Estimating Motors from a Variety of Geometric Data in 3D Conformal Geometric Algebra

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Abstract
The motion rotors, or *motors*, are used to model Euclidean motion in 3D conformal geometric algebra. In this chapter we present a technique for estimating the motor which best transforms one set of noisy geometric objects onto another. The technique reduces to an eigenrotator problem and has some advantages over matrix formulations. It allows motors to be estimated from a variety of geometric data such as points, spheres, circles, lines, planes, directions, and tangents; and the different types of geometric data are combined naturally in a single framework. Also, it excludes the possibility of a reflection unlike some matrix formulations. It returns the motor with the smallest translation and rotation angle when the optimal motor is not unique.

2.1 Introduction

The motion rotors or *motors*, denoted $\mathscr{M}$, are used to model Euclidean motions in 3D conformal geometric algebra (CGA). It is often useful to be able to estimate a motor which best maps one data set onto another in some sense. The canonical problem involves two sets of noisy points where one set is nominally a rotated and translated version of the other. This situation arises frequently, for example when two sets of reconstructed 3D points need to be merged and they share some common points. Several solutions exist to minimise the squared distance between the points, using matrix techniques based on SVD, polar decomposition, and quaternions [3].

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In addition to points, many other geometric objects such as lines, directions, and planes provide useful information which can be used to help estimate the rigid body relationship between the data sets.

In this chapter we present a technique for estimating the motor which best transforms one set of noisy geometric objects onto another. The technique reduces to an eigenrotator problem and has some advantages over matrix formulations. It allows motors to be estimated from a wide variety of geometric data such as points, spheres, circles, lines, planes, directions, and tangents; and the different types of geometric data to be combined naturally in a single framework. Also, it does not admit the possibility of a reflection as do some matrix formulations. It returns the motor with the smallest translation and rotation angle when the optimal motor is not unique. To assist the development, we will first examine some useful algebraic and differential properties of the motors.

The following geometric algebra conventions are used in this chapter. The geometric algebra over $\mathbb{R}$ with signature (p,q) (p positive and q negative basis elements) is denoted $\mathbb{R}_{p,q}$. When $q=0$ we write $\mathbb{R}_p$. A pure Euclidean multivector in $\mathbb{R}_3$ is usually represented in boldface, such as $\boldsymbol{V}$. The grade-r elements of a geometric algebra $\mathbb{R}_{p,q}$ are denoted $\mathbb{R}^r_{p,q}$. $\mathbb{R}^+_{p,q}$ and $\mathbb{R}^-_{p,q}$ refer to the even and odd elements of $\mathbb{R}_{p,q}$. The conformal geometric algebra (CGA) of the 3D space $\mathbb{R}^3$ is denoted by $\mathbb{R}_{4,1}$. The dual of X is denoted $X^* = X \cdot I^{-1}$. The CGA vector n_o represents the origin and the CGA vector n_∞ represents the point at infinity, with $n_o \cdot n_\infty = -1$. A CGA point or dual sphere s (which is an element of $\mathbb{R}^1_{4,1}$) is normalised if $s \cdot n_\infty = -1$, and a direct sphere (an element of $\mathbb{R}^4_{4,1}$) is normalised if $S \wedge n_\infty = -I_{4,1}$. A round R (including tangents) is normalised if $|R \wedge n_\infty| = 1$, a flat (line or plane) F is normalised if $|F| = 1$, and a direction Δ is normalised if $|n_o \wedge \Delta| = 1$. The notation $\langle X\rangle_{i,j,\ldots,k}$ is used as an abbreviation for $\langle X\rangle_i + \langle X\rangle_j + \cdots + \langle X\rangle_k$.

2.2 The Linear Spaces $\mathbb{M}$, $\mathbb{B}$, and $\mathbb{S}$

The 8D linear space $\mathbb{M} = \operatorname{span}\{1, e_{12}, e_{13}, e_{23}, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty, I_3 n_\infty\} \subset \mathbb{R}_{4,1}$ is the smallest linear space in which motors reside. It is convenient to restrict most of the analysis to elements in $\mathbb{M}$ because many simplifications arise. Most of these are consequences of the following split: if $X \in \mathbb{M}$, then $X = R + Q$ where $R \in \mathbb{R}^+_3$ and $Q \in \mathbb{R}^-_3 n_\infty = \{\boldsymbol{V} n_\infty : \boldsymbol{V} \in \mathbb{R}^-_3\}$. As $Q\widetilde{Q} = 0$, $\langle X\widetilde{X}\rangle = R\widetilde{R} \geq 0$, so $|X|^2 = |\langle X\widetilde{X}\rangle| = \langle X\widetilde{X}\rangle$, and we can drop the absolute value. We will use the property that $\mathbb{M}$ is closed under multiplication, so if $X, Y \in \mathbb{M}$, and then $XY \in \mathbb{M}$. This is clear by simply multiplying the basis elements. If $X, Y \in \mathbb{M}$, then $\langle XY\widetilde{Y}\widetilde{X}\rangle = \langle X\widetilde{X}\rangle\langle Y\widetilde{Y}\rangle$, so $|XY| = |X||Y|$. In addition, $X \in \mathbb{M}$ is invertible iff $|X| \neq 0$. If X is invertible, then $1 = |X^{-1}X| = |X^{-1}||X|$ and $|X| \neq 0$. Conversely, if $|X| \neq 0$, then

$$X^{-1} = \widetilde{X}\left(\frac{\langle X\widetilde{X}\rangle - \langle X\widetilde{X}\rangle_4}{\langle X\widetilde{X}\rangle^2}\right). \tag{2.1}$$

The denominator of this is simplified because $\langle X\widetilde{X}\rangle_4^2$ vanishes. It is also convenient to split $X \in \mathbb{M}$ into symmetric and antisymmetric parts $X = S + B$ where $S = \frac{1}{2}(X + \widetilde{X}) = \langle X\rangle_{0,4}$ and $B = \frac{1}{2}(X - \widetilde{X}) = \langle X\rangle_2$. The antisymmetric grade-2 elements of $\mathbb{M}$ will be denoted $\mathbb{B} = \text{span}\{e_{12}, e_{13}, e_{23}, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty\}$, and the symmetric grade 0 and 4 elements will be denoted $\mathbb{S} = \text{span}\{1, I_3 n_\infty\}$. The elements of $\mathbb{S}$ are "symmetric" in the sense that for $S \in \mathbb{S}$, we have $S = \widetilde{S}$. $\mathbb{S}$ is closed under multiplication: $S_1, S_2 \in \mathbb{S} \Rightarrow S_1 S_2 \in \mathbb{S}$. Note that if $X \in \mathbb{M}$, then $X\widetilde{X} = \langle X\widetilde{X}\rangle + \langle X\widetilde{X}\rangle_4$. Therefore the condition $X\widetilde{X} = 1$ encodes two constraints: $\langle X\widetilde{X}\rangle = 1$ and $\langle X\widetilde{X}\rangle_4 = 0$ (there is only one grade-4 basis element in $\mathbb{M}$). The following lemma uses these constraints to characterise how the 6D motor manifold $\mathscr{M}$ sits in the 8D linear space $\mathbb{M}$.

Lemma 2.1 *$X \in \mathscr{M} \Leftrightarrow X \in \mathbb{M}$ and $X\widetilde{X} = 1$.*

Proof Let $X = R + Q \in \mathbb{M}$ where $R \in \mathbb{R}_3^+$ and $Q \in \mathbb{R}_3^- n_\infty$. $X\widetilde{X} = 1$ implies $R\widetilde{R} = 1$ and $Q\widetilde{R} = \langle Q\widetilde{R}\rangle_2$. Thus, R is a rotator, and $X = R + Q\widetilde{R}R = (1 + \langle Q\widetilde{R}\rangle_2)R = TR$ where $T = 1 + \langle Q\widetilde{R}\rangle_2$ is a translator. $\square$

The space $\mathbb{M}$ is incomplete in the sense that, given a basis of $\mathbb{M}$, we cannot find a reciprocal basis that also lies in $\mathbb{M}$. We can enlarge $\mathbb{M}$ to a complete space such as $\mathbb{M} \cup \text{span}\{e_1 n_o, e_2 n_o, e_3 n_o, \widetilde{I}_3 n_o\}$ or $\mathbb{R}_{4,1}^+$ and then construct a reciprocal basis. The subspace spanned by reciprocal vectors associated with elements in $\mathbb{M}$ is denoted $\overline{\mathbb{M}} = \text{span}\{1, \widetilde{e}_{12}, \widetilde{e}_{13}, \widetilde{e}_{23}, e_1 n_o, e_2 n_o, e_3 n_o, \widetilde{I}_3 n_o\}$. Almost every result in $\mathbb{M}$ has a counterpart in $\overline{\mathbb{M}}$. An element $T = 1 + \mathbf{t} n_o = s(1 + \frac{1}{2}\mathbf{t} n_\infty)s$ represents a transversor (reflection in the unit sphere $s = n_o - \frac{1}{2}n_\infty$ followed by a translation $1 + \frac{1}{2}\mathbf{t} n_\infty$ and another reflection in the unit sphere). It is the product of an even number of vectors and satisfies $T\widetilde{T} = 1$, so it is a rotor. Let $\bar{\mathscr{M}}$ denote the rotors of the form $M = TR$ where T is a transversor and R a rotator. The counterpart to Lemma 2.1 takes the form:

Lemma 2.2 *$X \in \bar{\mathscr{M}} \Leftrightarrow X \in \overline{\mathbb{M}}$ and $X\widetilde{X} = 1$.*

The intersection of $\mathbb{M}$ and $\overline{\mathbb{M}}$ is $\mathbb{R}_3^+$. The rotators $\mathscr{R}$ lie in $\mathbb{R}_3^+$ and are a subset of both $\mathscr{M}$ and $\bar{\mathscr{M}}$. The relationship between the spaces $\mathbb{M}$, $\mathscr{M}$, $\overline{\mathbb{M}}$, $\bar{\mathscr{M}}$, $\mathbb{R}_3^+$, $\mathscr{R}$, and $\mathbb{R}_{4,1}^+$ is shown in Fig. 2.1. We will sometimes want to project an element $X \in \mathbb{R}_{4,1}$ on $\mathbb{M}$ or $\overline{\mathbb{M}}$. Let $\{e_J\}$ be a basis for $\mathbb{M}$, and $\{e^J\}$ be the associated reciprocal basis in $\overline{\mathbb{M}}$. The projection on $\mathbb{M}$ is defined by

$$\mathrm{P}_{\mathbb{M}}(X) = \sum_J \langle e^J X\rangle e_J.$$

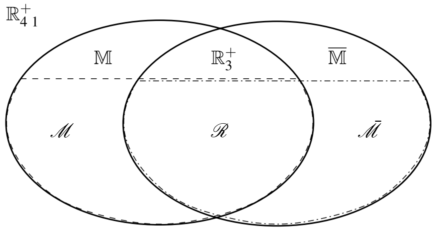

Fig. 2.1 The relationship between the manifolds of motors $\mathscr{M}$, rotators $\mathscr{R}$, and reciprocal motors $\bar{\mathscr{M}}$ and the linear spaces $\mathbb{M}$, $\mathbb{R}^+_3$, and $\overline{\mathbb{M}}$ they reside in

As $\langle \mathrm{P}_{\mathbb{M}}(X)Y\rangle = \sum_J \langle e^J X\rangle\langle e_J Y\rangle = \langle X\bar{\mathrm{P}}_{\mathbb{M}}(Y)\rangle$, the adjoint is the projection onto $\overline{\mathbb{M}}$ given by

$$\bar{\mathrm{P}}_{\mathbb{M}}(Y) = \mathrm{P}_{\overline{\mathbb{M}}}(Y) = \sum_J e^J \langle e_J Y\rangle.$$

This can also be expressed using the multivector derivative $\partial_X = \sum_J e^J\langle e_J \partial_X\rangle$: $\partial_X\langle XY\rangle = \sum_J e^J\langle e_J Y\rangle = \bar{\mathrm{P}}_{\mathbb{M}}(Y)$. When no ambiguity arises, it is convenient to use the terse notation P_X for the projection onto the basis of the linear space in which the element X resides. For example, if $X \in \mathbb{M}$, then $\mathrm{P}_X = \mathrm{P}_{\mathbb{M}}$, if $R \in \mathbb{R}^+_3$, then P_R is a projection onto $\mathbb{R}^+_3$, and if $Q \in \mathbb{R}^-_3 n_\infty$, then P_Q is the projection onto $\mathbb{R}^-_3 n_\infty = \mathrm{span}\{e_1 n_\infty, e_2 n_\infty e_3 n_\infty, I_3 n_\infty\}$. Using the split $X = R + Q \in \mathbb{M}$, where $R \in \mathbb{R}^+_3$ and $Q \in \mathbb{R}^-_3 n_\infty$, gives $\mathrm{P}_{\mathbb{M}} = \mathrm{P}_X = \mathrm{P}_R + \mathrm{P}_Q$, and $\bar{\mathrm{P}}_{\mathbb{M}} = \bar{\mathrm{P}}_X = \mathrm{P}_R + \bar{\mathrm{P}}_Q$ because $\mathrm{P}_R = \bar{\mathrm{P}}_R$.

2.3 Geometry of the Motors

The following constructions in $\mathbb{M}$ directly parallel constructions in matrix theory, where $\mathscr{M}$ plays the role of the $n \times p$ Stiefel manifold, and $\mathbb{S}$ the symmetric positive definite matrices [4]. Refer to Fig. 2.2 which illustrates some of the concepts introduced in this section. Consider the curve $\psi(t) \in \mathscr{M}$ with $M = \psi(0)$ and $\Delta = \psi'(0)$. Differentiating the constraint $\widetilde{\psi}(t)\psi(t) = 1$ and evaluating at $t = 0$ gives $\widetilde{M}\Delta = -\widetilde{\Delta}M$. As $\Delta \in \mathbb{M}$, $\widetilde{M}\Delta \in \mathbb{M}$, and it follows that $\Delta = MB$ where $B \in \mathbb{B}$. We define the *tangent space* of $\mathscr{M}$ at $M \in \mathscr{M}$ by $\mathscr{T}_M = M\mathbb{B} = \{MB : B \in \mathbb{B}\} \subset \mathbb{M}$. Any element $X \in \mathbb{M}$ can be split:

$$X = M(\widetilde{M}X) = M\langle\widetilde{M}X\rangle_2 + M\langle\widetilde{M}X\rangle_{0,4}.$$

The first term in this split is in $\mathscr{T}_M$, while the second term is of the form MS where $S \in \mathbb{S}$. We define the *normal space* of $\mathscr{M}$ at $M \in \mathscr{M}$ (restricted to $\mathbb{M}$) by $\mathscr{N}_M = M\mathbb{S} = \{MS : S \in \mathbb{S}\}$. If $X = MB \in \mathscr{T}_M$ and $Y = MS \in \mathscr{N}_M$, then $\langle X\widetilde{Y}\rangle = \langle MBS\widetilde{M}\rangle = \langle BS\rangle = 0$, and $\mathscr{T}_M$ is orthogonal to $\mathscr{N}_M$, so $\mathbb{M} = \mathscr{T}_M \oplus \mathscr{N}_M$. From the split, for $X \in \mathbb{M}$ we can define the projection on $\mathscr{T}_M$ along $\mathscr{N}_M$ by

$$\mathrm{P}_{\mathscr{T}_M}(X) = M\langle\widetilde{M}X\rangle_2.$$

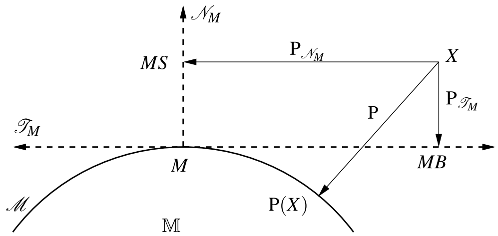

Fig. 2.2 An intuitive sketch of the geometry of motors $\mathscr{M}$ in $\mathbb{M}$ showing the tangent space $\mathscr{T}_M$ and the normal space $\mathscr{N}_M$ at M, and the projections onto $\mathscr{T}_M$, $\mathscr{N}_M$, and $\mathscr{M}$

It is clear that $\mathrm{P}_{\mathscr{T}_M}$ is idempotent, onto $\mathscr{T}_M$, and has null-space $\mathscr{N}_M$. Similarly, for $X \in \mathbb{M}$ the projection on $\mathscr{N}_M$ along $\mathscr{T}_M$ is defined by

$$\mathrm{P}_{\mathscr{N}_M}(X) = M\langle \widetilde{M} X\rangle_{0,4}.$$

It is also clear that $\mathrm{P}_{\mathscr{N}_M}$ is idempotent, along $\mathscr{T}_M$, and onto $\mathscr{N}_M$. Closely related to $\mathscr{N}_M$, we can define a polar decomposition for an element in $\mathbb{M}$.

Lemma 2.3 *An element $X \in \mathbb{M}$ with $|X| \neq 0$ has a unique polar decomposition $X = MS = SM$ where $M \in \mathscr{M}$, $S \in \mathbb{S}$, and $\langle S\rangle > 0$.*

Proof Suppose that $MS = M'S'$ are two such decompositions. Then $N = \widetilde{M}'M = S'S^{-1}$ is a symmetric motor ($N = \widetilde{N}$). Hence $N = \alpha + \beta I_3 n_\infty$ and $1 = N^2 = \alpha^2 + 2\alpha\beta I_3 n_\infty$, so $\beta = 0$ and $\alpha = 1$ because $\langle S\rangle > 0$ and $\langle S'\rangle > 0$. As $M I_3 n_\infty \widetilde{M} = I_3 n_\infty$, we have $MS = SM$. The polar decomposition is given by

$$S = |X|\left(1 + \frac{\langle X\widetilde{X}\rangle_4}{2\langle X\widetilde{X}\rangle}\right), \qquad M = XS^{-1} = \frac{X}{|X|}\left(1 - \frac{\langle X\widetilde{X}\rangle_4}{2\langle X\widetilde{X}\rangle}\right). \tag{2.2}$$

□

As shown, given $M \in \mathscr{M}$, any $X \in \mathbb{M}$ can be decomposed into components in $\mathscr{T}_M$ and $\mathscr{N}_M$ giving $X = MS + MB$. The polar decomposition can be interpreted as simply choosing M appropriately so that the component in $\mathscr{T}_M$ vanishes leaving $X = MS \in \mathscr{N}_M$. The polar decomposition is applied to more general elements $X \in \mathbb{R}^+_{4,1}$ in [1] (Chap. 5 in this book).

The polar decomposition on $\mathbb{M}$ provides a natural way to define the operation of projection onto $\mathscr{M}$ in the same way as the polar decomposition on $\mathbb{R}^{n\times p}$ defines a projection onto the $n \times p$ orthogonal matrices in matrix theory. If $X \in \mathbb{M}$ has polar decomposition $X = MS$, we define the projection onto $\mathscr{M}$ by

$$\mathrm{P}(X) = XS^{-1} \in \mathscr{M}.$$

The element $S^{-1} \in \mathbb{S}$ nudges X onto $\mathscr{M}$. It is interesting to note that several other situations arise where elements of $\mathbb{S}$ perform some useful transformation. The element $S^{-2} \in \mathbb{S}$ maps $\widetilde{X}$ onto $X^{-1} = \widetilde{X}S^{-2}$ (refer to (2.1)). An element $B \in \mathbb{B}$ can be split into two commuting blades using $S_- = \langle \widetilde{B}B\rangle_4/\langle 2\widetilde{B}B\rangle \in \mathbb{S}$ and $S_+ =$

$1 - S_- \in \mathbb{S}$. If $B_+ = BS_+$ and $B_- = BS_-$, then $B = B_+ + B_-$, and $B_+B_- = B_-B_+$. This split can be used to factor a motor in accordance with Chasles's decomposition

$$M = \exp\left(-\frac{1}{2}B\right) = \exp\left(-\frac{1}{2}B_+\right)\exp\left(-\frac{1}{2}B_-\right), \tag{2.3}$$

where $\exp(-B_+/2)$ is a generalized rotator about an axis, and $\exp(-B_-/2)$ is a translator along the axis. Using the polar decomposition, it is a simple matter to show that for any element $Y \in \mathbb{M}$ with $|Y| \neq 0$, we can find an element $X = \log(Y) \in \mathbb{M}$ such that $Y = \exp(X)$. Let Y have a polar decomposition $Y = MS'$. The motor M can be expressed $M = \exp(B)$ where $B \in \mathbb{B}$ (an expression for the motor logarithm may be found in [2]). Also note that if $S = \alpha + Q \in \mathbb{S}$, then $\exp(S) = \exp(\alpha)(1 + Q) = \exp(\alpha) + \exp(\alpha)Q$ because α and Q commute and $Q^2 = 0$. So if $S' = \alpha' + Q' = \exp(S)$, then we take $\alpha = \ln \alpha'$ and $Q = Q'/\alpha'$ giving $S = \ln \alpha' + Q'/\alpha'$. As B and S commute, we can take $X = B + S \in \mathbb{M}$.

There is an equivalent polar decomposition for an element $X \in \overline{\mathbb{M}}$ with $|X| > 0$, of the form $X = MS$, where $M \in \bar{\mathscr{M}}$ models a rotation and transversion, and $S \in \bar{\mathbb{S}} = \text{span}\{1, \widetilde{I}_3 n_o\}$.

The rotators $\mathscr{R}$ are used to model rotation about the origin and lie in the linear space $\mathbb{R}_3^+ = \text{span}\{1, e_{12}, e_{13}, e_{23}\}$. All the ideas above simplify when restricted to rotators. If $X \in \mathbb{R}_3^+$, then $X\widetilde{X} = \langle X\widetilde{X}\rangle$, so the equation $X\widetilde{X} = 1$ imposes only one constraint and is equivalent to the statement $\langle X\widetilde{X}\rangle = 1$. We will see that the absence of the constraint $\langle X\widetilde{X}\rangle_4 = 0$ is an important simplification for rotator estimation. If $R \in \mathscr{R}$, then $\mathscr{N}_R = \{Rs : s \in \mathbb{R}\}$ (i.e. just scalar multiples of R), and the projection on $\mathscr{N}_R$ is given by $\mathrm{P}_{\mathscr{N}_R}(X) = R\langle\widetilde{R}X\rangle$. The tangent space $\mathscr{T}_R = \{RB : B \in \text{span}\{e_{12}, e_{13}, e_{23}\}\}$, and $\mathrm{P}_{\mathscr{T}_R}(X) = R\langle\widetilde{R}X\rangle_2$. The polar decomposition takes the simple form $X = Rs$ where $R = X/|X|$ and $s = |X| \in \mathbb{R}$.

The translators are used to model translation and lie in the linear space $\mathbb{T} = \text{span}\{1, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty\}$. A translator $T = 1 - \frac{1}{2}\mathbf{t}n_\infty$ has a constant scalar coefficient, so there are only three degrees of freedom, as required. If $X \in \mathbb{T}$, then $\langle X\rangle^2 = X\widetilde{X} = \langle X\widetilde{X}\rangle$, so the equation $X\widetilde{X} = 1$ imposes only one constraint as for rotators. Because $\langle\mathbb{T}\rangle_2$ is made up of null bivectors, significant simplifications arise. If T is a translator, then $\mathscr{N}_T = \{Ts : s \in \mathbb{R}\}$ (i.e. just scalar multiples of T), and the projection on $\mathscr{N}_T$ is given by $\mathrm{P}_{\mathscr{N}_T}(X) = T\langle X\rangle$. The tangent space $\mathscr{T}_T = \text{span}\{e_1 n_\infty, e_2 n_\infty, e_3 n_\infty\}$, and $\mathrm{P}_{\mathscr{T}_T}(X) = \langle\widetilde{T}X\rangle_2$. The polar decomposition takes the simple form $X = Ts$ where $T = X/|X|$ and $s = |X| = |\langle X\rangle| \in \mathbb{R}$.

2.4 Estimating Motors

We have two sets of noisy geometric data and wish to estimate the motor that optimally maps one data set onto the other. To solve this problem, we need to be precise about what optimal means, so we will define a measure that is used to determine if two geometric objects are similar. For example, if P and Q are normalised points, then $\langle PQ\rangle = -\frac{1}{2}d^2$ where d is the distance between the points. Two points are considered similar if they are close together. We choose a similarity rather an error measure only because it avoids a sign change for the most common case of points. The inner product between points increases as the points get closer; hence it already has the correct sign. To set the problem up so it has a simple closed-form solution as an eigenrotator problem, we need to restrict the form of the similarity measure as described in the next section. However, even with this restriction, not all possibilities for object representation are admissible into the framework for estimating motors. This is because one of the constraints $\langle X\widetilde{X}\rangle_4 = 0$ for an element $X \in \mathbb{M}$ to be a motor (recall Lemma 2.1) is awkward to handle, and we will only want to consider object representations where it can be dropped so that we can estimate the motor using linear methods. Surprisingly, this occurs quite often as we will see later.

2.4.1 Similarity Measures in CGA

In order to set the problem up as a eigenrotator problem, we need to restrict the similarity measure between objects P and Q to the simple form $\langle P\check{Q}\rangle$, where the *check* operator $\check{Q}$ is a grade-dependent sign change defined by $\check{Q} = \langle Q\rangle_{0,1,3} - \langle Q\rangle_{2,4,5}$. Note that $\check{Q} = \widetilde{Q}$ if $Q = \langle Q\rangle_{0,1,2}$ and $\check{Q} = -\widetilde{Q}$ if $Q = \langle Q\rangle_{3,4,5}$. This operation is motivated by the requirements (i) $\langle p\check{q}\rangle = \langle P\check{Q}\rangle$ where $p = P^*$ and $q = Q^*$ and (ii) $\langle P\check{Q}\rangle = \cos(\theta)$ when P, Q are flats (see below). This simple form is not as much of a restriction as it may first seem. If we carefully consider the object representation, many physically meaningful quantities can be expressed in this way. Consider the following examples:

Points and Spheres We have already seen that if P and Q are normalised points (grade-1), then

$$\langle P\check{Q}\rangle = \langle PQ\rangle = -\frac{1}{2}d^2 \tag{2.4}$$

where d is the distance between them. Points can be considered as dual spheres with zero radius. When $P = p - \frac{1}{2}\rho_p^2 n_\infty$ and $Q = q - \frac{1}{2}\rho_q^2 n_\infty$ are dual spheres (grade-1), we get

$$\begin{aligned}\langle P\check{Q}\rangle = \langle PQ\rangle &= \langle pq\rangle + \frac{1}{2}\big(\rho_p^2 + \rho_q^2\big)\\ &= -\frac{1}{2}d^2 + \frac{1}{2}\big(\rho_p^2 + \rho_q^2\big).\end{aligned}$$

As the radii ρ_p and ρ_q are constant under rigid body motion, this effectively reduces to the point case, and two spheres are considered similar if their centres are close. When P and Q are normalised spheres (grade-4), we get exactly the same expression because of the way the check operator $\check{}$ is defined. A physical interpretation of $\langle P\check{Q}\rangle$ in terms of a line segment joining the spheres is given in [2, Fig. 14.8, p. 418].

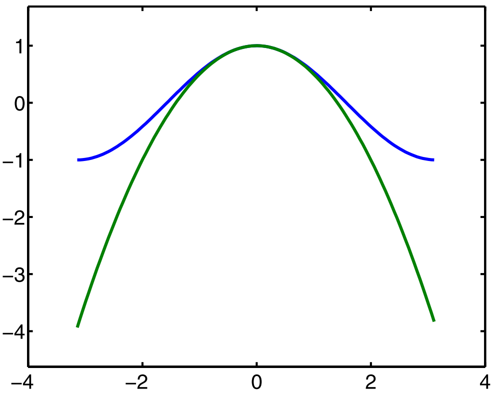

Fig. 2.3 Graph showing $\cos(\theta)$ and $-\frac{1}{2}\theta^2+1$. As $\cos(\theta)$ (and $\sin^2(\theta)$) turn up so frequently in geometric calculations, we should embrace there advantages over θ^2

Flats Flats are objects like planes and lines. A flat can be modelled $P = p \wedge \mathbf{V} \wedge n_\infty$, where p is a point on the flat and $\mathbf{V}$ is a Euclidean blade representing the direction of the flat. If $\mathbf{V}$ is a Euclidean vector, then $p \wedge \mathbf{V} \wedge n_\infty$ is grade-3 and represents a line. If $\mathbf{V}$ is a Euclidean bivector, then $p \wedge \mathbf{V} \wedge n_\infty$ is grade-4 and represents a plane. The other cases are less interesting in the current application: if $\mathbf{V} = 1$, then $p \wedge n_\infty$ is a flat point, and if $\mathbf{V}$ is a Euclidean trivector, then $p \wedge \mathbf{V} \wedge n_\infty$ represents a volume but is both translation and rotation invariant. If P and Q are normalised flats so that $|P| = 1$ and $|Q| = 1$, then

$$\langle P\check{Q}\rangle = \cos(\theta) \tag{2.5}$$

where θ is the dihedral angle between them. Two flats are considered similar if the angle between them is small. Note that for small θ, $\cos(\theta) \approx -\theta^2/2! + 1$ as shown in Fig. 2.3. There is no drawback in maximising $\cos(\theta)$ as opposed to $-\theta^2/2!$ for many practical situations. Using $\cos(\theta)$ can even have an added benefit. Because $\cos(\theta) \geq -1$, it restricts the influence of outliers, so we are more likely to get an acceptable solution even with significant outliers. If required, we can then reject outliers and refit until the fit is acceptable. One potential concern with the measure is that it does not capture the distance between lines, only the angle. The distance is usually regarded as the closest distance that the lines pass. It is a simple matter to determine this distance, for example, by forming the motor $P\widetilde{Q}$ and making use of Chasles's decomposition. It is not clear how to do this while keeping the simple form of a scalar product $\langle P\check{Q}\rangle$. This is not so much of a concern with planes as they will always intersect unless they are exactly parallel, so we are often only interested in the angle between them. When there is a specific point of interest on a line or plane, we should consider modelling it as a tangent instead of a flat as discussed below.

Directions Directions are used to model 1D direction and attitude and can be represented in CGA in the form $\varDelta = \mathbf{V}n_\infty$ where $\mathbf{V}$ is a Euclidean blade. They are translation invariant, so for translator T, we have $T\varDelta\widetilde{T} = \varDelta$. The case where $\mathbf{V}$ is grade-1 gives a 1D or line direction, and the case where $\mathbf{V}$ is grade-2 gives a 2D or plane direction. The other cases (scalar and grade-3) are of no practical interest here. For the scalar case, we get a scale multiple of n_∞, and for the grade-3 case, we get a scale multiple of $I_3 n_\infty$, both of which are translation and rotation invariant. If Δ_p and Δ_q are two directions, then $\langle \Delta_p \check{\Delta}_q \rangle = 0$, so we cannot use the directions directly. We can construct a meaningful quantity by representing the directions as flats $n_o \wedge \Delta$, dual flats $n_o \cdot \Delta^*$, or Euclidean directions $n_o \cdot \Delta$. If P and Q are two normalised directions represented in one of the above three forms, then

$$\langle P \check{Q} \rangle = \cos(\theta) \tag{2.6}$$

where θ is the dihedral angle between them. Two directions are considered similar if the angle between them is small.

Tangents Tangents have both location and direction and can be used to model various objects such as tangent planes on a surface, tangent lines on a curve, and rays leaving a camera where the optical centre is the location. A tangent at location p with normalised direction $\Delta = \mathbf{V} n_\infty$ can be represented in CGA as a blade $T' = p \wedge (p \cdot \hat{\Delta})$. If Δ is a bivector, then T' is a tangent line, and if Δ is a trivector, then T' is a tangent plane. When $\Delta = n_\infty$, then $T' = p$, and when $\Delta = I_3 n_\infty$, then $T' = p^*$, and we see that a point can be regarded as a degenerate tangent. Unfortunately, except in the case of points, taking the inner product between two tangents in this form does not give a particularly meaningful quantity. If we are prepared to consider a broader range of representations than blades, then we can construct a meaningful quantity using the measure. To be concrete, we will discuss the case of tangent lines first. Let $T = p + \Lambda$ be a flag (nested sequence of linear spaces) representation of the tangent with grade-1 and 3 parts, where p is the tangent location, and $\Lambda = T' \wedge n_\infty$ is the carrier line with $p \wedge \Lambda = 0$. The representations T and T' are equivalent with $T' = \langle T \rangle_1 \cdot \hat{T}$ and $T = (1 + \widetilde{T}')(T' \wedge n_\infty)$. If $P = p + \Lambda_p$ and $Q = q + \Lambda_q$ are two tangent lines, then

$$\begin{aligned}
\langle P \check{Q} \rangle &= \langle pq \rangle + \langle \Lambda_p \check{\Lambda}_q \rangle \\
&= -\frac{1}{2} d^2 + \cos(\theta) \\
&\approx -\frac{1}{2}\left(d^2 + \theta^2\right) + 1,
\end{aligned}$$

where d is the distance between the tangent locations, and θ is the dihedral angle between the tangent carriers. Two tangents are considered similar if their locations are close and the angle between them is small. We can adjust ratio of the locational and angular parts by encoding a weight in the line. For example, if $w = |\Lambda|$, then

$$\begin{aligned}
\langle P \check{Q} \rangle &= -\frac{1}{2} d^2 + w^2 \cos(\theta) \\
&\approx -\frac{1}{2}\left(d^2 + w^2\theta^2\right) + w^2.
\end{aligned}$$

Exactly the same construction works with tangent planes. Here we take $P = p + \Pi_p$ and $Q = q + \Pi_q$ to be two tangent planes where Π_p, Π_q are planes with $p \wedge \Pi_p = 0$ and $q \wedge \Pi_q = 0$.

Rounds Rounds are objects like spheres, circles, and point pairs. We have already discussed spheres above, and we will now generalise this to include the remaining round objects. A direct round can be represented in CGA as a blade of the form $R = s \wedge (s \cdot \hat{\Delta})$ where s is a dual sphere and $\Delta = \mathbf{V} n_\infty$ is the direction. This is the same expression as for tangents, and tangents can simply be regarded as rounds with zero radius. A normalised direct round object R can also be represented as a tangent-like flag object $T = s + F$ where s is a dual sphere and F is a carrier flat with $s \wedge F = 0$. Just as for tangents the two representations are equivalent with $T = (1 + \widetilde{R})(R \wedge n_\infty)$ and $R = \langle T \rangle_1 \cdot \hat{T}$. If $P = s_p + F_p$ and $Q = s_q + F_q$ are two rounds represented in this way, with radii ρ_p and ρ_q, respectively, then

$$\langle P \check{Q} \rangle = -\frac{1}{2} d^2 + \cos(\theta) + \frac{1}{2}\bigl(\rho_p^2 + \rho_q^2\bigr),$$

where d is the distance between the centres of the rounds, and θ is the dihedral angle between the carrier flats. As mentioned when discussing spheres, the radii are invariant under rigid body motion, so this effectively reduces to the tangent case. If P and Q are direct spheres, then $P \wedge n_\infty = -I_{4,1}$ and $Q \wedge n_\infty = -I_{4,1}$ and $\cos(\theta) = 1$, and it reduces further to the point case.

We have associated a physically meaningful measure with the basic objects available in CGA. Some objects, such as points, spheres, and flats, are represented in their basic blade form, and we will refer to these as primitive objects. Other objects, such as rounds and tangents, are represented in flag form and constructed using primitive objects. The directions, on the other hand, are converted to a primitive object representation. Other ways of representing the objects P and Q can be designed to give different measures. The only structural requirement is that they are expressed in the form $\langle P \check{Q} \rangle$.

2.4.2 Motor Estimation Problem Formulation

We are now in a position to formulate the estimation problem. Let P_k, $k = 1, \ldots, n$, be a set of normalised CGA objects before motion, and Q_k, $k = 1, \ldots, n$, be the set of objects after motion, $w_k \in \mathbb{R}$ be scalar weights, and $M \in \mathscr{M}$. The total similarity is given by the weighted sum of the symmetrised similarity between $M P_k \widetilde{M}$ and Q_k as follows:

$$E = \frac{1}{2} \sum_{k=1}^{n} w_k \bigl(\langle M P_k \widetilde{M} \check{Q}_k \rangle + \langle \widetilde{\check{Q}}_k M \widetilde{P}_k \widetilde{M} \rangle \bigr) = \langle \widetilde{M} \mathscr{L} M \rangle, \tag{2.7}$$

where

$$\mathcal{L}X = \frac{1}{2}\sum_{k=1}^{n} w_k(\check{Q}_k X P_k + \tilde{\check{Q}}_k X \widetilde{P}_k). \tag{2.8}$$

Note that $\mathcal{L}$ satisfies the useful symmetry property $\langle \widetilde{A}\mathcal{L}B\rangle = \langle \widetilde{B}\mathcal{L}A\rangle$ for all $A, B \in \mathbb{R}_{4,1}$. If P_k and Q_k have the same symmetry and are either both symmetric (i.e. $A = \widetilde{A}$) or both antisymmetric (i.e. $A = -\widetilde{A}$), then $\mathcal{L}X$ reduces to $\mathcal{L}X = \sum_{k=1}^{n} w_k \check{Q}_k X P_k$. This is clearly true when P_k and Q_k are homogeneous (and the same grade). However, in some mixed grade situations (e.g. for the flags $P = \langle P\rangle_1 + \langle P\rangle_3$ and $Q = \langle Q\rangle_1 + \langle Q\rangle_3$) we require the full form given by (2.8). The data P_k, $k = 1, \ldots, n$, need not all be of the same object type but could contain a variety of geometric objects such as points, spheres, flats, and directions. Clearly, for a given k, P_k and Q_k represent the same object type as one is simply a rotated and translated version of the other. The magnitude of the weights w_k can be used to adjust the contribution a data element makes based on its reliability, or to introduce attractive and repulsive contributions. We can now couch the problem of finding an optimal motor more precisely as maximising $\langle \widetilde{X}\mathcal{L}X\rangle$ subject to $X \in \mathcal{M}$. Using Lemma 2.1, we can rewrite this as

$$\max_{X\in\mathbb{M}} \langle \widetilde{X}\mathcal{L}X\rangle \text{ subject to } \langle X\widetilde{X}\rangle = 1 \text{ and } \langle X\widetilde{X}\rangle_4 = 0. \tag{2.9}$$

2.4.3 Optimal Rotator and Translator Estimation

First consider the simpler case of rotator estimation so that problem (2.9) reduces to

$$\max_{X\in\mathbb{R}_3^+} \langle \widetilde{X}\mathcal{L}X\rangle \text{ subject to } \langle X\widetilde{X}\rangle = 1 \tag{2.10}$$

This has a simple solution which is captured in the following theorem.

Theorem 2.1 *Let P_k and Q_k, $k = 1, \ldots, n$, be two sets of normalised conformal objects in $\mathbb{R}_{4,1}$, $w_k \in \mathbb{R}$ be scalar weights, and $\mathcal{L}$ be defined by*

$$\mathcal{L}X = \frac{1}{2}\sum_{k=1}^{n} w_k(\check{Q}_k X P_k + \tilde{\check{Q}}_k X \widetilde{P}_k).$$

Then the maximiser of $\langle \widetilde{R}\mathcal{L}R\rangle$ subject to $R \in \mathcal{R}$ is an eigenrotator of $\mathrm{P}_R\mathcal{L}$ associated with the largest eigenvalue, where P_R is the projection onto $\mathbb{R}_3^+$.

Proof The Lagrange function associated with problem (2.10) is given by $L(X) = \frac{1}{2}\langle \widetilde{X}\mathcal{L}X\rangle - \frac{\alpha}{2}(\langle \widetilde{X}X\rangle - 1)$ where $X \in \mathbb{R}_3^+$. Using the first-order optimality condition $\partial_{\widetilde{X}} L = 0$ and noting that $\mathrm{P}_R X = X$ gives $\mathrm{P}_R\mathcal{L}X = \alpha X$ at the maximiser. In addition, $\alpha = \langle \widetilde{X} \mathscr{L} X \rangle$, so X is the eigenrotator of $\mathrm{P}_R \mathscr{L}$ associated with the largest eigenvalue. □

The optimal rotator can be readily obtained by forming the matrix representative of $\mathrm{P}_R \mathscr{L}$ as outlined in the following procedure:

1. Form an orthonormal basis $e_k, k = 1, \ldots, 4$, of $\mathbb{R}_3^+$ (e.g. $\{1, e_{12}, e_{13}, e_{23}\}$).
2. Form the 4×4 symmetric matrix $L_{ij} = \langle \widetilde{e}_i \mathrm{P}_R \mathscr{L} e_j \rangle = \langle \widetilde{e}_i \mathscr{L} e_j \rangle$.
3. Calculate $r \in \mathbb{R}^4$, a unit eigenvector of L associated with the largest eigenvalue.
4. Calculate the optimal rotator $R = \sum_k r_k e_k \in \mathscr{R}$.

If the dimension d of the eigenspace associated with the largest eigenvalue is greater than one, then the optimal eigenrotator is not unique. This will happen in degenerate situations such as estimating a rotator from a single pair of planes. The planes will be made parallel, but any additional rotation about an axis normal to the planes is permissible and will not affect the measure. A specific solution can be returned at the expense of a small increase in complexity as follows. Let $V \in \mathbb{R}^{4\times d}$, $d \leq 4$, be an orthogonal matrix whose range is the eigenspace of L associated with the largest eigenvalue. Any maximum unit eigenvector can be expressed as $r = Vx$ for unit vector $x \in \mathbb{R}^d$. Note $\cos(\frac{\theta}{2}) = \langle R \rangle = \sum_k r_k \langle e_k \rangle = r^T z$ where $z \in \mathbb{R}^4$ with $z_k = \langle e_k \rangle$, and θ is the angle of rotation. With the natural basis above we get $z = (1 \quad 0 \quad 0 \quad 0)^T$. Hence $x^T V^T z$ can be identified with $\cos(\frac{\theta}{2})$. Maximising $x^T V^T z$ subject to $x^T x = 1$ gives the following enhancement to step 3 above:

3′. Calculate $r = \mathrm{unit}(V V^T z) \in \mathbb{R}^4$, the unit eigenvector of L associated with the largest eigenvalue and the smallest angle of rotation, where $z \in \mathbb{R}^4$ with $z_k = \langle e_k \rangle, k = 1, \ldots, 4$.

If $d = 1$, then there is no choice, and $r = V$ or $r = -V$, as expected.

This approach has an advantage over the standard methods of estimating an orthogonal 3×3 matrix using polar decomposition (or SVD) because improper rotations are excluded at the outset rather than removed at the end with a determinant check [3]. This advantage can be achieved with a matrix formulation based on quaternions [3]. However, the rotator formulation is also directly applicable to a wider range of objects than just points, including spheres, flats, and directions, and allows all these objects to be incorporated into a single framework.

The translator case is simpler because we can encode the constraint in the parameterisation of the translator. Let $T = 1 + Q$ where $Q = q_1 e_1 n_\infty + q_2 e_2 n_\infty + q_3 e_3 n_\infty \in \mathbb{R}_3^1 n_\infty$. Let $\mathscr{F}^+$ denote the Moore–Penrose pseudo-inverse of a linear transformation $\mathscr{F}$.

Theorem 2.2 *Let P_k and Q_k, $k = 1, \ldots, n$, be two sets of normalised conformal objects in $\mathbb{R}_{4,1}$, $w_k \in \mathbb{R}$ be scalar weights, and $\mathscr{L}$ be defined by*

$$\mathscr{L} X = \frac{1}{2} \sum_{k=1}^{n} w_k (\check{Q}_k X P_k + \widetilde{\check{Q}}_k X \widetilde{P}_k).$$

Then the maximiser of $\langle \widetilde{T} \mathscr{L} T \rangle$ subject to T being a translator is given by $T = 1 + Q$ where $Q = -(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^+ \mathscr{L} 1$.

Proof The objective function is given by $L(T) = \langle \widetilde{T} \mathscr{L} T \rangle = \langle 1 \mathscr{L} 1 \rangle + 2\langle \widetilde{Q} \mathscr{L} 1 \rangle + \langle \widetilde{Q} \mathscr{L} Q \rangle$ for $T \in \mathbb{T}$. The first-order optimality condition $\partial_{\widetilde{Q}} L = 0$ gives $\bar{\mathrm{P}}_Q \mathscr{L} Q + \bar{\mathrm{P}}_Q \mathscr{L} 1 = 0$, so $Q = -(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^+ \mathscr{L} 1$. □

The optimal translator can be obtained by forming the matrix representative of $\bar{\mathrm{P}}_{\mathbb{T}} \mathscr{L}$ as outlined in the following procedure:

1. Form a basis e_k, $k = 1, \dots, 4$, of $\mathbb{T}$, where the first basis vector is scalar (e.g. $\{1, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty\}$).
2. Form the 4×4 symmetric matrix $L_{ij} = \langle \widetilde{e}_i \bar{\mathrm{P}}_{\mathbb{T}} \mathscr{L} e_j \rangle = \langle \widetilde{e}_i \mathscr{L} e_j \rangle$ and break it into sub-matrices $L = \left(\begin{smallmatrix} L_{rr} & L_{rq} \\ L_{qr} & L_{qq} \end{smallmatrix}\right)$ where $L_{rr} \in \mathbb{R}$ and $L_{qq} \in \mathbb{R}^{3\times 3}$.
3. Calculate $q = -L_{qq}^+ L_{qr} \in \mathbb{R}^3$.
4. Form the full coefficient vector $t = \left(\begin{smallmatrix} 1 \\ q \end{smallmatrix}\right) \in \mathbb{R}^4$.
5. Calculate the optimal translator $T = \sum_k t_k e_k$.

The use of the Moore–Penrose pseudo-inverse will ensure that the smallest translation q is returned when there is not a unique maximiser of $\langle \widetilde{T} \mathscr{L} T \rangle$. This will happen when no locational information is provided, for example, finding the translator between two sets of directions. In such a case the above procedure will return an identity translator $T = 1$.

2.4.4 Optimal Motor Estimation as an Eigenrotator Problem

It is interesting to see that much of the structure for rotators and translators is preserved when we consider the more complex case of motor estimation. First note that the key difference between the full motor problem (2.9) and the rotator problem (2.10) is the addition of the extra constraint $\langle X \widetilde{X} \rangle_4 = 0$. We will show that by restricting the representation of CGA objects the constraint $\langle X \widetilde{X} \rangle_4 = 0$ can be dropped entirely, leaving a problem no more difficult than the rotator estimation problem. The other difference between problems (2.9) and (2.10) is the linear space involved. The motors lie in $\mathbb{M}$, while the rotators lie in $\mathbb{R}_3^+ \subset \mathbb{M}$. The only implication is that $\mathbb{M}$ is incomplete in the sense discussed previously: we cannot construct a reciprocal basis that also lies in $\mathbb{M}$. The following lemma characterises those elements which are nearly motors, where we have not enforced the constraint $\langle X \widetilde{X} \rangle_4 = 0$.

Lemma 2.4 $X \in \mathbb{M}$ *and* $\langle X \widetilde{X} \rangle = 1 \Leftrightarrow X = M + \beta M I_3 n_\infty$, $M \in \mathscr{M}$, *and* $\beta \in \mathbb{R}$.

Proof Let $X = MS$ be the polar decomposition of $X \in \mathbb{M}$ with $S = \alpha + \beta I_3 n_\infty$. Because $1 = \langle \widetilde{X} X \rangle = \langle S^2 \rangle = \alpha^2$ and $\alpha \geq 0$, we have $\alpha = 1$ and $X = M + \beta M I_3 n_\infty$. If $X = M + \beta M I_3 n_\infty$ where $M \in \mathscr{M} \subset \mathbb{M}$, then X is the sum of products of elements in $\mathbb{M}$, so $X \in \mathbb{M}$, and $\langle X \widetilde{X} \rangle = \langle M \widetilde{M} \rangle = 1$. □

On the LHS of Lemma 2.4 we have the 8D space $\mathbb{M}$ with one constraint imposed, and on the RHS we have the 6D motor manifold with an extra degree of freedom added through β. It is convenient to use the notation $\Psi = I_3 n_\infty = \widetilde{\Psi}$ for the quad-vector basis element of $\mathbb{M}$ as it is used frequently. In addition, we will denote the set

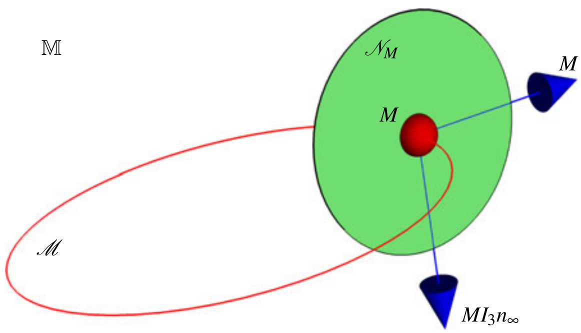

Fig. 2.4 Sketch showing the 2D normal space $\mathscr{N}_M$ of $\mathscr{M}$ at M (restricted to $\mathbb{M}$). Imposing the constraint $\langle X\widetilde{X}\rangle = 1$ restricts us the 1D subspace of $\mathscr{N}_M$ consisting of elements of the form $M(1+\beta I_3 n_\infty)$

defined in Lemma 2.4 by $\mathscr{M}' = \{M + \beta M\Psi : M \in \mathscr{M}, \beta \in \mathbb{R}\}$. One way to study the problem is to consider the behaviour of the objective function $\langle \widetilde{X}\mathscr{L}X\rangle$ with elements $X \in \mathscr{M}'$. This allows us to separate out the terms which result from relaxing the constraint $\langle X\widetilde{X}\rangle_4 = 0$. When $\beta = 0$, X lies on the motor manifold $\mathscr{M}$. As $|\beta|$ increases, X leaves $\mathscr{M}$ along a 1D subspace of $\mathscr{N}_M$. A sketch of the situation is shown in Fig. 2.4. Note that M is both a point on $\mathscr{M}$ and a direction vector in $\mathscr{N}_M$.

Expanding the objective function at $X = M + \beta M\Psi \in \mathscr{M}'$ gives

$$\langle \widetilde{X}\mathscr{L}X\rangle = \langle \widetilde{M}\mathscr{L}M\rangle + 2\beta\big\langle \widetilde{M}\mathscr{L}(M\Psi)\big\rangle + \beta^2\big\langle \widetilde{\Psi}\widetilde{M}\mathscr{L}(M\Psi)\big\rangle. \tag{2.11}$$

For a given $M \in \mathscr{M}$, this is a quadratic in β. We are interested in the cases where the coefficient of β vanishes and coefficient of β^2 is not positive, independently of M. When the coefficient of β^2 is negative, leaving $\mathscr{M}$ decreases the objective function, and maximising $\langle \widetilde{X}\mathscr{L}X\rangle$ subject to $X \in \mathscr{M}'$ will give us the optimal motor which solves problem (2.9). If the coefficient of β^2 vanishes, then the solution is not unique, and if $M \in \mathscr{M}$ is a solution, then so is $M(1+\beta\Psi)$. In such situations we can maximise $\langle \widetilde{X}\mathscr{L}X\rangle$ to give a solution, and then project the resulting X onto $\mathscr{M}$ to get the optimal motor. We first make some general observations which help to manipulate (2.11).

1. When $\mathscr{L}X = \frac{1}{2}\sum_{k=1}^{n} w_k(\check{Q}_k X P_k + \widetilde{\check{Q}}_k X \widetilde{P}_k)$ is substituted in the coefficients of β and β^2, the term $P_k' = P_k'(M) = \widetilde{M} Q_k M$ turns up which has the same grades as Q_k and P_k. (It represents the same kind of object.)
2. The coefficient of β is made up of terms $\langle \Psi(\check{P}_k' P_k + P_k \check{P}_k')\rangle$.
3. If P_k and Q_k have the same symmetry, the coefficient of β reduces to $2\langle \Psi \check{P}_k' P_k\rangle$.
4. The coefficient of β^2 is made up of terms $\langle \Psi \check{P}_k' \Psi P_k\rangle$.
5. $\langle \Psi \check{P}' \Psi P\rangle = -\langle n_\infty \check{P}' n_\infty P\rangle$ for all $P, P' \in \mathbb{R}_{4,1}$.
6. $\langle \widetilde{X}\check{Q} Y P\rangle = \langle \widetilde{X}\check{q} Y p\rangle$ where $p = P^*$ and $q = Q^*$ for all $X, Y, P, Q \in \mathbb{R}_{4,1}$.

We will examine what conditions need to be imposed on P_k, Q_k, and w_k so that we can ensure that the coefficient of β vanishes and the coefficient of β^2 is not positive. Let us first consider the cases where P_k and Q_k are homogeneous and then extend to mixed grade elements. We only need to provide proofs for scalars, vectors, and bivectors because the trivector, quadvector, and pseudoscalar cases follow by observation 6 above. First examine the case where P_k and Q_k are vectors or quadvectors.

Lemma 2.5 *Let P_k and Q_k, $k = 1, \dots, n$, be two sets of vectors or quadvectors. Then $\langle \widetilde{X} \mathscr{L} X \rangle = \langle \widetilde{M} \mathscr{L} M \rangle - \beta^2 \langle n_\infty \mathscr{L} n_\infty \rangle$.*

Proof As $\check{P}'_k P_k$ has no grade-4 part, the coefficient of β vanishes. Also note that for vector or quadvector Q_k, we have $n_\infty \widetilde{M} Q_k M n_\infty = \widetilde{M} n_\infty Q_k n_\infty M = n_\infty Q_k n_\infty$ so $\langle n_\infty \check{P}'_k n_\infty P_k \rangle = \langle n_\infty \check{Q}_k n_\infty P_k \rangle$, and the coefficient of β^2 is independent of M. □

Using Lemma 2.5, we can now provide the following useful results for normalised points, spheres, and dual spheres; and planes and dual planes:

Lemma 2.6 *Let P_k and Q_k, $k = 1, \dots, n$, be two sets of normalised conformal points, spheres, or dual spheres in $\mathbb{R}_{4,1}$, w_k be scalar weights with $\sum_k w_k > 0$, and*

$$\mathscr{L} X = \frac{1}{2} \sum_{k=1}^{n} w_k (\check{Q}_k X P_k + \tilde{\check{Q}}_k X \widetilde{P}_k).$$

Then the maximiser of $\langle \widetilde{X} \mathscr{L} X \rangle$ subject to $X \in \mathbb{M}$ and $\langle \widetilde{X} X \rangle = 1$ is a motor.

Proof Assume that X is not a motor, so $X = M(1 + \beta \Psi)$. For normalised spheres, dual spheres, and points, we have $\langle n_\infty \check{Q}_k n_\infty P_k \rangle = 2$, so $\langle n_\infty \mathscr{L} n_\infty \rangle = 2 \sum_k w_k > 0$. By Lemma 2.5 we have $\langle \widetilde{X} \mathscr{L} X \rangle = \langle \widetilde{M} \mathscr{L} M \rangle - 2\beta^2 \sum_k w_k$, and X cannot be a maximiser. □

It is interesting that only the sum of the weights $\sum_k w_k > 0$ need be positive. Some points can have a repulsive force as long as the sum of the attractive contribution is greater than the repulsive terms. The result for planes is as follows:

Lemma 2.7 *Let P_k and Q_k, $k = 1, \dots, n$, be two sets of normalised conformal planes or dual planes in $\mathbb{R}_{4,1}$, and $\mathscr{L}$ be defined by (2.8). The maximum value of $\langle \widetilde{X} \mathscr{L} X \rangle$ subject to $X \in \mathbb{M}$ and $\langle \widetilde{X} X \rangle = 1$ is obtained by a motor.*

Proof For planes or dual planes P_k and Q_k, we have $\langle n_\infty \check{Q}_k n_\infty P_k \rangle = 0$ so $\langle n_\infty \mathscr{L} n_\infty \rangle = 0$, and the coefficient of β^2 also vanishes. □

This is a weaker result than for points and spheres since we can only state that the maximum is obtained by *a* motor because the maximiser is not unique. If $M \in \mathscr{M}$ is a maximiser, then so is $X = M + \beta M \Psi$. The case of bivectors and trivectors is not quite as clean.

Lemma 2.8 *Let P_k and Q_k, $k = 1, \dots, n$, be two sets of bivectors or trivectors such that $n_\infty P_k n_\infty = n_\infty Q_k n_\infty = 0$. Then $\langle \widetilde{X} \mathscr{L} X \rangle = \langle \widetilde{M} \mathscr{L} M \rangle$.*

Proof $n_\infty P_k n_\infty = n_\infty Q_k n_\infty = 0$ iff Q_k and P_k have no terms of the form $\mathbf{V} n_o$ where $\mathbf{V}$ is a Euclidean blade. This precludes the appearance of a term $\widetilde{I}_3 n_o$ in the product $\check{P}' P$; hence $\langle \Psi \check{P}_k P_k \rangle = 0$, and the coefficient of β vanishes. As $n_\infty Q_k n_\infty = 0$, we have $\langle \Psi \check{P}_k \Psi P_k \rangle = 0$, and the coefficient of β^2 also vanishes. □

While Lemma 2.8 is somewhat restrictive, it is still sufficiently general to allow the following useful result for lines, which is analogous to Lemma 2.7 for planes.

Lemma 2.9 *Let P_k and Q_k, $k = 1, \dots, n$, be two sets of normalised conformal lines or dual lines in $\mathbb{R}_{4,1}$, and $\mathscr{L}$ be defined by* (2.8). *The maximum value of $\langle \widetilde{X} \mathscr{L} X \rangle$ subject to $X \in \mathbb{M}$ and $\langle \widetilde{X} X \rangle = 1$ is obtained by a motor.*

Proof If P_k and Q_k are lines, then $n_\infty P_k n_\infty = n_\infty Q_k n_\infty = 0$. □

For completeness, the case for scalars and grade-5 elements is also given but is of limited interest as these elements are invariant to rigid body motion.

Lemma 2.10 *If P_k and Q_k, $k = 1, \dots, n$, are scalars or grade-*5, *then $\langle \widetilde{X} \mathscr{L} X \rangle = \langle \widetilde{M} \mathscr{L} M \rangle = \sum_k w_k \langle P_k \check{Q}_k \rangle$.*

Proof If P_k and Q_k are scalar, $\langle \Psi \check{P}_k' P_k \rangle = \check{P}_k' P_k \langle \Psi \rangle = 0$, and the coefficient of β vanishes. Similarly $\langle \Psi \check{P}_k' \Psi P_k \rangle = \check{P}_k' P_k \langle \Psi^2 \rangle = 0$, so the coefficient of β^2 also vanishes. Also $\langle \widetilde{M} \mathscr{L} M \rangle = \sum_{k=1}^n w_k \langle \widetilde{M} \check{Q}_k M P_k \rangle = \sum_{k=1}^n w_k \langle \check{Q}_k P_k \rangle$. □

The cases where P and Q are mixed grade can now be expressed in terms of the homogeneous cases. We will only consider the mixed grade case where $P = \langle P \rangle_1 + \langle P \rangle_r$ and $Q = \langle Q \rangle_1 + \langle Q \rangle_r$ because this is all we currently require. The coefficient of β will have terms of the form

$$\begin{aligned}\big\langle \Psi \big(\check{P}' P + P \check{P}'\big)\big\rangle &= 2\big\langle \Psi \big\langle \check{P}' \big\rangle_1 \langle P \rangle_1 \big\rangle + 2\big\langle \Psi \big\langle \check{P}' \big\rangle_r \langle P \rangle_r \big\rangle \\ &\quad + 2\big\langle \Psi \big(\big\langle \check{P}' \big\rangle_1 \cdot \langle P \rangle_r + \langle P \rangle_1 \cdot \big\langle \check{P}' \big\rangle_r\big)\big\rangle. \end{aligned} \tag{2.12}$$

The first two terms involve a single grade and are handled by the homogeneous cases. The last term can only make a contribution when $r = 5$. The coefficient of β^2 will have terms of the form

$$\begin{aligned}\big\langle \Psi \check{P}' \Psi P \big\rangle &= -\big\langle n_\infty \big\langle \check{P}' \big\rangle_1 n_\infty \langle P \rangle_1 \big\rangle - \big\langle n_\infty \big\langle \check{P}' \big\rangle_r n_\infty \langle P \rangle_r \big\rangle \\ &\quad - \big\langle n_\infty \big\langle \check{P}' \big\rangle_1 n_\infty \langle P \rangle_r \big\rangle - \big\langle n_\infty \langle P \rangle_1 n_\infty \big\langle \check{P}' \big\rangle_r \big\rangle. \end{aligned} \tag{2.13}$$

Again the first two terms involve a single grade and are handled by the homogeneous cases. If $v = \langle v \rangle_1$, then $n_\infty v n_\infty = 2(v \cdot n_\infty) n_\infty$ is a scale multiple of n_∞. The last two terms can only make a contribution if $r = 1$, which has already been taken into consideration by the first two terms. Let

$$\mathscr{L}_r X = \sum_{k=1}^n w_k \big(\langle \check{Q}_k \rangle_r X \langle P_k \rangle_r + \widetilde{\langle \check{Q}_k \rangle_r} X \widetilde{\langle P_k \rangle_r} \big)$$

denote the restriction of $\mathscr{L}$ to the grade-r parts of P_k and Q_k. We can summarise the above discussion by stating that for mixed grade objects of the form $P_k = \langle P_k \rangle_1 + \langle P_k \rangle_r$, $Q_k = \langle Q_k \rangle_1 + \langle Q_k \rangle_r$, where $r \neq 5$, we have

$$\langle \widetilde{X} \mathscr{L} X \rangle = \langle \widetilde{X} \mathscr{L}_1 X \rangle + \langle \widetilde{X} \mathscr{L}_r X \rangle. \tag{2.14}$$

Lemma 2.5 to Lemma 2.10, together with the comments of mixed grade cases, tell us for which object representations we can ignore the constraint $\langle \widetilde{X} X \rangle_4 = 0$ during motor estimation. For convenience we, will refer to these objects as *admissible*, and we see immediately that all the objects represented earlier when discussing measures are admissible. We wish to maximise $\langle \widetilde{X} \mathscr{L} X \rangle$ where $X \in \mathscr{M}$ as stated in problem (2.9). For admissible objects, we can neglect the awkward condition $\langle \widetilde{X} X \rangle_4 = 0$ and solve

$$\max_{X \in \mathbb{M}} \langle \widetilde{X} \mathscr{L} X \rangle \text{ subject to } \langle X \widetilde{X} \rangle = 1. \tag{2.15}$$

Thus we can maximise $\langle \widetilde{X} \mathscr{L} X \rangle$ under the more relaxed constraints $X \in \mathbb{M}$ and $\langle X \widetilde{X} \rangle = 1$. This problem can be readily solved, and we can now present the generalisation of Lemma 2.1 and Lemma 2.2 to the case of motors:

Theorem 2.3 *Let P_k and Q_k, $k = 1, \ldots, n$ be two sets of admissible normalised conformal objects in $\mathbb{R}_{4,1}$, $w_k \in \mathbb{R}$ be scalar weights, and $\mathscr{L}$ be defined by*

$$\mathscr{L} X = \frac{1}{2} \sum_{k=1}^{n} w_k (\check{Q}_k X P_k + \tilde{\check{Q}}_k X \widetilde{P}_k).$$

Then the maximiser of $\langle \widetilde{M} \mathscr{L} M \rangle$ subject to $M \in \mathscr{M}$ is given by $M = R + Q$ where R is an eigenrotator of $\mathrm{P}_R \mathscr{L}'$ associated with the largest eigenvalue, $\mathscr{L}' = \mathscr{L} - \mathscr{L}(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^{+} \mathscr{L}$, and $Q = -(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^{+} \mathscr{L} R$.

Proof The Lagrange function associated with problem (2.15) is given by $L(X) = \frac{1}{2} \langle \widetilde{X} \mathscr{L} X \rangle - \frac{\alpha}{2} (\langle \widetilde{X} X \rangle - 1)$ for $X \in \mathbb{M}$. The first-order optimality condition $\partial_{\widetilde{X}} L = 0$ gives $\bar{\mathrm{P}}_{\mathbb{M}} \mathscr{L} X = \alpha \bar{\mathrm{P}}_{\mathbb{M}} X$. Let $X = R + Q \in \mathbb{M}$ where $R \in \mathbb{R}_3^+$ and $Q \in \mathbb{R}_3^- n_\infty$. Using $\bar{\mathrm{P}}_{\mathbb{M}} = \mathrm{P}_R + \bar{\mathrm{P}}_Q$, we can separate $\partial_{\widetilde{X}} L = 0$ into R and Q components as follows:

$$\begin{aligned} \mathrm{P}_R \mathscr{L} R + \mathrm{P}_R \mathscr{L} Q &= \alpha R, \\ \bar{\mathrm{P}}_Q \mathscr{L} R + \bar{\mathrm{P}}_Q \mathscr{L} Q &= 0. \end{aligned}$$

This is a standard form for quadratic minimisation with a homogeneous quadratic constraint, and we can calculate Q from the second equation and then eliminate Q from the first equation in the usual way. This gives $\mathrm{P}_R \mathscr{L}' R = \alpha R$ where $\mathscr{L}' = \mathscr{L} - \mathscr{L}(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^{+} \mathscr{L}$ and $Q = -(\bar{\mathrm{P}}_Q \mathscr{L} \mathrm{P}_Q)^{+} \mathscr{L} R$. At the maximum, α equals

$\langle \widetilde{X}\mathscr{L}X\rangle$; therefore R is the eigenrotator of $\mathrm{P}_R\mathscr{L}'$ associated with the largest eigenvalue. □

We see that, as for the rotator case, the problem reduces to an eigenrotator problem. This motor estimation method is easily implemented by forming the matrix representative of $\bar{\mathrm{P}}_{\mathbb{M}}\mathscr{L}$ as outlined in the following procedure:

1. Form a basis e_k, $k = 1, \ldots, 8$, of $\mathbb{M}$, where the first four basis vectors are associated with R and orthonormal, and the last four are associated with Q in the split $X = R + Q$ (e.g. $\{1, e_{12}, e_{13}, e_{23}, e_1 n_\infty, e_2 n_\infty, e_3 n_\infty, I_3 n_\infty\}$).
2. Form the 8×8 symmetric matrix $L_{ij} = \langle \widetilde{e}_i \bar{\mathrm{P}}_{\mathbb{M}}\mathscr{L} e_j\rangle = \langle \widetilde{e}_i \mathscr{L} e_j\rangle$ and break it into 4×4 sub-matrices $L = \left(\begin{smallmatrix} L_{rr} & L_{rq} \\ L_{qr} & L_{qq}\end{smallmatrix}\right)$.
3. Form the 4×4 matrix $L' = L_{rr} - L_{rq}(L_{qq}^{+} L_{qr})$.
4. Calculate $r = \mathrm{unit}(V V^T z) \in \mathbb{R}^4$, the unit eigenvector of L' associated with the largest eigenvalue and the smallest angle of rotation, where $z \in \mathbb{R}^4$ with $z_k = \langle e_k\rangle$, $k = 1, \ldots, 4$.
5. Calculate $q = -(L_{qq}^{+} L_{qr}) r \in \mathbb{R}^4$.
6. Form the full coefficient vector $m = \binom{r}{q} \in \mathbb{R}^8$.
7. Calculate the optimal motor $M = \sum_k m_k e_k \in \mathscr{M}$.

The key steps of estimating R and estimating Q are both robust in the sense that a reasonable value will be returned even if insufficient information is provided. Let $M = TR$ where $T = 1 - \frac{1}{2}\boldsymbol{t} n_\infty$ is a translator and $\boldsymbol{t}$ is the Euclidean translation vector. Note that $Q = -\frac{1}{2}\boldsymbol{t} R n_\infty$, so we have $|q| = |n_o \cdot Q| = \frac{1}{2}|\boldsymbol{t}|$. The use of the Moore–Penrose pseudo-inverse will ensure that the smallest translation $\boldsymbol{t}$ is returned when there is not a unique maximiser as discussed after the procedure for estimating translators. The estimated motor will maximise the measure and provide the motor with the smallest translation and rotation angle when there is not a unique maximiser.

2.5 Examples

In this section we provide some illustrations of the algorithm. The data is generated as follows. A random geometric object P_k is generated, such as a point, sphere, line, circle, or tangent. Noise is added to the data P_k by perturbing it with a small random motor $M_k \approx 1$ before applying the general fixed rigid body transformation M_o to give $Q_k = M_o M_k P_k \widetilde{M}_k \widetilde{M}_o$. The noise is sufficient to provide clear delineation between the objects in the figures presented. The motor estimation procedure is then applied to the data pairs (P_k, Q_k), $k = 1, \ldots, K$, to obtain an optimal estimate M of M_o. In the figures presented the dark data is the source data after the action of the estimated motor $Q'_k = M P_k \widetilde{M}$, and the light data is the target data Q_k. The difference between the sets is the error remaining after applying the estimation procedure and is due to the noise on the data. If no noise is present, the fit is perfect, and the data sits exactly on top of each other.

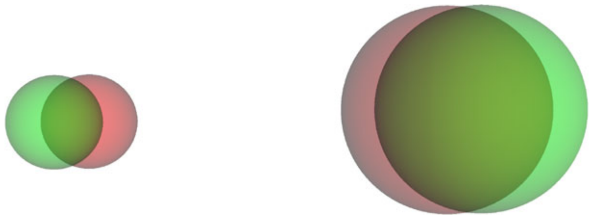

Fig. 2.5 Two pairs of spheres used to estimate the rigid body motion. The centres all lie on a line

First consider the problem of fitting spheres. As discussed earlier, the radius of the spheres plays no role, as it is invariant to rigid body transformations, and the situation is identical to the case of noisy points. With just one pair of spheres, the fit is perfect, and the centres of the spheres coincide. The rotational part vanishes because the smallest angle of rotation is zero and the estimated motor is a pure translator. With two pairs of spheres, the optimal motor makes their centres lie on the same axis with equal separation as shown in Fig. 2.5. This example is a typical situation where there is insufficient information to get a unique maximiser of $\langle \widetilde{M}\mathscr{L}M\rangle$. For the estimated motor, the rotation about the axis is zero, and the rotation is in a plane parallel with the axis through the points before and after motion. A more general situation is shown in Fig. 2.6, where there are five pairs of noisy spheres.

The more complex example in Fig. 2.7 shows the algorithm being applied to five pairs of different objects. We use spheres, lines, circles, and 1D and 2D tangents. We have not included planes simply because, unlike lines, it is hard to visualise the separation between planes in a figure.

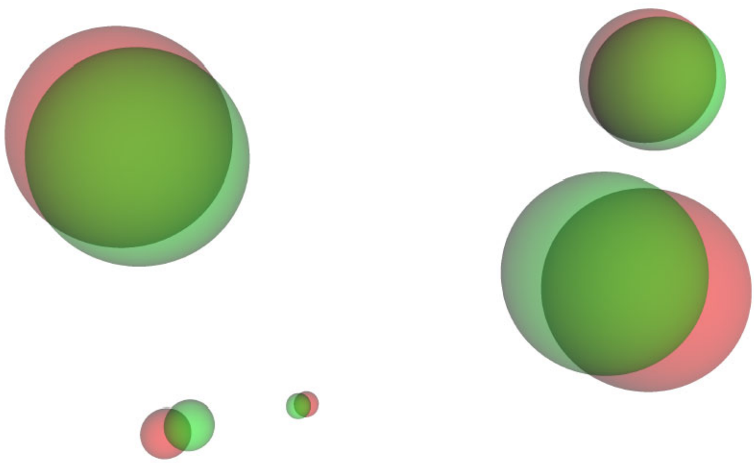

Fig. 2.6 Five pairs of spheres used to estimate the rigid body motion

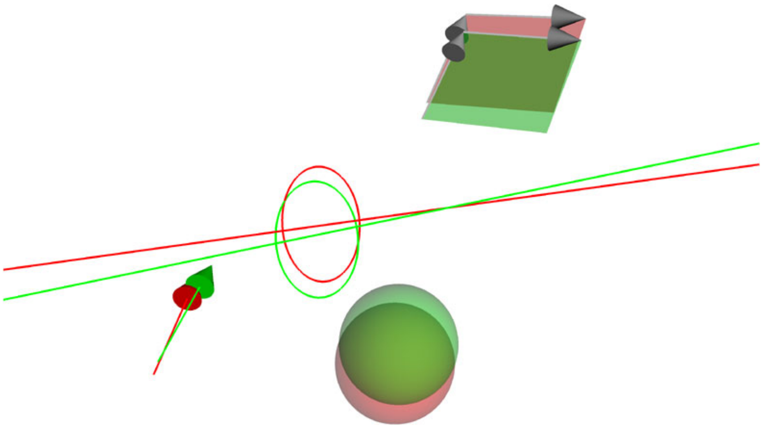

Fig. 2.7 Five pairs of different objects (spheres, lines, circles, 1D and 2D tangents) used to estimate the rigid body motion

2.6 Discussion

We have presented a technique for estimating motors from noisy geometric data. The data may comprise a variety of objects including points, rounds (point pairs, circles, spheres), flats (lines, planes), tangents, and directions. To assist the developments, we first studied the geometry of the motors in the smallest linear embedding space $\mathbb{M}$. The estimation technique reduced to a small eigenrotator problem and allowed the different types of geometric data to be combined naturally in a single framework while excluding reflection. In order to formulate the problem, we restricted the similarity measure between geometric objects P and Q to the simple form $\langle P\check{Q}\rangle$ (with the aid of a grade dependent sign operator). In addition we restricted the representation of objects to what we referred to as the admissible objects. These are representations that allow us to ignore the motor constraint $\langle M\widetilde{M}\rangle_4 = 0$ during optimisation. With these restrictions, we are able to associate a physically meaningful measure to the primitive objects: points, spheres, lines, and planes. Other objects such as circles, point pairs, and tangents were incorporated by representing them as flags using sums of primitive objects. Directions were incorporated by representing them as associated flats. The estimation procedure reduced to a standard constrained optimisation problem with a closed-form solution, which could be expressed as an eigenrotator problem.

2.7 Exercises

2.1 Find a representation for spheres so that if P and Q are two spheres we get the following measure $\langle P\check{Q}\rangle = -\frac{1}{2}d^2 - \frac{1}{2}(\rho_p - \rho_q)^2$ where d is the distance between the centres, and $\rho_p - \rho_q$ the difference in radii.

2.2 Show that for $X, Y \in \mathbb{M}$ we have $|XY| = |X||Y|$.

2.3 Consider an object of the form $F = p + \Lambda + \Pi$, where p is a point, Λ a line through p, and Π a plane through p. Show that if $P = p + \Lambda_p + \Pi_p$ and $Q = q + \Lambda_q + \Pi_q$ have this form then $\langle P\check{Q}\rangle = -\frac{1}{2}d^2 + \cos(\theta) + \cos(\phi)$ where d is the distance between the points, θ is the dihedral angle between the lines, and ϕ the dihedral angle between the planes. Are objects of this form admissible? What if Λ and Π are perpendicular?

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References

1. Dorst, L., Valkenburg, R.: Square root and logarithm of rotors in 3D conformal geometric algebra using polar decomposition. In: Dorst, L., Lasenby, J. (eds.) Guide to Geometric Algebra in Practice. Springer, London (2011), Chap. 5 in this book
2. Dorst, L., Fontijne, D., Mann, S.: Geometric Algebra for Computer Science: An Object-Oriented Approach to Geometry. Morgan Kaufman, San Mateo (2007/2009)
3. Kanatani, K.: Geometric Computation for Machine Vision. Oxford University Press, Oxford (1993)
4. Valkenburg, R.J., Kakarala, R.: Lower bounds for the divergence of orientational estimators. IEEE Trans. Inf. Theory **47**(6) (2001)

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