

Contents

1	Introduction	1
1.1	Loomings	1
1.1.1	The Special Nature of Mathematical Reasoning Within Human Reason in General	5
1.2	Proof Verifiers	6
1.3	Informal Introduction to the Formalism in which We Will Work	7
1.3.1	(A) Immediate Deduction	7
1.3.2	(B) Proof by ‘Supposition’ and ‘Discharge’ (‘Natural Deduction’)	8
1.3.3	(C) Use of Definitions	8
1.4	More About Our Formalism	10
1.4.1	Propositional and Predicate Calculus	10
1.4.2	Set Theory: The Third Main Ingredient of Our Formalism	13
1.5	An Informal Overview of the Sequence of Formal Set-Theoretic Proofs to Be Given Later	25
1.5.1	Basic Elementary Results	25
1.5.2	Ordinals	25
1.5.3	Well Ordering: The Principle of Transfinite Enumerability	27
1.5.4	Cardinal Numbers	29
1.5.5	Survey of the Major Sequence of Definitions and Proofs Considered in This Text	32
2	Propositional- and Predicate-Calculus Preliminaries	37
2.1	The Propositional Calculus	37
2.2	The Predicate Calculus	44
2.2.1	Proof Rules of the Predicate Calculus	51
2.2.2	The Gödel Completeness Theorem	52
2.2.3	Working with Universally Valid Predicate Formulae. A Few Simple Examples of Predicate Proof	54
2.2.4	The Prenex Normal Form of Predicate Formulae	60
2.2.5	The Deduction Theorem	60

2.2.6	Definitions in Predicate Calculus; the Notion of ‘Conservative Extension’	62
2.2.7	Proof of the Gödel Completeness Theorem	65
2.3	Predicate Calculus with Equality as a Built-in	75
2.4	Set Theory as an Axiomatic Extension of Predicate Calculus	77
2.4.1	Zermelo–Fraenkel Theory with the Axiom of Choice	77
2.4.2	Concerning the Consistency of ZFC and Various Interesting Extensions of It	79
	References	91
3	A Survey of Inference Mechanisms	93
3.1	The Davis–Putnam Propositional Decision Algorithm	93
3.1.1	Horn Formulae and Sets of Formulae	95
3.1.2	Reducing Collections of Propositional Formulae to Collections of Standardized Disjunctions	96
3.2	Elementary Boolean Theory of Sets	97
3.2.1	Elementary Boolean Theory of Sets, Plus the Predicates ‘Finite’ and ‘Countable’	100
3.2.2	Elementary Boolean Operators on Sets, with the Cardinality Operator and Additive Arithmetic on Integers . .	102
3.2.3	Quantified Predicate Formulae Involving Predicates of One Argument Only	104
3.3	MLSS: Multilevel Syllogistic with Singletons	109
3.4	MLSS Plus the Predicates ‘Finite’ and ‘Countable’	113
3.5	The Tableau Method	115
3.6	Elementary Booleans Plus Map Primitives	120
3.7	Various Commonly Occurring Decidable Extensions of MLSS . . .	122
3.7.1	Extension Conditions in the Other Cases Listed Above . . .	126
3.7.2	The Case of Mutually Inverse Functions	129
3.8	More Examples of Decidable Sublanguages	131
3.8.1	Presburger’s Decidable Quantified Language of Additive Arithmetic	131
3.8.2	A Decidable Quantified Theory Involving Ordinals	134
3.8.3	A Language of Additive Infinite Cardinal Arithmetic	148
3.8.4	Behmann’s Quantified Language of Elementary Set-Theoretic Formulae	151
3.9	A Decision Algorithm for the Theory of Totally Ordered Sets . . .	157
3.10	A Decision Algorithm for Ordered Abelian Groups	159
3.11	A Fragment of Analysis: Theory of Reals and Single-Valued Continuous Functions with Predicates ‘Monotone’, ‘Convex’, ‘Concave’, Real Addition, and Comparison	165
3.11.1	Syntax of RMCF^+	165
3.11.2	Semantics of RMCF^+	166
3.11.3	Preparing a Set of RMCF^+ Statements for Satisfiability Testing	169
3.12	The Resolution Method for Pure Predicate-Calculus Proving	177

3.12.1	Resolution in the Propositional Calculus	179
3.12.2	Resolution and Syntactic Unification in the Predicate Calculus	180
3.13	Universally Quantified Predicate Sentences Involving Function Symbols of One Argument Only	190
3.14	The Knuth–Bendix Equational Method	193
3.14.1	Overview of the Method	193
3.14.2	Details	195
3.14.3	Testing Completeness by Superposition of Reductions: The Knuth–Bendix Completion Process	199
3.14.4	More Details	200
3.14.5	Examples of the Knuth–Bendix Procedure	200
	References	202
4	More on the Structure of the Verifier System	205
4.1	Introduction to the General Syntax and Overall Structure of Proofs	205
4.1.1	The Syntax of Proofs	205
4.1.2	The ELEM Primitive and ‘Blobbing’	208
4.1.3	The Suppose_not, QED, Suppose, Discharge Primitives	210
4.1.4	THEORY Application	211
4.1.5	Context of an Inference Step	213
4.2	The Syntax and Semantics of Definitions	215
4.3	Other Techniques Used in the Verifier as Implemented	218
4.3.1	Supplementary Proof Mechanisms for the ELEM Rule	218
4.3.2	Limited Predicate Proof	219
4.3.3	Proof by Equality	224
4.3.4	Proof by Monotonicity	224
4.3.5	Algebraic Deduction	227
4.3.6	Proof by Closure	229
4.3.7	The Behind-the-Scenes Activity of Proof by Structure	230
4.3.8	‘Blobbing’ More General Formulae Down to a Specified Decidable or Semi-decidable Sublanguage of Set Theory	235
4.3.9	Accelerated Instantiation of Quantifiers and Set Formers	236
4.3.10	Computation with Hereditarily Finite Sets	238
4.4	Dividing Long Proof Verifications into Multiple Separate ‘Sessions’	253
	References	255
5	A Closer Examination of the Sequence of Definitions and Theorems Presented in this Book	257
5.1	Basic Operations of Set Theory and the Theory of Ordinals	258
5.1.1	Pairs, Set Formers, and Maps	258
5.1.2	Transfinite Induction	260
5.1.3	Ordinals	260
5.1.4	The Ordinal Enumerability Theorem	261
5.2	Elementary Laws on Map Constructs	262
5.3	Cardinality of a Set; Cardinal Numbers	268

5.3.1	Finiteness	270
5.4	The Set of All Integers, Basic Arithmetic of Integers and Cardinals	273
5.5	The Cardinal Product Theorem	279
5.6	The Signed Integers	281
5.7	Induction Principles for Ordinals	285
5.7.1	Mathematical Induction for Integers	287
5.8	Equivalence Relationships and Classes; the General Summation Operator; Recursion	287
5.9	Formal Fractions and Rational Numbers	289
5.10	Real Numbers	295
5.11	Complex Numbers	300
5.12	Functions of Real and Complex Variables	302
	References	311
6	Undecidability and Unsolvability	313
6.1	Chaitin's Theorem	313
6.1.1	Undecidability Results Derivable from Chaitin's Theorem	315
6.2	The Two Gödel Theorems	319
6.2.1	Programming Considerations	320
6.2.2	Programming and Proof; 'Mirroring' Programmable Set-Theoretic Functions	323
6.2.3	Additional Comments on the Legitimacy of Recursive Definitions	328
6.2.4	Properties of Integers	329
6.2.5	A Final Remark on Proof and Computation	336
6.2.6	A Technical Adjustment	336
6.2.7	The 'Provability' Predicate $\text{Pr}(s)$	337
6.2.8	Proof Visibility Lemma	339
6.2.9	Gödel's Trick Sentence	343
6.2.10	Rosser's Variant of Gödel's Trick Sentence	344
6.2.11	Proof of Rosser's Variant of Gödel's First Theorem	345
6.2.12	Proof of Gödel's Second Theorem	346
6.3	Axioms of Reflection	346
6.3.1	Statement of the Axioms of Reflection	359
6.4	A Digression Concerning Foundations	367
	References	371
7	A Self-contained Beginning for Ref's Main Proof Scenario	373
7.1	Axioms of Set Theory	373
7.2	Pairs and Maps	374
7.3	From Reachability to Transfinite Induction	378
7.3.1	Reachability in a Big Graph	378
7.3.2	Full Sets and Ordinals	386
7.3.3	The Transitive Closure Operation	390
7.3.4	A Basic Form of the Principle of Transfinite Induction	392
7.3.5	Some Basic Facts on Ordinal Numbers	393

Contents	xvii
7.4 Zorn's Lemma	398
7.5 Finiteness	405
References	409
Index	411



<http://www.springer.com/978-0-85729-807-2>

Computational Logic and Set Theory
Applying Formalized Logic to Analysis
Schwartz, J.T.; Cantone, D.; Omodeo, E.G.
2011, XVII, 416 p., Hardcover
ISBN: 978-0-85729-807-2