2.5.11.** *Assume that (♣′) holds. Let $\mathfrak{T}$ be a two-sided Kazhdan–Lusztig cell and $a \in \Gamma_{\geqslant 0}$ be the common value of $\mathbf{a}_\lambda$, where $\lambda \in \Lambda$ is such that $\mathscr{F}_\lambda \subseteq \mathfrak{T}$. Then*

$$\varepsilon^a h_{x,y,z} \in \mathbb{Z}[\Gamma_{\geqslant 0}] \qquad \textit{and} \qquad \tilde{\gamma}_{x,y,z^{-1}} \equiv \varepsilon^a h_{x,y,z} \bmod \mathbb{Z}[\Gamma_{>0}]$$

*for all $x \in W$ and $y, z \in \mathfrak{T}$. In particular, $\tilde{\gamma}_{x,y,z^{-1}} \in \mathbb{Z}$.*

*Proof.* We have $h_{x,y,z} = \tau(C_x C_y D_{z^{-1}})$ and $\tau = \sum_{\lambda \in \Lambda} \mathbf{c}_\lambda^{-1} \chi^\lambda$. Furthermore, as in the proof of Lemma 2.3.10, $\mathbf{c}_\lambda^{-1} = f_\lambda^{-1} \varepsilon^{2\mathbf{a}_\lambda}/(1 + g_\lambda)$, where $g_\lambda \in F[\Gamma_{>0}]$. This yields

$$\varepsilon^a h_{x,y,z} = \sum_{\lambda \in \Lambda} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} \frac{f_\lambda^{-1}}{1 + g_\lambda} \big(\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{st}}^\lambda(C_x)\big)\big(\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{tu}}^\lambda(C_y)\big)\big(\varepsilon^a \rho_{\mathfrak{us}}^\lambda(D_{z^{-1}})\big).$$

Now assume that $\lambda \in \Lambda$ and $\mathfrak{s}, \mathfrak{t}, \mathfrak{u} \in M(\lambda)$ are such that all three terms

$$\varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{st}}^\lambda(C_x), \qquad \varepsilon^{\mathbf{a}_\lambda} \rho_{\mathfrak{tu}}^\lambda(C_y), \qquad \varepsilon^a \rho_{\mathfrak{us}}^\lambda(D_{z^{-1}})$$

in the above sum are non-zero. Let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell such that $m(\mathfrak{C}, \lambda) > 0$. Then, by Lemma 2.3.9, there exist $y', z' \in \mathfrak{C}$ such that $y' \leqslant_{\mathscr{R}} y$ and

$z \leqslant_{\mathscr{R}} z'$. Since $y, z \in \mathfrak{T}$, we deduce that $\mathfrak{C} \subseteq \mathfrak{T}$. By Lemma 2.2.4, there exists some $w \in \mathfrak{C} \cap \mathscr{F}_\lambda$. Since $\mathfrak{C} \subseteq \mathfrak{T}$, this implies that $\mathscr{F}_\lambda \subseteq \mathfrak{T}$ and so $a = \mathbf{a}_\lambda$. Consequently, all of the above three terms lie in $\mathscr{O}_0$. Hence, the whole sum lies in $\mathscr{O}_0$ and its constant term can be computed term by term. Thus, we obtain

$$\varepsilon^a h_{x,y,z} \equiv \sum_{\lambda \in \Lambda} \sum_{\mathfrak{s},\mathfrak{t},\mathfrak{u} \in M(\lambda)} c_{x,\lambda}^{\mathfrak{st}} c_{y,\lambda}^{\mathfrak{tu}} c_{z^{-1},\lambda}^{\mathfrak{us}} \equiv \tilde{\gamma}_{x,y,z^{-1}} \bmod \mathfrak{m}.$$

Since $h_{x,y,z} \in \mathbb{Z}[\Gamma]$, we have $\tilde{\gamma}_{x,y,z^{-1}} \in \mathbb{Z}$ and the congruences are modulo $\mathbb{Z}[\Gamma_{>0}]$. $\square$

**Proposition 2.5.12.** *Assume that we are in the equal-parameter case where $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Then property ($\spadesuit$) in 2.5.3 is a consequence of ($\clubsuit'$).*

*Proof.* By Remark 2.5.4, it is enough to prove ($\spadesuit$) assuming that $w = s \in S$. Thus, we must show that

(a) $$\sum_{z \in W} \tilde{\gamma}_{x,w',z^{-1}} h_{s,z,y} = \sum_{z \in W} h_{s,x,z} \tilde{\gamma}_{z,w',y^{-1}} \qquad \text{for all } s \in S,$$

where $w', x, y \in W$ are such that $x \sim_{\mathscr{LR}} y$ (in the Kazhdan–Lusztig pre-order). Let $\mathfrak{T}$ denote the two-sided Kazhdan–Lusztig cell such that $x, y \in \mathfrak{T}$. First we note that, by Lemmas 1.6.5 and 1.6.6, we can assume that $z \in \mathfrak{T}$ on both sides of the above identity; furthermore, we can also assume that $w' \in \mathfrak{T}$. Now we argue as follows.

If $L(s) = 0$, then $h_{s,z,y} = \delta_{zy}$; see Theorem 2.1.8. Hence, the left-hand side of (a) reduces to $\tilde{\gamma}_{x,w',y^{-1}}$. Similarly, since $h_{s,x,z} = \delta_{xz}$, the right-hand side reduces to $\tilde{\gamma}_{x,w',y^{-1}}$. Hence, the assertion is true in this case. We can assume from now on that $L(s) > 0$.

*Case 1*: $sx < x$. Then, by Theorem 2.1.8, we have $C_s C_x = -(v_s + v_s^{-1}) C_x$ and so the right-hand side of (a) reduces to $-(v_s + v_s^{-1}) \tilde{\gamma}_{x,w',y^{-1}}$. Now let $z \in W$ and assume that the corresponding term on the left-hand side of (a) is non-zero; that is, $\tilde{\gamma}_{x,w',z^{-1}} \neq 0$ and $h_{s,z,y} \neq 0$. By Lemma 1.6.5, this implies that $z \sim_{\mathscr{R}} x$ and so $sz < z$; see Remark 2.1.16. Hence, the left hand side also reduces to $-(v_s + v_s^{-1}) \tilde{\gamma}_{x,w',y^{-1}}$. Thus, the identity (a) holds in this case.

*Case 2*: $sx > x$. Let again $z \in W$ and assume that the corresponding term on the left-hand side of (a) is non-zero; that is, we have $\tilde{\gamma}_{x,w',z^{-1}} \neq 0$ and $h_{s,z,y} \neq 0$. Again, this implies that $z \sim_{\mathscr{R}} x$ and so $sz > z$. Hence, the sum on the left-hand side only needs to be extended over all $z \in W$ such that $sz > z$. But then, since we are in the equal-parameter case, we have $h_{s,z,y} \in \mathbb{Z}$; see Example 2.1.10(b). Now consider the usual associativity rule in $\mathbf{H}$: the identity $C_s(C_x C_{w'}) = (C_s C_x) C_{w'}$ yields

$$\sum_{z \in W} h_{x,w',z} h_{s,z,y} = \sum_{z \in W} h_{z,w',y} h_{s,x,z} \qquad \text{for all } y \in W.$$

Let $z \in W$ be such that the corresponding term on the left-hand side is non-zero. Then $h_{x,w',z} \neq 0$ and so $z \leqslant_{\mathscr{R}} x$; furthermore, $h_{s,z,y} \neq 0$ and so $y \leqslant_{\mathscr{L}} z$. Since $x \sim_{\mathscr{LR}} y$, we deduce that $z \sim_{\mathscr{LR}} x$. Thus, we can assume that $z \in \mathfrak{T}$ in the sum on the left-hand side. Now we use ($\clubsuit'$). Let $a \in \Gamma_{\geqslant 0}$ be the common value of $\mathbf{a}_\lambda$, where $\lambda \in \Lambda$ is such that $\mathscr{F}_\lambda \subseteq \mathfrak{T}$. By Lemma 2.5.11, we have

$$\varepsilon^a h_{x,w',z} \equiv \tilde{\gamma}_{x,w,z^{-1}} \mod \mathbb{Z}[\Gamma_{>0}].$$

Hence, since $h_{s,z,y} \in \mathbb{Z}$ for all $z$ such that $sz > z$, we have

$$\varepsilon^a \Big( \sum_{z \in W} h_{x,w',z} h_{s,z,y} \Big) = \sum_{z \in \mathfrak{T}} \big( \varepsilon^a h_{x,w',z} \big) h_{s,z,y} \in \mathbb{Z}[\Gamma_{>0}]$$

and the constant term of this expression equals the left hand side of (a). A similar argument applies to the right-hand side of the above associativity identity: the sum only needs to be extended over all $z \in \mathfrak{T}$. Furthermore, we have

$$\varepsilon^a h_{z,w',y} \equiv \tilde{\gamma}_{z,w',y^{-1}} \mod \mathbb{Z}[\Gamma_{>0}].$$

Hence, since $sx > x$ and $h_{s,x,z} \in \mathbb{Z}$, we have

$$\varepsilon^a \Big( \sum_{z \in W} h_{z,w',y} h_{s,x,z} \Big) = \sum_{z \in \mathfrak{T}} \big( \varepsilon^a h_{z,w',y} \big) h_{s,x,z} \in \mathbb{Z}[\Gamma_{>0}]$$

and the constant term equals the right hand side of (a). Thus, (a) is proved. $\square$

**Lemma 2.5.13.** *Assume that (♣) holds. Then we have $\tilde{\gamma}_{x,y,z} \in \mathbb{Z}$ for all $x,y,z \in W$ and $\tilde{n}_w \in \mathbb{Z}$ for all $w \in W$.*

*Proof.* If $y, z^{-1}$ belong to the same two-sided Kazhdan–Lusztig cell, then we have $\tilde{\gamma}_{x,y,z} \in \mathbb{Z}$ by Lemma 2.5.11. Otherwise, we have $\tilde{\gamma}_{x,y,z} = 0$ by Proposition 2.1.20. It remains to consider $\tilde{n}_w$. Let $\lambda_0 \in \Lambda$ be such that $E^{\lambda_0} \leftrightsquigarrow_L w$. We have $\overline{P}^*_{1,w} = (-1)^{l(w)} \tau(C_w)$. Expressing $\tau$ as in the proof of Lemma 2.5.11, we obtain

$$(-1)^{l(w)} \varepsilon^{-\mathbf{a}_{\lambda_0}} \overline{P}^*_{1,w} = \sum_{\lambda \in \Lambda} \sum_{\mathfrak{s} \in M(\lambda)} \frac{f_\lambda^{-1}}{1+g_\lambda} \varepsilon^{\mathbf{a}_\lambda - \mathbf{a}_{\lambda_0}} \big( \varepsilon^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{s}\mathfrak{s}}(C_w) \big).$$

Let $\lambda \in \Lambda$ be such that $\rho^\lambda(C_w) \neq 0$. Then we claim that $\mathbf{a}_{\lambda_0} \leqslant \mathbf{a}_\lambda$. Indeed, let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell such that $m(\mathfrak{C}, \lambda) > 0$. By Lemma 2.3.9(a), we have $y \leqslant_{\mathscr{R}} w$ for some $y \in \mathfrak{C}$. By Lemma 2.2.4, there also exists some $y' \in \mathfrak{C}$ such that $E^\lambda \leftrightsquigarrow_L y'$. In particular, we now have $y' \sim_{\mathscr{LR}} y \leqslant_{\mathscr{LR}} w$ and so $E^\lambda \preceq_L E^{\lambda_0}$. Since (♣) is assumed to hold, we can conclude that $\mathbf{a}_{\lambda_0} \leqslant \mathbf{a}_\lambda$, as required. This shows that the above sum lies in $\mathscr{O}_0$ and we have

$$(-1)^{l(w)} \varepsilon^{-\mathbf{a}_{\lambda_0}} \overline{P}^*_{1,w} \equiv \sum_{\substack{\lambda \in \Lambda \\ \mathbf{a}_\lambda = \mathbf{a}_{\lambda_0}}} \sum_{\mathfrak{s} \in M(\lambda)} f_\lambda^{-1} c^{\mathfrak{s}\mathfrak{s}}_{w,\lambda} \mod \mathfrak{m}.$$

But then the first sum can be extended over all $\lambda \in \Lambda$: just note that if $c^{\mathfrak{s}\mathfrak{s}}_{w,\lambda} \neq 0$, then $E^\lambda \leftrightsquigarrow_L w$ and so $E^\lambda \sim_L E^{\lambda_0}$; hence, $\mathbf{a}_\lambda = \mathbf{a}_{\lambda_0}$ in this case. So we conclude that

$$(-1)^{l(w)} \varepsilon^{-\mathbf{a}_{\lambda_0}} \overline{P}^*_{1,w} \equiv \tilde{n}_w \mod \mathfrak{m}.$$

Since the left-hand side lies in $\mathbb{Z}[\Gamma]$, we deduce that $\tilde{n}_w \in \mathbb{Z}$, as required. $\square$

**Example 2.5.14.** Assume that (♠), (♣), (♦) are satisfied. Then Conjectures 1.5.12 and 1.6.18 hold. (Indeed, part (a) of Conjecture 1.5.12 holds by Lemma 2.5.13; using Lemma 2.5.9 and (♦), we see that Conjecture 1.6.18 holds; then part (b) of Conjecture 1.5.12 follows by the argument in Remark 1.6.19.) Now let $\mathfrak{C}$ be a left Kazhdan–Lusztig cell of $W$. Then, by Examples 1.8.6 and 2.5.10, we have

$$\sum_{\lambda \in \Lambda} f_\lambda^{-1} m(\mathfrak{C}, \lambda) = 1.$$

This is a quite powerful statement. (It can also be easily deduced from [231, 21.4].) For example, it directly shows that if $f_\lambda = 1$ for all $\lambda \in \Lambda$, then $[\mathfrak{C}]_1 \in \mathrm{Irr}_{\mathbb{K}}(W)$.

*Remark 2.5.15.* Following Lusztig [223, 2.8], we can now also give a more direct proof of the fact that $\phi_K \colon \mathbf{H}_K \to \tilde{\mathbf{J}}_K$ is an isomorphism. Indeed, let us assume that both (♠) and (♣) hold. For each $w \in W$, choose some $\lambda \in \Lambda$ such that $E^\lambda \leftsquigarrow\!\!\rightsquigarrow_L w$ and set $\mathbf{a}_w := \mathbf{a}_\lambda$. (This does not depend on the choice of $\lambda$, thanks to (♣).) Then

$$\varepsilon^{\mathbf{a}_w} \phi(C_w) = \sum_{\substack{z \in W, d \in \tilde{\mathscr{D}} \\ z \sim_{\mathscr{LR}} d}} \varepsilon^{\mathbf{a}_w} \tilde{n}_d h_{w,d,z} t_z = \sum_{z \in W} \Big( \sum_{d \in \tilde{\mathscr{D}} : z \sim_{\mathscr{LR}} d} \varepsilon^{\mathbf{a}_w} \tilde{n}_d h_{w,d,z} \Big) t_z.$$

Assume that $z$ and $d$ in the above sum are such that $h_{w,d,z} \neq 0$. Let $\mu \in \Lambda$ be such that $E^\mu \leftsquigarrow\!\!\rightsquigarrow_L z$. Since $h_{w,d,z} \neq 0$ and $z \sim_{\mathscr{LR}} d$, we have $z \leqslant_{\mathscr{LR}} w$ and so $E^\mu \preceq_L E^\lambda$. Since (♣) holds, this implies that $\mathbf{a}_w \leqslant \mathbf{a}_z$, with equality only if $w \sim_{\mathscr{LR}} z$. Assume first that $w \sim_{\mathscr{LR}} z$. Then $\mathbf{a}_w = \mathbf{a}_z$ and Lemma 2.5.11 shows that $\varepsilon^{\mathbf{a}_w} h_{w,d,z}$ lies in $\mathbb{Z}[\Gamma_{\geqslant 0}]$ and has constant term $\tilde{\gamma}_{w,d,z^{-1}}$. Thus, we have

$$\sum_{d \in \tilde{\mathscr{D}} : z \sim_{\mathscr{LR}} d} \varepsilon^{\mathbf{a}_w} \tilde{n}_d h_{w,d,z} \equiv \sum_{d \in \tilde{\mathscr{D}} : z \sim_{\mathscr{LR}} d} \tilde{n}_d \tilde{\gamma}_{w,d,z^{-1}} \mod \mathbb{Z}[\Gamma_{>0}],$$

where $w \sim_{\mathscr{LR}} z$. Now, if $\tilde{\gamma}_{w,d,z^{-1}} \neq 0$, then $w, d, z$ belong to the same two-sided Kazhdan–Lusztig cell; see Proposition 2.1.20. So we can omit the condition $z \sim_{\mathscr{LR}} d$ in the above sum. Then Lemma 1.5.3 shows that

$$\sum_{d \in \tilde{\mathscr{D}} : z \sim_{\mathscr{LR}} d} \tilde{n}_d \tilde{\gamma}_{w,d,z^{-1}} = \sum_{d \in \tilde{\mathscr{D}}} \tilde{n}_d \tilde{\gamma}_{z^{-1},w,d} = \delta_{zw}.$$

Since $\tilde{n}_d \in \mathbb{Z}$ by Lemma 2.5.13, we finally obtain that

$$\begin{aligned} \varepsilon^{\mathbf{a}_w} \phi(C_w) = t_w &+ \mathbb{Z}[\Gamma_{>0}]\text{-combination of terms } t_z, \text{ where } z \sim_{\mathscr{LR}} w \\ &+ \mathbb{Z}[\Gamma]\text{-combination of terms } t_z, \text{ where } z \leqslant_{\mathscr{LR}} w,\ z \not\sim_{\mathscr{LR}} w. \end{aligned}$$

So, for a suitable ordering of the elements of $W$, the matrix of $\phi$ with respect to the basis $\{\varepsilon^{\mathbf{a}_w} C_w \mid w \in W\}$ of $\mathbf{H}$ and the basis $\{t_w \mid w \in W\}$ of $\tilde{\mathbf{J}}_A$ has a block triangular shape where the determinant of each diagonal block lies in $1 + \mathbb{Z}[\Gamma_{>0}]$. Hence, the determinant of the whole matrix of $\phi$ lies in $1 + \mathbb{Z}[\Gamma_{>0}]$. In particular, it is non-zero.

## 2.6 A Cellular Basis for H

We are now ready to define a new basis of **H** which will turn out to be a "cellular basis" in the sense of Graham and Lehrer [144]. We recall the basic definitions first.

**2.6.1.** Let $k$ be a commutative ring (with 1) and $\underline{H}$ be an associative $k$-algebra (with identity) which is finitely generated and free over $k$. Following Graham and Lehrer [144, Def. 1.1], a *cell datum* for $\underline{H}$ is a quadruple $(\underline{\Lambda}, \underline{M}, \underline{C}, *)$ satisfying the following conditions.

(C1) $\underline{\Lambda}$ is a partially ordered set, $\{\underline{M}(\lambda) \mid \lambda \in \underline{\Lambda}\}$ is a collection of finite sets and $\underline{C} = \{C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \underline{\Lambda},\, \mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)\}$ is a $k$-basis for $\underline{H}$.
(C2) There is a $k$-linear anti-involution, $h \mapsto h^*$, on $\underline{H}$ such that $(C^\lambda_{\mathfrak{s},\mathfrak{t}})^* = C^\lambda_{\mathfrak{t},\mathfrak{s}}$ for all $\lambda \in \underline{\Lambda}$ and all $\mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)$.
(C3) Denote by $\preceq$ the partial order on $\underline{\Lambda}$. If $\lambda \in \underline{\Lambda}$ and $\mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)$, then

$$hC^\lambda_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}' \in \underline{M}(\lambda)} r^\lambda_h(\mathfrak{s}',\mathfrak{s})\, C^\lambda_{\mathfrak{s}',\mathfrak{t}} \mod \underline{H}(\prec \lambda) \qquad \text{for all } h \in \underline{H},$$

where $r^\lambda_h(\mathfrak{s}',\mathfrak{s}) \in k$ is independent of $\mathfrak{t}$ and where $\underline{H}(\prec \lambda)$ is the $k$-submodule of $\underline{H}$ generated by $\{C^\mu_{\mathfrak{s}'',\mathfrak{t}''} \mid \mu \prec \lambda; \mathfrak{s}'',\mathfrak{t}'' \in \underline{M}(\mu)\}$.

If these conditions hold, we say that $\{C^\lambda_{\mathfrak{s},\mathfrak{t}}\}$ is a *cellular basis* of $\underline{H}$. Assume now that this is the case. Given $\lambda \in \underline{\Lambda}$, we can define a corresponding *cell representation* (or *cell module*) of $\underline{H}$ as follows. Let $W(\lambda)$ be a free $k$-module with basis $\{C_\mathfrak{s} \mid \mathfrak{s} \in \underline{M}(\lambda)\}$. Then, using (C3), $W(\lambda)$ is seen to be an $\underline{H}$-module with action given by

$$h.C_\mathfrak{s} = \sum_{\mathfrak{s}' \in M(\lambda)} r^\lambda_h(\mathfrak{s}',\mathfrak{s})\, C_{\mathfrak{s}'} \qquad \text{for } h \in \underline{H} \text{ and } \mathfrak{s} \in \underline{M}(\lambda).$$

This module is equipped with a canonical invariant bilinear form; see the following lemma.

**Lemma 2.6.2 (Graham and Lehrer [144, 2.4]).** *Let $\lambda \in \underline{\Lambda}$. Then there is a well-defined symmetric bilinear form $\langle\ ,\ \rangle_\lambda : W(\lambda) \times W(\lambda) \to k$ such that*

$$C^\lambda_{\mathfrak{u},\mathfrak{t}} C^\lambda_{\mathfrak{s},\mathfrak{v}} \equiv \langle C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda\, C^\lambda_{\mathfrak{u},\mathfrak{v}} \mod \underline{H}(\prec \lambda) \qquad \textit{for all } \mathfrak{s},\mathfrak{t},\mathfrak{u},\mathfrak{v} \in \underline{M}(\lambda).$$

*Furthermore, we have $\langle h.C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda = \langle C_\mathfrak{s}, h^*.C_\mathfrak{t}\rangle_\lambda$ for all $\mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)$ and $h \in \underline{H}$.*

*Proof.* This is a good exercise to see how the axioms are used. By (C3), we have

$$C^\lambda_{\mathfrak{u},\mathfrak{t}} C^\lambda_{\mathfrak{s},\mathfrak{v}} \equiv \sum_{\mathfrak{s}' \in \underline{M}(\lambda)} r^\lambda_{h_1}(\mathfrak{s}',\mathfrak{s})\, C^\lambda_{\mathfrak{s}',\mathfrak{v}} \mod \underline{H}(\prec \lambda), \qquad \text{where } h_1 = C^\lambda_{\mathfrak{u},\mathfrak{t}}.$$

On the other hand, by (C2), we have $C^\lambda_{\mathfrak{u},\mathfrak{t}} C^\lambda_{\mathfrak{s},\mathfrak{v}} = \left(C^\lambda_{\mathfrak{t},\mathfrak{u}}\right)^* \left(C^\lambda_{\mathfrak{v},\mathfrak{s}}\right)^* = \left(C^\lambda_{\mathfrak{v},\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{u}}\right)^*$. Applying (C3) to the product $C^\lambda_{\mathfrak{v},\mathfrak{s}} C^\lambda_{\mathfrak{t},\mathfrak{u}}$ and using (C2), we obtain that

$$C^\lambda_{\mathfrak{u},\mathfrak{t}} C^\lambda_{\mathfrak{s},\mathfrak{v}} \equiv \sum_{\mathfrak{t}' \in \underline{M}(\lambda)} r^\lambda_{h_2}(\mathfrak{t}',\mathfrak{t}) C^\lambda_{\mathfrak{u},\mathfrak{t}'} \bmod \underline{H}(\prec \lambda), \qquad \text{where } h_2 = C^\lambda_{\mathfrak{v},\mathfrak{s}}.$$

Hence, we deduce that $C^\lambda_{\mathfrak{u},\mathfrak{t}} C^\lambda_{\mathfrak{s},\mathfrak{v}} \equiv \alpha C^\lambda_{\mathfrak{u},\mathfrak{v}} \bmod \underline{H}(\prec \lambda)$, where $\alpha := r^\lambda_{h_1}(\mathfrak{u},\mathfrak{s}) = r^\lambda_{h_2}(\mathfrak{v},\mathfrak{t})$. Note that $r^\lambda_{h_1}(\mathfrak{u},\mathfrak{s})$ does not depend on $\mathfrak{v}$, and $r^\lambda_{h_2}(\mathfrak{v},\mathfrak{t})$ does not depend on $\mathfrak{u}$. Consequently, $\alpha$ does not depend on $\mathfrak{u}$ and not on $\mathfrak{v}$. Now choose $\mathfrak{u} = \mathfrak{v}$. Then we also see that $\alpha$ is not affected if we exchange the roles of $\mathfrak{s}$ and $\mathfrak{t}$. Thus, we obtain a well-defined symmetric bilinear form $\langle\ ,\ \rangle_\lambda$, as required. It remains to show that this form has the desired invariance property. Let $h \in \underline{H}$ and $\mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)$. Then

$$\langle h.C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda = \sum_{\mathfrak{s}' \in \underline{M}(\lambda)} r^\lambda_h(\mathfrak{s}',\mathfrak{s}) \langle C_{\mathfrak{s}'}, C_\mathfrak{t}\rangle_\lambda.$$

Now let $\mathfrak{u},\mathfrak{v} \in \underline{M}(\lambda)$. Multiplying the above identity by $C^\lambda_{\mathfrak{u},\mathfrak{v}}$ and using the defining formula for $\langle\ ,\ \rangle_\lambda$, we obtain

$$\langle h.C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda C^\lambda_{\mathfrak{u},\mathfrak{v}} \equiv \sum_{\mathfrak{s}' \in \underline{M}(\lambda)} r^\lambda_h(\mathfrak{s}',\mathfrak{s}) C^\lambda_{\mathfrak{u},\mathfrak{s}'} C^\lambda_{\mathfrak{t},\mathfrak{v}} \quad \bmod \underline{H}(\prec \lambda).$$

On the other, by (C2) and (C3), we have

$$\sum_{\mathfrak{s}' \in \underline{M}(\lambda)} r^\lambda_h(\mathfrak{s}',\mathfrak{s}) C^\lambda_{\mathfrak{u},\mathfrak{s}'} \equiv \big(h.C^\lambda_{\mathfrak{s},\mathfrak{u}}\big)^* \equiv C^\lambda_{\mathfrak{u},\mathfrak{s}} h^* \quad \bmod \underline{H}(\prec \lambda)$$

and so

$$\begin{aligned}\langle h.C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda C^\lambda_{\mathfrak{u},\mathfrak{v}} &\equiv C^\lambda_{\mathfrak{u},\mathfrak{s}} \big(h^* C^\lambda_{\mathfrak{t},\mathfrak{v}}\big) \equiv \sum_{\mathfrak{t}' \in \underline{M}(\lambda)} r^\lambda_{h^*}(\mathfrak{t}',\mathfrak{t}) C^\lambda_{\mathfrak{u},\mathfrak{s}} C^\lambda_{\mathfrak{t}',\mathfrak{v}} \\ &\equiv \sum_{\mathfrak{t}' \in \underline{M}(\lambda)} r^\lambda_{h^*}(\mathfrak{t}',\mathfrak{t}) \langle C_\mathfrak{s}, C_{\mathfrak{t}'}\rangle_\lambda C^\lambda_{\mathfrak{u},\mathfrak{v}} \equiv \langle C_\mathfrak{s}, h^*.C_\mathfrak{t}\rangle_\lambda C^\lambda_{\mathfrak{u},\mathfrak{v}} \quad \bmod \underline{H}(\prec \lambda),\end{aligned}$$

as required. □

**Corollary 2.6.3.** *Let* $\lambda \in \underline{\Lambda}$ *and* $\mathfrak{s},\mathfrak{s}',\mathfrak{t},\mathfrak{u} \in M(\lambda)$. *Then*

$$r_h(\mathfrak{s}',\mathfrak{s}) = \delta_{\mathfrak{u}\mathfrak{s}'} \langle C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda, \qquad \textit{where} \qquad h = C^\lambda_{\mathfrak{u},\mathfrak{t}}.$$

*Proof.* This is clear by Lemma 2.6.2 and the definiton of $r_h(\mathfrak{s}',\mathfrak{s})$ in (C3). □

**Definition 2.6.4.** Let $L^\lambda := W(\lambda)/\mathrm{rad}(\langle\ ,\ \rangle_\lambda)$ for any $\lambda \in \underline{\Lambda}$. Then $L^\lambda$ is a left $\underline{H}$-module since, by Lemma 2.6.2, the radical of $\langle\ ,\ \rangle_\lambda$ is an $\underline{H}$-submodule of $W(\lambda)$. Note that we may have $L^\lambda = \{0\}$; this happens if and only if $\langle\ ,\ \rangle_\lambda$ is identically zero.

Now we have the following two fundamental results of Graham and Lehrer [144] whose proof we will not give here. (See also Mathas [245, Chap. 2].)

**Theorem 2.6.5 (Graham and Lehrer [144, 3.4, 3.8]).** *Assume that $k$ is a field. If* $\langle\ ,\ \rangle_\lambda \neq 0$, *then* $L^\lambda$ *is an absolutely irreducible* $\underline{H}$*-module; furthermore,*

$$\operatorname{Irr}(\underline{H}) = \{L^\mu \mid \mu \in \underline{\Lambda}^\circ\}, \quad \textit{where} \quad \underline{\Lambda}^\circ = \{\lambda \in \underline{\Lambda} \mid \langle\ ,\ \rangle_\lambda \neq 0\}.$$

*In particular, the algebra $\underline{H}$ is split. Finally, $\underline{H}$ is semisimple if and only if $\underline{\Lambda} = \underline{\Lambda}^\circ$ and $\langle\ ,\ \rangle_\lambda$ is non-degenerate for all $\lambda \in \underline{\Lambda}$.*

Recall that an algebra is called *split* if the endomorphism algebra of any irreducible representation consists just of the scalar multiples of the identity.

**Theorem 2.6.6 (Graham and Lehrer [144, 3.6]).** *Assume that $k$ is a field. For $\lambda \in \underline{\Lambda}$ and $\mu \in \underline{\Lambda}^\circ$, denote by $(W(\lambda) : L^\mu)$ the multiplicity of $L^\mu$ as a composition factor of $W(\lambda)$. Then*

$$(\Delta) \qquad (W(\mu) : L^\mu) = 1 \quad \textit{and} \quad (W(\lambda) : L^\mu) = 0 \quad \textit{unless} \quad \lambda \preceq \mu.$$

*Thus, the decomposition matrix $D = \big((W(\lambda) : L^\mu)\big)_{\lambda \in \underline{\Lambda}, \mu \in \underline{\Lambda}^\circ}$ has a lower unitriangular shape, if the rows and columns are ordered according to the order relation $\preceq$.*

Let us now also assume that $\underline{H}$ is a *symmetric algebra*, with trace form $\tau \colon \underline{H} \to k$. Then, given a basis $\{C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \underline{\Lambda}, \mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)\}$ as above, we have a corresponding dual basis $\underline{\hat{C}} := \{\hat{C}^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \underline{\Lambda}, \mathfrak{s},\mathfrak{t} \in \underline{M}(\lambda)\}$. We choose the notation such that

$$\tau\big(C^\lambda_{\mathfrak{s},\mathfrak{t}} \hat{C}^\mu_{\mathfrak{u},\mathfrak{v}}\big) = \begin{cases} 1 & \text{if } \lambda = \mu,\ \mathfrak{s} = \mathfrak{v},\ \mathfrak{t} = \mathfrak{u}, \\ 0 & \text{otherwise.} \end{cases}$$

To state the following result, note that if $V$ is a left $\underline{H}$-module, then $\operatorname{Hom}_k(V,k)$ also is a left $\underline{H}$-module where the action is given by $h.f(v) = f(h^*.v)$ for $h \in \underline{H}$, $f \in \operatorname{Hom}_k(V,k)$ and $v \in V$.

**Proposition 2.6.7 (Graham [143, 4.12]).** *Assume that $\underline{H}$ is symmetric with trace form $\tau \colon \underline{H} \to k$ such that $\tau(h^*) = \tau(h)$ for all $h \in \underline{H}$. Then, with the above notation, the following hold.*

*(a) The quadruple $(\underline{\Lambda}^{\mathrm{op}}, \underline{M}, \underline{\hat{C}}, *)$ also is a cell datum for $\underline{H}$, where $\underline{\Lambda}^{\mathrm{op}}$ is the set $\underline{\Lambda}$ endowed with the opposite partial order $\preceq_{\mathrm{op}}$ (that is, $\lambda \preceq_{\mathrm{op}} \mu \Leftrightarrow \mu \preceq \lambda$).*
*(b) Let $\lambda \in \underline{\Lambda}$ and $\hat{W}(\lambda)$ be the cell module with respect to the cell datum in (a). Then there is an isomorphism of left $\underline{H}$-modules $\hat{W}(\lambda) \cong \operatorname{Hom}_k(W(\lambda),k)$.*
*(c) If $k$ is a field and $\underline{H}$ is semisimple, then $W(\lambda) \cong \hat{W}(\lambda)$ for all $\lambda \in \Lambda$.*

We shall call $(\underline{\Lambda}^{\mathrm{op}}, \underline{M}, \underline{\hat{C}}, *)$ the *opposite cell datum* to $(\underline{\Lambda}, \underline{M}, \underline{C}, *)$.

*Proof.* Let us verify that (C1), (C2), (C3) hold for the quadruple in (a). First note that (C1) is clear and (C2) is easily seen to hold thanks to the assumption on $\tau$. To prove (C3), let $h \in \underline{H}$ and consider the product $h\hat{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$. Let $\mu \in \underline{\Lambda}$ and $\mathfrak{u},\mathfrak{v} \in \underline{M}(\mu)$ be such that $\hat{C}^\mu_{\mathfrak{u},\mathfrak{v}}$ appears with a non-zero coefficient in the expansion of $h\hat{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$ with respect to the basis $\underline{\hat{C}}$. Note that this coefficient is given by

$$\tau\big((h\hat{C}^\lambda_{\mathfrak{s},\mathfrak{t}})C^\mu_{\mathfrak{v},\mathfrak{u}}\big) = \tau\big(((h\hat{C}^\lambda_{\mathfrak{s},\mathfrak{t}})C^\mu_{\mathfrak{v},\mathfrak{u}})^*\big) = \tau\big((h^*C^\mu_{\mathfrak{u},\mathfrak{v}})\hat{C}^\lambda_{\mathfrak{t},\mathfrak{s}}\big).$$

Hence, by (C3) for the original cell datum, we must have $\lambda \preceq \mu$; furthermore, if $\lambda = \mu$, then the above expression evaluates to $\delta_{\mathfrak{v},\mathfrak{t}} r^{\lambda}_{h^*}(\mathfrak{s},\mathfrak{u})$. Thus, we have

$$h\hat{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}} = \sum_{\mathfrak{u}\in \underline{M}(\lambda)} r^{\lambda}_{h^*}(\mathfrak{s},\mathfrak{u})\,\hat{C}^{\lambda}_{\mathfrak{u},\mathfrak{t}} \mod \underline{H}(\prec_{\mathrm{op}} \lambda).$$

This shows that (C3) holds. The above formula also proves (b). More precisely, if $\rho^\lambda : \underline{H} \to M_{d_\lambda}(k)$ is the matrix representation afforded by $W(\lambda)$ (with respect to its standard basis), then the matrix representation afforded by $\hat{W}(\lambda)$ (with respect to its standard basis) is given by $h \mapsto \rho(h^*)^{\mathrm{tr}}$ ($h \in \underline{H}$).

Finally, to prove (c), assume that $k$ is a field and $\underline{H}$ is semisimple. Let $\lambda \in \Lambda$ and $G^\lambda$ be the Gram matrix of the bilinear form $\langle\ ,\ \rangle_\lambda$ with respect to the standard basis of $W(\lambda)$. Then $G^\lambda$ is invertible by Theorem 2.6.5. On the other hand, the invariance condition in Lemma 2.6.2 implies that $G^\lambda \rho^\lambda(h) = \rho(h^*)^{\mathrm{tr}} G^\lambda$ for all $h \in \underline{H}$. Hence, the two representations are equivalent; that is, $W(\lambda) \cong \hat{W}(\lambda)$. □

We return to the situation where we consider the generic Iwahori–Hecke algebra $\mathbf{H} = \mathbf{H}_A(W,S,L)$ associated with a finite Coxeter group $W$ and a weight function $L\colon W \to \Gamma$. Recall that $\mathbf{H}$ is defined over $A = R[\Gamma]$, where $R \subseteq \mathbb{C}$ is a subring such that $\mathbb{Z}_W \subseteq R$; furthermore, we assume that there is a monomial order $\leqslant$ on $\Gamma$ such that $L(s) \geqslant 0$ for all $s \in S$. Let $\{C_w \mid w \in W\}$ be the associated Kazhdan–Lusztig basis of $\mathbf{H}$; see Section 2.1. Write

$$\mathrm{Irr}_{\mathbb{K}}(W) = \{E^\lambda \mid \lambda \in \Lambda\}, \qquad d_\lambda = \dim E^\lambda,$$

and let $M(\lambda)$ be an indexing set for a basis of $E^\lambda$, as in Section 1.2; for each $E^\lambda \in \mathrm{Irr}_{\mathbb{K}}(W)$, we have a corresponding invariant $\mathbf{a}_\lambda \in \Gamma_{\geqslant 0}$. In Section 1.5, we used the leading matrix coefficients $c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda}$ to construct the ring $\tilde{\mathbf{J}}$.

**Definition 2.6.8 (Cf. [111, §3]).** Assume that $R \subseteq \mathbb{C}$ is $L$-good in the sense of Definition 1.5.9. Let $\underline{\Lambda} := \Lambda$ and $\underline{M}(\lambda) := M(\lambda)$ for all $\lambda \in \Lambda$. Let $\bar{\rho}^\lambda$ and $B^\lambda$ be as in Proposition 1.5.11. Let us write

$$\bar{\rho}^\lambda(t_w) = \big(c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda}\big)_{\mathfrak{s},\mathfrak{t}\in M(\lambda)} \qquad \text{and} \qquad B^\lambda = \big(\beta^\lambda_{\mathfrak{s}\mathfrak{t}}\big)_{\mathfrak{s},\mathfrak{t}\in M(\lambda)}.$$

Then, for any $\lambda \in \Lambda$ and $\mathfrak{s},\mathfrak{t} \in M(\lambda)$, we define

$$\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} := \sum_{w\in W} \sum_{\mathfrak{u}\in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}}\, c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, C_w \quad \in \mathbf{H}.$$

We now show in several steps that (C1), (C2), (C3) hold for these data.

*Remark 2.6.9.* In the defining formula for $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$, we can assume that the first sum runs over all $w \in \mathscr{F}_\lambda$ (where $\mathscr{F}_\lambda$ is defined in Proposition 1.6.11). Indeed, if $C_w$ appears with a non-zero coefficient in that sum, then $c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda} \neq 0$ for some $\mathfrak{u},\mathfrak{s} \in M(\lambda)$, and so $w^{-1} \in \mathscr{F}_\lambda$. But then Lemma 1.6.6 also shows that $w \in \mathscr{F}_\lambda$, as required.

**Lemma 2.6.10.** *The elements* $\{\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda,\ \mathfrak{s},\mathfrak{t} \in M(\lambda)\}$ *form an* $A$*-basis of* $\mathbf{H}$. *In fact, let* $y \in W$ *and* $\mathscr{F}$ *be the two-sided* $\tilde{\mathbf{J}}$*-cell containing* $y$. *Then* $C_y$ *is an* $R$*-linear combination of elements* $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$, *where* $\lambda \in \Lambda$ *is such that* $E^\lambda \leftrightsquigarrow_L y$.

*Proof.* By the Artin–Wedderburn theorem, $|W| = \sum_{\lambda\in\Lambda} |M(\lambda)|^2$. Hence, the above set has the correct cardinality. It is now sufficient to show that the elements $\{\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}\}$ span $\mathbf{H}$ as an $A$-module. Let us fix $y \in W$. We claim that

$$C_y = \sum_{\lambda\in\Lambda} \sum_{\mathfrak{s},\mathfrak{s}',\mathfrak{t}\in M(\lambda)} f_\lambda^{-1} c_{y,\lambda}^{\mathfrak{s}\mathfrak{s}'} \hat{\beta}^\lambda_{\mathfrak{s}'\mathfrak{t}} \mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}, \quad \text{where } (B^\lambda)^{-1} = \big(\hat{\beta}_{\mathfrak{s}\mathfrak{t}}\big).$$

Note that the coefficients in the above sum lie in $R$, since $f_\lambda$ and $\det(B^\lambda)$ are invertible in $R$ (since $R$ is $L$-good and by Proposition 1.5.11(b)). Furthermore, we have $E^\lambda \leftrightsquigarrow_L y$ if $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$ occurs with non-zero coefficient in the above sum. Thus, it remains to prove the above identity. For this purpose, we insert the defining formula for $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$ into the right-hand side; this yields

$$\begin{aligned}
&\sum_{w\in W} \sum_{\lambda\in\Lambda} \sum_{\mathfrak{s},\mathfrak{s}',\mathfrak{u}\in M(\lambda)} f_\lambda^{-1} c_{y,\lambda}^{\mathfrak{s}\mathfrak{s}'} \Big(\sum_{\mathfrak{t}\in M(\lambda)} \hat{\beta}^\lambda_{\mathfrak{s}'\mathfrak{t}} \beta^\lambda_{\mathfrak{t}\mathfrak{u}}\Big) c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}} C_w \\
&\qquad = \sum_{w\in W} \Big(\sum_{\lambda\in\Lambda} \sum_{\mathfrak{s},\mathfrak{u}\in M(\lambda)} f_\lambda^{-1} c_{y,\lambda}^{\mathfrak{s}\mathfrak{u}} c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}}\Big) C_w = C_y
\end{aligned}$$

as desired, where the last equality holds by Proposition 1.4.10(b). □

**Lemma 2.6.11.** *We have* $(\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}})^\flat = \mathbf{C}^\lambda_{\mathfrak{t},\mathfrak{s}}$ *for all* $\lambda \in \Lambda$ *and* $\mathfrak{s},\mathfrak{t} \in M(\lambda)$, *where* $\flat$ *is the anti-involution in 2.1.14.*

*Proof.* By 2.1.14, we have $C_w^\flat = C_{w^{-1}}$ for all $w \in W$. Thus, we obtain

$$(\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}})^\flat = \sum_{w\in W} \sum_{\mathfrak{u}\in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}} c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}} C_{w^{-1}} = \sum_{w\in W} \big(B^\lambda.\bar{\rho}^\lambda(t_{w^{-1}})\big)_{\mathfrak{t},\mathfrak{s}} C_{w^{-1}}.$$

By Proposition 1.5.11, we have $B^\lambda.\bar{\rho}^\lambda(t_{w^{-1}}) = \bar{\rho}^\lambda(t_w)^{\mathrm{tr}}.B^\lambda$. This yields

$$(\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}})^\flat = \sum_{w\in W} \big(\bar{\rho}^\lambda(t_w)^{\mathrm{tr}}.B^\lambda\big)_{\mathfrak{t},\mathfrak{s}} C_{w^{-1}} = \sum_{w\in W} \sum_{\mathfrak{u}\in M(\lambda)} c_{w,\lambda}^{\mathfrak{u}\mathfrak{t}} \beta^\lambda_{\mathfrak{u}\mathfrak{s}} C_{w^{-1}} = \mathbf{C}^\lambda_{\mathfrak{t},\mathfrak{s}},$$

as required. (Recall that $B^\lambda$ is symmetric.) □

We can now state the main result of this chapter.

**Theorem 2.6.12 (Cf. [111, §3] [112, §5]).** *Assume that* $R$ *is* $L$*-good and that* (♠) *in 2.5.3 holds. Then the elements* $\{\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}\}$ *introduced in Definition 2.6.8 form a cellular basis of* $\mathbf{H}$ *with respect to the anti-involution* $T_w \mapsto T_w^\flat = T_{w^{-1}}$ *(see 2.1.14), and the partial order* $\trianglelefteq_L$ *on* $\Lambda$ *defined by*

$$\mu \trianglelefteq_L \lambda \quad \stackrel{def}{\Leftrightarrow} \quad \mu = \lambda \quad \textit{or} \quad E^\mu \preceq_L E^\lambda,\ E^\mu \not\sim_L E^\lambda,$$

*where $\preceq_L$ and $\sim_L$ are as in Definition 2.2.1. If property (♣) in 2.5.3 also holds, then we have*

$$\mu \trianglelefteq_L \lambda \quad \Rightarrow \quad \lambda = \mu \quad \textit{or} \quad \mathbf{a}_\mu > \mathbf{a}_\lambda.$$

*Proof.* Recall that (C1) holds by Lemma 2.6.10; (C2) holds by Lemma 2.6.11. In order to prove (C3), we need to consider a product $h\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$ where $h \in \mathbf{H}$ and $\lambda \in \Lambda$, $\mathfrak{s},\mathfrak{t} \in M(\lambda)$. It is sufficient to consider the case where $h = C_x$ for some $x \in W$. Now, by the definition of $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}$ and Remark 2.6.9, we have

$$h\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} = \sum_{y\in W} r_y C_y \qquad \text{where} \qquad r_y = \sum_{w\in\mathscr{F}_\lambda}\sum_{\mathfrak{u}\in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}}\, c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, h_{x,w,y}.$$

Let $\mathfrak{T}_\lambda$ be the two-sided Kazhdan–Lusztig cell such that $\mathscr{F}_\lambda \subseteq \mathfrak{T}_\lambda$. Note that if $r_y \neq 0$, then there is some $w \in \mathscr{F}_\lambda \subseteq \mathfrak{T}_\lambda$ such that $h_{x,w,y} \neq 0$ and so $y \leqslant_{\mathscr{LR}} w$ (in the Kazhdan–Lusztig pre-order).

Assume first that $r_y \neq 0$ and $y \notin \mathfrak{T}_\lambda$. By Lemma 2.6.10, $C_y$ is a linear combination of elements $\mathbf{C}^\mu_{\mathfrak{u},\mathfrak{v}}$, where $E^\mu \leftsquigarrow\!\!\rightsquigarrow_L y$. Hence, since $y \notin \mathfrak{T}_\lambda$, we conclude that $C_y \in \mathbf{H}(\triangleleft_L \lambda)$ and so we do not need to consider these terms in any more detail.

Thus, we can now assume that $y \in \mathfrak{T}_\lambda$. Then, by Lemma 2.5.8, we have

$$h_{x,w,y} = \sum_{\substack{z\in W,\, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{x,d,z}\, \tilde{\gamma}_{z,w,y^{-1}} \qquad \text{for any } w \in \mathscr{F}_\lambda \subseteq \mathfrak{T}_\lambda.$$

We insert this formula for $h_{x,w,y}$ into the above expression for $r_y$; this yields

$$\begin{aligned} r_y &= \sum_{w\in\mathscr{F}_\lambda} \sum_{\substack{z\in W,\, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \sum_{\mathfrak{u}\in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}}\, c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, \tilde{n}_d\, h_{x,d,z}\, \tilde{\gamma}_{z,w,y^{-1}} \\ &= \sum_{\substack{z\in W,\, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{x,d,z} \sum_{\mathfrak{u}\in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}} \Big(\sum_{w\in\mathscr{F}_\lambda} c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, \tilde{\gamma}_{z,w,y^{-1}}\Big). \end{aligned}$$

Now, by 1.6.10 and Proposition 1.6.11, the sum over $w \in \mathscr{F}_\lambda$ can be extended to a sum over all $w \in W$. Using the defining equation for $\tilde{\gamma}_{z,w,y^{-1}}$, we obtain

$$\begin{aligned} \sum_{w\in\mathscr{F}_\lambda} c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, \tilde{\gamma}_{z,w,y^{-1}} &= \sum_{w\in W} c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda} \Big(\sum_{\mu\in\Lambda} \sum_{\mathfrak{s}',\mathfrak{v},\mathfrak{v}'\in M(\mu)} f_\mu^{-1}\, c^{\mathfrak{s}'\mathfrak{v}}_{z,\mu}\, c^{\mathfrak{v}\mathfrak{v}'}_{w,\mu}\, c^{\mathfrak{v}'\mathfrak{s}'}_{y^{-1},\mu}\Big) \\ &= \sum_{\mu\in\Lambda} \sum_{\mathfrak{s}',\mathfrak{v},\mathfrak{v}'\in M(\mu)} f_\mu^{-1}\, c^{\mathfrak{s}'\mathfrak{v}}_{z,\mu}\, c^{\mathfrak{v}'\mathfrak{s}'}_{y^{-1},\mu} \Big(\sum_{w\in W} c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, c^{\mathfrak{v}\mathfrak{v}'}_{w,\mu}\Big) = \sum_{\mathfrak{s}'\in M(\lambda)} c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda}\, c^{\mathfrak{u}\mathfrak{s}'}_{y^{-1},\lambda} \end{aligned}$$

where the last equality holds by Proposition 1.4.10(a). This yields

$$hC^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{y\in\mathfrak{T}_\lambda} r_y C_y \equiv \sum_{\substack{z\in W, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{x,d,z} \sum_{\mathfrak{s}'\in M(\lambda)} c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda}\Big(\sum_{y\in W}\sum_{\mathfrak{u}\in M(\lambda)} \beta^{\lambda}_{\mathfrak{tu}}\, c^{\mathfrak{us}'}_{y^{-1},\lambda}\, C_y\Big)$$

$\operatorname{mod}\mathbf{H}(\lhd_L \lambda)$. Since the parenthesised sum equals $\mathbf{C}^{\lambda}_{\mathfrak{s}',\mathfrak{t}}$, we see that

$$h\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}} \equiv \sum_{\mathfrak{s}'\in M(\lambda)} \Big(\sum_{\substack{z\in W, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{x,d,z}\, c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda}\Big)\,\mathbf{C}^{\lambda}_{\mathfrak{s}',\mathfrak{t}} \mod \mathbf{H}(\lhd_L \lambda).$$

Thus, we have shown that, for $h = C_x$ $(x \in W)$, we have

$$r^{\lambda}_h(\mathfrak{s}',\mathfrak{s}) = \sum_{\substack{z\in W, d\in\tilde{\mathscr{D}}\\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{x,d,z}\, c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda} \qquad \text{for all } \mathfrak{s},\mathfrak{s}'\in M(\lambda);$$

in particular, this expression does not depend on $\mathfrak{t}$, as required. □

The model for this theorem, namely the case where $W$ is the symmetric group $\mathfrak{S}_n$, will be considered in detail in Section 2.8.

*Remark 2.6.13.* Note that the ingredients for a cellular basis of **H** (that is, the elements $\{\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$ and the partial order $\unlhd_L$) are defined without reference to (♠); this property is only required for the proof.

*Remark 2.6.14.* Assume that we are in the equal-parameter case. Then we have seen in Proposition 2.5.12 that (♠) is a consequence of the following implication:

$$(♣') \qquad E^{\lambda} \sim_L E^{\mu} \quad \Rightarrow \quad \mathbf{a}_\mu = \mathbf{a}_\lambda.$$

Thus, in order to prove Theorem 2.6.12 in the equal-parameter case, we only need to assume that (♣′) holds. Recall that (♣′) does hold in type $I_2(m)$ (any $m \geqslant 2$), $H_3$, $H_4$ by Examples 2.2.8 and 2.2.9. Furthermore, (♣′) was already established by Lusztig [220, 5.27] (around 1985) for all finite Weyl groups.

*Remark 2.6.15.* Recall that, by Remark 2.2.11, we have the implication

$$E^{\lambda} \preceq_L E^{\mu} \quad \Rightarrow \quad E^{\mu^\dagger} \preceq_L E^{\lambda^\dagger}.$$

Now, by Examples 1.2.7 and 1.3.4, the following relation holds between $\mathbf{a}_\lambda$ and $\mathbf{a}_{\lambda^\dagger}$:

$$\mathbf{a}_{\lambda^\dagger} - \mathbf{a}_\lambda = N_\lambda = \sum_{s\in S'} \Big(\frac{N_s\, \mathrm{trace}(s, E^{\lambda})}{\dim E^{\lambda}}\Big) L(s).$$

It follows that if (♣) is satisfied, then we have the implication

$$\lambda \unlhd_L \mu \quad \Rightarrow \quad \lambda = \mu \quad \text{or} \quad N_\lambda < N_\mu.$$

Thus, Theorem 2.6.12 could be alternatively formulated using $N_\lambda$ instead of the invariants $\mathbf{a}_\lambda$. Note that $N_\lambda$ is much easier to define than $\mathbf{a}_\lambda$; also, $N_\lambda$ does not depend on the monomial order on $\Gamma$. The idea of using the invariants $N_\lambda$ appears, in a somewhat different context in Ginzburg et al. [137, §6]; see also Gordon [140].

The following result shows that, for any Iwahori–Hecke algebra associated with a finite Coxeter group, there does exist at least some cellular structure.

**Corollary 2.6.16.** *Let $k$ be a commutative ring (with 1) and $\{\xi_s \mid s \in S\} \subseteq k^\times$ a collection of elements such that $\xi_s = \xi_t$ whenever $s,t \in S$ are conjugate in $W$. Let $H_k = H_k(W,S,\{\xi_s\})$ be the corresponding Iwahori–Hecke algebra; see 1.1.2. Assume that $k$ is $L_0$-good (see Definition 1.5.9) for the "universal" weight function $L_0$ in Example 1.1.9. Then $H_k$ admits a cellular basis $\{C^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda, \mathfrak{s},\mathfrak{t} \in M(\lambda)\}$ with respect to the anti-involution $T_w \mapsto T_w^\flat = T_{w^{-1}}$ and some partial order on $\Lambda$.*

*Proof.* Let $\Gamma_0$, $A_0$ and $\mathbf{H}_0$ be "universal", as in Example 1.1.9. Choose a monomial order on $\Gamma_0$ such that, on every irreducible component of type $B_n$, $F_4$ or $I_2(m)$ ($m$ even), we are in the "asymptotic case" in Example 1.1.11. Then, by Corollary 2.4.2, we know that **P1–P15** hold. Hence, as discussed in 2.5.3, the properties (♣) and (♠) also hold and so Theorem 2.6.12 applies. Thus, we obtain a cellular basis $\{\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}}\}$ for $\mathbf{H}_0$, where the partial order on $\Lambda$ is given by $\trianglelefteq_{L_0}$.

Since $k$ is $L_0$-good, there is a ring homomorphism $R \to k$. This extends to a ring homomorphism $\theta\colon A_0 \to k$ such that $\theta(v_s^\circ) = \xi_s$ for all $s \in S'$, where $\{v_s^\circ\}$ are the parameters of $\mathbf{H}_0$. Thus, $H_k = k \otimes_{A_0} \mathbf{H}_0$, where $k$ is regarded as an $A_0$-module via $\theta$. Since the elements $\{\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda,\ \mathfrak{s},\mathfrak{t} \in M(\lambda)\}$ in $\mathbf{H}$ satisfy (C1), (C2), (C3), it is clear that the elements $\{C^\lambda_{\mathfrak{s},\mathfrak{t}} := 1 \otimes \mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} \mid \lambda \in \Lambda,\ \mathfrak{s},\mathfrak{t} \in M(\lambda))\}$ satisfy (C1), (C2), (C3) in $H_k$. Thus, we have constructed a cell datum for $H_k$. □

**Example 2.6.17.** Let $W$ be of type $I_2(4) = B_2$, where $S = \{s_1, s_2\}$ and $(s_1s_2)^4 = 1$. Assume that we are in the equal-parameter case, where $\Gamma = \mathbb{Z}$ and $L(s_1) = L(s_2) = 1$. Then $A = R[v, v^{-1}]$ is the ring of Laurent polynomials in one indeterminate $v = \varepsilon$. Now Theorem 2.6.12 applies where $R \subseteq \mathbb{C}$ can be any subring in which 2 is invertible. In order to determine a cellular basis, we need to work out the leading matrix coefficients of the irreducible representations of $\mathbf{H}_K$. The two-sided cells are given by $\{1_0\}$, $\{1_4\}$ and $W \setminus \{1_0, 1_4\}$, where we use the notation in Example 1.7.3. First consider the representation $\sigma_1$. By Example 1.3.7, we have $\mathbf{a}_{\sigma_1} = 1$ and so

$$\begin{aligned}
v\sigma_1^\varepsilon(T_{1_1}) &= \begin{pmatrix} -1 & 0 \\ 2v & v^2 \end{pmatrix} &&\equiv \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \bmod \mathfrak{m},\\
v\sigma_1^\varepsilon(T_{2_1}) &= \begin{pmatrix} v^2 & v \\ 0 & -1 \end{pmatrix} &&\equiv \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bmod \mathfrak{m},\\
v\sigma_1^\varepsilon(T_{1_2}) &= \begin{pmatrix} -v & -1 \\ 2v^2 & v \end{pmatrix} &&\equiv \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \bmod \mathfrak{m},\\
v\sigma_1^\varepsilon(T_{2_2}) &= \begin{pmatrix} v & v^2 \\ -2 & -v \end{pmatrix} &&\equiv \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} \bmod \mathfrak{m},\\
v\sigma_1^\varepsilon(T_{1_3}) &= \begin{pmatrix} -1 & -v \\ 0 & v^2 \end{pmatrix} &&\equiv \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \bmod \mathfrak{m},\\
v\sigma_1^\varepsilon(T_{2_3}) &= \begin{pmatrix} v^2 & 0 \\ -2v & -1 \end{pmatrix} &&\equiv \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \bmod \mathfrak{m}.
\end{aligned}$$

A corresponding symmetric matrix is given by

$$B_{\sigma_1} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}; \qquad \text{see Example 1.4.6.}$$

Performing similar (but much simpler) computations for the one-dimensional representations, we obtain the following expressions for $\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}$ (see also [111, Exp. 4.3]):

$$\begin{aligned}
\mathbf{C}^{1_W}_{1,1} &= C_{1_0}, & \mathbf{C}^{\sigma_1}_{1,1} &= -2C_{1_1} - 2C_{1_3},\\
\mathbf{C}^{\mathrm{sgn}}_{1,1} &= C_{1_4}, & \mathbf{C}^{\sigma_1}_{1,2} &= -2C_{1_2},\\
\mathbf{C}^{\mathrm{sgn}_1}_{1,1} &= -C_{2_1} + C_{2_3}, & \mathbf{C}^{\sigma_1}_{2,1} &= -2C_{2_2},\\
\mathbf{C}^{\mathrm{sgn}_2}_{1,1} &= -C_{1_1} + C_{1_3}, & C^{\sigma_1}_{2,2} &= -C_{2_1} - C_{2_3}.
\end{aligned}$$

For the case of unequal parameters, see 2.8.19.

**Example 2.6.18.** Let $W$ be of type $I_2(6) = G_2$, where $S = \{s_1, s_2\}$ and $(s_1s_2)^6 = 1$.
(a) Assume that we are in the equal-parameter case where $L(s_1) = L(s_2) > 0$. Then, again, Theorem 2.6.12 applies and so we have a cellular basis $\{\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$. In this case, we can take for $R$ any subring of $\mathbb{C}$ in which 2, 3 are invertible. Expressions for $\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}$ have been worked out in [129, Exp. 2.7] (using computations similar to those in the previous example):

$$\begin{aligned}
\mathbf{C}^{1_W}_{1,1} &= C_{1_0}, & \mathbf{C}^{\mathrm{sgn}}_{1,1} &= C_{1_6},\\
\mathbf{C}^{\mathrm{sgn}_1}_{1,1} &= C_{2_1} - C_{2_3} + C_{2_5}, & \mathbf{C}^{\mathrm{sgn}_2}_{1,1} &= C_{1_1} - C_{1_3} + C_{1_5},\\
\mathbf{C}^{\sigma_1}_{1,1} &= 3C_{1_1} + 6C_{1_3} + 3C_{1_5}, & \mathbf{C}^{\sigma_2}_{1,1} &= C_{1_1} - C_{1_5},\\
\mathbf{C}^{\sigma_1}_{1,2} &= -3C_{1_2} - 3C_{1_4}, & \mathbf{C}^{\sigma_2}_{1,2} &= -C_{1_2} + C_{1_4},\\
\mathbf{C}^{\sigma_1}_{2,1} &= -3C_{2_1} - 3C_{2_4}, & \mathbf{C}^{\sigma_2}_{2,1} &= -C_{2_1} + C_{2_4},\\
\mathbf{C}^{\sigma_1}_{2,2} &= C_{2_1} + 2C_{2_3} + C_{2_5}, & \mathbf{C}^{\sigma_2}_{2,2} &= C_{2_1} - C_{2_5}.
\end{aligned}$$

(b) Assume that the monomial order on $\Gamma$ is such that $L(s_1) > L(s_2) > 0$. By 2.4.1, **P1**–**P15** hold and, hence, by 2.5.3, the hypothesis of Theorem 2.6.12 is satisfied. In this case, we can take for $R$ any subring of $\mathbb{C}$ in which 2 is invertible. We find the following expressions for the cellular basis:

$$\begin{aligned}
\mathbf{C}^{1_W}_{1,1} &= C_{1_0}, & \mathbf{C}^{\sigma_1}_{1,1} &= C_{1_1} + C_{1_3}, & \mathbf{C}^{\sigma_2}_{1,1} &= C_{1_1} - C_{1_3},\\
\mathbf{C}^{\mathrm{sgn}}_{1,1} &= C_{1_6}, & \mathbf{C}^{\sigma_1}_{1,2} &= -C_{1_2} - C_{1_4}, & \mathbf{C}^{\sigma_2}_{1,2} &= -C_{1_2} + C_{1_4},\\
\mathbf{C}^{\mathrm{sgn}_1}_{1,1} &= C_{2_1}, & \mathbf{C}^{\sigma_1}_{2,1} &= -C_{2_2} - C_{2_4}, & \mathbf{C}^{\sigma_2}_{2,1} &= -C_{2_2} + C_{2_4},\\
\mathbf{C}^{\mathrm{sgn}_2}_{1,1} &= C_{1_5}, & \mathbf{C}^{\sigma_1}_{2,2} &= C_{2_3} + C_{2_5}, & \mathbf{C}^{\sigma_2}_{2,2} &= C_{2_3} - C_{2_5}.
\end{aligned}$$

**Example 2.6.19.** Let $W$ be of type $I_2(5)$, where $S = \{s_1, s_2\}$ and $(s_1s_2)^5 = 1$. We are in the equal-parameter case; assume that $L(s_1) = L(s_2) > 0$. Then Theorem 2.6.12

applies and so we have a cellular basis $\{\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$. In this case, we can take for $R$ any subring of $\mathbb{C}$ in which 5 is invertible and $\alpha := \frac{1}{2}(-1+\sqrt{5}) \in R$. We find the following expressions for $\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}$:

$$
\begin{array}{ll}
\mathbf{C}^{1_W}_{1,1} = C_{1_0}, & C^{\mathrm{sgn}}_{1,1} = C_{1_5}, \\
\mathbf{C}^{\sigma_1}_{1,1} = (2+\alpha)C_{1_1} + (3+2\alpha)C_{1_3}, & \mathbf{C}^{\sigma_2}_{1,1} = (1-\alpha)C_{1_1} + (1-2\alpha)C_{1_3} \\
\mathbf{C}^{\sigma_1}_{1,2} = -(2+\alpha)C_{1_2} - (1+\alpha)C_{1_4}, & \mathbf{C}^{\sigma_2}_{1,2} = (\alpha-1)C_{1_2} + \alpha C_{1_4} \\
\mathbf{C}^{\sigma_1}_{2,1} = -(2+\alpha)C_{2_2} - (1+\alpha)C_{2_4}, & \mathbf{C}^{\sigma_2}_{2,1} = (\alpha-1)C_{2_2} + \alpha C_{2_4} \\
\mathbf{C}^{\sigma_1}_{2,2} = C_{2_1} + (1+\alpha)C_{2_3}, & \mathbf{C}^{\sigma_2}_{2,2} = C_{2_1} - \alpha C_{2_3}.
\end{array}
$$

## 2.7 Further Properties of the Cellular Basis of H

Throughout this section (except for Corollary 2.7.14 at the very end), we assume that we are in the setting of Theorem 2.6.12, where properties (♣) and (♠) in 2.5.3 hold. Thus, we have a cellular basis $\{\mathbf{C}^{\lambda}_{\mathfrak{s},\mathfrak{t}}\}$ of $\mathbf{H}$, and the partial order on $\Lambda$ satisfies

$$\mu \trianglelefteq_L \lambda \qquad \Rightarrow \qquad \mu = \lambda \quad \text{or} \quad \mathbf{a}_\mu > \mathbf{a}_\lambda.$$

Let $\{W(\lambda) \mid \lambda \in \Lambda\}$ be the cell modules constructed from the cellular basis; see 2.6.1. By extension of scalars from $A$ to $K$, we obtain modules for $\mathbf{H}_K$ which we denote by $W_K(\lambda)$ $(\lambda \in \Lambda)$. Since $\mathbf{H}_K$ is semisimple, Theorem 2.6.5 shows that

$$\mathrm{Irr}(\mathbf{H}_K) = \{W_K(\lambda) \mid \lambda \in \Lambda\}.$$

**Proposition 2.7.1.** *Let $\lambda \in \Lambda$. For any $h = C_w$ $(w \in W)$ and $\mathfrak{s}, \mathfrak{s}' \in M(\lambda)$, we have*

$$\varepsilon^{\mathbf{a}_\lambda}\, r^{\lambda}_h(\mathfrak{s}',\mathfrak{s}) \in R[\Gamma_{\geqslant 0}], \quad \textit{with constant term equal to } c^{\mathfrak{s}'\mathfrak{s}}_{w,\lambda}.$$

*In particular, the representation of $\mathbf{H}_K$ afforded by $W_K(\lambda)$ (with respect to its standard basis) is balanced, and we have $W_K(\lambda) \cong E^{\lambda}_{\varepsilon}$.*

*Proof.* At the end of the proof of Theorem 2.6.12, we obtained the formula

$$r^{\lambda}_h(\mathfrak{s}',\mathfrak{s}) = \sum_{\substack{z\in W,\, d\in\tilde{\mathscr{D}} \\ z\sim_{\mathscr{LR}} d}} \tilde{n}_d\, h_{w,d,z}\, c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda}.$$

Now assume $z \in W$ and $d \in \tilde{\mathscr{D}}$ are such that $z \sim_{\mathscr{LR}} d$ and the corresponding terms in the above sum are non-zero; that is, $h_{w,d,z} \neq 0$ and $c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda} \neq 0$. Since $c^{\mathfrak{s}'\mathfrak{s}}_{z,\lambda} \neq 0$, we have $E^{\lambda} \leftrightsquigarrow_L z$. Hence, Lemma 2.5.11 shows that $\varepsilon^{\mathbf{a}_\lambda} h_{w,d,z}$ lies in $\mathbb{Z}[\Gamma_{\geqslant 0}]$ and has constant term $\tilde{\gamma}_{w,d,z^{-1}}$. Thus, we have $\varepsilon^{\mathbf{a}_\lambda}\, r^{\lambda}_h(\mathfrak{s}',\mathfrak{s}) \in R[\Gamma_{\geqslant 0}]$ and

$$\varepsilon^{\mathbf{a}_\lambda}\, r_h^\lambda(\mathfrak{s}',\mathfrak{s}) \equiv \sum_{z\in W}\Big(\sum_{d\in\tilde{\mathscr{D}}: z\sim_{\mathscr{LR}} d} \tilde{n}_d\, \tilde{\gamma}_{w,d,z^{-1}}\Big) c_{z,\lambda}^{\mathfrak{s}'\mathfrak{s}} \equiv c_{w,\lambda}^{\mathfrak{s}'} \mod R[\Gamma_{>0}],$$

where the last congruence follows as in Remark 2.5.15. Once this is established, it is clear that the representation afforded by $W_K(\lambda)$ is balanced. Furthermore, by Proposition 1.5.7, we also see that $W_K(\lambda) \cong E_\varepsilon^\lambda$. □

**Corollary 2.7.2.** *Let $\lambda \in \Lambda$ and denote by $G^\lambda = \big(g_{\mathfrak{s}\mathfrak{t}}^\lambda\big)_{\mathfrak{s},\mathfrak{t}\in M(\lambda)}$ the Gram matrix of the bilinear form $\langle\ ,\ \rangle_\lambda : W(\lambda)\times W(\lambda) \to A$. Then*

$$\varepsilon^{\mathbf{a}_\lambda}\, g_{\mathfrak{s}\mathfrak{t}}^\lambda \in R[\Gamma_{\geqslant 0}] \qquad \textit{and} \qquad \varepsilon^{\mathbf{a}_\lambda}\, g_{\mathfrak{s}\mathfrak{t}}^\lambda \equiv f_\lambda \beta_{\mathfrak{s}\mathfrak{t}}^\lambda \mod R[\Gamma_{>0}].$$

*Proof.* Recall that $g_{\mathfrak{s}\mathfrak{t}}^\lambda = g_{\mathfrak{t}\mathfrak{s}}^\lambda = r_h^\lambda(\mathfrak{s},\mathfrak{s})$, where $h = \mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda$. Hence, using the defining formula for $\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda$, we obtain

$$\varepsilon^{\mathbf{a}_\lambda}\, g_{\mathfrak{s}\mathfrak{t}}^\lambda = \sum_{w\in W}\sum_{u\in M(\lambda)} \beta_{\mathfrak{t}\mathfrak{u}}^\lambda\, c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}} \big(\varepsilon^{\mathbf{a}_\lambda}\, r_{C_w}^\lambda(\mathfrak{s},\mathfrak{s})\big).$$

By Proposition 2.7.1, this expression lies in $R[\Gamma_{\geqslant 0}]$ and has constant term

$$\sum_{w\in W}\sum_{u\in M(\lambda)} \beta_{\mathfrak{t}\mathfrak{u}}^\lambda\, c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}}\, c_{w,\lambda}^{\mathfrak{s}\mathfrak{s}} = \sum_{u\in M(\lambda)} \beta_{\mathfrak{t}\mathfrak{u}}^\lambda \Big(\sum_{w\in W} c_{w^{-1},\lambda}^{\mathfrak{u}\mathfrak{s}}\, c_{w,\lambda}^{\mathfrak{s}\mathfrak{s}}\Big) = \beta_{\mathfrak{t}\mathfrak{s}}^\lambda f_\lambda,$$

where we used the "Schur relations" in Proposition 1.4.10(a). □

**Example 2.7.3.** Let $\lambda \in \Lambda$ and consider the representation $\bar{\rho}^\lambda : \tilde{\mathbf{J}} \to M_{d_\lambda}(\mathbb{K})$, where $\bar{\rho}^\lambda(t_w) \in M_{d_\lambda}(R)$ for all $w \in W$. By first restricting $\bar{\rho}^\lambda$ to $\tilde{\mathbf{J}}_R$ and then extending scalars from $R$ to $A$, we can also regard $\bar{\rho}^\lambda$ as an $A$-algebra homomorphism

$$\bar{\rho}^\lambda : \tilde{\mathbf{J}}_A \to M_{d_\lambda}(A).$$

With this convention, the formula at the end of Theorem 2.6.12 means that

$$\big(r_h^\lambda(\mathfrak{s}',\mathfrak{s})\big)_{\mathfrak{s}',\mathfrak{s}\in M(\lambda)} = \bar{\rho}^\lambda\big(\phi(C_w)\big) \qquad (h = C_w).$$

By Proposition 2.7.1, we have $E_\varepsilon^\lambda \cong W_K(\lambda)$. Hence, the above formula shows that, for a suitable basis of $E_\varepsilon^\lambda$, the action of $C_w$ on $E_\varepsilon^\lambda$ is given by the matrix $\bar{\rho}^\lambda(\phi(C_w))$. We express this by saying that the action of $\mathbf{H}$ on $E_\varepsilon^\lambda$ factors through $\phi$.

**Example 2.7.4.** Let $\lambda \in \Lambda$ and assume that there exists a left Kazhdan–Lusztig cell $\mathfrak{C}$ such that $E_\varepsilon^\lambda \cong [\mathfrak{C}]_K$. (This is a very special situation, but we will see in Section 2.8 that it holds, for example, when $W \cong \mathfrak{S}_n$.) Let us write $\mathfrak{C} = \{x_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\}$. Then we have a corresponding representation $\rho_\mathfrak{C} : \mathbf{H}_K \to M_{d_\lambda}(K)$ such that

$$\rho_\mathfrak{C}(C_w) = \big(h_{w,x_\mathfrak{t},x_\mathfrak{s}}\big)_{\mathfrak{s},\mathfrak{t}\in M(\lambda)} \qquad \text{for all } w \in W.$$

By Lemma 2.2.4, there exists some $w \in \mathfrak{C}$ such that $E^\lambda \leftrightsquigarrow_L w$. Since (♣) is assumed to hold, we can apply Lemma 2.5.11, which yields that $\rho_{\mathfrak{C}}$ is balanced and, for all $w \in W$ and $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$, we have

$$\varepsilon^{\mathbf{a}_\lambda} h_{w,x_\mathfrak{t},x_\mathfrak{s}} \equiv \tilde{\gamma}_{w,x_\mathfrak{t},x_\mathfrak{s}^{-1}} \bmod \mathbb{Z}[\Gamma_{>0}].$$

Thus, we can assume that $\rho^\lambda = \rho_{\mathfrak{C}}$ and the leading matrix coefficients are given by

$$c_{w,\lambda}^{\mathfrak{s}\,\mathfrak{t}} = \tilde{\gamma}_{w,x_\mathfrak{t},x_\mathfrak{s}^{-1}} \qquad \text{for all } w \in W \text{ and } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

Now, by Lemma 1.5.3, we have $\tilde{\gamma}_{w,x_\mathfrak{t},x_\mathfrak{s}^{-1}} = \tilde{\gamma}_{x_\mathfrak{t}^{-1},w^{-1},x_\mathfrak{s}} = \tilde{\gamma}_{w^{-1},x_\mathfrak{s},x_\mathfrak{t}^{-1}}$. This means that $\bar{\rho}^\lambda(t_w)^{\mathrm{tr}} = \bar{\rho}^\lambda(t_{w^{-1}})$ for all $w \in W$. Consequently, the conditions in Proposition 1.5.11 are satisfied where we take $B^\lambda$ to be the identity matrix. The formula for $r_h^\lambda(\mathfrak{s}', \mathfrak{s})$ in the proof of Proposition 2.7.1 now reads

$$r_h^\lambda(\mathfrak{s}', \mathfrak{s}) = \sum_{\substack{z \in W, d \in \tilde{\mathscr{D}} \\ z \sim_{\mathscr{LR}} d}} \tilde{n}_d \, h_{w,d,z} \, \tilde{\gamma}_{z,x_\mathfrak{s},x_{\mathfrak{s}'}^{-1}} = h_{w,x_\mathfrak{s},x_{\mathfrak{s}'}} \qquad (h = C_w),$$

where the last equality holds by Lemma 2.5.8. Thus, we have shown that $W(\lambda)$ is nothing but the left cell module $[\mathfrak{C}]_A$. Furthermore, the action of $\mathbf{H}$ on $[\mathfrak{C}]_A$ factors through $\phi$, as in Example 2.7.3.

**2.7.5.** One important feature of the definition of a "cell datum" is that it behaves well with respect to a *specialisation*; see [144, (1.8)]. Let $\theta \colon A \to k$ be a homomorphism into a field $k$. Let $\mathbf{H}_k = k \otimes_A \mathbf{H}$ be the corresponding specialised algebra over $k$. Assume that $\{\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda\}$ satisfy (C1), (C2), (C3) in $\mathbf{H}$. Then the elements $\{1 \otimes \mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda\}$ will satisfy (C1), (C2), (C3) in $\mathbf{H}_k$. Hence, a cell datum for $\mathbf{H}$ automatically gives rise to a cell datum for $\mathbf{H}_k$. Note that then the cell representations of $\mathbf{H}_k$ are given by $W_k(\lambda) = k \otimes_A W(\lambda)$ ($\lambda \in \Lambda$), and the bilinear form $\langle\;,\;\rangle_\lambda$ on $W(\lambda)$ induces the corresponding form $\langle\;,\;\rangle_{\lambda,k}$ on $W_k(\lambda)$. In particular, we have the following:

(a) Extending scalars from $A$ to $K$, we obtain a cell datum for $\mathbf{H}_K$. As already mentioned (see Proposition 2.7.1), since $\mathbf{H}_K$ is semisimple, we have

$$\operatorname{Irr}(\mathbf{H}_K) = \{W_K(\lambda) \mid \lambda \in \Lambda\}, \qquad \text{where} \qquad W_K(\lambda) \cong E_\varepsilon^\lambda \text{ for all } \lambda \in \Lambda.$$

(b) In general, given any map $\theta \colon A \to k$ as above, we set $L_k^\lambda = W_k(\lambda)/\operatorname{rad}(\langle\;,\;\rangle_{\lambda,k})$ for $\lambda \in \Lambda$. Then Theorem 2.6.5 implies that

$$\operatorname{Irr}(\mathbf{H}_k) = \{L_k^\mu \mid \mu \in \Lambda_k^\circ\}, \qquad \text{where} \qquad \Lambda_k^\circ := \{\lambda \in \Lambda \mid \langle\;,\;\rangle_{\lambda,k} \neq 0\}.$$

Furthermore, the composition multiplicities $(W_k(\lambda) : L_k^\mu)$ satisfy the conditions $(\Delta)$ in Theorem 2.6.6. Hence, since (♣) is assumed to hold, this means

$$(\Delta^a) \qquad \begin{cases} (W_k(\mu) : L_k^\mu) = 1 & \text{for all } \mu \in \Lambda_k^\circ, \\ (W_k(\lambda) : L_k^\mu) = 0 & \text{unless } \lambda = \mu \text{ or } \mathbf{a}_\lambda > \mathbf{a}_\mu. \end{cases}$$

Thus, our "fundamental problem" (p. 3) of determining $\mathrm{Irr}(\mathbf{H}_k)$ now takes the following more precise form (and this will be addressed in the subsequent chapters).

**Fundamental Problem (revised).** *Given a cell datum for* $\mathbf{H}_k$*, describe the subset* $\Lambda_k^\circ \subseteq \Lambda$ *and determine the dimension of* $L_k^\mu$ *for* $\mu \in \Lambda_k^\circ$.

Our next result provides an alternative characterisation of the subset $\Lambda_k^\circ \subseteq \Lambda$. In particular, this shows that $\Lambda_k^\circ$ does not depend on the choices involved in the definition of $\{\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda\}$. (Recall that, for example, this definition relies on the balanced representations $\rho^\lambda$, and these are not uniquely determined.)

**Proposition 2.7.6.** *In the setting of 2.7.5, let* $\lambda \in \Lambda$*. Then the following three conditions are equivalent.*

*(a)* $\lambda \in \Lambda_k^\circ$.
*(b)* $\theta\big(\chi^\lambda(C_w)\big) \neq 0$ *for some* $w \in W$ *such that* $E^\lambda \leftrightsquigarrow_L w$.
*(c)* $C_w.W_k(\lambda) \neq \{0\}$ *for some* $w \in W$ *such that* $E^\lambda \leftrightsquigarrow_L w$.

*Proof.* "(a) $\Rightarrow$ (b)" If $\lambda \in \Lambda_k^\circ$, then $\langle\ ,\ \rangle_{\lambda,k} \neq 0$ and so $\langle C_\mathfrak{u}, C_\mathfrak{t}\rangle_{\lambda,k} \neq 0$ for some $\mathfrak{u},\mathfrak{t} \in M(\lambda)$. Now, by Proposition 2.7.1, we have $E_\varepsilon^\lambda \cong W_K(\lambda)$ and so

$$\chi^\lambda(h) = \mathrm{trace}(h, E_\varepsilon^\lambda) = \mathrm{trace}\big(h, W_K(\lambda)\big) = \sum_{\mathfrak{s}\in M(\lambda)} r_h^\lambda(\mathfrak{s},\mathfrak{s}) \quad \text{for all } h \in \mathbf{H}.$$

Let $h = \mathbf{C}_{\mathfrak{u},\mathfrak{t}}^\lambda$ and apply Corollary 2.6.3. This yields $r_h^\lambda(\mathfrak{s},\mathfrak{s}) = \delta_{\mathfrak{u}\mathfrak{s}}\langle C_\mathfrak{s}, C_\mathfrak{t}\rangle_\lambda$ for all $\mathfrak{s} \in M(\lambda)$ and so $\chi^\lambda(h) = \langle C_\mathfrak{u}, C_\mathfrak{t}\rangle_\lambda$. Since $\langle C_\mathfrak{u}, C_\mathfrak{t}\rangle_{\lambda,k} \neq 0$, this shows that $\theta(\chi^\lambda(h)) \neq 0$. Finally, by Remark 2.6.9, $h$ is an $R$-linear combination of elements $C_w$, where $E^\lambda \leftrightsquigarrow_L w$. Hence, (b) follows.

"(b) $\Rightarrow$ (c)" As above, we have $\chi^\lambda(h) = \sum_{\mathfrak{s}\in M(\lambda)} r_h^\lambda(\mathfrak{s},\mathfrak{s})$ for all $h \in \mathbf{H}$. Since $W_k(\lambda) = k \otimes_A W(\lambda)$, this implies

$$\theta\big(\chi^\lambda(C_w)\big) = \mathrm{trace}\big(C_w, W_k(\lambda)\big) \qquad \text{for all } w \in W,$$

where $C_w$ is regarded as an element of $\mathbf{H}_k$ on the right-hand side. Hence, if (b) holds, then $C_w$ does not act as zero on $W_k(\lambda)$ and so (c) holds.

"(c) $\Rightarrow$ (a)" Let $w \in W$ be such that $E^\lambda \leftrightsquigarrow_L w$ and $C_w.W_k(\lambda) \neq \{0\}$. By Lemma 2.6.10, $C_w$ is an $R$-linear combination of elements $\mathbf{C}_{\mathfrak{u},\mathfrak{v}}^\mu$, where $\mu \in \Lambda$ is such that $E^\mu \leftrightsquigarrow_L w$. Hence, we also have $h.W_k(\lambda) \neq \{0\}$, where $h = \mathbf{C}_{\mathfrak{u},\mathfrak{v}}^\mu$ for some $\mu \in \Lambda$ and $\mathfrak{u},\mathfrak{v} \in M(\mu)$ such that $E^\lambda \sim_L E^\mu$. In particular, this implies that $h.W(\lambda) \neq \{0\}$. By the definition of the action of $\mathbf{H}$ on $W(\lambda)$, this means that there exist some $\mathfrak{s},\mathfrak{s}',\mathfrak{t} \in M(\lambda)$ such that $\mathbf{C}_{\mathfrak{s}',\mathfrak{t}}^\lambda$ appears with a non-zero coefficient in the decomposition of $h\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda = \mathbf{C}_{\mathfrak{u},\mathfrak{v}}^\mu \mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda$. By (C2) and (C3), this implies that $\lambda \trianglelefteq_L \mu$. Since also $E^\lambda \sim_L E^\mu$, we conclude that $\lambda = \mu$.

Thus, we have $h.W_k(\lambda) \neq \{0\}$, where $h = \mathbf{C}^\lambda_{\mathfrak{u},\mathfrak{v}}$ for some $\mathfrak{u},\mathfrak{v} \in M(\lambda)$. By the definition of the action of $\mathbf{H}_k$ on $W_k(\lambda)$ and Corollary 2.6.3, this implies that

$$\langle C_\mathfrak{s}, C_\mathfrak{v}\rangle_{\lambda,k} = \theta\big(r_h^\lambda(\mathfrak{u},\mathfrak{s})\big) \neq 0 \quad \text{for some } \mathfrak{s} \in M(\lambda).$$

Thus, $\langle\ ,\ \rangle_{\lambda,k} \neq 0$ and so (a) holds. □

**Proposition 2.7.7.** *Let $\lambda \in \Lambda_k^\circ$. Then the following hold.*

*(a) We have $\theta(\chi^\lambda(C_w)) = \mathrm{trace}(C_w, L_k^\lambda)$ for all $w \in W$ such that $E^\lambda \leftrightsquigarrow_L w$.*
*(b) We have $C_w.L_k^\lambda \neq \{0\}$ for some $w \in W$ such that $E^\lambda \leftrightsquigarrow_L w$.*

*Proof.* (a) Let $w \in W$ be such that $E^\lambda \leftrightsquigarrow_L w$. As in the above proof, $\theta(\chi^\lambda(C_w)) = \mathrm{trace}(C_w, W_k(\lambda))$. Considering a composition series for $W_k(\lambda)$, we obtain

$$\theta(\chi^\lambda(C_w)) = \sum_{\mu\in\Lambda_k^\circ} (W_k(\lambda) : L_k^\mu)\,\mathrm{trace}(C_w, L_k^\mu).$$

Let $\mu \in \Lambda_k^\circ$ and assume that the corresponding terms in the sum are non-zero; that is, we have $(W_k(\lambda) : L_k^\mu) \neq 0$ and $\mathrm{trace}(C_w, L_k^\mu) \neq 0$. In particular, this means that $C_w.L_k^\mu \neq \{0\}$. We claim that this implies that $E^\mu \preceq_L E^\lambda$. Indeed, since $C_w.L_k^\mu \neq \{0\}$, we also have $C_w.W_k(\mu) \neq \{0\}$ and, hence, $C_w.W(\mu) \neq 0$. By Lemma 2.6.10, $C_w$ is an $R$-linear combination of elements $\mathbf{C}^\nu_{\mathfrak{u},\mathfrak{v}}$, where $\nu \in \Lambda$ is such that $E^\nu \leftrightsquigarrow_L w$. Hence, we also have $h.W(\mu) \neq \{0\}$, where $h = \mathbf{C}^\nu_{\mathfrak{u},\mathfrak{v}}$ for some $\nu \in \Lambda$ and $\mathfrak{u},\mathfrak{v} \in M(\nu)$ such that $E^\lambda \sim_L E^\nu$. Arguing as in the above proof, this implies that $\mu \trianglelefteq_L \nu$. Since also $E^\lambda \sim_L E^\nu$, we conclude that $E^\mu \preceq_L E^\lambda$, as claimed. On the other hand, since $(W_k(\lambda) : L_k^\mu) \neq 0$, we have $\lambda \trianglelefteq_L \mu$; see Theorem 2.6.6. In combination with $E^\mu \preceq_L E^\lambda$, this implies that $\lambda = \mu$. Thus, since $(W_k(\lambda) : L_k^\lambda) = 1$, we have shown that $\theta(\chi^\lambda(C_w)) = \mathrm{trace}(C_w, L_k^\lambda)$.

(b) By Proposition 2.7.6, we have $\theta(\chi^\lambda(C_w)) \neq 0$ for some $w \in W$ such that $E^\lambda \leftrightsquigarrow_L w$. By (a), this implies that $\mathrm{trace}(C_w, L_k^\lambda) \neq 0$ and so $C_w.L_k^\lambda \neq \{0\}$. □

*Remark 2.7.8.* Once a cellular structure for **H** is available, it also natural to consider the "Jantzen filtration" on cell modules; for recent results and open problems in this direction, see James and Mathas [183], [184], Shan [277] (type $A$), Bonnafé and Jacon [27] and Policzew [266] (exceptional types).

Finally, we discuss the existence of $W$-graph representations, as already briefly mentioned at the end of Section 1.4. We begin with a preliminary result about the structure and the representations of $\tilde{\mathbf{J}}$.

**Lemma 2.7.9.** *In addition to (♣) and (♠), also assume that (♦) holds. Then:*

*(a) We have $\tilde{\gamma}_{x,y,z} \in \mathbb{Z}$ and $\tilde{n}_d = \pm 1$ for all $d \in \tilde{\mathscr{D}}$.*
*(b) The elements $\{\tilde{n}_d t_d \mid d \in \tilde{\mathscr{D}}\}$ are orthogonal idempotents.*

*Furthermore, for each $\lambda \in \Lambda$, the balanced representation $\rho^\lambda$ of $\mathbf{H}_K$ can be chosen such that the following holds for the corresponding representation $\bar{\rho}^\lambda$ of $\tilde{\mathbf{J}}$:*

*(c) The conditions in Proposition 1.5.11 hold where* $\bar{\rho}^\lambda(t_w) = \big(c_{w,\lambda}^{\mathfrak{st}}\big) \in M_{d_\lambda}(\mathbb{Z}_W)$ *for all* $w \in W$.

*(d) For any* $d \in \tilde{\mathscr{D}}$*, the matrix* $\bar{\rho}^\lambda(t_d)$ *is diagonal with* $0, \pm 1$ *on the diagonal.*

*Proof.* (a), (b) See Example 2.5.14. Once we know that $\tilde{\gamma}_{x,y,z}$ and $\tilde{n}_w$ are integers, the fact that $\tilde{n}_d = \pm 1$ follows from the formula in Remark 1.6.19(a).

(c), (d) We slightly refine the argument in Proposition 1.5.11. We can assume that $(W,S)$ is irreducible. First let $W$ be of type $I_2(m)$. In the proof of Proposition 1.5.11, we have seen that the representations in Example 1.3.7 satisfy (c). By Example 1.7.4, these representations also satisfy (d). Now assume that $W$ is not of type $I_2(m)$. Then $\mathbb{Z}_W$ is a principal ideal domain. As in the proof of Proposition 1.5.11, a general argument shows that $\rho^\lambda$ can be chosen such that (c) holds. Let $\tilde{\mathbf{J}}_{\mathbb{Z}_W} = \langle t_w \mid w \in W \rangle_{\mathbb{Z}_W}$ and let $\bar{E}^\lambda$ be a $\tilde{\mathbf{J}}_{\mathbb{Z}_W}$-module (finitely generated and free over $\mathbb{Z}_W$) which affords the representation $\bar{\rho}^\lambda$. Since the idempotents $\tilde{n}_d t_d$ ($d \in \tilde{\mathscr{D}}$) lie in $\tilde{\mathbf{J}}_{\mathbb{Z}_W}$ and since $\mathbb{Z}_W$ is a principal ideal domain, we have a direct sum decomposition $\bar{E}^\lambda = \bigoplus_{d \in \tilde{\mathscr{D}}} \bar{E}_d^\lambda$, where $\bar{E}_d^\lambda := \tilde{n}_d t_d.\bar{E}^\lambda$ is a $\mathbb{Z}_W$-submodule of $\bar{E}^\lambda$ which is finitely generated and free over $\mathbb{Z}_W$. Now choose a $\mathbb{Z}_W$-basis of $\bar{E}^\lambda$ which is adapted to this decomposition and perform a base change (over $\mathbb{Z}_W$) to this new basis. We replace $\rho^\lambda$ by the representation obtained via this base change (as in the proof of Proposition 1.5.11). This new representation is balanced, and it satisfies (c) and (d). $\square$

**2.7.10.** We keep the assumptions of Lemma 2.7.9. We shall consider the effect of Lusztig's homomorphism $\phi\colon \mathbf{H} \to \tilde{\mathbf{J}}_A$ (see Theorem 2.5.5) on $C_s$, where $s \in S$ is such that $L(s) > 0$. For any $d \in \tilde{\mathscr{D}}$, we have $C_s C_d = -(v_s + v_s^{-1})C_d$ if $sd < d$; furthermore, if $sd > d$, then $h_{s,d,z} = 0$ unless $sz < z$, in which case we have $h_{s,d,z} = (-1)^{l(d)+l(z)+1}\mu_{z,d}^s$. Thus, the formula in Theorem 2.5.5 can be written as

$$\phi(C_s) = -(v_s + v_s^{-1})\Big(\sum_{\substack{d \in \tilde{\mathscr{D}} \\ sd<d}} \tilde{n}_d t_d\Big) + \Big(\sum_{\substack{z \in W,\, d \in \tilde{\mathscr{D}} \\ sz<z<d<sd \text{ and } z \sim_{\mathscr{LR}} d}} (-1)^{l(z)+l(d)+1} \tilde{n}_d \mu_{z,d}^s t_z\Big).$$

Following Gyoja [150] (where this is discussed in the equal-parameter case, in a somewhat different setting), we define elements of $\tilde{\mathbf{J}}_A$ as follows:

$$\tilde{s}_0 := \sum_{\substack{d \in \tilde{\mathscr{D}} \\ sd<d}} \tilde{n}_d t_d \qquad \text{and} \qquad \tilde{s}_1 := \sum_{\substack{z \in W,\, d \in \tilde{\mathscr{D}} \\ sz<z<d<sd \text{ and } z \sim_{\mathscr{LR}} d}} (-1)^{l(z)+l(d)+1} \tilde{n}_d \mu_{z,d}^s t_z.$$

Thus, $\phi(C_s) = -(v_s + v_s^{-1})\tilde{s}_0 + \tilde{s}_1$. Now recall that $C_s = T_s - v_s T_1$. Then we have

$$\phi(T_s) = \phi(C_s) + v_s 1_{\tilde{\mathbf{J}}} = -v_s^{-1}\tilde{s}_0 + v_s(1_{\tilde{\mathbf{J}}} - \tilde{s}_0) + \tilde{s}_1 \qquad \text{for any } s \in S.$$

The collection of elements $\{\tilde{s}_0, \tilde{s}_1 \mid s \in S, L(s) > 0\}$ satisfies the following properties.

**Lemma 2.7.11 (Cf. Gyoja [150, 2.4]).** *In the above setting, we have*

$$\tilde{s}_0^2 = \tilde{s}_0, \qquad \tilde{s}_0\tilde{t}_0 = \tilde{t}_0\tilde{s}_0, \qquad \tilde{s}_0\tilde{s}_1 = \tilde{s}_1, \qquad \tilde{s}_1\tilde{s}_0 = 0$$

*for all* $s,t \in S$ *such that* $L(s) > 0$ *and* $L(t) > 0$.

*Proof.* By Lemma 2.7.9(b), the elements $\{\tilde{n}_d t_d \mid d \in \tilde{\mathscr{D}}\}$ are orthogonal idempotents in $\tilde{\mathbf{J}}$. This yields that $\tilde{s}_0^2 = \tilde{s}_0$ and $\tilde{s}_0\tilde{t}_0 = \tilde{t}_0\tilde{s}_0$ for all $s, t \in S$. Now consider

$$\tilde{s}_0\tilde{s}_1 = \sum_{\substack{z\in W, d'\in\tilde{\mathscr{D}} \\ sz<z<d'<sd' \text{ and } z\sim_{\mathscr{LR}} d'}} (-1)^{l(z)+l(d')+1}\tilde{n}_{d'}\,\mu^s_{z,d'}\Big(\sum_{\substack{d\in\tilde{\mathscr{D}} \\ sd<d}} \tilde{n}_d t_d t_z\Big).$$

In order to show that this equals $\tilde{s}_1$, it will be enough to show that $\sum_{d\in\tilde{\mathscr{D}}, sd<d} \tilde{n}_d t_d t_z = t_z$ for all $z \in W$ such that $sz < z$. Now, given any $d \in \tilde{\mathscr{D}}$, we have

$$\tilde{n}_d t_d t_z = \sum_{x\in W} \tilde{n}_d \tilde{\gamma}_{d,z,x^{-1}}\, t_x = \sum_{x\in W} \tilde{n}_d \tilde{\gamma}_{z,x^{-1},d}\, t_x = \begin{cases} t_z & \text{if } z^{-1} \sim_{\mathscr{L}} d, \\ 0 & \text{otherwise,} \end{cases}$$

where the second equality holds by Lemma 1.5.3(a) and the third equality holds by Remark 1.6.19(a). Hence, it remains to show that the unique $d \in \tilde{\mathscr{D}}$ such that $z^{-1} \sim_{\mathscr{L}} d$ satisfies $sd < d$. Now recall that $sz < z$. Using Lemma 2.1.16, we conclude that $s \in \mathscr{L}(z) = \mathscr{R}(z^{-1}) = \mathscr{R}(d)$. But, by Remark 1.6.19(a), we have $d^2 = 1$ and so $\mathscr{R}(d) = \mathscr{L}(d)$. Thus, $sd < d$, as required. Finally, consider

$$\tilde{s}_1\tilde{s}_0 = \sum_{\substack{d\in\tilde{\mathscr{D}} \\ sd<d}} \sum_{\substack{z\in W, d'\in\tilde{\mathscr{D}} \\ sz<z<d'<sd' \text{ and } z\sim_{\mathscr{LR}} d'}} (-1)^{l(z)+l(d')+1}\tilde{n}_{d'}\,\mu^s_{z,d'}\tilde{n}_d t_z t_d.$$

Assume, if possible, that this is non-zero. Then for some $z, d, d'$ in the above sum, we have $\mu^s_{z,d'} \neq 0$ and $t_z t_d \neq 0$. Arguing as above, the latter condition implies that $z \sim_{\mathscr{L}} d$. On the other hand, since $\mu^s_{z,d'} \neq 0$, we have $z \leqslant_{\mathscr{L}} d'$. Since we also have $z \sim_{\mathscr{LR}} d'$, we can conclude that $z \sim_{\mathscr{L}} d'$; see Lemma 2.5.9. Hence, (♦) yields that $d = d'$. But we have $sd < d$ and $sd' > d'$, which is a contradiction. Hence, the assumption was wrong and so we must have $\tilde{s}_1\tilde{s}_0 = 0$, as claimed. □

A version of the following result (for equal parameters, and without taking into account the cellular structure) was first shown by Gyoja [150, §2]; subsequently, Lusztig [224, 3.8] gave a slightly different argument based on his asymptotic algebra. Our proof is a combination of the techniques used in [150] and [224].

**Theorem 2.7.12.** *Recall our standing assumption that (♠) and (♣) hold; also assume that (♦) holds and that $L(s) > 0$ for all $s \in S$. Then the data in Definition 2.6.8 can be chosen such that the cell modules $\{W(\lambda) \mid \lambda \in \Lambda\}$ are afforded by $W$-graphs where the elements $\{m^s_{x,y}\} \subseteq A$ (see Definition 1.4.11) all lie in the subring $\mathbb{Z}_W[\Gamma] \subseteq A$.*

*Proof.* By Lemma 2.7.11, the elements $\{\tilde{s}_0 \mid s \in S\}$ pairwise commute with each other. Hence, we can define

$$F_I := \Big(\prod_{s\in I}\tilde{s}_0\Big)\Big(\prod_{s\in S\setminus I}(1_{\tilde{\mathbf{J}}} - \tilde{s}_0)\Big) \in \tilde{\mathbf{J}} \qquad \text{for any subset } I \subseteq S.$$

These elements have the following properties:

(a) $$1_{\tilde{\mathbf{J}}} = \sum_{I \subseteq S} F_I, \qquad F_I^2 = F_I \quad (I \subseteq S), \qquad F_I F_J = 0 \quad (I \neq J).$$

Indeed, by Lemma 2.7.11, the elements $\{\tilde{s}_0 \mid s \in S\}$ do not only commute with each other, but they are also idempotents. Hence, each $F_I$ is an idempotent (possibly zero). Furthermore, suppose that $I \neq J$. If $s \in I \setminus J$, then the factor $\tilde{s}_0$ occurs in $F_I$ and the factor $1_{\tilde{\mathbf{J}}} - \tilde{s}_0$ occurs in $F_J$. Hence, we have $F_I F_J = 0$. The argument is analogous if $s \in J \setminus I$. Finally, notice that $1_{\tilde{\mathbf{J}}} = 1_{\tilde{\mathbf{J}}}^{|S|} = \prod_{s \in S} \big(\tilde{s}_0 + (1_{\tilde{\mathbf{J}}} - \tilde{s}_0)\big)$. Expanding the product yields that $1_{\tilde{\mathbf{J}}} = \sum_{I \subseteq S} F_I$, as required. Thus, (a) is proved.

Now assume that the balanced representations $\{\rho^\lambda \mid \lambda \in \Lambda\}$ are chosen such that they satisfy the properties in Lemma 2.7.9. In particular, for any $\lambda \in \Lambda$ and $d \in \tilde{\mathscr{D}}$, the matrix $\bar{\rho}^\lambda(t_d)$ is diagonal with $0, \pm 1$ on the diagonal. We conclude that $\bar{\rho}^\lambda(\tilde{s}_0)$ is a diagonal matrix for any $s \in S$ and, hence, $\bar{\rho}^\lambda(F_I)$ is a diagonal matrix for any $I \subseteq S$. Since $F_I$ is an idempotent, the diagonal coefficients of $\bar{\rho}^\lambda(F_I)$ will be $0, 1$. Since the elements $\{F_I \mid I \subseteq S\}$ are orthogonal idempotents whose sum is $1_{\tilde{\mathbf{J}}}$, we conclude that there is a well-defined partition

(b) $$M(\lambda) = \bigsqcup_{I \subseteq S} M_I(\lambda) \qquad \text{such that} \qquad F_I.e_{\mathfrak{t}} = e_{\mathfrak{t}} \Leftrightarrow \mathfrak{t} \in M_I(\lambda).$$

Let us now extend scalars from $R$ to $A$. Then $\bar{E}_A^\lambda := A \otimes_R \bar{E}^\lambda$ is a $\tilde{\mathbf{J}}_A$-module but it also becomes an $\mathbf{H}$-module via Lusztig's homomorphism $\phi \colon \mathbf{H} \to \tilde{\mathbf{J}}_A$. The formula at the end of the proof of Theorem 2.6.12 shows that $h \in \mathbf{H}$ acts on $W(\lambda)$ in the same way as $\phi(h) \in \tilde{\mathbf{J}}_A$ acts on $\bar{E}_A^\lambda$. So let us identify $W(\lambda) = \bar{E}_A^\lambda$. Then, using the formula for $\phi(T_s)$ ($s \in S$) in 2.7.10, we see that the action of $\mathbf{H}$ on $W(\lambda)$ is given by

$$T_s.e_{\mathfrak{t}} = -v_s^{-1}\tilde{s}_0 e_{\mathfrak{t}} + v_s(1_{\tilde{\mathbf{J}}} - \tilde{s}_0)e_{\mathfrak{t}} + \tilde{s}_1 e_{\mathfrak{t}}, \qquad \text{where } \mathfrak{t} \in M(\lambda).$$

We will now show that this formula comes from a $W$-graph structure on $W(\lambda)$. First of all, the definition of a $W$-graph requires a map from $M(\lambda)$ to the set of all subsets of $S$. We define such a map as follows. Given $\mathfrak{t} \in M(\lambda)$, let $I(\mathfrak{t})$ be the unique subset $I \subseteq S$ such that $\mathfrak{t} \in M_I(\lambda)$; see (b). This definition implies that

(c) $$F_I.e_{\mathfrak{t}} = \begin{cases} e_{\mathfrak{t}} & \text{if } I = I(\mathfrak{t}), \\ 0 & \text{otherwise.} \end{cases}$$

Next we consider the action of $T_s$, where $s \in S$. If $s \in I(\mathfrak{t})$, then $\tilde{s}_1\tilde{s}_0 = 0$ and so $\tilde{s}_1 F_{I(\mathfrak{t})} = 0$; since $e_{\mathfrak{t}} = F_{I(\mathfrak{t})}.e_{\mathfrak{t}}$, we conclude that $\tilde{s}_1.e_{\mathfrak{t}} = 0$. Furthermore, $\tilde{s}_0.e_{\mathfrak{t}} = e_{\mathfrak{t}}$ (since $\tilde{s}_0^2 = \tilde{s}_0$ and so $\tilde{s}_0 F_{I(\mathfrak{t})} = F_{I(\mathfrak{t})}$). Hence, we obtain in this case

$$T_s.e_{\mathfrak{t}} = -v_s^{-1} e_{\mathfrak{t}},$$

as required in the definition of a $W$-graph. Now assume that $s \notin I(\mathfrak{t})$. Then $\tilde{s}_0.e_{\mathfrak{t}} = 0$ (since $\tilde{s}_0 F_{I(\mathfrak{t})} = 0$) and so

$$T_s.e_{\mathfrak{t}} = v_s e_{\mathfrak{t}} + \tilde{s}_1.e_{\mathfrak{t}} = v_s e_{\mathfrak{t}} + \sum_{\mathfrak{u} \in M(\lambda)} m^s_{\mathfrak{u},\mathfrak{t}} e_{\mathfrak{u}},$$

where the terms $m^s_{\mathfrak{u},\mathfrak{t}} \in A$ are such that $\bar{m}^s_{\mathfrak{u},\mathfrak{t}} = m^s_{\mathfrak{u},\mathfrak{t}}$ and $v_s m^s_{\mathfrak{u},\mathfrak{t}} \in \mathbb{Z}_W[\Gamma_{>0}]$ (by the defining formula for $\tilde{s}_1$, Example 2.1.10 and Lemma 2.7.9(a) and (c)). So it remains to show that $m^s_{\mathfrak{u},\mathfrak{t}} = 0$ unless $s \in I(\mathfrak{u})$. But, for any $I \subseteq S$ such that $s \notin I$, we have $(1_{\tilde{\mathbf{J}}} - \tilde{s}_0)\tilde{s}_1 = 0$ by Lemma 2.7.11 and so $F_I \tilde{s}_1 = 0$. Hence, we have

$$\tilde{s}_1.e_{\mathfrak{t}} = \Big(\sum_{I \subseteq S} F_I\Big)\tilde{s}_1.e_{\mathfrak{t}} = \sum_{I \subseteq S, s \in I} F_I \tilde{s}_1.e_{\mathfrak{t}} \subseteq \langle F_I.e_{\mathfrak{u}} \mid \mathfrak{u} \in M(\lambda), s \in I \subseteq S\rangle_A.$$

By (c), the latter submodule is contained in $\langle e_{\mathfrak{u}} \mid s \in I(\mathfrak{u})\rangle_A$, as required. So the above formulae show that $W(\lambda)$ is afforded by a $W$-graph. $\square$

The above result shows that the cell modules $\{W(\lambda) \mid \lambda \in \Lambda\}$ arising from our construction of a cellular basis of $\mathbf{H}$ are afforded by $W$-graphs. The following conjecture is a kind of converse to this statement.

**Conjecture 2.7.13 (Geck and Müller [129, 4.5]).** *Assume that $L(s) > 0$ for all $s \in S$ and that, for every $\lambda \in \Lambda$, we are given a $W$-graph affording an $\mathbf{H}$-module $V^\lambda$ such that $K \otimes_A V^\lambda \cong E^\lambda_\varepsilon$. Then the data in Definition 2.6.8 can be chosen such that $\{V^\lambda \mid \lambda \in \Lambda\}$ are the corresponding cell modules.*

In order to prove this conjecture, it would be sufficient to show that every $W$-graph representation of $\mathbf{H}$ factors through Lusztig's homomorphism $\phi\colon \mathbf{H} \to \tilde{\mathbf{J}}_A$. Somewhat related open problems are formulated by Gyoja [150, Remark 2.18].

The final result in this section holds without any assumptions on $W, L$.

**Corollary 2.7.14.** *Let $\mathbf{H} = \mathbf{H}_A(W,S,L)$ be any generic Iwahori–Hecke algebra, where $W$ is finite and the general assumptions in 1.2.1 hold; that is, $\Gamma$ admits a monomial ordering and $A = R[\Gamma]$, where $\mathbb{Z}_W \subseteq R \subseteq \mathbb{C}$. Then every irreducible representation of $\mathbf{H}_K$ can be realised over $\mathbb{Z}_W[\Gamma]$.*

*Proof.* Let $\mathbb{K}$ be the field of fractions of $R$ and set $k := \mathbb{K}[\Gamma]$. Then $k$ certainly is $L_0$-good and so, as in the proof of Corollary 2.6.16, there exists some cell datum for $\mathbf{H}_k$ which is obtained by extension of scalars from a cell datum in a "universal" algebra $\mathbf{H}_0$ over $A_0 = \mathbb{K}[\Gamma_0]$, where **P1–P15** are known to hold. Let $\{W_0(\lambda) \mid \lambda \in \Lambda\}$ be the corresponding cell modules of $\mathbf{H}_0$. We have a unique $\mathbb{K}$-linear ring homomorphism $\theta\colon A_0 \to k$ such that $\theta(v_s^\circ) = v_s$ for all $s \in S$, where $\{v_s^\circ\}$ are the parameters of $\mathbf{H}_0$ and $\{v_s \mid s \in S\}$ are the parameters of $\mathbf{H}$. Thus, $\mathbf{H}_k = k \otimes_{A_0} \mathbf{H}_0$, where $k$ is regarded as an $A_0$-module via $\theta$. Now $K$ (the field of fractions of $A$) equals the field of fractions of $k$. Since $\mathbf{H}_K = K \otimes_k \mathbf{H}_k$ is semisimple, we have

$$\mathrm{Irr}(\mathbf{H}_K) = \{W_K(\lambda) := K \otimes_{A_0} W_0(\lambda) \mid \lambda \in \Lambda\};$$

see Theorem 2.6.5 and 2.7.5. Since each $W_0(\lambda)$ is realised over $\mathbb{Z}_W[\Gamma_0]$ by Theorem 2.7.12, we conclude that $W_K(\lambda)$ is realised over the image of $\mathbb{Z}_W[\Gamma_0]$ under $\theta_0$; that is, over $\mathbb{Z}_W[\Gamma]$, as required. $\square$

## 2.8 The Case of the Symmetric Group

The aim of this section is to give an elementary proof of the properties (♣), (♠) and (♦) in 2.5.3 when $W \cong \mathfrak{S}_n$. We will then see that the Kazhdan–Lusztig basis $\{C_w\}$ itself is a cellular basis in this case. Note that even if we were willing to admit from the beginning that **P1**–**P15** hold for $W$, then there would still be a substantial piece of work to do in order to determine the partial order $\trianglelefteq_L$ in Theorem 2.6.12.

We begin with a few general (and more or less well-known) remarks related to longest elements in parabolic subgroups. In 2.8.1–2.8.7, $W$ may be any finite Coxeter group $W$ and $L\colon W \to \Gamma$ a weight function such that $L(s) > 0$ for all $s \in S$. (Here, we explicitly exclude the possibility that $L(s) = 0$ for some $s \in S$.)

**2.8.1.** Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$. Let $X_I$ be the set of distinguished left coset representatives of $W_I$ in $W$; we have

$$X_I = \{w \in W \mid w \text{ has minimal length in } wW_I\}.$$

The map $X_I \times W_I \to W$, $(x,u) \mapsto xu$, is a bijection and we have $l(xu) = l(x) + l(u)$ for $u \in W_I$ and $x \in X_I$; see [132, §2.1]. Let $\mathbf{H}_I = \langle T_w \mid w \in W_I \rangle_A \subseteq \mathbf{H}$ be the corresponding parabolic subalgebra of $\mathbf{H}$. For any $w \in W_I$, we have $C_w \in \mathbf{H}_I$ and $C'_w \in \mathbf{H}_I$; hence, $\{C_w \mid w \in W_I\}$ and $\{C'_w \mid w \in W_I\}$ are the Kazhdan–Lusztig bases of $\mathbf{H}_I$.

**Lemma 2.8.2.** *Let $w_I \in W_I$ be the unique element of maximal length. We have*

$$C_{w_I} = (-1)^{l(w_I)} \varepsilon^{L(w_I)} \sum_{w \in W_I} (-1)^{l(w)} \varepsilon^{-L(w)} T_w.$$

*Furthermore, the following hold.*

*(a) For any $w \in W_I$, we have $T_w C_{w_I} = (-1)^{l(w)} \varepsilon^{-L(w)} C_{w_I}$.*
*(b) We have $C_{w_I}^2 = (-1)^{l(w_I)} \varepsilon^{-L(w_I)} P_I C_{w_I}$, where $P_I = \sum_{w \in W_I} \varepsilon^{2L(w)}$; cf. 1.2.11(c).*
*(c) The set $X_I w_I$ is a union of left cells in $W$; we have $X_I w_I = \{w \in W \mid w \leqslant_{\mathscr{L}} w_I\}$.*

*Proof.* The formula for $C_{w_I}$ already appears in Example 2.1.17. Next, we prove (a), by induction on $l(w)$. First assume that $w = s \in I$. Then we have $sw_I < w_I$ and so the multiplication rule in Theorem 2.1.8 shows that $C_s C_{w_I} = -(v_s + v_s^{-1}) C_{w_I}$. Since $C_s = T_s - v_s T_1$, this yields $T_s C_{w_I} = -v_s^{-1} C_{w_I}$. If $l(w) > 1$, then write $w = w's$, where $s \in I$, $w' \in W_I$ and $l(w) = l(w') + 1$. We have $T_w = T_{w'} T_s$, and so the desired formula follows by induction. Once (a) is established, we compute

$$\begin{aligned} C_{w_I}^2 &= (-1)^{l(w_I)} \varepsilon^{L(w_I)} \sum_{w \in W_I} (-1)^{l(w)} \varepsilon^{-L(w)} T_w C_{w_I} \\ &= (-1)^{l(w_I)} \varepsilon^{L(w_I)} \sum_{w \in W_I} \varepsilon^{-2L(w)} C_{w_I} = (-1)^{l(w_I)} \varepsilon^{-L(w_I)} P_I C_{w_I}. \end{aligned}$$

To obtain the last equality, we used the formula $l(ww_I) = l(w_I) - l(w)$ for all $w \in W_I$. Thus, (b) is proved. Finally, consider (c). Let $w \in W$ be such that $w \leqslant_{\mathscr{L}} w_I$. Then $\mathscr{R}(w_I) \subseteq \mathscr{R}(w)$; see Lemma 2.1.16. Hence, since $\mathscr{R}(w_I) = I$, we can write $w = xw_I$, where $x \in X_I$. Conversely, if $x \in X_I$, then $l(xw_I) = l(x) + l(w_I)$ and so $xw_I \leqslant_{\mathscr{L}} w_I$. (This follows since $sw \leftarrow_{\mathscr{L}} w$ if $s \in S$, $w \in W$ are such that $sw > w$.) Thus, we obtain $X_I w_I = \{w \in W \mid w \leqslant_{\mathscr{L}} w_I\}$. This also shows that $X_I w_I$ is a union of left cells. $\square$

**Lemma 2.8.3.** *Let $I \subseteq S$ and $\mathfrak{I}_{w_I}^{\mathscr{L}} \subseteq \mathbf{H}$ be the left ideal defined by the general procedure in 1.6.2, with respect to the basis $\{C_w \mid w \in W\}$ of $\mathbf{H}$. Then we have*

$$\mathfrak{I}_{w_I}^{\mathscr{L}} = \langle\, C_{xw_I} \mid x \in X_I \,\rangle_A = \langle\, T_x C_{w_I} \mid x \in X_I \,\rangle_A.$$

*Proof.* By definition, we have $\mathfrak{I}_{w_I}^{\mathscr{L}} = \langle\, C_w \mid w \leqslant_{\mathscr{L}} w_I \rangle_A$; this equals $\langle C_{xw_I} \mid x \in X_I \rangle_A$ by Lemma 2.8.2(c). Now set $\mathscr{M}_I := \langle T_x C_{w_I} \mid x \in X_I \rangle_A$. Since $\mathfrak{I}_{w_I}^{\mathscr{L}}$ is a left ideal, it is clear that $\mathscr{M}_I \subseteq \mathfrak{I}_{w_I}^{\mathscr{L}}$. Both $\mathscr{M}_I$ and $\mathfrak{I}_{w_I}^{\mathscr{L}}$ are free of the same rank over $A$; this already implies that $K \otimes_A \mathscr{M}_I = K \otimes_A \mathfrak{I}_{w_I}^{\mathscr{L}}$. But we also have that $\mathbf{H}/\mathfrak{I}_{w_I}^{\mathscr{L}}$ is free over $A$; furthermore, $\mathbf{H}$ is free as an $\mathbf{H}_I$-module and so $\mathbf{H}/\mathscr{M}_I$ is free as an $A$-module. Hence, we must have $\mathbf{H}_I = \mathfrak{I}_{w_I}^{\mathscr{L}}$. $\square$

**Lemma 2.8.4.** *Let $I \subseteq S$ and $\mu \in \Lambda$ be such that $E^\mu$ is a constituent of the induced representation $\mathrm{Ind}_{W_I}^W(\mathrm{sgn}_I)$. Then $E^\mu \leftrightsquigarrow_L xw_I$ for some $x \in X_I$.*

*Proof.* Let $\mathrm{sgn}_I^\varepsilon$ denote the sign representation of $\mathbf{H}_I$. By Example 1.3.3, we have $\mathrm{sgn}_I^\varepsilon(T_w) = (-1)^{l(w)} \varepsilon^{-L(w)}$ for $w \in W_I$. So Lemma 2.8.2(a) shows that $\langle C_{w_I} \rangle_A \subseteq \mathbf{H}_I$ affords $\mathrm{sgn}_I^\varepsilon$. Now, the induction of representations is also defined on the level of $\mathbf{H}$; see [132, §9.1]. Hence, by Lemma 2.8.3, we have an isomorphism of left $\mathbf{H}$-modules

$$\mathfrak{I}_{w_I}^{\mathscr{L}} \xrightarrow{\sim} \mathrm{Ind}_{\mathbf{H}_I}^{\mathbf{H}}(\langle C_{w_I} \rangle_A), \qquad T_x C_{w_I} \mapsto T_x \otimes C_{w_I} \quad (x \in X_I).$$

By a specialisation argument (see Example 1.2.4), our assumption implies that $E_\varepsilon^\mu$ is a constituent of $\mathfrak{I}_{w_I,K}^{\mathscr{L}} := K \otimes_A \mathfrak{I}_{w_I}^{\mathscr{L}}$. Now, for any $w \in W$, we have $\mathrm{trace}(C_w, \mathfrak{I}_{w_I}^{\mathscr{L}}) = \sum_{x \in X_I} h_{w,xw_I,xw_I}$. Furthermore, by Lemma 2.8.2(a), we can write $X_I w_I = \mathfrak{C}_1 \cup \ldots \cup \mathfrak{C}_m$, where $\mathfrak{C}_1, \ldots, \mathfrak{C}_m$ are (pairwise distinct) left cells of $W$. Consequently, we have

$$\mathrm{trace}(C_w, \mathfrak{I}_{w_I}^{\mathscr{L}}) = \sum_{1 \leqslant i \leqslant m} \sum_{z \in \mathfrak{C}_i} h_{w,z,z} = \sum_{1 \leqslant i \leqslant m} \mathrm{trace}(C_w, [\mathfrak{C}_i]_A) \quad \text{for all } w \in W.$$

Since $\mathbf{H}_K$ is split semisimple, this implies that $\mathfrak{I}_{w_I,K}^{\mathscr{L}} \cong [\mathfrak{C}_1]_K \oplus \ldots \oplus [\mathfrak{C}_m]_K$. It follows that $E_\varepsilon^\mu$ is a constituent of $[\mathfrak{C}_i]_K$ for some $i$ and so $m(\mathfrak{C}_i, \mu) > 0$; see 2.2.2(b). Hence, by Lemma 2.2.4, there exists some $w \in X_I w_I$ such that $E^\mu \leftrightsquigarrow_L w$. $\square$

**Lemma 2.8.5.** *Let $I \subseteq S$ and $\mu \in \Lambda$ be such that $\rho^\mu(C_{w_I}) \neq 0$. Then $E^\mu$ is a constituent of $\mathrm{Ind}_{W_I}^W(\mathrm{sgn}_I)$.*

*Proof.* As in the above proof, let $\mathrm{sgn}_I^\varepsilon$ denote the sign representation of $\mathbf{H}_I$. Using the formula for $C_{w_I}$ in Lemma 2.8.2, we obtain

$$\chi^\mu(C_{w_I}) = (-1)^{l(w_I)} \varepsilon^{L(w_I)} \sum_{w \in W_I} \mathrm{sgn}_I^\varepsilon(T_{w^{-1}})\, \chi^\mu(T_w),$$

where we also used the fact that $\mathrm{sgn}_I^\varepsilon(T_{w^{-1}}) = \mathrm{sgn}_I^\varepsilon(T_w)$ for all $w \in W_I$. Now, writing the restriction of $\chi^\mu$ to $\mathbf{H}_{I,K}$ as a sum of irreducible characters of $\mathbf{H}_{I,K}$ and using the Schur relations in Proposition 1.2.12, we conclude that

$$\chi^\mu(C_{w_I}) = (-1)^{l(w_I)}\varepsilon^{L(w_I)}\mathbf{c}_{\mathrm{sgn}_I}\, m(I,\mu),$$

where $m(I,\mu)$ denotes the multiplicity of $\mathrm{sgn}_I^\varepsilon$ in the restriction of $\chi^\mu$ to $\mathbf{H}_{I,K}$. By a specialisation argument (see Example 1.2.4), $m(I,\mu)$ also equals the multiplicity of $\mathrm{sgn}_I$ in the restriction of $E^\mu$ from $W$ to $W_I$. And by Frobenius reciprocity, this is the same as the multiplicity of $E^\mu$ as a constituent of $\mathrm{Ind}_I^W(\mathrm{sgn}_I)$.

Thus, it remains to show that $\chi^\mu(C_{w_I}) \neq 0$. Now, by Lemma 2.8.2(b), $C_{w_I}$ is a non-zero scalar multiple of an idempotent. Hence, $\rho^\mu(C_{w_I})$ will be conjugate to a non-zero scalar multiple of a diagonal matrix with 0 and 1 on the diagonal. Since $\rho^\mu(C_{w_I}) \neq 0$, we conclude that $\chi^\mu(C_{w_I}) = \mathrm{trace}(\rho^\mu(C_{w_I})) \neq 0$, as required. ☐

**Corollary 2.8.6.** *Let $I \subseteq S$ and $\lambda \in \Lambda$ be such that $\mathbf{a}_\lambda = L(w_I)$ and $E^\lambda$ is a constituent of $\mathrm{Ind}_{W_I}^W(\mathrm{sgn}_I)$. Then $w_I \in \mathscr{F}_\lambda$ (where $\mathscr{F}_\lambda$ is defined in Proposition 1.6.11).*

*Proof.* As in the above proof, $\chi^\lambda(C_{w_I}) = \pm\varepsilon^{L(w_I)}\mathbf{c}_{\mathrm{sgn}_I} m(I,\lambda)$. Now, we have

$$\mathbf{c}_{\mathrm{sgn}_I} = \sum_{w\in W_I} \mathrm{sgn}_I^\varepsilon(T_w)\mathrm{sgn}_I^\varepsilon(T_{w^{-1}}) = \sum_{w\in W_I} \varepsilon^{-2L(w)}.$$

Since $\mathbf{a}_\lambda = L(w_I)$ and $L(w) < L(w_I)$ for $w \in W$, $w \neq w_I$, we obtain that

$$\varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(C_{w_I}) \equiv \pm m(I,\lambda) \sum_{w\in W_I} \varepsilon^{2(L(w_I)-L(w))} \equiv \pm m(I,\lambda) \mod \mathfrak{m}.$$

Since we also have $\varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(T_{w_I}) \equiv \varepsilon^{\mathbf{a}_\lambda}\chi^\lambda(C_{w_I}) \mod \mathfrak{m}$ by 2.1.19, we conclude that $c_{w_I,\lambda} = \pm m(I,\lambda) \neq 0$ and so $w_I \in \mathscr{F}_\lambda$. ☐

**Corollary 2.8.7 (Cf. [107, 4.7, 4.8]).** *Let $I \subseteq S$ and define*

$$Z_{x,y}^I := P_I^{-1}\varepsilon^{L(w_I)}C_{xw_I}C_{w_Iy^{-1}} \in \mathbf{H}_K \qquad \textit{for any } x,y \in X_I.$$

*Then the following hold:*

*(a) We have $Z_{x,y}^I = \overline{Z}_{x,y}^I \in \mathbf{H}$. Furthermore, $Z_{x,y}^I \in \mathfrak{I}_{w_I}^{\mathscr{LR}}$, where $\mathfrak{I}_{w_I}^{\mathscr{LR}}$ is defined by the general procedure in 1.6.2, with respect to the basis $\{C_w \mid w \in W\}$ of $\mathbf{H}$.*
*(b) If $\mu \in \Lambda$ is such that $\rho^\mu(Z_{x,y}^I) \neq 0$, then $E^\mu$ is a constituent of $\mathrm{Ind}_{W_I}^W(\mathrm{sgn}_I)$.*

*Proof.* (a) By Lemma 2.8.3, $C_{xw_I}$ is an $A$-linear combination of terms $T_{x_1}C_{w_I}$, where $x_1 \in X_I$. By 2.1.14, we have $C_{w_Iy^{-1}} = C_{yw_I}^\flat$ and so $C_{w_Iy^{-1}}$ is an $A$-linear combination of terms $C_{w_I}T_{y_1^{-1}}$, where $y_1 \in X_I$. Consequently, by Lemma 2.8.2(b), $C_{xw_I}C_{w_Iy^{-1}}$ is an $A$-linear combination of terms $P_IT_{x_1}C_{w_I}T_{y_1^{-1}}$, where $x_1, y_1 \in X_I$. Hence, we have $Z_{x,y}^I \in \mathbf{H}$ and $Z_{x,y}^I \in \mathfrak{I}_{w_I}^{\mathscr{LR}}$. Since $\overline{P}_I = \varepsilon^{-2L(w_I)}P_I$, we also see that $Z_{x,y}^I = \overline{Z}_{x,y}^I$.

(b) Assume that $\rho^\mu(Z^I_{x,y}) \neq 0$. In the proof of (a), we have seen that $Z^I_{x,y}$ is an $A$-linear combination of terms $P_I T_{x_1} C_{w_I} T_{y_1^{-1}}$, where $x_1, y_1 \in X_I$. Hence, we must have $\rho^\mu(C_{w_I}) \neq 0$ and so the assertion follows from Lemma 2.8.5. $\square$

**2.8.8.** From now on, let $W = \mathfrak{S}_n$ be the symmetric group where $S = \{s_1, \ldots, s_{n-1}\}$ and $s_i = (i, i+1)$ for $1 \leqslant i \leqslant n-1$. The set $\Lambda$ consists of all partitions of $n$; we write $\lambda \vdash n$ to denote that $\lambda$ is a partition of $n$. Furthermore, we assume that $\Gamma = \mathbb{Z}$ and $L(s_i) = 1$ for $1 \leqslant i \leqslant n-1$. By 1.7.6, we have $f_\lambda = 1$ for all $\lambda \vdash n$. So $R = \mathbb{Z}$ is an $L$-good subring of $\mathbb{C}$; in particular, $A = \mathbb{Z}[v, v^{-1}]$, where $v = \varepsilon$ is an indeterminate. By Corollary 1.7.9, the balanced representations $\{\rho^\lambda \mid \lambda \vdash n\}$ can be chosen such that the corresponding leading matrix coefficients satisfy the following condition:

(a) $$c^{\mathfrak{st}}_{w,\lambda} = c^{\mathfrak{ts}}_{w^{-1},\lambda} \in \{0, \pm 1\} \qquad \text{for all } w \in W \text{ and } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$$

Consequently, we have a bijection

(b) $$\bigsqcup_{\lambda \in \Lambda} \big(M(\lambda) \times M(\lambda)\big) \xrightarrow{1-1} W, \qquad (\mathfrak{s}, \mathfrak{t}) \mapsto w_\lambda(\mathfrak{s}, \mathfrak{t}),$$

satisfying the properties in Theorem 1.7.10; in particular, we have

(c) $$\mathscr{F}_\lambda = \{w_\lambda(\mathfrak{s}, \mathfrak{t}) \mid \mathfrak{s}, \mathfrak{t} \in M(\lambda)\} \qquad \text{for all } \lambda \vdash n.$$

By Proposition 2.1.20, we already know that $\mathscr{F}_\lambda$ is contained in a two-sided Kazhdan–Lusztig cell. One of our aims is to show that the converse also holds.

**2.8.9.** For any subset $I \subseteq \{1, \ldots, n-1\}$, denote by $W_I \subseteq W$ the parabolic subgroup generated by $\{s_i \mid i \in I\}$. We now define, for any partition $\lambda \vdash n$, a particular parabolic subgroup of $W$. For this purpose, we set

$$I(\lambda) := \{1, \ldots, n\} \setminus \{\lambda_1^*, \lambda_1^* + \lambda_2^*, \lambda_1^* + \lambda_2^* + \lambda_3^*, \ldots\},$$

where $\lambda_1^* \geqslant \lambda_2^* \geqslant \lambda_3^* \geqslant \ldots \geqslant 0$ are the parts of $\lambda^*$, the conjugate partition of $\lambda$. (Thus, $W_{I(\lambda)} \subseteq W$ is the Young subgroup $\mathfrak{S}_{\lambda^*}$.) For example, if $\lambda = (1^n)$, then $\lambda^* = (n)$ and so $W_I = W$. Then *Young's rule* shows that, for any $\mu \vdash n$, we have

(a) $$E^\mu \text{ is a constituent of } \mathrm{Ind}_{W_{I(\lambda)}}^W(\mathrm{sgn}_{I(\lambda)}) \quad \Leftrightarrow \quad \kappa_{\mu^*\lambda^*} \neq 0 \quad \Leftrightarrow \quad \mu \trianglelefteq \lambda,$$

where $\kappa_{\mu^*\lambda^*}$ is a *Kostka number* and $\trianglelefteq$ denotes the dominance order on partitions, as defined in Example 2.2.13. (For the first equivalence in (a), see Macdonald [236, p. 115]; the second equivalence is a combination of [236, I.6.5 and I.7.9]. Note also that $\lambda \trianglelefteq \mu$ if and only if $\mu^* \trianglelefteq \lambda^*$; see [236, I.1.11].) Consequently, we have

(b) $$\mathbf{a}_\lambda = l(w_{I(\lambda)}) \qquad \text{and} \qquad w_{I(\lambda)} \in \mathscr{F}_\lambda \qquad \text{for all } \lambda \vdash n,$$

where we use Corollary 2.8.6 and [132, 5.4.1, 5.4.3] to relate $\mathbf{a}_\lambda$ and $l(w_{I(\lambda)})$. Now we define a two-sided ideal of $\mathbf{H}_K$ by

(c) $\mathscr{N}_\lambda := \{h \in \mathbf{H}_K \mid \rho^\mu(h) = 0 \text{ for all } \mu \vdash n \text{ such that } \mu \not\trianglelefteq \lambda\}.$

We also set $\hat{\mathscr{N}_\lambda} := \{h \in \mathscr{N}_\lambda \mid \rho^\lambda(h) = 0\}$. Note that $\mathscr{N}_\lambda$ is the sum of all Wedderburn components of the split semisimple algebra $\mathbf{H}_K$ which correspond to the irreducible representations $E_\varepsilon^\mu$ where $\mu \trianglelefteq \lambda$.

**2.8.10.** We have just seen that $w_{I(\lambda)} \in \mathscr{F}_\lambda$ for any $\lambda \vdash n$. In particular, $w_{I(\lambda)} = w_\lambda(\mathfrak{t}_0, \mathfrak{t}_0)$ for a unique $\mathfrak{t}_0 \in M(\lambda)$. By Theorem 1.7.10(b) and Proposition 2.1.20, the set $\mathfrak{C}_0 := \{w_\lambda(\mathfrak{s}, \mathfrak{t}_0) \mid \mathfrak{s} \in M(\lambda)\}$ is contained in a left Kazhdan–Lusztig cell. Hence, by Lemma 2.8.2(c), we have $\mathfrak{C}_0 \subseteq X_{I(\lambda)} w_{I(\lambda)}$. Consequently, there is a subset $\{x_\mathfrak{s} \mid \mathfrak{s} \in M(\lambda)\} \subseteq X_{I(\lambda)}$ such that $w_\lambda(\mathfrak{s}, \mathfrak{t}_0) = x_\mathfrak{s} w_{I(\lambda)}$ for all $\mathfrak{s} \in M(\lambda)$. We now set

(a) $Z_w := Z^{I(\lambda)}_{x_\mathfrak{s}, x_\mathfrak{t}}, \qquad \text{where} \qquad w = w_\lambda(\mathfrak{s}, \mathfrak{t}) \text{ for } \mathfrak{s}, \mathfrak{t} \in M(\lambda).$

By Corollary 2.8.7(a), we have $Z_w = \overline{Z}_w \in \mathbf{H}$. We claim that

(b) $Z_w \in \mathscr{N}_\lambda \qquad \text{for all } w \in \mathscr{F}_\lambda.$

This is seen as follows. Let $\mu \vdash n$ and assume that $\rho^\mu(Z_w) \neq 0$. We must show that $\mu \trianglelefteq \lambda$. Now, by Corollary 2.8.7(b), $E^\mu$ is a constituent of $\mathrm{Ind}^W_{W_{I(\lambda)}}(\mathrm{sgn}_{I(\lambda)})$. By 2.8.9(a), this implies that $\mu \trianglelefteq \lambda$, as claimed.

**Lemma 2.8.11.** *Let $\lambda \in \Lambda$ and $\mathfrak{u}, \mathfrak{v} \in M(\lambda)$. Then, for any $w \in \mathscr{F}_\lambda$, we have*

$$v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w) \in \mathscr{O}_0 \qquad \textit{and} \qquad v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w) \equiv \pm \delta_{\mathfrak{s}\mathfrak{u}} \delta_{\mathfrak{t}\mathfrak{v}} \mod \mathfrak{m},$$

*where $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ are such that $w = w_\lambda(\mathfrak{s}, \mathfrak{t})$.*

*Proof.* We have $\mathbf{a}_\lambda = l(w_{I(\lambda)})$ by 2.8.9(b). Hence we obtain

$$\begin{aligned} v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w) &= P^{-1}_{I(\lambda)} v^{2\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}\big(C_{x_\mathfrak{s} w_{I(\lambda)}} C_{w_{I(\lambda)} x_\mathfrak{t}^{-1}}\big) \\ &= P^{-1}_{I(\lambda)} v^{2\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}\big(C_{w_\lambda(\mathfrak{s},\mathfrak{t}_0)} C_{w_\lambda(\mathfrak{t}_0,\mathfrak{t})}\big) \\ &= P^{-1}_{I(\lambda)} \sum_{\mathfrak{r} \in M(\lambda)} \big(v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{r}}(C_{w_\lambda(\mathfrak{s},\mathfrak{t}_0)})\big) \big(v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{r}\mathfrak{v}}(C_{w_\lambda(\mathfrak{t}_0,\mathfrak{t})})\big). \end{aligned}$$

First of all, this shows that the above expression lies in $\mathscr{O}_0$; note that $P_{I(\lambda)} \in 1 + \mathfrak{m}$. Furthermore, its constant term can be expressed by the leading matrix coefficients of $w_\lambda(\mathfrak{s}, \mathfrak{t}_0)$ and $w_\lambda(\mathfrak{t}_0, \mathfrak{t})$. Indeed, by 2.1.19 and Theorem 1.7.10, we have

$$\begin{aligned} v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{r}}(C_{w_\lambda(\mathfrak{s},\mathfrak{t}_0)}) &\equiv c^{\mathfrak{u}\mathfrak{r}}_{w_\lambda(\mathfrak{s},\mathfrak{t}_0),\lambda} \equiv \pm \delta_{\mathfrak{s}\mathfrak{u}} \delta_{\mathfrak{r}\mathfrak{t}_0} \quad \mod \mathfrak{m}, \\ v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{r}\mathfrak{v}}(C_{w_\lambda(\mathfrak{t}_0,\mathfrak{t})}) &\equiv c^{\mathfrak{r}\mathfrak{v}}_{w_\lambda(\mathfrak{t}_0,\mathfrak{t}),\lambda} \equiv \pm \delta_{\mathfrak{t}\mathfrak{v}} \delta_{\mathfrak{r}\mathfrak{t}_0} \quad \mod \mathfrak{m}. \end{aligned}$$

Since $P_{I(\lambda)} \in 1 + \mathfrak{m}$, we obtain $v^{\mathbf{a}_\lambda} \rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w) \equiv \pm \delta_{\mathfrak{s}\mathfrak{u}} \delta_{\mathfrak{t}\mathfrak{v}} \mod \mathfrak{m}$, as desired. ☐

**Theorem 2.8.12 (Cf. [107, 4.10]).** *Recall that we are in the setting of 2.8.8, where $W = \mathfrak{S}_n$. Then the following hold for any partition $\lambda \vdash n$.*

(a) $\pm C_w \in Z_w + \hat{\mathscr{N}_\lambda} \subseteq \mathscr{N}_\lambda$ *for all* $w \in \mathscr{F}_\lambda$,

(b) $\mathscr{N}_\lambda = \langle C_w \mid w \in \mathscr{F}_\mu$ *for some* $\mu \vdash n$ *such that* $\mu \trianglelefteq \lambda\rangle_K$,

(c) $\hat{\mathscr{N}_\lambda} = \langle C_w \mid w \in \mathscr{F}_\mu$ *for some* $\mu \vdash n$ *such that* $\mu \triangleleft \lambda\rangle_K$,

*where* $\hat{\mathscr{N}_\lambda} \subseteq \mathscr{N}_\lambda$ *are the two-sided ideals of* $\mathbf{H}_K$ *defined in 2.8.9.*

*Proof.* We prove (a) by induction on the dominance order on partitions. The unique minimal element in this order is the partition $(1^n)$. We have $\mathscr{F}_{(1^n)} = \{w_0\}$ (where $w_0$ is the longest element of $W$), $I(1^n) = \{1,\ldots,n-1\}$, $X_{I(1^n)} = \{1\}$ and

$$Z_{w_0} = P_{I(1^n)}^{-1}\varepsilon^{l(w_0)}C_{w_0}^2 = (-1)^{l(w_0)}C_{w_0}; \qquad \text{see Lemma 2.8.2(b).}$$

Hence, (a) holds in this case. Now assume that $\lambda \neq (1^n)$ and that (a) holds for all partitions $\mu \vdash n$, where $\mu \triangleleft \lambda$. Let $w \in \mathscr{F}_\lambda$. By 2.8.10(b), we already know that $Z_w \in \mathscr{N}_\lambda$. Since $Z_w = \overline{Z}_w \in \mathbf{H}$ (see Corollary 2.8.7(a)), we can write

$$Z_w = \sum_{y\in W} \eta_y C_y, \qquad \text{where } \eta_y = \overline{\eta}_y \in \mathbb{Z}[v,v^{-1}] \text{ for all } y \in W.$$

Let $y \in W$ be such that $\eta_y \neq 0$; we have $y \in \mathscr{F}_\mu$ for a unique $\mu \vdash n$. If $\mu \triangleleft \lambda$, then $\pm C_y \in Z_y + \hat{\mathscr{N}_\mu}$ by induction. Furthermore, by 2.8.10(b), we have $Z_y \in \mathscr{N}_\mu$. By definition, it is also clear that $\mathscr{N}_\mu \subseteq \hat{\mathscr{N}_\lambda}$. Hence, we conclude that $C_y \in \mathscr{N}_\mu \subseteq \hat{\mathscr{N}_\lambda}$. So it remains to consider those $y$ where $y \in \mathscr{F}_\mu$, $\mu \ntriangleleft \lambda$. Let us write

$$\mathscr{C} := \{y \in W \mid \eta_y \neq 0 \text{ and } y \in \mathscr{F}_\mu \text{ where } \mu \ntriangleleft \lambda\}$$

and set $m := \max\{\deg(\eta_y) \mid y \in \mathscr{C}\}$. We claim that

$$(*) \qquad \{y \in \mathscr{C} \mid \deg(\eta_y) = m\} \subseteq \mathscr{F}_\lambda \qquad \text{and} \qquad m = 0.$$

This is seen as follows. Let $y_0 \in \mathscr{C}$ be such that $\deg(\eta_{y_0}) = m$; then $y_0 = w_\mu(\mathfrak{u},\mathfrak{v})$, where $\mu \vdash n$, $\mathfrak{u},\mathfrak{v} \in M(\mu)$ and $\mu \ntriangleleft \lambda$. By 2.1.19 and Theorem 1.7.10, we have

$$v^{\mathbf{a}_\mu}\rho^\mu_{\mathfrak{u}\mathfrak{v}}(C_y) \equiv c^{\mathfrak{u}\mathfrak{v}}_{y,\mu} \equiv \pm\delta_{yy_0} \bmod \mathfrak{m} \qquad \text{for any } y \in W.$$

It follows that

$$v^{\mathbf{a}_\mu+m}\rho^\mu_{\mathfrak{u}\mathfrak{v}}(Z_w) \equiv \sum_{y\in W}\big(v^m\eta_y\big)\big(v^{\mathbf{a}_\mu}\rho^\mu_{\mathfrak{u}\mathfrak{v}}(C_y)\big) \equiv \big(v^m\eta_{y_0}\big)c^{\mathfrak{u}\mathfrak{v}}_{y_0,\mu} \not\equiv 0 \bmod \mathfrak{m}.$$

In particular, this yields that $\rho^\mu(Z_w) \neq 0$ and so $\mu \trianglelefteq \lambda$, since $Z_w \in \mathscr{N}_\lambda$ by 2.8.10(b). Combined with $\mu \ntriangleleft \lambda$, we conclude that $\mu = \lambda$ and so $y_0 \in \mathscr{F}_\lambda$, which proves the first part of $(*)$. Now the above congruence reads

$$v^m\big(v^{\mathbf{a}_\lambda}\rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w)\big) \not\equiv 0 \bmod \mathfrak{m}.$$

But then Lemma 2.8.11 implies that $m = 0$, as required. Thus, $(*)$ is proved. Consequently, we can now write

$$Z_w \equiv \sum_{y \in \mathscr{F}_\lambda} \eta_y C_y \mod \hat{\mathscr{N}_\lambda}, \qquad \text{where } \eta_y \in \mathbb{Z} \text{ for all } y \in \mathscr{F}_\lambda.$$

Let $\mathfrak{s}, \mathfrak{t} \in M(\lambda)$ be such that $w = w_\lambda(\mathfrak{s}, \mathfrak{t})$. Since $\rho^\lambda(h) = 0$ for all $h \in \hat{\mathscr{N}_\lambda}$, we have

$$\rho^\lambda_{\mathfrak{u}\mathfrak{v}}(Z_w) = \sum_{y \in \mathscr{F}_\lambda} \eta_y \rho^\lambda_{\mathfrak{u}\mathfrak{v}}(C_y) \qquad \text{for any } \mathfrak{u}, \mathfrak{v} \in M(\lambda).$$

We multiply this identity by $v^{\mathbf{a}_\lambda}$ and take constant terms. Using Theorem 1.7.10, Lemma 2.8.11 and 2.1.19, we deduce that

$$\pm\delta_{\mathfrak{s}\mathfrak{u}}\delta_{\mathfrak{t}\mathfrak{v}} = \sum_{y \in \mathscr{F}_\lambda} \eta_y c^{\mathfrak{u}\mathfrak{v}}_{y,\lambda} = \eta_{y_0} c^{\mathfrak{u}\mathfrak{v}}_{y_0,\lambda}, \qquad \text{where} \qquad y_0 = w_\lambda(\mathfrak{u}, \mathfrak{v}).$$

It follows that $\eta_w = \pm c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda}$ and $\eta_y = 0$ for all $y \in \mathscr{F}_\lambda \setminus \{w\}$, as required. Thus, (a) is proved. Now let $\mathscr{M}_\lambda$ be the $K$-subspace of $\mathbf{H}_K$ defined by the right-hand side of the desired identity in (b). We claim that $\mathscr{M}_\lambda \subseteq \mathscr{N}_\lambda$. Indeed, let $w \in \mathscr{F}_\mu$, where $\mu \trianglelefteq \lambda$. By 2.8.10(b), we have $Z_w \in \mathscr{N}_\mu$. Hence, using (a), we see that $C_w \in \mathscr{N}_\mu$. Furthermore, by definition, it is clear that $\mathscr{N}_\mu \subseteq \mathscr{N}_\lambda$. Hence, we have $C_w \in \mathscr{N}_\lambda$, as claimed. Now notice that $\dim \mathscr{M}_\lambda = \sum_{\mu \trianglelefteq \lambda} |\mathscr{F}_\mu|$; furthermore, by 2.8.8(c), we have $|\mathscr{F}_\mu| = |M(\mu)|^2$. On the other hand, as already noted in 2.8.9, the ideal $\mathscr{N}_\lambda$ is the sum of all Wedderburn components of $\mathbf{H}_K$ which correspond to the irreducible representations $E^\mu_\varepsilon$ where $\mu \trianglelefteq \lambda$. Hence, we also have $\dim \mathscr{N}_\lambda = \sum_{\mu \trianglelefteq \lambda} |M(\mu)|^2$ and, consequently, $\mathscr{N}_\lambda = \mathscr{M}_\lambda$. Thus, (b) is proved. This also implies (c) since, by definition, $\hat{\mathscr{N}_\lambda} = \sum_\mu \mathscr{N}_\mu$, where the sum runs over all $\mu \vdash n$ such that $\mu \triangleleft \lambda$. $\square$

*Remark 2.8.13.* The above proof essentially follows [107, Theorem 4.10]. However, in [107], we referred to the results of Murphy [256], [257] in order to define the ideals $\mathscr{N}_\lambda$ and $\hat{\mathscr{N}_\lambda}$. The discussion here avoids that reference and, thus, is considerably more self-contained than that in [107].

**Corollary 2.8.14.** *Let $\lambda, \mu \vdash n$. Then we have $E^\mu \preceq_L E^\lambda$ if and only if $\mu \trianglelefteq \lambda$. In particular, the equivalence classes of* $\mathrm{Irr}_{\mathbb{K}}(\mathfrak{S}_n)$ *under $\sim_L$ are singleton sets. Consequently, the properties (♣) and (♠) (see 2.5.3) hold.*

*Proof.* Assume first that $\mu \trianglelefteq \lambda$. By 2.8.9(a) (Young's rule) and Lemma 2.8.4, this implies that $xw_{I(\lambda)} \in \mathscr{F}_\mu$ for some $x \in X_{I(\lambda)}$. But then we have $xw_{I(\lambda)} \leqslant_{\mathscr{L}} w_{I(\lambda)}$ and so $E^\mu \preceq_L E^\lambda$; recall that $w_{I(\lambda)} \in \mathscr{F}_\lambda$ by 2.8.9(b). Conversely, assume that $E^\mu \preceq_L E^\lambda$. This means that $y \leqslant_{\mathscr{LR}} w$, where $y \in \mathscr{F}_\mu$ and $w \in \mathscr{F}_\lambda$. By definition, we can find a sequence $y = y_0, y_1, \ldots, y_m = w$ such that, for each $i \in \{1, \ldots, m\}$, there exist some $x_i \in W$ such that $h_{x_i,y_i,y_{i-1}} \neq 0$ or $h_{y_i,x_i,y_{i-1}} \neq 0$. Now, by Theorem 2.8.12(b), we have $C_{y_m} = C_w \in \mathscr{N}_\lambda$. Since $\mathscr{N}_\lambda$ is a two-sided ideal, we have $C_{x_m}C_{y_m} \in \mathscr{N}_\lambda$ and $C_{y_m}C_{x_m} \in \mathscr{N}_\lambda$. Hence, we have $y_{m-1} \in \mathscr{F}_{\mu_{m-1}}$, where $\mu_{m-1} \trianglelefteq \lambda$; see Theorem 2.8.12(b). We repeat the argument with $x_{m-1}$, $y_{m-1}$ and find that

$y_{m-2} \in \mathscr{F}_{\mu_{m-2}}$, where $\mu_{m-2} \trianglelefteq \mu_{m-1} \trianglelefteq \lambda$. Continuing in this way, we eventually obtain that $\mu = \mu_0 \trianglelefteq \lambda$, as required. It is known that this implies $\mathbf{a}_\lambda \leqslant \mathbf{a}_\mu$, with equality only if $\lambda = \mu$; see 2.2.13. Consequently, (♣) holds. But then the weaker property (♣′) holds and so (♠) also holds; see Proposition 2.5.12. □

**Theorem 2.8.15.** *The algebra* $\mathbf{H}$ *admits a cellular basis as in Theorem 2.6.12. The data in Definition 2.6.8 can be chosen such that, for* $\lambda \vdash n$ *and* $\mathfrak{s},\mathfrak{t} \in M(\lambda)$*, we have*

$$\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} = c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda}\, C_w = \pm C_w, \qquad \textit{where} \qquad w = w_\lambda(\mathfrak{s},\mathfrak{t}).$$

*Furthermore, the partial order* $\trianglelefteq_L$ *is given by the dominance order on partitions; we have* $\mathscr{N}_\lambda = \mathbf{H}_K(\trianglelefteq_L)$ *and* $\hat{\mathscr{N}_\lambda} = \mathbf{H}_K(\triangleleft_L)$.

*Proof.* Since (♣) and (♠) hold, we can apply Theorem 2.6.12 and so $\mathbf{H}$ admits a cellular basis where, by Definition 2.6.8, we have

$$\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} = \sum_{w \in W} \sum_{\mathfrak{u} \in M(\lambda)} \beta^\lambda_{\mathfrak{t}\mathfrak{u}}\, c^{\mathfrak{u}\mathfrak{s}}_{w^{-1},\lambda}\, C_w.$$

Now, since $c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda} = c^{\mathfrak{t}\mathfrak{s}}_{w^{-1},\lambda}$ for all $\mathfrak{s},\mathfrak{t} \in M(\lambda)$, we have $\bar{\rho}^\lambda(t_{w^{-1}}) = \bar{\rho}^\lambda(t_w)^{\mathrm{tr}}$ for all $w \in W$. Hence, we can take for $B^\lambda = (\beta^\lambda_{\mathfrak{s}\mathfrak{t}})$ the identity matrix. Then the above sum reduces to $\mathbf{C}^\lambda_{\mathfrak{s},\mathfrak{t}} = c^{\mathfrak{s}\mathfrak{t}}_{w,\lambda}\, C_w$, as required. Finally, by Corollary 2.8.14, the partial order $\trianglelefteq_L$ in Theorem 2.6.12 coincides with the dominance order. The identities $\mathscr{N}_\lambda = \mathbf{H}_K(\trianglelefteq_L)$, $\hat{\mathscr{N}_\lambda} = \mathbf{H}_K(\triangleleft_L)$ now follow from Theorem 2.8.12(b) and (c). □

*Remark 2.8.16.* This result was first stated by Graham and Lehrer [144, Example 1.2], but the argument is very sketchy, especially concerning the ordering $\trianglelefteq_L$. Some more details are contained in Graham's thesis [143, Example 4.3] and Williamson's Honours essay [295]. As far as we are aware, the first elementary proof of the characterisation of $\trianglelefteq_L$ in terms of the dominance order appeared in [107]. In Remark 2.8.18, we show how the signs in Theorem 2.8.15 can be fixed. A completely different construction of a cellular basis (with respect to the above ordering on $\Lambda$) is due to Murphy [256], [257]; the equivalence of the two constructions is shown in [107].

**2.8.17.** By Corollary 2.8.14 and Lemma 2.5.9, the Kazhdan–Lusztig cells of $W = \mathfrak{S}_n$ are given by Theorem 1.7.10. So, for any $\lambda \vdash n$, the following hold.

(a1) The set $\mathscr{F}_\lambda = \{w_\lambda(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s},\mathfrak{t} \in M(\lambda)\}$ is a two-sided Kazhdan–Lusztig cell.
(a2) For $\mathfrak{t} \in M(\lambda)$, the set $\{w_\lambda(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{s} \in M(\lambda)\}$ is a left Kazhdan–Lusztig cell.
(a3) For $\mathfrak{s} \in M(\lambda)$, the set $\{w_\lambda(\mathfrak{s},\mathfrak{t}) \mid \mathfrak{t} \in M(\lambda)\}$ is a right Kazhdan–Lusztig cell.

In particular, we see that (♦) holds. We also obtain the following result originally due to Kazhdan and Lusztig [195, Theorem 1.4]: for any left Kazhdan–Lusztig cell $\mathfrak{C}$ of $W$, we have

(b) $$[\mathfrak{C}]_1 \in \mathrm{Irr}_{\mathbb{K}}(W) \qquad \text{and} \qquad [\mathfrak{C}]_1 \cong E^\lambda \;\Leftrightarrow\; \mathfrak{C} \subseteq \mathscr{F}_\lambda.$$

Indeed, let $\mathfrak{C}$ be a left cell and $\lambda \in \Lambda$ be such that $m(\mathfrak{C},\lambda) > 0$. Then $\mathfrak{C} \subseteq \mathscr{F}_\lambda$ by Lemma 2.2.4. Hence, $\mathfrak{C}$ is equal to a set as in (a2). But then $|\mathfrak{C}| = |M(\lambda)| = d_\lambda = \dim E^\lambda$ and so we must have $E^\lambda \cong [\mathfrak{C}]_1$. Finally, assume that $\mathfrak{C}$ is a left cell such that $\mathfrak{C} \subseteq \mathscr{F}_\lambda$. Then the same argument shows that $[\mathfrak{C}]_1 \cong E^\lambda$. (If we had $[\mathfrak{C}]_1 \cong E^\mu$, where $\mu \neq \lambda$, then $\mathfrak{C} \subseteq \mathscr{F}_\mu$, which is a contradiction.) Thus, (b) is proved. We can now also apply the discussion in Example 2.7.4, which shows that, for each $\lambda \vdash n$, the balanced representation $\rho^\lambda$ can be chosen such that

(c) $\quad W(\lambda) = [\mathfrak{C}_\lambda]_A, \qquad$ where $\mathfrak{C}_\lambda \subseteq \mathscr{F}_\lambda$ is a fixed left Kazhdan–Lusztig cell.

See also McDonough and Pallikaros [251], where the above cell modules are identified with the original "Specht modules" of Dipper and James [62].

*Remark 2.8.18.* Once Theorem 2.8.12, Corollary 2.8.14 and 2.8.17 are established, it is actually not too difficult to show that **P1**–**P15** hold for $W = \mathfrak{S}_n$; see [107, §5], [121, §4]. Furthermore, one can even show a tiny piece of "positivity" by elementary methods; namely, the fact that $\gamma_{x,y,z} \geqslant 0$ for all $x,y,z \in W$; see [107, Theorem 5.10]. (Recall that $\gamma_{x,y,z} = (-1)^{l(x)+l(y)+l(z)} c_{x,y,z}$; see Remark 2.3.6.) The argument relies on basic properties of the "Knuth–Robinson–Schensted correspondence" and the Kazhdan–Lusztig "star operations"; see Kazhdan and Lusztig [195, §5], Knuth [205, §5.1.4] and Ariki [8]. We will not go into any more detail here, as these results are not needed for the further discussions in this book.

Let us just explain how the signs in Theorem 2.8.15 can be fixed, *assuming* that **P1** and **P4** hold and that $\gamma_{x,y,z} \geqslant 0$ for all $x,y,z \in W$ (which is also known to be the case by 2.4.1(a)). This is done as follows. Choosing the balanced representation $\rho^\lambda$ as in Proposition 1.8.9, each coefficient $c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}}$ is equal to a structure constant $\tilde{\gamma}_{w,x,y}$ for suitable $x,y \in \mathscr{F}_\lambda$. Now, by Proposition 2.3.16, Remark 2.3.5 and Proposition 2.3.14, we have

$$\tilde{\gamma}_{w,x,y} = (-1)^{l(w)+l(x)+l(y)} \gamma_{w,x,y} = (-1)^{\mathbf{a}(y)} \gamma_{w,x,y} \qquad \text{and} \qquad \mathbf{a}(y) = \mathbf{a}_\lambda.$$

Since $\gamma_{w,x,y} \geqslant 0$ and $c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}} \in \{0, \pm 1\}$, we deduce that

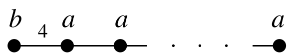
$$c_{w,\lambda}^{\mathfrak{s}\mathfrak{t}} = (-1)^{\mathbf{a}_\lambda}, \qquad \text{where} \qquad w = w_\lambda(\mathfrak{s},\mathfrak{t}).$$

Arguing as in the proof of Theorem 2.8.15, we now let $B^\lambda = (\beta_{\mathfrak{s}\mathfrak{t}}^\lambda)$ be equal to $(-1)^{\mathbf{a}_\lambda}$ times the identity matrix. Then we obtain $\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^\lambda = C_w$, as required.

For a further discussion of the combinatorics involved in the above constructions (Knuth–Robinson–Schensted correspondence, etc.), we refer the reader to the references cited in Remark 2.8.18. In a somewhat different context, we will have more to say about the combinatorics of Young tableaux in Section 3.5.

**2.8.19.** Having dealt with $W = \mathfrak{S}_n$, it is natural to ask what happens with the other cases in 1.7.6. So, let $W$ be of type $B_n$ and $L\colon W \to \Gamma$ a weight function given by

$B_n$



where $a,b>0$ and $b \notin \{a,2a,\ldots,(n-1)a\}$. Recall that $\Lambda$ is the set of all pairs of partitions $(\lambda,\mu)$ such that $|\lambda|+|\mu|=n$; see Example 1.3.9. As in the proof of Theorem 2.8.15, one sees that, for any $(\lambda,\mu)\in\Lambda$ and $\mathfrak{s},\mathfrak{t}\in M(\lambda,\mu)$, we have

$$\mathbf{C}_{\mathfrak{s},\mathfrak{t}}^{(\lambda,\mu)} = \pm C_w, \qquad \text{where} \qquad w = w_{(\lambda,\mu)}(\mathfrak{s},\mathfrak{t}).$$

However, property (♠) is not known in general, so we cannot conclude that the above elements form a cellular basis of $\mathbf{H}$.

Complete results are available in the *asymptotic case*, where $b>(n-1)a>0$; denote by $L_{\text{asy}}$ the weight function in this case. Then **P1–P15** hold for $W,L_{\text{asy}}$ by the series of papers by Bonnafé, Geck, and Iancu [21], [26], [108], [114], [121]. Furthermore, as already mentioned in Example 2.2.17, we have

$$(\lambda,\mu) \trianglelefteq_{L_{\text{asy}}} (\lambda',\mu') \qquad \Leftrightarrow \qquad (\lambda,\mu) \trianglelefteq (\lambda',\mu').$$

Arguing as in 2.8.17, we obtain the following result originally due to Bonnafé and Iancu [26, Prop. 7.9]: for any left Kazhdan–Lusztig cell $\mathfrak{C}$ of $W$ (with respect to $L_{\text{asy}}$), we have

$$[\mathfrak{C}]_1 \in \mathrm{Irr}_{\mathbb{K}}(W) \qquad \text{and} \qquad [\mathfrak{C}]_1 \cong E^{\lambda} \Leftrightarrow \mathfrak{C} \subseteq \mathscr{F}_{(\lambda,\mu)}.$$

Furthermore, for $(\lambda,\mu)\in\Lambda$, the balanced representation $\rho^{(\lambda,\mu)}$ can be chosen such that $W(\lambda,\mu)=[\mathfrak{C}_{(\lambda,\mu)}]_A$, where $\mathfrak{C}_{(\lambda,\mu)} \subseteq \mathscr{F}_{(\lambda,\mu)}$ is a fixed left Kazhdan–Lusztig cell.

A completely different construction of a cellular basis is due to Dipper, James and Murphy [68]; but, by [124], the above cell modules in the asymptotic case are naturally isomorphic to the "Specht modules" of [68]. See also Chlouveraki, Gordon and Griffeth [49] for further realisations of these modules. The construction of [68] has been further generalised to Ariki–Koike algebras; see Dipper, James and Mathas [67] (and also Graham and Lehrer [144, §5] for a slightly different approach). We will describe these results on Ariki–Koike algebras in Section 5.3.

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