Chapter 2
Splitting of the Main Problem into Four Sub-cases

2.1 Introduction

Each of Chaps. 2–5 of this book is self-contained. Therefore the definitions in the different parts of the book may sometimes appear with different signs.

The concept of Table Algebra was introduced by Z. Arad and H. Blau in [AB] in order to study in a uniform way properties of products of conjugacy classes and of irreducible characters of a finite group.

Definition 2.1 A Table Algebra (A, B) is a finite dimensional, commutative algebra A with identity element 1 over the complex numbers C, and a distinguished base $B = \{b_1 = 1, b_2, \dots, b_k\}$ such that the following properties hold (where (b_i, a) denotes the coefficient of b_i in $a \in A$, a written as a linear combination of B; and where R^* denotes $R^+ \cup \{0\}$, the set of non-negative real numbers):

(I) For all i, j, m, $b_i b_j = \sum_m \delta_{ijm} b_m$ with δ_{ijm} a nonnegative real number.

(II) There is an algebra automorphism (denoted by $^-$) of A, whose order divides 2, such that $b_i \in B$ implies that $\bar{b}_i \in B$. (Then $\bar{i}$ is defined by $\bar{b}_i = b_{\bar{i}}$.)

(III) Hypothesis (II) holds and there is a function $g : B \times B \to R^+$ (the positive reals) such that

$$(b_m, b_i b_j) = g(b_i, b_m) \cdot (b_i, \bar{b}_j b_m),$$

where $g(b_i, b_m)$ is independent of j for all i, j, m.

B is called the table basis of (A, B). Clearly $1 \in B$, we always use b_1 to denote base element 1, and $B^\sharp$ to denote $B \setminus \{b_1\}$. The elements of B are called the irreducible components of (A, B), and nonzero nonnegative linear combinations of elements of B with coefficients in R^+ are called components. If $a = \sum_{m=1}^{k} \lambda_m b_m$ is a component ($\lambda_m \in R^*$) then $Supp\{a\} = \{b_m | \lambda_m \neq 0\}$ is called the set of irreducible constituents of a. An element $a \in A$ is called a real element if $a = \bar{a}$.

Two Table Algebras (A, B) and (A', B') are called isomorphic (denoted $B \cong B'$) when there exists an algebra isomorphism $\psi : A \to A'$ such that $\psi(B)$ is a rescaling of B'; and the algebras are called exactly isomorphic (denoted $B \cong_x B'$) when $\psi(B) = B'$. So $B \cong_x B'$ means that B and B' yield the same structure constants.

Z. Arad et al., *On Normalized Integral Table Algebras (Fusion Rings)*,
Algebra and Applications 16, DOI 10.1007/978-0-85729-850-8_2,

Proposition 2.2 of [AB] shows that if (A, B) is a Table Algebra, then there exists a basis B', which consists of suitable positive real scalar multiples b'_i of the elements b_i of B, $g(b'_i, b'_j) = 1$ for any $b'_i, b'_j \in B'$. Such a basis B' is called a normalized basis. Now $Supp(b'_{i_1} b'_{i_2} \cdots b'_{i_t})$ consists of the corresponding scalar multiples of $Supp(b_{i_1} b_{i_2} \cdots b_{i_t})$, for any sequence $i_1, i_2, \ldots, i_t$, of indices. So in the proof of any proposition which identifies the irreducible constituents of a product of basis elements, we may assume that B is normalized.

Suppose that B is normalized. It follows from (III) and [AB, Sect. 2] that A has a positive definite Hermitian form, with B as an orthonormal basis, and such that

$$(a, bc) = (a\bar{b}, c)$$

for all $a, b, c \in \mathcal{R}B$.

It follows from Sect. 2 in [AB] that there exists an algebra homomorphism f from A to C such that $f(B) \subseteq R^+$. For an element $b \in B$, $f(b)$ is called the degree of b.

Each finite group G yields two natural Table Algebras: the Table Algebra of conjugacy classes (denoted by $(ZC(G), Cl(G))$) and the Table Algebra of generalized characters (denoted by $(Ch(G), Irr(G))$).

Both Table Algebras arising from group theory have an additional property: their structure constants and degrees are non-negative integers. Such algebras were defined in [B1] as Integral Table Algebras (denoted ITA).

Each of the elements of a Table Algebra is contained in a unique table subalgebra which may be considered as a table subalgebra generated by this element. So it is natural to start the study of Integral Table Algebras from those which are generated by a single element. Normalized Integral Table Algebras (denoted NITA) generated by an element of degree 2 were completely classified by H. Blau in [B].

The finite linear groups in dimension $n \leq 7$ have been completely classified. See Feit [F]. Representation theory and properties of finite groups are heavily used for the proofs. For $n = 2, 3, 4$ see Blichfeldt [Bl]. For $n = 5$ see Brauer [Br]. For $n = 6$ see Lindsey [L]. For $n = 7$ see Wales [W]. In order to generalize these results to Normalized Integral Table Algebras, the authors began to classify Normalized Integral Table Algebras (A, B) generated by a faithful nonreal element of degree 3 and without nonidentity irreducible elements of degree 1 or 2 in [CA], and arrived at the following theorem:

Theorem 2.1 *Let (A, B) be NITA generated by a nonreal element $b_3 \in B$ of degree 3 and without non-identity basis element of degree 1 or 2. Then $b_3\bar{b}_3 = 1 + b_8$ and one of the following holds*:

(1) $(A, B) \cong_x (Ch(PSL(2, 7)), Irr(PSL(2, 7)))$;
(2) $b_3^2 = b_4 + b_5$, *where* $b_4 \in B$ *and* $b_5 \in B$;
(3) $b_3^2 = \bar{b}_3 + b_6$, *where* $b_6 \in B$ *is a nonreal element*;
(4) $b_3^2 = c_3 + b_6$, *where* $c_3, b_6 \in B$ *and* $c_3 \neq b_3, \bar{b}_3$.

The purpose of this chapter is to investigate NITA generated by a nonreal element b_3 that satisfies one of the conditions (2), (3) and (4) in Theorem 2.1. The following theorem is proved:

Main Theorem 1 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ of degree 3 and without non-identity basis element of degree 1 or 2. Then $b_3\bar{b}_3 = 1 + b_8$, $b_8 \in B$ and one of the following holds*:

(1) *There exists a real element $b_6 \in B$ such that $b_3^2 = \bar{b}_3 + b_6$ and $(A, B) \cong_x (Ch(PSL(2,7)), Irr(PSL(2,7)))$.*
(2) *There exists $b_4, b_5 \in B$ such that $b_3^2 = b_4 + b_5$, $(b_3b_8, b_3b_8) = 3$ and $(b_4^2, b_4^2) = 3$.*
(3) *There exists $b_6, b_{10}, b_{15} \in B$, where b_6 is nonreal such that*

$$b_3^2 = \bar{b}_3 + b_6, \qquad \bar{b}_3 b_6 = b_3 + b_{15}, \qquad b_3 b_6 = b_8 + b_{10}$$

and $(b_3b_8, b_3b_8) = 3$. Moreover if b_{10} is real, then $(A, B) \cong_x (CH(3 \cdot A_6), Irr(3 \cdot A_6))$.
(4) *There exists $c_3, b_6 \in B$, $c_3 \neq b_3, \bar{b}_3$, such that $b_3^2 = c_3 + b_6$ and either $(b_3b_8, b_3b_8) = 3$ or 4.*

When $(b_3b_8, b_3b_8) = 3$. If c_3 is nonreal, then (A, B) is exactly isomorphic to $(A(3 \cdot A_6 \cdot 2), B_{32})$ (see Theorem 2.9). If c_3 is real, then (A, B) is exactly isomorphic to the NITA $(A(7 \cdot 5 \cdot 10), B_{22})$ of dimension 22 (see Theorem 2.10).

Proof The theorem follows from Theorem 2.1 and Theorems 2.2, 2.7, 2.9 and 2.10 below. □

2.2 Two NITA Generated by a Non-real Element of Degree 3 not Derived from a Group and Lemmas

Here we give two examples of NITA not induced from group theory, which has either 32 or 22 basis elements. The NITA of dimension 32 contains a subalgebra which is strictly isomorphic to the algebra of dimension 17 of characters of the group $3 \cdot A_6$, we denote it as $(A(3 \cdot A_6 \cdot 2), B_{32})$, where its basis is B_{32}.

For B_{32} there are three table subsets: $\{1\} \subseteq C \subseteq D \subseteq B$, where

$$C = \{1, b_8, x_{10}, b_5, c_5, c_8, x_9\},$$
$$D = C \cup \{c_3, \bar{c}_3, d_3, \bar{d}_3, c_9, \bar{c}_9, b_6, \bar{b}_6, y_5, \bar{y}_{15}\},$$
$$B_{32} = D \cup \{b_3, \bar{b}_3, r_3, x_6, \bar{x}_6, s_6, b_9, \bar{b}_9, x_{15}, \bar{x}_{15}, t_{15}, d_9, y_3, z_3, \bar{z}_3\}.$$

From the equations below one can check that B_{32} has the following table subsets: $\{1\} \subseteq C \subseteq E = C \cup \{r_3, s_6, t_{15}, d_9, y_3\} \subseteq B_{32}$. The table subsets E and D are two maximal table subsets of B_{32}.

The structure algebra constants of C:

$$\begin{aligned}
b_5^2 &= 1 + x_9 + x_{10} + b_5,\\
b_5c_5 &= c_8 + b_8 + x_9,\\
b_5c_8 &= c_5 + c_8 + b_8 + x_9 + x_{10},\\
b_5b_8 &= c_5 + c_8 + b_8 + x_9 + x_{10},\\
b_5x_9 &= c_8 + c_5 + b_8 + b_5 + x_9 + x_{10},\\
b_5x_{10} &= c_8 + b_8 + x_9 + b_5 + 2x_{10},
\end{aligned}$$

$$\begin{aligned}
c_5^2 &= 1 + x_9 + x_{10} + c_5,\\
c_5c_8 &= b_5 + c_8 + b_8 + x_9 + x_{10},\\
c_5b_8 &= b_5 + c_8 + b_8 + x_9 + x_{10},\\
c_5x_9 &= b_8 + c_8 + b_5 + c_5 + x_9 + x_{10},\\
c_5x_{10} &= b_8 + c_8 + x_9 + c_5 + 2x_{10},
\end{aligned}$$

$$\begin{aligned}
b_8^2 &= 1 + b_5 + c_5 + c_8 + 2b_8 + x_9 + 2x_{10},\\
b_8x_9 &= b_5 + c_5 + 2c_8 + b_8 + 2x_9 + 2x_{10},\\
b_8x_{10} &= b_5 + c_5 + 2c_8 + 2b_8 + 2x_9 + 2x_{10},
\end{aligned}$$

$$\begin{aligned}
c_8^2 &= 1 + b_5 + c_5 + 2c_8 + b_8 + x_9 + 2x_{10},\\
c_8b_8 &= b_5 + c_5 + c_8 + b_8 + 2x_9 + 2x_{10},\\
c_8x_9 &= b_5 + c_5 + c_8 + 2b_8 + 2x_9 + 2x_{10},\\
c_8x_{10} &= b_5 + c_5 + 2c_8 + 2b_8 + 2x_9 + 2x_{10},
\end{aligned}$$

$$\begin{aligned}
x_9^2 &= 1 + b_5 + c_5 + 2c_8 + 2b_8 + 2x_9 + 2x_{10},\\
x_9x_{10} &= b_5 + c_5 + 2c_8 + 2b_8 + 2x_9 + 3x_{10},\\
x_{10}^2 &= 1 + 2b_5 + 2c_5 + 2c_8 + 2b_8 + 3x_9 + 2x_{10}.
\end{aligned}$$

The structure algebra constants of D (since $C \subseteq D$ we will not repeat the equations of C as shown above):

$$\begin{aligned}
c_3\bar{c}_3 &= 1 + b_8,\\
c_3^2 &= \bar{c}_3 + \bar{b}_6,\\
c_3d_3 &= x_9,\\
c_3\bar{d}_3 &= \bar{c}_9,\\
c_3b_6 &= \bar{c}_3 + \bar{y}_{15},\\
c_3\bar{b}_6 &= b_8 + x_{10},\\
c_3c_9 &= d_3 + \bar{c}_9 + \bar{y}_{15},\\
c_3\bar{c}_9 &= c_8 + x_9 + x_{10},\\
c_3y_{15} &= \bar{b}_6 + 2\bar{y}_{15} + \bar{c}_9,\\
c_3\bar{y}_{15} &= b_5 + c_5 + c_8 + b_8 + x_9 + x_{10},\\
c_3b_8 &= c_3 + b_6 + y_{15},\\
c_3x_{10} &= b_6 + c_9 + y_{15},\\
c_3b_5 &= y_{15},\\
c_3c_5 &= y_{15},\\
c_3c_8 &= y_{15} + c_9,\\
c_3x_9 &= y_{15} + c_9 + \bar{d}_3,
\end{aligned}$$

$$
\begin{aligned}
d_3\bar{d}_3 &= 1 + c_8,\\
d_3^2 &= \bar{d}_3 + b_6,\\
d_3b_6 &= c_8 + x_{10},\\
d_3\bar{b}_6 &= \bar{d}_3 + y_{15},\\
d_3c_9 &= b_8 + x_9 + x_{10},\\
d_3\bar{c}_9 &= c_3 + y_{15} + c_9,\\
d_3y_{15} &= b_5 + c_5 + c_8 + b_8 + x_9 + x_{10},\\
d_3\bar{y}_{15} &= 2y_{15} + b_6 + c_9,\\
d_3b_8 &= \bar{c}_9 + \bar{y}_{15},\\
d_3x_{10} &= \bar{y}_{15} + \bar{b}_6 + \bar{c}_9,\\
d_3b_5 &= \bar{y}_{15},\\
d_3c_5 &= \bar{y}_{15},\\
d_3c_8 &= \bar{y}_{15} + b_6 + d_3,\\
d_3x_9 &= \bar{y}_{15} + \bar{c}_3 + \bar{c}_9,
\end{aligned}
$$

$$
\begin{aligned}
b_6\bar{b}_6 &= 1 + b_5 + c_5 + c_8 + b_8 + x_9,\\
b_6^2 &= 2\bar{b}_6 + \bar{y}_{15} + \bar{c}_9,\\
b_6c_9 &= 2\bar{y}_{15} + 2\bar{c}_9 + \bar{b}_6,\\
b_6\bar{c}_9 &= b_5 + c_5 + c_8 + b_8 + 2x_9 + x_{10},\\
b_6\bar{y}_{15} &= b_5 + c_5 + 2c_8 + 2b_8 + 2x_9 + 3x_{10},\\
b_6y_{15} &= 4\bar{y}_{15} + \bar{b}_6 + \bar{c}_3 + d_3 + 2\bar{c}_9,\\
b_6b_5 &= b_6 + y_{15} + c_9,\\
b_6c_5 &= y_{15} + b_6 + c_9,\\
b_6c_8 &= \bar{d}_3 + b_6 + 2y_{15} + c_9,\\
b_6b_8 &= c_3 + b_6 + c_9 + 2y_{15},\\
b_6x_9 &= 2y_{15} + b_6 + 2c_9,\\
b_6x_{10} &= 3y_{15} + c_3 + \bar{d}_3 + c_9,
\end{aligned}
$$

$$
\begin{aligned}
y_{15}\bar{y}_{15} &= 1 + 3b_5 + 3c_5 + 5c_8 + 5b_8 + 6x_9 + 6x_{10},\\
y_{15}^2 &= 9\bar{y}_{15} + 4\bar{b}_6 + 2\bar{c}_3 + 2d_3 + 6\bar{c}_9,\\
y_{15}b_5 &= 3y_{15} + b_6 + c_3 + \bar{d}_3 + 2c_9,\\
y_{15}c_5 &= 3y_{15} + b_6 + c_3 + \bar{d}_3 + 2c_9,\\
y_{15}c_8 &= 5y_{15} + c_3 + \bar{d}_3 + 2b_6 + 3c_9,\\
y_{15}b_8 &= 5y_{15} + c_3 + \bar{d}_3 + 2b_6 + 3c_9,\\
y_{15}x_9 &= 6y_{15} + 2b_6 + \bar{d}_3 + c_3 + 3c_9,\\
y_{15}x_{10} &= 6y_{15} + 3b_6 + c_3 + \bar{d}_3 + 4c_9,
\end{aligned}
$$

$$
\begin{aligned}
c_9\bar{c}_9 &= 1 + 2c_8 + 2b_8 + 2x_9 + 2x_{10} + c_5 + b_5,\\
c_9^2 &= 3\bar{y}_{15} + d_3 + \bar{c}_3 + 2\bar{b}_6 + 2\bar{c}_9,\\
c_9 y_{15} &= 6\bar{y}_{15} + 2\bar{b}_6 + \bar{c}_3 + d_3 + 3\bar{c}_9,\\
c_9\bar{y}_{15} &= 2b_5 + 2c_5 + 3c_8 + 3b_8 + 3x_9 + 4x_{10},\\
c_9 b_5 &= 2y_{15} + b_6 + c_9,\\
c_9 c_5 &= 2y_{15} + b_6 + c_9,\\
c_9 b_8 &= 3y_{15} + \bar{d}_3 + b_6 + 2c_9,\\
c_9 c_8 &= 3y_{15} + b_6 + c_3 + 2c_9,\\
c_9 x_9 &= c_3 + 2c_9 + 3y_{15} + \bar{d}_3 + 2b_6,\\
c_9 x_{10} &= c_3 + 2c_9 + 4y_{15} + \bar{d}_3 + b_6.
\end{aligned}
$$

From the equations one can see that C and D are table subsets of B_{32}. Furthermore, the NITA subalgebra $(\langle D\rangle, D)$ of dimension 17 is strictly isomorphic to $(CH(3\cdot A_6), Irr(3\cdot A_6))$ while our NITA of dimension 32 is not induced from a finite group G.

The following equations describe other products of basis elements in B_{32}:

$$
\begin{aligned}
b_3\bar{b}_3 &= 1 + b_8,\\
b_3^2 &= c_3 + b_6,\\
b_3 r_3 &= \bar{c}_3 + \bar{b}_6,\\
b_3 x_6 &= c_3 + y_{15},\\
b_3\bar{x}_6 &= b_8 + x_{10},\\
b_3 s_6 &= \bar{c}_3 + \bar{y}_{15},\\
b_3 b_9 &= y_{15} + c_9 + \bar{d}_3,\\
b_3\bar{b}_9 &= x_9 + c_8 + x_{10},\\
b_3 x_{15} &= b_6 + 2y_{15} + c_9,\\
b_3\bar{x}_{15} &= b_5 + c_5 + b_8 + c_8 + x_9 + x_{10},\\
b_3 t_{15} &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15},\\
b_3 d_9 &= \bar{c}_9 + \bar{y}_{15} + d_3,\\
b_3 y_3 &= \bar{c}_9,\\
b_3 z_3 &= x_9,\\
b_3\bar{z}_3 &= c_9,\\
b_3 b_8 &= b_3 + x_6 + x_{15},\\
b_3 x_{10} &= x_6 + x_{15} + b_9,\\
b_3 b_5 &= x_{15},\\
b_3 c_5 &= x_{15},\\
b_3 c_8 &= x_{15} + b_9,\\
b_3 x_9 &= \bar{z}_3 + b_9 + x_{15},\\
b_3 c_3 &= r_3 + s_6,\\
b_3\bar{c}_3 &= \bar{b}_3 + \bar{x}_6,\\
b_3 d_3 &= \bar{b}_9,\\
b_3\bar{d}_3 &= d_9,\\
b_3 c_9 &= t_{15} + d_9 + y_3,\\
b_3\bar{c}_9 &= \bar{b}_9 + \bar{x}_{15} + z_3,\\
b_3 b_6 &= r_3 + t_{15},\\
b_3\bar{b}_6 &= \bar{b}_3 + \bar{x}_{15},\\
b_3 y_{15} &= s_6 + 2t_{15} + d_9,\\
b_3\bar{y}_{15} &= \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9,
\end{aligned}
$$

$$
\begin{aligned}
r_3^2 &= 1 + b_8,\\
r_3 x_6 &= \bar{c}_3 + \bar{y}_{15},\\
r_3 s_6 &= b_8 + x_{10},\\
r_3 b_9 &= d_3 + \bar{c}_9 + \bar{y}_{15},\\
r_3 x_{15} &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15},\\
r_3 t_{15} &= b_5 + c_5 + b_8 + c_8 + x_9 + x_{10},\\
r_3 d_9 &= c_8 + x_9 + x_{10},\\
r_3 y_3 &= x_9,\\
r_3 z_3 &= c_9,\\
r_3 b_8 &= s_6 + r_3 + t_{15},\\
r_3 x_{10} &= s_6 + t_{15} + d_9,\\
r_3 b_5 &= t_{15},\\
r_3 c_5 &= t_{15},\\
r_3 c_8 &= t_{15} + d_9,\\
r_3 x_9 &= t_{15} + d_9 + y_3,\\
r_3 c_3 &= \bar{b}_3 + \bar{x}_6,\\
r_3 d_3 &= b_9,\\
r_3 c_9 &= \bar{b}_9 + \bar{x}_{15} + z_3,\\
r_3 b_6 &= \bar{x}_{15} + \bar{b}_3,\\
r_3 y_{15} &= \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9,
\end{aligned}
$$

$$
\begin{aligned}
x_6^2 &= 2b_6 + y_{15} + c_9,\\
x_6 \bar{x}_6 &= 1 + b_8 + b_5 + c_5 + c_8 + x_9,\\
x_6 s_6 &= 2\bar{b}_6 + \bar{c}_9 + \bar{y}_{15},\\
x_6 b_9 &= \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15},\\
x_6 \bar{b}_9 &= b_5 + c_5 + x_{10} + b_8 + c_8 + 2x_9,\\
x_6 x_{15} &= 4y_{15} + b_6 + 2c_9 + d_3 + c_3,\\
x_6 \bar{x}_{15} &= b_5 + c_5 + 2b_8 + 3x_{10} + 2c_8 + 2x_9,\\
x_6 t_{15} &= \bar{c}_3 + d_3 + 2\bar{c}_9 + 4\bar{y}_{15} + \bar{b}_6,\\
x_6 d_9 &= \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15},\\
x_6 y_3 &= d_3 + \bar{y}_{15},\\
x_6 z_3 &= x_{10} + c_8,\\
x_6 \bar{z}_3 &= \bar{d}_3 + y_{15},\\
x_6 b_8 &= b_3 + x_6 + 2x_{15} + b_9,\\
x_6 x_{10} &= b_9 + b_3 + \bar{z}_3 + 3x_{15},\\
x_6 b_5 &= x_6 + b_9 + x_{15},\\
x_6 c_5 &= x_6 + b_9 + x_{15},\\
x_6 c_8 &= x_6 + b_9 + 2x_{15} + \bar{z}_3,\\
x_6 x_9 &= 2b_9 + x_6 + 2x_{15},\\
x_6 c_3 &= r_3 + t_{15},\\
x_6 \bar{c}_3 &= \bar{b}_3 + \bar{x}_{15},\\
x_6 d_3 &= z_3 + \bar{x}_{15},\\
x_6 \bar{d}_3 &= y_3 + t_{15},\\
x_6 c_9 &= 2t_{15} + 2d_9 + s_6,\\
x_6 \bar{c}_9 &= 2\bar{x}_{15} + \bar{x}_6 + 2\bar{b}_9,\\
x_6 b_6 &= 2s_6 + t_{15} + d_9,\\
x_6 \bar{b}_6 &= 2\bar{x}_6 + \bar{b}_9 + \bar{x}_{15},\\
x_6 y_{15} &= r_3 + 4t_{15} + s_6 + 2d_9 + y_3,\\
x_6 \bar{y}_{15} &= \bar{b}_3 + z_3 + \bar{x}_6 + 4\bar{x}_{15} + 2\bar{b}_9,
\end{aligned}
$$

$$\begin{aligned}
s_6^2 &= 1 + b_5 + c_5 + b_8 + c_8 + x_9,\\
s_6 b_9 &= \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15},\\
s_6 x_{15} &= \bar{c}_3 + d_3 + \bar{b}_6 + 2\bar{c}_9 + 4\bar{y}_{15},\\
s_6 t_{15} &= b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 3x_{10},\\
s_6 d_9 &= b_5 + c_5 + 2x_9 + b_8 + c_8 + x_{10},\\
s_6 y_3 &= c_8 + x_{10},\\
s_6 z_3 &= \bar{d}_3 + y_{15},\\
s_6 b_8 &= r_3 + s_6 + 2t_{15} + d_9,\\
s_6 x_{10} &= r_3 + y_3 + d_9 + 3t_{15},\\
s_6 b_5 &= s_6 + d_9 + t_{15},\\
s_6 c_5 &= s_6 + t_{15} + d_9,\\
s_6 c_8 &= y_3 + s_6 + d_9 + 2t_{15},\\
s_6 x_9 &= s_6 + 2d_9 + 2t_{15},\\
s_6 c_3 &= \bar{b}_3 + \bar{x}_{15},\\
s_6 d_3 &= \bar{z}_3 + x_{15},\\
s_6 c_9 &= \bar{x}_6 + 2\bar{b}_9 + 2\bar{x}_{15},\\
s_6 b_6 &= 2\bar{x}_6 + \bar{b}_9 + \bar{x}_{15},\\
s_6 y_{15} &= \bar{b}_3 + z_3 + 4\bar{x}_{15} + \bar{x}_6 + 2\bar{b}_9,
\end{aligned}$$

$$\begin{aligned}
b_9 \bar{b}_9 &= 1 + b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 2x_{10},\\
b_9^2 &= \bar{d}_3 + c_3 + 2b_6 + 2c_9 + 3y_{15},\\
b_9 x_{15} &= 6y_{15} + 3c_9 + \bar{d}_3 + 2b_6 + c_3,\\
b_9 \bar{x}_{15} &= 2b_5 + 2c_5 + 3c_8 + 3b_8 + 4x_{10} + 3x_9,\\
b_9 t_{15} &= d_3 + \bar{c}_3 + 6\bar{y}_{15} + 3\bar{c}_9 + 2\bar{b}_6,\\
b_9 d_9 &= \bar{c}_3 + d_3 + 2\bar{b}_6 + 2\bar{c}_9 + 3\bar{y}_{15},\\
b_9 y_3 &= \bar{c}_3 + \bar{y}_{15} + \bar{c}_9,\\
b_9 z_3 &= b_8 + x_9 + x_{10},\\
b_9 \bar{z}_3 &= c_9 + c_3 + y_{15},\\
b_9 b_8 &= \bar{z}_3 + x_6 + 2b_9 + 3x_{15},\\
b_9 x_{10} &= \bar{z}_3 + b_3 + 2b_9 + x_6 + 4x_{15},\\
b_9 b_5 &= x_6 + b_9 + 2x_{15},\\
b_9 c_5 &= b_9 + 2x_{15} + x_6,\\
b_9 c_8 &= r_3 + t_{15} + 2b_9 + x_6 + 2x_{15},
\end{aligned}$$

$$\begin{aligned}
b_9 x_9 &= 2b_9 + 3x_{15} + \bar{z}_3 + b_3 + 2x_6,\\
b_9 c_3 &= t_{15} + d_9 + \bar{y}_3,\\
b_9 \bar{c}_3 &= z_3 + \bar{b}_9 + \bar{x}_{15},\\
b_9 d_3 &= \bar{b}_3 + \bar{b}_9 + \bar{x}_{15},\\
b_9 \bar{d}_3 &= r_3 + t_{15} + d_9,\\
b_9 c_9 &= r_3 + y_3 + 2s_6 + 2d_9 + 3t_{15},\\
b_9 \bar{c}_9 &= \bar{b}_3 + z_3 + 2\bar{x}_6 + 3\bar{x}_{15} + 2\bar{b}_9,\\
b_9 b_6 &= s_6 + 2d_9 + 2t_{15},\\
b_9 \bar{b}_6 &= 2\bar{b}_9 + 2\bar{x}_{15} + \bar{x}_6,\\
b_9 \bar{y}_{15} &= 6\bar{x}_{15} + \bar{b}_3 + z_3 + 2\bar{x}_6 + 3\bar{b}_9,\\
b_9 y_{15} &= r_3 + y_3 + 2s_6 + 6t_{15} + 3d_9,
\end{aligned}$$

$$
\begin{aligned}
x_{15}^2 &= 2c_3 + 2\bar{d}_3 + 4b_6 + 6c_9 + 9y_{15},\\
x_{15}\bar{x}_{15} &= 1 + 6x_9 + 3b_5 + 3c_5 + 5b_8 + 5c_8 + 6x_{10},\\
x_{15}t_{15} &= 4\bar{b}_6 + 6\bar{c}_9 + 9\bar{y}_{15} + 2\bar{c}_3 + 2d_3,\\
x_{15}d_9 &= \bar{c}_3 + d_3 + 2\bar{b}_6 + 3\bar{c}_9 + 6\bar{y}_{15},\\
x_{15}y_3 &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15},\\
x_{15}\bar{z}_3 &= b_6 + c_9 + 2y_{15},\\
x_{15}z_3 &= b_5 + c_5 + b_8 + c_8 + x_9 + x_{10},\\
x_{15}b_8 &= b_3 + \bar{z}_3 + 2x_6 + 5x_{15} + 3b_9,\\
x_{15}x_{10} &= b_3 + \bar{z}_3 + 3x_6 + 4b_9 + 6x_{15},\\
x_{15}b_5 &= \bar{z}_3 + b_3 + x_6 + 2b_9 + 3x_{15},\\
x_{15}c_5 &= b_3 + \bar{z}_3 + x_6 + 2b_9 + 3x_{15},\\
x_{15}c_8 &= b_3 + \bar{z}_3 + 2x_6 + 3b_9 + 5x_{15},\\
x_{15}x_9 &= b_3 + \bar{z}_3 + 2x_6 + 3b_9 + 6x_{15},\\
x_{15}c_3 &= 2t_{15} + s_6 + d_9,\\
x_{15}\bar{c}_3 &= 2\bar{x}_{15} + \bar{x}_6 + \bar{b}_9,\\
x_{15}d_3 &= \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9,\\
x_{15}\bar{d}_3 &= s_6 + 2t_{15} + d_9,\\
x_{15}c_9 &= r_3 + y_3 + 2s_6 + 6t_{15} + 3d_9,\\
x_{15}\bar{c}_9 &= \bar{b}_3 + z_3 + 3\bar{b}_9 + 2\bar{x}_6 + 6\bar{x}_{15},\\
x_{15}b_6 &= r_3 + s_6 + y_3 + 2d_9 + 4t_{15},\\
x_{15}\bar{b}_6 &= \bar{b}_3 + z_3 + 2\bar{b}_9 + 4\bar{x}_{15} + \bar{x}_6,\\
x_{15}y_{15} &= 2y_3 + 2r_3 + 4s_6 + 6d_9 + 9t_{15},\\
x_{15}\bar{y}_{15} &= 2z_3 + 9\bar{x}_{15} + 2\bar{b}_3 + 4\bar{x}_6 + 6\bar{b}_9,
\end{aligned}
$$

$$
\begin{aligned}
t_{15}^2 &= 1 + 5b_8 + 5c_8 + 3b_5 + 3c_5 + 6x_{10} + 6x_9,\\
t_{15}d_9 &= 2b_5 + 2c_5 + 3b_8 + 3c_8 + 3x_9 + 4x_{10},\\
t_{15}y_3 &= b_5 + c_5 + b_8 + c_8 + x_9 + x_{10},\\
t_{15}z_3 &= b_6 + c_9 + 2y_{15},\\
t_{15}b_8 &= r_3 + y_3 + 5t_{15} + 2s_6 + 3d_9,\\
t_{15}x_{10} &= r_3 + y_3 + 4d_9 + 3s_6 + 6t_{15},\\
t_{15}b_5 &= r_3 + y_3 + s_6 + 2d_9 + 3t_{15},\\
t_{15}c_5 &= r_3 + y_3 + s_6 + 2d_9 + 3t_{15},\\
t_{15}c_8 &= r_3 + y_3 + 2s_6 + 3d_9 + 5t_{15},\\
t_{15}x_9 &= r_3 + y_3 + 3d_9 + 2s_6 + 6t_{15},\\
t_{15}c_3 &= \bar{x}_6 + \bar{b}_9 + 2\bar{x}_{15},\\
t_{15}d_3 &= x_6 + b_9 + 2x_{15},\\
t_{15}c_9 &= \bar{b}_3 + z_3 + 3\bar{b}_9 + 2\bar{x}_6 + 6\bar{x}_{15},\\
t_{15}b_6 &= \bar{b}_3 + z_3 + \bar{x}_6 + 2\bar{b}_9 + 4\bar{x}_{15},\\
t_{15}y_{15} &= 2\bar{b}_3 + 2z_3 + 4\bar{x}_6 + 6\bar{b}_9 + 9\bar{x}_{15},
\end{aligned}
$$

$$
\begin{aligned}
d_9^2 &= 1 + b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 2x_{10},\\
d_9 y_3 &= x_9 + b_8 + x_{10},\\
d_9 z_3 &= c_3 + c_9 + y_{15},\\
d_9 b_8 &= y_3 + s_6 + 2d_9 + 3t_{15},\\
d_9 x_{10} &= y_3 + r_3 + s_6 + 2d_9 + 4t_{15},\\
d_9 b_5 &= s_6 + d_9 + 2t_{15},\\
d_9 c_5 &= s_6 + d_9 + 2t_{15},\\
d_9 c_8 &= r_3 + s_6 + 2d_9 + 3t_{15},\\
d_9 x_9 &= r_3 + y_3 + 2s_6 + 2d_9 + 3t_{15},\\
d_9 c_3 &= \bar{b}_9 + z_3 + \bar{x}_{15},\\
d_9 d_3 &= b_3 + b_9 + x_{15},\\
d_9 c_9 &= \bar{b}_3 + z_3 + 2\bar{x}_6 + 2\bar{b}_9 + 3\bar{x}_{15},\\
d_9 b_6 &= \bar{x}_6 + 3\bar{b}_9 + 2\bar{x}_{15},\\
d_9 y_{15} &= \bar{b}_3 + z_3 + 2\bar{x}_6 + 3\bar{b}_9 + 6\bar{x}_{15},
\end{aligned}
$$

$$
\begin{aligned}
y_3^2 &= 1 + c_8,\\
y_3 z_3 &= \bar{d}_3 + b_6,\\
y_3 b_8 &= d_9 + t_{15},\\
y_3 x_{10} &= s_6 + t_{15} + d_9,\\
y_3 b_5 &= t_{15},\\
y_3 c_5 &= t_{15},\\
y_3 c_8 &= s_6 + t_{15} + y_3,\\
y_3 x_9 &= t_{15} + d_9 + r_3,\\
y_3 c_3 &= \bar{b}_9,\\
y_3 d_3 &= x_6 + \bar{z}_3,\\
y_3 c_9 &= \bar{b}_3 + \bar{b}_9 + \bar{x}_{15},\\
y_3 b_6 &= \bar{x}_{15} + z_3,\\
y_3 y_{15} &= \bar{x}_6 + \bar{b}_9 + 2\bar{x}_{15},
\end{aligned}
$$

$$
\begin{aligned}
z_3 \bar{z}_3 &= 1 + c_8,\\
z_3^2 &= d_3 + \bar{b}_6,\\
z_3 b_8 &= \bar{b}_9 + \bar{x}_{15},\\
z_3 x_{10} &= \bar{x}_6 + \bar{b}_9 + \bar{x}_{15},\\
z_3 b_5 &= \bar{x}_{15},\\
z_3 c_5 &= \bar{x}_{15},\\
z_3 c_8 &= \bar{x}_6 + \bar{x}_{15} + z_3,\\
z_3 x_9 &= \bar{b}_3 + \bar{b}_9 + \bar{x}_{15},\\
z_3 c_3 &= b_9,\\
z_3 \bar{c}_3 &= d_9,\\
z_3 d_3 &= s_6 + y_3,\\
z_3 \bar{d}_3 &= \bar{z}_3 + x_6,\\
z_3 c_9 &= b_3 + b_9 + x_{15},\\
z_3 \bar{c}_9 &= r_3 + d_9 + t_{15},\\
z_3 b_6 &= \bar{z}_3 + x_{15},\\
z_3 \bar{b}_6 &= y_3 + t_{15},\\
z_3 y_{15} &= x_6 + b_9 + 2x_{15},\\
z_3 \bar{y}_{15} &= s_6 + 2t_{15} + d_9.
\end{aligned}
$$

Now let us see the following example of NITA of dimension 22, which satisfies $b_3^2 = r_3 + s_6$, r_3 is real, denoted as $(A(7 \cdot 5 \cdot 10), B_{22})$, where $B_{22} = \{1, b_8, x_{10}, b_5, c_5, c_8, x_9, r_3, y_3, s_6, t_{15}, d_9, b_3, \bar{b}_3, t_6, \bar{t}_6, b_{15}, \bar{b}_{15}, y_9, \bar{y}_9, x_3, \bar{x}_3\}$. The interesting thing is that this example also contains table subsets:

$$\{1\} \subseteq C \subseteq E \subseteq B_{22},$$

where C and E are as defined in the previous example of dimension 32. In particular both examples of dimensions 22 and 32 each contain the same sub-table algebra of dimension 12 generated by the basis E. Also E is a maximal table subset of B_{22}. The equations of products of the basis elements of C and E are as in the example of $A(3 \cdot A_6 \cdot 2, 32)$ described above

$$\begin{aligned}
b_3\bar{b}_3 &= 1 + b_8,\\
b_3^2 &= r_3 + s_6,\\
b_3t_6 &= b_8 + x_{10},\\
b_3\bar{t}_6 &= r_3 + t_{15},\\
b_3b_{15} &= 2t_{15} + s_6 + d_9,\\
b_3\bar{b}_{15} &= b_8 + x_{10} + b_5 + c_5 + c_8 + x_9,\\
b_3y_9 &= t_{15} + d_9 + y_3,\\
b_3\bar{y}_9 &= c_8 + x_9 + x_{10},\\
b_3x_3 &= d_9,\\
b_3\bar{x}_3 &= x_9,\\
b_3b_8 &= b_3 + \bar{t}_6 + b_{15},\\
b_3x_{10} &= b_{15} + \bar{t}_6 + y_9,\\
b_3b_5 &= b_{15},\\
b_3c_5 &= b_{15},\\
b_3c_8 &= b_{15} + y_9,\\
b_3x_9 &= b_{15} + y_9 + x_3,\\
b_3r_3 &= \bar{b}_3 + t_6,\\
b_3y_3 &= \bar{y}_9,\\
b_3s_6 &= \bar{b}_3 + \bar{b}_{15},\\
b_3t_{15} &= 2\bar{b}_{15} + t_6 + \bar{y}_9,\\
b_3d_9 &= \bar{b}_{15} + \bar{y}_9 + \bar{x}_3,
\end{aligned}$$

$$
\begin{aligned}
t_6\bar{t}_6 &= 1 + b_5 + c_5 + b_8 + c_8 + x_9,\\
t_6^2 &= 2s_6 + t_{15} + d_9,\\
t_6 b_{15} &= 2b_8 + 2c_8 + b_5 + c_5 + 3x_{10} + 2x_9,\\
t_6\bar{b}_{15} &= c_3 + b_6 + 4t_{15} + y_3 + 2d_9,\\
t_6 y_9 &= b_8 + b_5 + c_5 + c_8 + 2x_9 + x_{10},\\
t_6\bar{y}_9 &= s_6 + 2d_9 + 2t_{15},\\
t_6 x_3 &= x_{10} + c_8,\\
t_6\bar{x}_3 &= t_{15} + y_3,\\
t_6 b_8 &= \bar{b}_3 + 2\bar{b}_{15} + t_6 + \bar{y}_9,\\
t_6 x_{10} &= 3\bar{b}_{15} + \bar{y}_9 + \bar{b}_3 + \bar{x}_3,\\
t_6 b_5 &= \bar{b}_{15} + t_6 + \bar{y}_9,\\
t_6 c_5 &= \bar{b}_{15} + t_6 + \bar{y}_9,\\
t_6 c_8 &= 2\bar{b}_{15} + t_6 + \bar{y}_9 + \bar{x}_3,\\
t_6 x_9 &= 2\bar{b}_{15} + t_6 + 2\bar{y}_9,\\
t_6 r_3 &= b_{15} + b_3,\\
t_6 y_3 &= b_{15} + x_3,\\
t_6 s_6 &= 2\bar{t}_6 + b_{15} + y_9,\\
t_6 t_{15} &= 2y_9 + x_3 + b_3 + \bar{t}_6 + 4b_{15},\\
t_6 d_9 &= 2y_9 + 2b_{15} + \bar{t}_6,
\end{aligned}
$$

$$
\begin{aligned}
b_{15}\bar{b}_{15} &= 1 + 3b_5 + 3c_5 + 5b_8 + 5c_8 + 6x_{10} + 6x_9,\\
b_{15}^2 &= 2r_3 + 2y_3 + 4s_6 + 6d_9 + 9t_{15},\\
b_{15} y_9 &= r_3 + y_3 + 2s_6 + 3d_9 + 6t_{15},\\
b_{15}\bar{y}_9 &= 2b_5 + 2c_5 + 3b_8 + 3c_8 + 3x_9 + 4x_{10},\\
b_{15} x_3 &= s_6 + 2t_{15} + d_9,\\
b_{15}\bar{x}_3 &= c_5 + c_8 + x_9 + b_8 + x_{10} + b_5,\\
b_{15} b_8 &= 5b_{15} + 3y_9 + x_3 + b_3 + 2\bar{t}_6,\\
b_{15} x_{10} &= 6b_{15} + 4y_9 + b_3 + x_3 + 3\bar{t}_6,\\
b_{15} b_5 &= b_3 + x_3 + \bar{t}_6 + 2y_9 + 3b_{15},\\
b_{15} c_5 &= 3b_{15} + 2y_9 + b_3 + x_3 + \bar{t}_6,\\
b_{15} c_8 &= 5b_{15} + 3y_9 + x_3 + b_3 + 2\bar{t}_6,\\
b_{15} x_9 &= 6b_{15} + 3y_9 + x_3 + b_3 + 2\bar{t}_6,\\
b_{15} r_3 &= 2\bar{b}_{15} + t_6 + \bar{y}_9,\\
b_{15} y_3 &= 2\bar{b}_{15} + \bar{y}_9 + t_6,\\
b_{15} s_6 &= 4\bar{b}_{15} + 2\bar{y}_9 + \bar{b}_3 + \bar{x}_3 + t_6,\\
b_{15} t_{15} &= 9\bar{b}_{15} + 6\bar{y}_9 + 2\bar{b}_3 + 2\bar{x}_3 + 4t_6,\\
b_{15} d_9 &= 6\bar{b}_{15} + 3\bar{y}_9 + \bar{b}_3 + \bar{x}_3 + 2t_6,
\end{aligned}
$$

$$
\begin{aligned}
y_9\bar{y}_9 &= 1 + b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 2x_{10},\\
y_9^2 &= r_3 + y_3 + 2s_6 + 2d_9 + 3t_{15},\\
y_9x_3 &= r_3 + d_9 + t_{15},\\
y_9\bar{x}_3 &= b_8 + x_{10} + x_9,\\
y_9b_8 &= \bar{t}_6 + 2y_9 + 3b_{15} + x_3,\\
y_9x_{10} &= \bar{t}_6 + 2y_9 + b_3 + x_3 + 4b_{15},\\
y_9b_5 &= \bar{t}_6 + y_9 + 2b_{15},\\
y_9c_5 &= \bar{t}_6 + y_9 + 2b_{15},\\
y_9c_8 &= 3b_{15} + b_3 + 2y_9 + \bar{t}_6,\\
y_9x_9 &= 2\bar{t}_6 + 2y_9 + 3b_{15} + x_3 + b_3,\\
y_9r_3 &= \bar{b}_{15} + \bar{y}_9 + \bar{x}_3,\\
y_9y_3 &= \bar{b}_{15} + \bar{b}_3 + \bar{y}_9,\\
y_9s_6 &= 2\bar{b}_{15} + t_6 + 2\bar{y}_9,\\
y_9t_{15} &= 6\bar{b}_{15} + 2t_6 + \bar{b}_3 + 3\bar{y}_9 + \bar{x}_3,\\
y_9d_9 &= 3\bar{b}_{15} + 2t_6 + \bar{b}_3 + 2\bar{y}_9 + \bar{x}_3,
\end{aligned}
$$

$$
\begin{aligned}
x_3\bar{x}_3 &= 1 + c_8,\\
x_3^2 &= s_6 + y_3,\\
x_3b_8 &= b_{15} + y_9,\\
x_3x_{10} &= b_{15} + \bar{t}_6 + y_9,\\
x_3b_5 &= b_{15},\\
x_3c_5 &= b_{15},\\
x_3c_8 &= x_3 + b_{15} + \bar{t}_6,\\
x_3x_9 &= b_3 + y_9 + b_{15},\\
x_3r_3 &= \bar{y}_9,\\
x_3y_3 &= \bar{x}_3 + t_6,\\
x_3s_6 &= \bar{b}_{15} + \bar{x}_3,\\
x_3t_{15} &= 2\bar{b}_{15} + t_6 + \bar{y}_9,\\
x_3d_9 &= \bar{b}_{15} + \bar{b}_3 + \bar{y}_9.
\end{aligned}
$$

If we define a Table Algebra (A, B) for $R, S \subseteq B$,

$$R * S = \{\cup Supp(R \cdot S) | r \in R, s \in S\},$$

then: $Cb_3 * C\bar{b}_3 = Cr_3^2 = Cc_3 * C\bar{c}_3 = C$, C is an identity. $(Cb_3)^2 = C\bar{c}_3$, $Cb_3 * Cc_3 = Cr_3$, $Cb_3 * C\bar{c}_3 = C\bar{b}_3$ and $Cb_3Cr_3 = C\bar{c}_3$. For the definitions of abelian table algebra, quotient of table algebra and its basis and the sub-table algebra generated by an element $b \in B$ denoted by $\langle b \rangle = B_b$, see [AB] and [B1]. The definitions of a linear and faithful element $b \in B$ can be also found there.

One can derive from Table 2.1 that $B_{32} = \bigcup_{i=1}^{10} Bb_3^i = \bigcup_{i=0}^{\infty} Bb_3^i$ and b_3 is a faithful element of B_{32} that generates B_{32}. Also the quotient table algebra with basis B_{32}/C is strictly isomorphic to the Table Algebra induced by the cyclic group Z_6 of order 6.

One can derive from Table 2.2 that $B_{22} = \bigcup_{i=1}^{7} Bb_3^i = \bigcup_{i=0}^{\infty} Bb_3^i$ and b_3 is a faithful element of B_{22} generates B_{22}. Also the quotient table algebra with basis

Table 2.1 NITA $(A(3\cdot A_6\cdot 2), B_{32})$

The powers of b_3	Constituents of the powers	Basis of quotient of B_{32}/C
b_3^2	c_3, b_6	$C=\{1, b_8, x_{10}, b_5, c_5, c_8, x_9\}$
b_3^3	r_3, s_6, t_{15}	$Cb_3=\{b_3, x_6, x_{15}, b_9, \bar{z}_3\}$
b_3^4	$\bar{c}_3, \bar{b}_6, \bar{y}_{15}, \bar{c}_9$	$C\bar{b}_3=\{\bar{b}_3, \bar{x}_6, \bar{x}_{15}, \bar{b}_9, z_3\}$
b_3^5	$\bar{b}_3, \bar{x}_6, \bar{x}_{15}, \bar{b}_9, z_3$	$Cr_3=\{r_3, t_{15}, d_9, y_3, s_6\}$
b_3^6	$1, b_8, x_{10}, b_5, c_5, c_8, x_9$	$Cc_3=\{c_3, b_6, y_{15}, c_9, \bar{d}_3\}$
b_3^7	$b_3, x_6, x_{15}, b_9, \bar{z}_3$	$C\bar{c}_3=\{\bar{c}_3, \bar{b}_6, \bar{y}_{15}, \bar{c}_9, d_3\}$
b_3^8	$c_3, b_6, y_{15}, c_9, \bar{d}_3$	
b_3^9	$r_3, s_6, t_{15}, d_9, y_3$	
b_3^{10}	$\bar{c}_3, \bar{b}_6, \bar{y}_{15}, \bar{c}_9, d_3$	

Table 2.2 NITA $(A(7\cdot 5\cdot 10), B_{22})$

The powers of b_3	New constituents of the powers	Basis of quotient of B_{32}/C
b_3^2	r_3, s_6	$C=\{1, b_8, x_{10}, b_5, c_5, c_8, c_9\}$
b_3^3	$\bar{b}_3, t_6, \bar{b}_{15}$	$C\bar{b}_3=\{\bar{b}_3, t_6, \bar{b}_{15}, \bar{y}_9, \bar{x}_3\}$
b_3^4	$1, b_8, x_{10}, b_5, c_5, c_8, x_9$	$Cr_3=\{r_3, s_6, t_{15}, d_9, y_3\}$
b_3^5	$b_3, \bar{t}_6, b_{15}, y_9, x_3$	$Cb_3=\{b_3, \bar{t}_6, b_{15}, y_9, x_3\}$
b_3^6	$r_3, s_6, t_{15}, d_9, y_{15}, y_3$	
b_3^7	$\bar{b}_3, t_6, \bar{b}_{15}, \bar{y}_9, \bar{x}_3$	

B_{22}/C is strictly isomorphic to the Table Algebra induced by the cyclic group Z_n of order 4.

In order to prove the Main Theorem 2.1, the following fact is frequently used:

Lemma 2.1 *Let $e_m, f_m, u_n, v_n \in B$ such that $e_m\bar{e}_m = f_m\bar{f}_m$. Then $(e_m u_n, e_m u_n) = (\bar{e}_m u_n, \bar{e}_m u_n) = (f_m v_n, f_m v_n) = (\bar{f}_m v_n, \bar{f}_m v_n)$.*

Proof (1) follows from $(e_m u_n, e_m u_n) = (e_m\bar{e}_m, u_n\bar{u}_n) = (f_m\bar{f}_m, u_n\bar{u}_n) = (f_m u_n, f_m u_n)$.

Note that $(t_3\bar{t}_3, b_8), (s_4\bar{s}_4, b_8) \leq 1$, and we obtain (2). □

Lemma 2.2 *Let $b_3, t_3, s_4 \in B$, $b_3\bar{b}_3 = 1 + b_8$. Then $(b_3t_3, b_3t_3), (b_3s_4, b_3s_4) \leq 2$. If $(b_3t_3, b_3t_3) = 2$, then $t_3\bar{t}_3 = 1 + b_8$.*

Proof If $(b_3x_4, b_3x_4) \geq 3$, then $(x_4\bar{x}_4, b_8) \geq 2$ for $b_3\bar{b}_3 = 1 + b_8$, a contradiction. (1) follows.

If $(b_3c_3, b_3c_3) = 2$, then $(b_3\bar{b}_3, c_3\bar{c}_3) = 2$. (2) follows from $b_3\bar{b}_3 = 1 + b_8$. □

2.3 NITA Generated by b_3 and Satisfying $b_3^2 = b_4 + b_5$

In this section, we shall investigate NITA generated by b_3 and satisfying $b_3^2 = b_4 + b_5$, and obtain the following theorem:

Theorem 2.2 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ of degree 3 and without non-identity basis element of degree 1 or 2. Then $b_3\bar{b}_3 = 1 + b_8$, $b_8 \in B$. Assume that $b_3^2 = b_4 + b_5$, $b_4 \in B$ and $b_5 \in B$. Then $(\bar{b}_5 b_5, b_8) = 1$, $(b_4^2, b_4^2) = 3$ and $(b_3 b_8, b_3 b_8) = 3$.*

Now we start to investigate NITA satisfying

$$b_3\bar{b}_3 = 1 + b_8, \tag{2.1}$$

$$b_3^2 = b_4 + b_5. \tag{2.2}$$

Proof Based on the above equations, one may set

$$\bar{b}_3 b_4 = b_3 + b_9, \quad \text{some } b_9 \in N^*B. \tag{2.3}$$

Since $(b_4\bar{b}_4, b_8) \leq 1$, we have that $(\bar{b}_3 b_4, \bar{b}_3 b_4) = 2$, $b_9 \in B$ and

$$b_4\bar{b}_4 = 1 + b_8 + c_7, \quad \text{where } c_7 \in N^*B. \tag{2.4}$$

□

Collecting the above equations, checking the associative laws for basis elements and making the necessary calculations and assumptions, we have the following lemma.

Lemma 2.3 *The following equations always hold*:

$$\begin{array}{ll}
\bar{b}_3 b_4 = b_3 + b_9, & \\
b_4\bar{b}_4 = 1 + b_8 + c_7, \quad c_7 \in N^*B, & \\
\bar{b}_3 b_5 = b_3 + x_{12}, \quad x_{12} \in N^*B & \textit{assumption}, \\
b_3 b_8 = b_3 + b_9 + x_{12} & \textit{calculating } b_3^2\bar{b}_3 = b_3(b_3\bar{b}_3), \\
b_3 c_7 = b_9 + f_{12}, \quad f_{12} \in N^*B & \textit{assumption}, \\
b_4\bar{b}_9 = b_3 + b_9 + x_{12} + f_{12} & \textit{calculating } b_3(\bar{b}_4 b_4) = (b_3\bar{b}_4)b_4, \\
b_4 b_8 = b_5 + b_3 b_9 & \textit{calculating } b_3(\bar{b}_3 b_4) = (b_3\bar{b}_3)b_4, \\
\bar{b}_4 b_5 = b_8 + z_{12}, \quad z_{12} \in N^*B & \textit{assumption}, \\
\bar{b}_3 b_9 = c_7 + b_8 + \bar{z}_{12} & \textit{calculating } \bar{b}_3^2 b_4 = \bar{b}_3(\bar{b}_3 b_4), \\
b_8^2 = 1 + c_7 + b_8 + \bar{z}_{12} + \bar{b}_3 x_{12} & \textit{calculating } (b_3\bar{b}_3)b_8 = (b_3 b_8)\bar{b}_3, \\
b_8 c_7 = b_8 + \bar{z}_{12} + \bar{b}_3 f_{12} & \textit{calculating } (b_3\bar{b}_3)c_7 = (b_3 b_7)\bar{b}_3, \\
b_5\bar{b}_9 = b_9 + b_3 z_{12} & \textit{calculating } (\bar{b}_3 b_4)\bar{b}_5 = \bar{b}_3(b_4\bar{b}_5), \\
b_9\bar{b}_9 = 1 + b_8 - z_{12} + \bar{z}_{12} + \bar{b}_3 f_{12} + \bar{b}_3 x_{12} & \textit{calculating } (b_4\bar{b}_4)b_8 = \bar{b}_4(b_4 b_8).
\end{array}$$

Proof Based on the known equations, checking associative laws one by one and making necessary assumptions, one can easily obtain the equations above. The proof of the last equation follows from

$$\begin{aligned}(b_4\bar{b}_4)b_8 &= b_8 + b_8^2 + b_8c_7\\ &= b_8 + 1 + c_7 + b_8 + \bar{z}_{12} + \bar{b}_3x_{12} + b_8 + \bar{z}_{12} + \bar{b}_3f_{12},\\ \bar{b}_4(b_4b_8) &= \bar{b}_4b_5 + (\bar{b}_4b_3)b_9\\ &= \bar{b}_4b_5 + \bar{b}_3b_9 + b_9\bar{b}_9\\ &= b_8 + z_{12} + c_7 + b_8 + \bar{z}_{12} + b_9\bar{b}_9.\end{aligned}$$

□

Obviously $(b_5\bar{b}_5, b_8) \leq 3$, c_7 is either $c_3 + c_4$ or irreducible.

If $c_7 \notin B$, then there exist $c_3, c_4 \in B$ such that $c_7 = c_3 + c_4$. First we have the following lemma:

Lemma 2.4 *There exists no NITA generated by b_3 and satisfying $b_3\bar{b}_3 = 1 + b_8$, $b_3^2 = b_4 + b_5$ and $c_7 = c_3 + c_4$.*

Proof By the assumption, we have that

$$b_4\bar{b}_4 = 1 + b_8 + c_3 + c_4. \tag{2.5}$$

Then

$$\begin{aligned}b_3^2\bar{b}_4 &= b_4\bar{b}_4 + b_5\bar{b}_4\\ &= 1 + c_3 + c_4 + b_8 + \bar{b}_4b_5,\\ b_3(b_3\bar{b}_4) &= b_3\bar{b}_3 + b_3\bar{b}_9\\ &= 1 + b_8 + b_3\bar{b}_9.\end{aligned}$$

Hence $(c_3, b_3\bar{b}_9) = (c_4, b_3\bar{b}_9) = 1$. Therefore

$$b_3c_3 = b_9, \tag{2.6}$$

$$b_3c_4 = d_3 + b_9, \quad \text{some } d_3 \in B,\ d_3 \neq b_3. \tag{2.7}$$

Now one has $(b_3\bar{d}_3,\ c_4) = 1$. Then

$$b_3\bar{d}_3 = c_4 + d_5, \quad \text{some } d_5 \in B, \tag{2.8}$$

from which $(b_3\bar{b}_3, d_3\bar{d}_3) = 2$ follows. Hence

$$d_3\bar{d}_3 = 1 + b_8. \tag{2.9}$$

By (2.5), we have that $(b_4c_3, b_4) = 1$. Let

$$b_4c_3 = b_4 + y_8, \quad y_8 \in N^*B. \tag{2.10}$$

Since $(b_3\bar{b}_3)c_3 = c_3 + c_3b_8$, $\bar{b}_3(b_3c_3) = \bar{b}_3b_9$, one has that $\bar{b}_3b_9 = c_3 + c_3b_8$. But $(c_4, \bar{b}_3b_9) = 1$. Then $(c_4, c_3b_8) = 1$ and so $(c_4c_3, b_8) = 1$. Thus

$$c_3c_4 = x_4 + b_8, \quad \text{some } x_4 \in B. \tag{2.11}$$

By (2.10) we have that

$$2 \le (b_4c_3, b_4c_3) = (b_4\bar{b}_4, c_3^2) = (1 + c_3 + c_4 + b_8, c_3^2),$$

which implies that c_3^2 equals one of

$$1 + b_8, \qquad 1 + c_3 + x_5, \qquad 1 + c_4 + y_4, \qquad 1 + 2c_4.$$

By (2.11), we know that c_3^2 cannot be $1 + c_4 + y_4$ or $1 + 2c_4$. And by (2.6), $c_3^2 \ne 1 + b_8$. Hence

$$c_3^2 = 1 + c_3 + x_5. \tag{2.12}$$

Therefore $(b_4c_3, b_4c_3) = (c_3^2, b_4\bar{b}_4) = 2$, which implies that $y_8 \in B$.

By Lemmas 2.3, (2.8) and (2.9), we have

$$\begin{aligned} b_3(b_3\bar{b}_3) &= b_3 + b_3b_8 = 2b_3 + b_9 + x_{12}, \\ \bar{b}_3(b_3^2) &= \bar{b}_3b_4 + \bar{b}_3b_5 = b_3 + b_9 + \bar{b}_3b_5, \\ (b_3\bar{d}_3)d_3 &= c_4d_3 + d_5d_3. \end{aligned}$$

Hence $b_3b_8 = b_9 + \bar{b}_3b_5$, and $b_9 \in Supp\{c_4d_3 + d_5d_3\}$.

Assume that $b_9 \in Supp\{d_3d_5\}$. By (2.8), it follows that

$$\begin{aligned} d_3d_5 &= b_9 + b_3 + l_3, \quad l_3 \in B, \\ \bar{b}_3b_5 &= c_4d_3 + l_3, \end{aligned}$$

so that $b_3b_8 = b_9 + l_3 + c_4d_3$, which implies that $b_8 \in Supp\{b_3\bar{l}_3\}$. Hence $l_3 = b_3$. So $(d_3d_5, b_3) = 3$, which implies that $(b_4\bar{d}_3, d_5) = 2$, a contradiction to (2.8). Thus $b_9 \in Supp\{c_4d_3\}$ and

$$c_4d_3 = b_3 + b_9. \tag{2.13}$$

Thus $(d_3\bar{d}_3, c_4^2) = 2$, we may assume that

$$c_4^2 = 1 + b_8 + f_7, \quad \text{some } f_7 \in N^*B. \tag{2.14}$$

Then $c_4 \notin Supp\{c_4x_5\}$. Otherwise, $c_4 \in Supp\{c_4x_5\}$, which implies that $x_5 \in Supp\{c_4^2\}$, and so $f_7 = x_5 + x_2$, a contradiction.

Since

$$\begin{aligned} c_3^2c_4 &= c_4 + c_3c_4 + x_5c_4 \\ &= c_4 + x_4 + b_8 + c_4x_5, \\ c_3(c_3c_4) &= c_3b_8 + c_3x_4, \end{aligned}$$

we have that

$$c_3b_8 + c_3x_4 = c_4 + x_4 + b_8 + c_4x_5.$$

But the left hand side of the above equation contains two c_4 by (2.11). Hence $c_4 = x_4$ by $c_4 \notin Supp\{c_4x_5\}$. Therefore

$$c_3c_4 = b_8 + c_4, \tag{2.15}$$

which implies that $(c_4^2, c_3) = 1$ and

$$c_4^2 = 1 + b_8 + c_3 + e_4, \quad \text{some } e_4 \in B.$$

Now we have the following equations:

$$\begin{aligned} b_3c_4^2 &= b_3 + b_3b_8 + b_3c_3 + b_3e_4 \\ &= 2b_3 + 2b_9 + x_{12} + b_3e_4, \\ \text{by (2.7) and (2.13)} \quad c_4(b_3c_4) &= c_4b_9 + c_4d_3 \\ &= c_4b_9 + b_9 + b_3. \end{aligned}$$

Then $c_4b_9 = b_3 + b_9 + x_{12} + b_3e_4$. Since $d_3 \in Supp\{c_4b_9\}$, one has that $d_3 \in Supp\{x_{12}\}$ or b_3e_4.

If $d_3 \in Supp\{x_{12}\}$, then $d_3 \in Supp\{b_3b_8\}$, which implies that $b_3\bar{d}_3 = 1 + b_8$. Thus $b_3 = d_3$, which implies that $b_3\bar{b}_3 = c_4 + d_5$, a contradiction. Hence $d_3 \in Supp\{b_3e_4\}$, $e_4 \in Supp\{b_3\bar{d}_3\}$. Therefore by (2.8) $e_4 = c_4$ and

$$\begin{aligned} \text{by (2.7)} \quad c_4^2 &= 1 + c_3 + c_4 + b_8 = b_4\bar{b}_4, \\ c_4b_9 &= b_3 + d_3 + 2b_9 + x_{12}. \end{aligned}$$

Furthermore

$$\begin{aligned} b_3c_4^2 &= b_3 + b_3b_8 + b_3c_4 + b_3c_3 \\ &= b_3 + b_3 + b_9 + x_{12} + d_3 + b_9 + b_9, \\ (b_3\bar{b}_4)b_4 &= \bar{b}_3b_4 + \bar{b}_9b_4 \\ &= b_3 + b_9 + b_4\bar{b}_9, \\ (b_3c_4)c_4 &= c_4b_9 + d_3c_4 \\ &= c_4b_9 + b_3 + b_9. \end{aligned}$$

Thus

$$b_4\bar{b}_9 = b_3 + d_3 + 2b_9 + x_{12} = c_4 b_9.$$

Hence by (2.6) $(b_3c_3)c_4 = b_9c_4 = b_3 + d_3 + 2b_9 + x_{12}$. But by (2.7) $(b_3c_4)c_3 = c_3b_9 + c_3d_3$.

Since $d_3 \notin Supp\{c_3d_3\}$, we have that $d_3 \in Supp\{c_3b_9\}$

$$\begin{aligned} c_3d_3 &= b_9, \\ c_3b_9 &= b_3 + d_3 + b_9 + x_{12}. \end{aligned} \tag{2.16}$$

Therefore $(b_3c_3, c_3d_3) = 1$, so $(b_3\bar{d}_3, c_3^2) = 1$. By (2.8) and (2.12), we have that $(b_3\bar{d}_3, x_5) = 1$, which implies that $x_5 = d_5$. Consequently d_5 is real. Then

$$\begin{aligned} b_3\bar{d}_3 &= c_4 + d_5 = \bar{b}_3d_3, \\ c_3^2 &= 1 + c_3 + d_5. \end{aligned}$$

Moreover $(b_3^2, d_3^2) = (b_3\bar{d}_3, \bar{b}_3d_3) = 2$. So

$$d_3^2 = 1 + 2b_4 \text{ or } b_4 + b_5.$$

Since (2.9) implies that d_3 is nonreal. So

$$d_3^2 = b_4 + b_5.$$

By (2.8), we may state that

$$\bar{b}_3d_5 = \bar{d}_3 + u_{12}, \quad u_{12} \in N^*B.$$

Then

$$\begin{aligned} \bar{b}_3c_3^2 &= \bar{b}_3 + \bar{b}_3c_3 + \bar{b}_3d_5 \\ &= \bar{b}_3 + \bar{b}_9 + \bar{d}_3 + u_{12}, \\ (\bar{b}_3c_3)c_3 &= c_3\bar{b}_9. \end{aligned}$$

Thus

$$c_3b_9 = b_3 + d_3 + b_9 + \bar{u}_{12}.$$

Hence $x_{12} = \bar{u}_{12}$ by (2.16) and so $b_3d_5 = d_3 + x_{12}$. Since $c_4c_3^2 = c_4 + c_3c_4 + d_5c_4$ and $c_3(c_3c_4) = c_3c_4 + c_3b_8$ by (2.15), we have that

$$c_3b_8 = c_4 + c_4d_5.$$

Now, by $c_4^2 = b_4\bar{b}_4$, we have that

$$\begin{aligned} c_3c_4^2 &= c_3 + c_3^2 + c_3c_4 + c_3b_8 \\ &= c_3 + 1 + c_3 + d_5 + c_4 + b_8 + c_4 + c_4d_5, \end{aligned}$$

$$\begin{aligned}(c_3b_4)\bar{b}_4 &= b_4\bar{b}_4 + y_8\bar{b}_4\\ &= 1 + c_3 + c_4 + b_8 + \bar{b}_4 y_8.\end{aligned}$$

So

$$\bar{b}_4 y_8 = c_3 + c_4 + d_5 + c_4 d_5.$$

We know that y_8 is irreducible, then $(c_4b_4, y_8) = 1$. But $(b_4c_4, b_4c_4) = 4$. Hence $(c_4b_4 - y_8, c_4b_4 - y_8) = 3$, which is impossible for $L_1(B) = \{1\}$ and $L_2(B) = \emptyset$. The lemma follows. □

Now we begin to investigate NITA such that $c_7 \in B$ is based on the value of $(b_5\bar{b}_5, b_8)$.

Lemma 2.5 *There exists no NITA such that $c_7 \in B$ and $(b_5\bar{b}_5, b_8) = 3$.*

Proof It is easy to see that $(b_5\bar{b}_5, b_8) = 3$ implies that $(\bar{b}_3b_5, \bar{b}_3b_5) = 4$. By Lemma 2.3, we may assume that

$$\begin{aligned}x_{12} &= x + y + z, \quad x, y, z \in B,\\ \bar{b}_3b_5 &= b_3 + x + y + z,\\ b_3b_8 &= b_3 + b_9 + x + y + z.\end{aligned}$$

From the last equation above, one has that b_8 is contained in $\bar{b}_3x$, $\bar{b}_3y$ and $\bar{b}_3z$. Hence no one of x, y and z has degree 3. Therefore

$$\begin{aligned}x_{12} &= x_4 + y_4 + z_4,\\ \bar{b}_3x_4 &= r_4 + b_8, \quad \text{some } r_4 \in B,\\ \bar{b}_3y_4 &= s_4 + b_8, \quad \text{some } s_4 \in B,\\ \bar{b}_3z_4 &= t_4 + b_8, \quad \text{some } t_4 \in B.\end{aligned}$$

Furthermore, b_3x_4, b_3y_4 and b_3z_4 have exactly two constituents. Let

$$\begin{aligned}b_3x_4 &= b_5 + r_7, \quad \text{some } r_7 \in B,\\ b_3y_4 &= b_5 + s_7, \quad \text{some } s_7 \in B,\\ b_3z_4 &= b_5 + t_7, \quad \text{some } t_7 \in B.\end{aligned}$$

Since

$$\begin{aligned}\bar{b}_3(b_3x_4) &= \bar{b}_3b_5 + \bar{b}_3r_7\\ &= b_3x_4 + y_4 + z_4 + \bar{b}_3r_7,\\ b_3(\bar{b}_3x_4) &= b_3b_8 + b_3r_4\\ &= b_3 + b_9 + x_4 + y_4 + z_4 + b_3r_4,\end{aligned}$$

we have that $\bar{b}_3 r_7 = b_9 + b_3 r_4$. By the same reasoning we have that $\bar{b}_3 s_7 = b_9 + b_3 s_4$ and $\bar{b}_3 t_7 = b_9 + b_3 t_4$. Hence $r_7, s_7, t_7 \in Supp\{b_3 b_9\}$.

If r_7, s_7 and t_7 are distinct, then

$$b_3 b_9 = b_4 + r_7 + s_7 + t_7 + e_2, \quad \text{some } e_2 \in B,$$

a contradiction. Hence at least two of r_7, s_7 and t_7 are equal, without loss of generality, let $r_7 = s_7$. Then $(b_3 x_4, b_3 y_4) = (\bar{b}_3 x_4, \bar{b}_3 y_4) = 2$. Thus $r_4 = s_4$. By Lemma 2.3, we have

$$b_8^2 = 1 + 4b_8 + c_7 + z_{12} + 2r_4 + t_4.$$

So $\bar{z}_{12} = 2r_4 + t_4$. Lemma 2.3 implies that

$$\bar{b}_3 b_9 = c_7 + b_8 + 2r_4 + t_4.$$

Then $(\bar{b}_3 b_9, r_4) = (b_9, b_3 r_4) \geq 2$, a contradiction, from which the lemma follows. □

Lemma 2.6 *There exists no NITA such that $c_7 \in B$ and $(b_5 \bar{b}_5, b_8) = 2$.*

Suppose that $(b_5 \bar{b}_5, b_8) = 2$. Let

$$b_5 \bar{b}_5 = 1 + 2b_8 + d_8, \quad \text{some } d_8 \in N^* B.$$

Since $(\bar{b}_3 b_5, \bar{b}_3 b_5) = (b_3 \bar{b}_3, b_5 \bar{b}_5) = 3$, by Lemma 2.3, we have that $x_{12} = x + y$ and

$$\begin{aligned} \bar{b}_3 b_5 &= b_3 + x + y, \\ b_3 b_8 &= b_3 + b_9 + x + y. \end{aligned}$$

Thus $b_8 \in Supp\{\bar{b}_3 x\}, Supp\{\bar{b}_3 y\}$, which implies that degrees of x and y are ≥ 4. We have three possibilities: $(f(x), f(y)) = (4, 8), (5, 7), (6, 6)$.

We shall divide the proof into three propositions, Propositions 2.1, 2.2, 2.3 from which the lemma follows.

Proposition 2.1 *There exists no NITA such that $b_5 \bar{b}_5 = 1 + 2b_8 + d_8$ and $(f(x), f(y)) = (4, 8)$.*

Proof Assume $(f(x), f(y)) = (4, 8)$. Then $x_{12} = x_4 + y_8$ and we may set

$$\begin{aligned} \bar{b}_3 x_4 &= b_8 + x_4', \quad x_4' \in B, \\ \bar{b}_3 y_8 &= b_8 + y_{16}, \quad y_{16} \in N^* B, \end{aligned} \tag{2.17}$$

Then

$$b_8^2 = 1 + c_7 + 3b_8 + x_4' + \bar{z}_{12} + y_{16}.$$

Hence

$$\begin{aligned}(b_3\bar{b}_3)^2 &= 2 + 5b_8 + c_7 + x_4' + \bar{z}_{12} + y_{16},\\ b_3^2\bar{b}_3^2 &= b_4\bar{b}_4 + b_4\bar{b}_5 + \bar{b}_4 b_5 + b_5\bar{b}_5\\ &= 1 + b_8 + c_7 + b_4\bar{b}_5 + \bar{b}_4 b_5 + 1 + 2b_8 + d_8,\end{aligned}$$

so $z_{12} + d_8 = x_4' + y_{16}$.

Now we have two cases, $x_4' \in Supp\{d_8\}$ or $x_4' \notin Supp\{d_8\}$.

Case I Suppose that $x_4' \in Supp\{z_{12}\}$. Then there exists s_8 such that

$$z_{12} = x_4' + s_8,$$
$$y_{16} = d_8 + s_8.$$

Since $(\bar{b}_4 b_5, \bar{b}_4 b_5) = (b_4\bar{b}_4, b_5\bar{b}_5) = 3$, we have that $s_8 \in B$ and

$$\bar{b}_3 b_9 = c_7 + b_8 + \bar{x}'_4 + \bar{s}_8.$$

Hence $b_9 \in Supp\{b_3 x_4'\}$, which means that $(b_3\bar{x}_4', b_3\bar{x}_4') = 2$. So $(b_3 x_4', b_3 x_4') = 2$. Since $\bar{b}_3 x_4 = b_8 + x_4'$, we may set $b_3\bar{x}_4' = a_3 + b_9$ and $b_3 x_4' = x_4 + e_8$, for some $a_3, e_8 \in B$.

Suppose that $b_3\bar{a}_3 = x_4' + l_5$, for some $l_5 \in B$. Then

$$\begin{aligned}b_3(\bar{b}_3 x_4') &= b_3\bar{b}_9 + b_3\bar{a}_3\\ &= c_7 + b_8 + x_4' + s_8 + x_4' + l_5,\\ \bar{b}_3(b_3 x_4') &= \bar{b}_3 x_4 + \bar{b}_3 e_8\\ &= b_8 + x_4' + \bar{b}_3 e_8.\end{aligned}$$

Hence

$$\bar{b}_3 e_8 = c_7 + s_8 + x_4' + l_5.$$

Therefore $e_8 \in Supp\{b_3 c_7\}$. Let $b_3 c_7 = b_9 + e_8 + e_4$, $e_4 \in B$. By $(b_3\bar{b}_3)\bar{a}_3 = \bar{b}_3(b_3\bar{a}_3)$, we obtain

$$\bar{a}_3 b_8 = \bar{b}_9 + \bar{b}_3 l_5.$$

Since $(\bar{a}_3 b_8, \bar{a}_3 b_8) = (a_3\bar{a}_3, b_8^2) = (b_3\bar{b}_3, b_8^2) = 4$, we have that $(\bar{b}_3 l_5, \bar{b}_3 l_5) = 3$. So $(b_3 l_5, b_3 l_5) = 3$. Let

$$b_3 l_5 = e_8 + \alpha_3 + \beta_4,$$

where $\alpha_3, \beta_4 \in B$. Obviously $\alpha_3 \neq b_3$.

$$\begin{aligned}b_3(\bar{b}_3 e_8) &= b_3 c_7 + b_3 b_8 + b_3 x_4' + b_3 l_5\\ &= 2b_9 + 3e_8 + e_4 + b_3 + 2x_4 + y_8 + \alpha_3 + \beta_4,\\ (b_3\bar{b}_3)e_8 &= e_8 + e_8 b_8.\end{aligned}$$

Then

$$e_8 b_8 = 2b_9 + 2e_8 + e_4 + b_3 + 2x_4 + y_8 + \alpha_3 + \beta_4.$$

Hence $e_8 \in Supp\{b_3 b_8\}$, which implies that $e_8 = y_8$. Hence $b_3 c_7 = b_9 + y_8 + e_4$, so that $c_7 \in Supp\{\bar{b}_3 y_8\}$, which implies that $c_7 \in Supp\{y_{16}\} = \{d_8, s_8\}$, a contradiction.

Case II NITA satisfying $x_4' \in Supp\{d_8\}$.

If $x_4' \in d_8$, then $d_8 = x_4' + \bar{x}_4'$ (x_4' maybe real). Then

$$b_5 \bar{b}_5 = 1 + 2b_8 + x_4' + \bar{x}_4'.$$

Hence $y_{16} = \bar{x}_4' + z_{12}$ and

$$b_8^2 = 1 + c_7 + 3b_8 + x_4' + \bar{x}_4' + z_{12} + \bar{z}_{12}, \tag{2.18}$$

$$\bar{b}_3 y_8 = b_8 + \bar{x}_4' + z_{12}, \tag{2.19}$$

$$b_9 \bar{b}_9 = 1 + 3b_8 + x_4' + z_{12} + \bar{x}_4' + \bar{b}_3 f_{12}, \tag{2.20}$$

by (2.17). Therefore $(y_8, b_3 \bar{x}_4') = 1$. Let

$$b_3 \bar{x}_4' = y_8 + t_4, b_3 x_4' = x_4 + j_8, \quad \text{where } t_4, j_8 \in B. \tag{2.21}$$

Since

$$\begin{aligned} b_3(b_5 \bar{b}_5) &= b_3 + 2b_3 b_8 + b_3 x_4' + b_3 \bar{x}_4' \\ &= 3b_3 + 3b_9 + 2x_4 + 3y_8 + t_4 + x_4 + j_8, \\ b_5(b_3 \bar{b}_5) &= b_5 \bar{b}_3 + b_5 \bar{x}_4 + b_5 \bar{y}_8 \\ &= b_3 + x_4 + y_8 + b_5 \bar{x}_4 + b_5 \bar{y}_8, \end{aligned}$$

we have that

$$\bar{x}_4 b_5 + \bar{y}_8 b_5 = 2b_3 + 2b_9 + 2x_4 + 2y_8 + t_4 + j_8.$$

Since

$$\begin{aligned} x_4'(\bar{b}_3 b_4) &= b_3 x_4' + b_9 x_4' \\ &= x_4 + j_8 + b_9 x_4', \\ (\bar{b}_3 x_4') b_4 &= \bar{y}_8 b_4 + \bar{t}_4 b_4, \end{aligned}$$

we have that $x_4 \in Supp\{b_4 \bar{y}_8\}$ or $Supp\{\bar{t}_4 b_4\}$.

We assert that $x_4 \in Supp\{b_4 \bar{t}_4\}$. Otherwise $x_4 \in Supp\{b_4 \bar{y}_8\}$, and $y_8 \in Supp\{b_4 \bar{x}_4\}$. But $b_9 \in Supp\{b_4 \bar{x}_4\}$ by Lemma 2.3, a contradiction. Hence $x_4 \in Supp\{b_4 \bar{t}_4\}$, which implies that $t_4 \in Supp\{b_4 \bar{x}_4\}$. Therefore

$$b_4 \bar{x}_4 = t_4 + b_9 + r_3, \quad r_3 \in B. \tag{2.22}$$

Since

$$\begin{aligned}
\bar{b}_3(b_4b_8) &= \bar{b}_3b_5 + \bar{b}_3(b_3b_9) \\
&= b_3 + x_4 + y_8 + b_9 + b_3(\bar{b}_3b_9) \\
&= b_3 + x_4 + y_8 + b_9 + b_3c_7 + b_3b_8 + b_3\bar{z}_{12} \\
&= b_3 + x_4 + y_8 + b_9 + b_9 + f_{12} + b_3 + x_4 + y_8 + b_9 + b_3\bar{z}_{12}, \\
b_4(\bar{b}_3b_8) &= b_4\bar{b}_3 + b_4\bar{x}_4 + b_4\bar{y}_8 + b_4\bar{b}_9 \\
&= b_3 + b_9 + t_4 + b_9 + r_3 + b_4\bar{y}_8 + b_3 + b_9 + x_4 + y_8 + f_{12}, \\
(\bar{b}_3b_4)b_8 &= b_3b_8 + b_8b_9 \\
&= b_3 + x_4 + y_8 + b_9 + b_8b_9,
\end{aligned}$$

we have that

$$\begin{aligned}
x_4'b_8 &= b_8 + z_{12} + b_3\bar{t}_4, \\
b_8b_9 &= b_3 + x_4 + y_8 + b_9 + y_{12} + b_3\bar{z}_{12}, \\
b_4\bar{y}_8 + b_9 + t_4 + r_3 &= x_4 + y_8 + b_3\bar{z}_{12}.
\end{aligned}$$

We continue the proof based on $t_4 = x_4$ or $t_4 \neq x_4$.

Subcase I There exists no NITA such that $t_4 = x_4$.

If $t_4 = x_4$, then $(x_4, b_3\bar{x}_4') = 1$ by $b_3\bar{x}_4' = y_8 + t_4$, which implies that $x_4' \in Supp\{b_3\bar{x}_4\} = \{b_8, \bar{x}_4'\}$. And $\bar{b}_3t_4 = \bar{b}_3x_4 = b_8 + x_4'$. Hence x_4' is real and

$$b_5\bar{b}_5 = 1 + 2b_8 + 2x_4', \tag{2.23}$$

$$b_3x_4' = x_4 + y_8. \tag{2.24}$$

Thus

$$\begin{aligned}
\bar{b}_3(b_3x_4') &= \bar{b}_3y_8 + \bar{b}_3x_4 \\
&= b_8 + x_4' + \bar{z}_{12} + b_8 + x_4', \\
(b_3\bar{b}_3)x_4' &= x_4' + x_4'b_8.
\end{aligned}$$

So

$$x_4'b_8 = x_4' + 2b_8 + z_{12}.$$

Since

$$\begin{aligned}
b_3(b_5\bar{b}_5) &= b_3 + 2b_3b_8 + 2b_3x_4' \\
&= b_3 + 2b_3 + 2b_9 + 2x_4 + 2y_8 + 2x_4 + 2y_8, \\
b_5(b_3\bar{b}_5) &= \bar{b}_3b_5 + \bar{x}_4b_5 + \bar{y}_8b_5 \\
&= b_3 + x_4 + y_8 + \bar{x}_4b_5 + \bar{y}_8b_5,
\end{aligned}$$

we have that $\bar{x}_4 b_5 + \bar{y}_8 b_5 = 2b_3 + 2b_9 + 3x_4 + 3y_8$. Since $b_3 \in Supp\{\bar{x}_4 b_5\}$ and $b_3 \in Supp\{\bar{y}_8 b_5\}$, there are exactly two possibilities:

$$\bar{x}_4 b_5 = b_3 + b_9 + y_8, \qquad \bar{y}_8 b_5 = b_3 + b_9 + 3x_4 + 2y_8,$$
$$\bar{x}_4 b_5 = b_3 + b_9 + 2x_4, \qquad \bar{y}_8 b_5 = b_3 + b_9 + x_4 + 3y_8.$$

If $\bar{x}_4 b_5 = b_3 + b_9 + y_8$ and $\bar{y}_8 b_5 = b_3 + b_9 + 3x_4 + 2y_8$, then $(\bar{y}_8 b_5, x_4) = 3$, which implies that $(b_5 \bar{x}_4, y_8) = 3$, a contradiction.

If $\bar{x}_4 b_5 = b_3 + b_9 + 2x_4$ and $\bar{y}_8 b_5 = b_3 + b_9 + x_4 + 3y_8$, then $(\bar{x}_4 b_5, x_4) = 3$. So $(x_4^2, b_5) = 2$. It follows that $(x_4^2, x_4^2) \geq 5$, which is impossible for either $x_4 \bar{x}_4 = 1 + b_8 + r_7, r_7 \in B$ or $x_4 \bar{x}_4 = 1 + b_8 + r_4 + r_3, r_3, r_4 \in B$. Hence Subcase I follows.

Subcase II There exists no NITA such that $t_4 \neq x_4$.

If $t_4 \neq x_4$, by $b_4 \bar{y}_8 + b_9 + t_4 + r_3 = x_4 + y_8 + b_3 \bar{z}_{12}$, we have that $r_3 + t_4 + b_9$ is a part of $b_3 \bar{z}_{12}$, $x_4 + y_8$ is a part of $b_4 \bar{y}_8$ and $b_3 + x_4 + y_8 + b_9 + y_{12} + r_3 + t_4 + b_9$ is a part of $b_8 b_9$.

If $b_9 \in Supp\{y_{12}\}$, then there exists $h_3 \in B$ such that $y_{12} = h_3 + b_9$. So $b_3 c_7 = h_3 + 2b_9$, $b_3 \bar{h}_3$ contains c_7, which is impossible for $L_2(B) = \emptyset$. Hence $b_9 \notin Supp\{y_{12}\}$.

Since $(b_9 \bar{b}_9, b_8) = 3$, so $(b_8 b_9, b_9) = 3$, which implies that $(b_3 \bar{z}_{12}, b_9) = 2$ and there exists $x_{11} \in N^* B$ of degrees 11 such that

$$b_3 \bar{z}_{12} = 2b_9 + t_4 + r_3 + x_{11},$$
$$b_4 \bar{y}_8 = x_4 + y_8 + b_9 + x_{11}.$$

The following leads to a contradiction based on the z_{12} representation. Since $(z_{12}, z_{12}) = 2$, we have that $z_{12} = \alpha + \beta$, where $(f(\alpha), f(\beta)) = (3, 9), (4, 8), (5, 7)$ or $(6, 6)$.

Step 1 There exists no NITA such that $z_{12} = \alpha_3 + \beta_9$.

If $z_{12} = \alpha_3 + \beta_9$, then $b_3(\bar{\alpha}_3 + \bar{\beta}_9) = r_3 + t_4 + b_9 + x_{11}$, $b_3 \bar{\alpha}_3 = b_9$ and

$$\bar{b}_3(b_3 \bar{\alpha}_3) = c_7 + b_8 + \bar{\alpha}_3 + \bar{\beta}_9,$$
$$(\bar{b}_3 b_3)\bar{\alpha}_3 = \bar{\alpha}_3 + \bar{\alpha}_3 b_8.$$

Thus

$$\alpha_3 b_8 = c_7 + b_8 + \beta_9,$$

which implies that $(\alpha_3 b_8, \alpha_3 b_8) = 3$. Hence $(\alpha_3 \bar{\alpha}_3, b_8^2) = 3$. It follows that $\alpha_3 \bar{\alpha}_3 = 1 + x'_4 + \bar{x}'_4$ by the expression b_8^2 (see (2.18)), which contradicts Theorem 2.1.

Step 2 There exists no NITA such that $z_{12} = \alpha_4 + \beta_8$.

If $z_{12} = \alpha_4 + \beta_8$, then $b_3(\bar{\alpha}_4 + \bar{\beta}_8) = r_3 + t_4 + 2b_9 + x_{11}$. Since $b_9 \in Supp\{b_3 \bar{\alpha}_4\}$, we have that

$$b_3 \bar{\alpha}_4 = r_3 + b_9, \qquad b_3 \bar{\beta}_8 = t_4 + b_9 + x_{11}.$$

Let $\bar{b}_3 r_3 = \bar{\alpha}_4 + e_5$. Then

$$\begin{aligned}\bar{b}_3(b_3\bar{\alpha}_4) &= \bar{b}_3 b_9 + \bar{b}_3 r_3\\ &= c_7 + b_8 + \bar{\alpha}_4 + \bar{\beta}_8 + \bar{\alpha}_4 + e_5,\\ \bar{\alpha}_4(b_3\bar{b}_3) &= \bar{\alpha}_4 + \bar{\alpha}_4 b_8,\end{aligned}$$

which means that $\alpha_4 b_8 = c_7 + b_8 + \beta_8 + e_5$. Since $\bar{b}_3 y_8 = \alpha_4 + \beta_8 + \bar{x}'_4 + b_8$, we have that $b_3\alpha_4 = y_8 + \gamma_4,\ \gamma_4 \in B$.

Since

$$\begin{aligned}\bar{b}_3(b_3\alpha_4) &= \bar{b}_3 y_8 + b_3\gamma_4\\ &= b_8 + \alpha_4 + \beta_8 + \bar{x}'_4 + b_3\gamma_4,\\ (\bar{b}_3 b_3)\alpha_4 &= \alpha_4 + \alpha_4 b_8\\ &= 2\alpha_4 + c_7 + b_8 + \beta_8 + e_5,\end{aligned}$$

we have that $\bar{x}'_4 + b_3\gamma_4 = \alpha_4 + c_7 + e_5$, which implies that $\bar{x}'_4 = \alpha_4$. But by (2.21) $b_3 x'_4 = j_8 + x_4,\ j_8 \in B$ and $b_3\bar{\alpha}_4 = b_9 + r_3$, a contradiction.

Step 3 There exists no NITA such that $z_{12} = \alpha_5 + \beta_7$.

If $z_{12} = \alpha_5 + \beta_7$, then

$$b_3(\bar{\alpha}_5 + \bar{\beta}_7) = 2b_9 + r_3 + t_4 + 2b_9 + x_{11}.$$

We need a sum of degree 6 to make up $b_3\bar{\alpha}_5$ for $b_9 \in Supp\{b_3\bar{\alpha}_5\}$.

If $r_3 \in Supp\{b_3\bar{\beta}_7\}$, then $\bar{b}_3 r_3$ contains an element of degree 2, a contradiction. Hence $r_3 \in Supp\{b_3\bar{\alpha}_5\}$. We may set

$$b_3\bar{\alpha}_5 = b_9 + r_3 + \delta_3, \quad \delta_3 \in Supp\{x_{11}\}.$$

Furthermore $\bar{b}_3 r_3 = \bar{\alpha}_5 + d_4,$ for some $d_4 \in B$. It follows that $r_3\bar{r}_3 = 1 + b_8$. Now we have that $(\alpha_5\bar{\alpha}_5, b_8) = 2$ and

$$2b_3 + x_4 + y_8 + b_9 = b_3(b_3\bar{b}_3) = r_3(b_3\bar{r}_3) = r_3\alpha_5 + r_3\bar{d}_4,$$

from which the following holds:

$$r_3\alpha_5 + r_3\bar{d}_4 = 2b_3 + x_4 + y_8 + b_9.$$

Comparing the two sides, the following equations hold:

$$\begin{aligned}r_3\bar{d}_4 &= b_3 + b_9,\\ r_3\alpha_5 &= b_3 + x_4 + y_8.\end{aligned}$$

Therefore

$$\begin{aligned}\bar{b}_3(r_3\alpha_5) &= b_3\bar{b}_3 + x_4\bar{b}_3 + y_8\bar{b}_3\\ &= 1 + b_8 + b_8 + x_4' + b_8 + \alpha_5 + \beta_7 + \bar{x}_4',\\ \alpha_5(\bar{b}_3 r_3) &= \alpha_5\bar{\alpha}_5 + d_4\alpha_5.\end{aligned}$$

Since $(\alpha_5\bar{\alpha}_5, b_8) = 2$, we can only have that

$$\begin{aligned}\alpha_5\bar{\alpha}_5 &= 1 + 2b_8 + x_4' + \bar{x}_4',\\ d_4\alpha_5 &= b_8 + \alpha_5 + \beta_7.\end{aligned}$$

Hence $d_4 = x_4'$ or $\bar{x}_4'$. So $\bar{b}_3 r_3 = \alpha_5 + x_4'$ or $\alpha_5 + \bar{x}_4'$.

If $\bar{b}_3 r_3 = \alpha_5 + \bar{x}_4'$, then $r_3 \in Supp\{b_3\bar{x}_4'\} = \{y_8, t_4\}$ by (2.21), a contradiction. Hence $\bar{b}_3 r_3 = \alpha_5 + x_4'$, which implies that $r_3 \in Supp\{b_3 x_4'\} = \{x_4, j_8\}$, a contradiction.

Step 4 There exists no NITA such that $z_{12} = \alpha_6 + \beta_6$.

If $z_{12} = \alpha_6 + \beta_6$, then

$$b_3(\bar{\alpha}_6 + \bar{\beta}_6) = 2b_9 + t_4 + r_3 + x_{11}.$$

Hence $\alpha_6 \neq \beta_6$. Without loss of generality, let $r_3 \in Supp\{b_3\alpha_6\}$. We may set $\bar{b}_3 r_3 = \bar{\alpha}_6 + d_3$, which implies that $r_3\bar{r}_3 = 1 + b_8$. Therefore we have the following equation:

$$\begin{aligned}\alpha_6 r_3 + \bar{d}_3 r_3 &= (b_3\bar{r}_3)r_3\\ &= (b_3\bar{b}_3)b_3\\ &= 2b_3 + b_9 + x_4 + y_8.\end{aligned}$$

It is easy to see that we cannot sum up $\bar{d}_3 r_3$, a contradiction. Proposition 2.1 follows. □

Proposition 2.2 *There exists no NITA such that* $(f(x), f(y)) = (5, 7)$.

Proof If $(f(x), f(y)) = (5, 7)$, that is to say, $x_{12} = x_5 + y_7$, and

$$\begin{aligned}\bar{b}_3 b_5 &= b_3 + x_5 + y_7,\\ b_3 b_8 &= b_3 + b_9 + x_5 + y_7.\end{aligned}$$

So we may set

$$\begin{aligned}\bar{b}_3 x_5 &= b_8 + e_7, && e_7 \in N^*B,\\ b_3 x_5 &= b_5 + e_{10}, && e_{10} \in N^*B,\\ \bar{b}_3 y_7 &= b_8 + y_{13}, && y_{13} \in N^*B,\\ b_3 y_7 &= b_5 + y_{16}, && y_{16} \in N^*B.\end{aligned}$$

Hence

$$\begin{aligned}
\bar{b}_3(b_3b_8) &= \bar{b}_3b_3 + \bar{b}_3b_9 + \bar{b}_3x_5 + \bar{b}_3y_7 \\
&= 1 + b_8 + c_7 + b_8 + \bar{z}_{12} + b_8 + e_7 + b_8 + y_{13}, \\
(b_3\bar{b}_3)b_8 &= b_8 + b_8^2, \\
(\bar{b}_3b_5)b_3 &= b_3^2 + b_3x_5 + b_3y_7 \\
&= b_4 + 3b_5 + e_{10} + y_{16}, \\
(b_3\bar{b}_3)b_5 &= b_5 + b_5b_8,
\end{aligned}$$

and one has that

$$\begin{aligned}
b_8^2 &= 1 + c_7 + e_7 + 3b_8 + \bar{z}_{12} + y_{13}, \\
b_5b_8 &= b_4 + 2b_5 + e_{10} + y_{16}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(b_3\bar{b}_3)^2 &= 1 + 2b_8 + 1 + c_7 + e_7 + 3b_8 + \bar{z}_{12} + y_{13}, \\
b_3^2\bar{b}_3^2 &= 1 + b_8 + c_7 + b_8 + z_{12} + b_8 + \bar{z}_{12} + 1 + 2b_8 + d_8,
\end{aligned}$$

which implies that $z_{12} + d_8 = e_7 + y_{13}$.

Step 1 There exists no NITA such that $e_7 \notin Supp\{z_{12}\}$.

Suppose that $e_7 \notin Supp\{z_{12}\}$. Since $e_7 \notin Supp\{d_8\}$, there exist $\alpha_3 \in Supp\{z_{12}\}$, $g_4 \in Supp\{d_8\}$ or $\alpha_4 \in Supp\{z_{12}\}$, $g_3 \in Supp\{d_8\}$ such that

$$e_7 = \alpha_3 + g_4 \text{ or } \alpha_4 + g_3.$$

Substep 1 There exists no NITA such that $e_7 = \alpha_3 + g_4$.

If $e_7 = \alpha_3 + g_4$, then there exist $z_9 \in N^*B$ and $h_4 \in B$ such that $z_{12} = \alpha_3 + z_9$ and $d_8 = g_4 + h_4$. Furthermore,

$$\begin{aligned}
\bar{b}_4b_5 &= \alpha_3 + z_9 + b_8, \\
\bar{b}_3b_9 &= c_7 + b_8 + \bar{\alpha}_3 + \bar{z}_9, \\
\bar{b}_3x_5 &= b_8 + \alpha_3 + g_4,
\end{aligned}$$

Hence

$$\begin{aligned}
b_3\alpha_3 &= x_5 + i_4, \quad i_4 \in B, \\
b_3\bar{\alpha}_3 &= b_9,
\end{aligned}$$

which is impossible for $(b_3\bar{\alpha}_3, b_3\bar{\alpha}_3) = 1$ and $(b_3\alpha_3, b_3\alpha_3) = 2$.

Substep 2 There exists no NITA such that $e_7 = \alpha_4 + g_3$.

If $e_7 = \alpha_4 + g_3$, then there exist $z_8 \in Supp\{N^*B\}$ and $h_5 \in B$ such that $\bar{z}_{12} = \alpha_4 + z_8$, $d_8 = g_3 + h_5$. Hence

$$\bar{b}_3 x_5 = g_3 + \alpha_4 + b_8,$$
$$b_5 \bar{b}_5 = 1 + 2b_8 + g_3 + h_5,$$
$$\bar{b}_3 y_7 = b_8 + z_8 + h_5.$$

So there exists $l_4 \in B$ and $n_8 \in N^*B$ such that $b_3 g_3 = x_5 + l_4$ and $b_3 h_5 = y_7 + n_8$. Hence $g_3 \bar{g}_3 = 1 + b_8$ and $\bar{b}_3 l_4 = g_3 + m_9$. Here $m_9 \in B$, otherwise $\bar{l}_4 l_4$ will contains two b_8, which is a contradiction.

Since

$$b_3(b_3\bar{b}_3) = 2b_3 + b_9 + x_5 + y_7,$$
$$(b_3 g_3)\bar{g}_3 = \bar{g}_3 x_5 + \bar{g}_3 l_4,$$

it follows that $\bar{g}_3 l_4 = b_3 + b_9$ and

$$\bar{g}_3 x_5 = b_3 + x_5 + y_7 = \bar{b}_3 b_5. \tag{2.25}$$

We assert that $Supp\{n_8\} \cap \{b_3, x_5, y_7, l_4\} = \emptyset$. Obviously $y_7 \notin Supp\{n_8\}$. By the expressions of $b_3 h_5$ and $b_3 \bar{b}_3$, we have that $b_3 \notin Supp\{n_8\}$. If $x_5 \in Supp\{n_8\}$, then $h_5 \in Supp\{\bar{b}_3 x_5\}$, a contradiction. If $l_4 \in Supp\{n_8\}$, then $h_5 \in Supp\{\bar{b}_3 l_4\}$, a contradiction. Since

$$\begin{aligned} b_3(b_5\bar{b}_5) &= b_3 + 2b_3 b_8 + b_3 g_3 + b_3 h_5 \\ &= 3b_3 + 2b_9 + 2x_5 + 2y_7 + x_5 + l_4 + y_7 + n_8, \\ (b_3\bar{b}_5)b_5 &= b_5(\bar{b}_3 + \bar{x}_5 + \bar{y}_7) \\ &= b_3 + x_5 + y_7 + \bar{x}_5 b_5 + \bar{y}_7 b_5, \end{aligned}$$

it holds that

$$b_5\bar{x}_5 + b_5\bar{y}_7 = 2b_3 + 2b_9 + 2x_5 + 2y_7 + l_4 + n_8. \tag{2.26}$$

Now we assert that $(b_5\bar{y}_7, b_9) = 1$. In fact, it follows from that $b_5\bar{b}_9 = b_9 + b_3\alpha_4 + b_3 z_8$ by Lemma 2.3 and $(b_3\alpha_4, y_7) = 0$, $(b_3 z_8, y_7) = 1$. Therefore $(b_5\bar{x}_5, b_3) = (b_5\bar{x}_5, b_9) = 1$ and $(b_5\bar{y}_7, b_3) = (b_5\bar{y}_7, b_9) = 1$.

By (2.25), $(\bar{g}_3 x_5, \bar{b}_3 b_5) = 3$, so $(b_3\bar{g}_3, b_5\bar{x}_5) = 3$. By (2.26), $b_5\bar{x}_5$ cannot contain three multiples of a constituent of degree 3 or two multiples of a constituent of degree 6. Then the common constituents of $b_3\bar{g}_3$ and $b_5\bar{x}_5$ are degree 5 or 4, since $b_3\bar{g}_3$ can now only have constituents of degree 5 and 4. From $(b_3\bar{g}_3, b_5\bar{x}_5) = 3$ and (2.26), we have that $b_5\bar{x}_5$ must contain $2x_5$ and some element v_4 of degree 4 from $l_4 + n_8$. This leads to

$$b_5\bar{x}_5 = b_3 + b_9 + 2x_5 + v_4,$$

a contradiction for the left hand side having degree 26.

Step 2 There exists no NITA such that $e_7 \in Supp\{z_{12}\}$.

If $e_7 \in Supp\{z_{12}\}$, then there exists $w_5 \in B$ such that $z_{12} = e_7 + w_5$. So $y_{13} = d_8 + w_5$ and

$$\bar{b}_3 x_5 = b_8 + e_7, \tag{2.27}$$

$$\bar{b}_3 y_7 = b_8 + d_8 + w_5, \tag{2.28}$$

$$\bar{b}_4 b_5 = w_5 + e_7 + b_8, \tag{2.29}$$

$$\bar{b}_3 b_9 = \bar{w}_5 + c_7 + \bar{e}_7 + b_8. \tag{2.30}$$

Since $(\bar{b}_4 b_5, \bar{b}_4 b_5) = 3$, we have that $e_7, w_5 \in B$. By Lemma 2.3

$$b_5 \bar{b}_9 = b_9 + b_3 e_7 + b_3 w_5. \tag{2.31}$$

Since $y_7 \in Supp\{b_3 w_5\}$, $b_9 \in Supp\{b_3 \bar{w}_5\}$, and $x_5 \in Supp\{b_3 e_7\}$, we may set

$$b_3 w_5 = y_7 + v_8, \quad v_8 \in N^* B, \tag{2.32}$$

$$b_3 \bar{w}_5 = b_9 + d_6, \quad d_8 \in N^* B, \tag{2.33}$$

$$b_3 c_7 = b_9 + f_{12}, \quad f_{12} \in N^* B, \tag{2.34}$$

$$b_3 e_7 = x_5 + h_{16}, \quad h_{16} \in N^* B, \tag{2.35}$$

$$b_3 \bar{e}_7 = b_9 + g_{12}, \quad g_{12} \in N^* B. \tag{2.36}$$

Case 1 There exists no NITA such that either v_8 or d_6 is reducible.

It is easy to see that when either v_8 or d_6 is reducible, this will imply that the other one is also reducible. Let $d_6 = d_3 + g_3$. Then

$$b_3 \bar{w}_5 = d_3 + g_3 + b_9.$$

It is easy to show that $d_3 \neq g_3$. Hence we may set

$$b_3 \bar{d}_3 = w_5 + \delta_4,$$

$$b_3 \bar{g}_3 = w_5 + \psi_4,$$

which implies that $d_3 \bar{d}_3 = g_3 \bar{g}_3 = 1 + b_8$. If $\delta_4 = \psi_4$, then $b_3 \bar{d}_3 = b_3 \bar{g}_3$, $(b_3 \bar{d}_3, b_3 \bar{g}_3) = 2$, which implies that $d_3 \bar{g}_3 = 1 + b_8$. Thus $g_3 = d_3$ and $b_3 \bar{w}_5 = b_9 + 2g_3$, so that $(b_3 \bar{g}_3, w_5) = 2$, a contradiction. Hence $\delta_4 \neq \psi_4$. We calculate $b_3^2 \bar{b}_3 = b_3 d_3 \bar{d}_3 = b_3 g_3 \bar{g}_3$. We have that

$$(b_3 \bar{b}_3) b_3 = 2b_3 + b_9 + x_5 + y_7,$$

$$(b_3 \bar{d}_3) d_3 = w_5 d_3 + \delta_4 d_3,$$

$$(b_3 \bar{g}_3) g_3 = w_5 g_3 + \psi_4 g_3.$$

It is easy to see that

$$\delta_4 d_3 = \psi_4 g_3 = b_3 + b_9,$$
$$w_5 d_3 = w_5 g_3 = b_3 + x_5 + y_7.$$

On the other hand, we have that

$$\begin{aligned} w_5\bar{w}_5 &= 1 + 2b_8 + u_8, \\ \bar{b}_3(g_3\psi_4) &= \bar{b}_3(d_3\delta_4) = \bar{b}_3 b_3 + \bar{b}_3 b_9 \\ &= 1 + b_8 + c_7 + \bar{e}_7 + \bar{w}_5 + b_8, \\ (\bar{b}_3 g_3)\psi_4 &= \bar{w}_5\psi_4 + \bar{\psi}_4\psi_4, \\ (\bar{b}_3 d_3)\delta_4 &= \bar{w}_5\delta_4 + \bar{\delta}_4\delta_4, \end{aligned}$$

so that

$$\bar{w}_5\psi_4 = \bar{w}_5\delta_4 = \bar{e}_7 + \bar{w}_5 + b_8,$$
$$\psi_4\bar{\psi}_4 = \delta_4\bar{\delta}_4 = 1 + b_8 + c_7.$$

Hence $\psi_4, \delta_4 \in Supp\{w_5\bar{w}_5\}$. Therefore

$$w_5\bar{w}_5 = 1 + 2b_8 + \delta_4 + \psi_4.$$

But $(g_3\bar{d}_3, w_5\bar{w}_5) = (g_3 w_5, d_3 w_5) = 3$. We have that $1 \in Supp\{g_3\bar{d}_3\}$, which implies that $g_3 = d_3$, a contradiction.

Case 2 There exists no NITA such that v_8 and d_6 are irreducible.

Suppose v_8 and d_6 are irreducible. We assert that f_{12} is irreducible. Otherwise, let $f_{12} = r + s$. Suppose that $c_7 \in Supp\{\bar{b}_3 r\}$, which implies that $f(r) \geq 4$.

By Lemma 2.3, one has that

$$b_8 c_7 = e_7 + w_5 + b_8 + b_3\bar{f}_{12},$$
$$\bar{b}_9 b_9 = 1 + d_8 + 3b_8 + e_7 + w_5 + b_3\bar{f}_{12}.$$

Subcase 1 w_5 is not real.

If w_5 is real, then $y_7 + v_8 = b_3 w_5 = b_3\bar{w}_5 = b_9 + d_6$ by (2.32) and (2.33), a contradiction.

Subcase 2 $f_{12} \notin B$.

By the expression $b_8 c_7$, we have that $\bar{w}_5 \in Supp\{b_3\bar{f}_{12}\}$. If f_{12} is irreducible, then $f_{12} \in Supp\{b_3 w_5\} = \{y_7, v_8\}$ by (2.32), a contradiction. So f_{12} is reducible.

Subcase 3 e_7 is not real.

If e_7 is real, then there exists an element $u_7 \in N^*B$ by (2.27) and (2.30), such that

$$b_3e_7 = x_5 + b_9 + u_7.$$

Hence $(e_7^2, b_8) \geq 2$, so $(e_7b_8, e_7) \geq 2$. By the expressions of b_8c_7 and b_8^2, we have that $(e_7b_8, c_7) = 1$, $(e_7b_8, b_8) \geq 2$. Thus there exists an element $x_{19} \in N^*B$ of degree 19 such that

$$e_7b_8 = 2e_7 + 2b_8 + c_7 + x_{19}.$$

On the other hand, from (2.27) and (2.30), we have that

$$\begin{aligned}
\bar{b}_3(b_3e_7) &= \bar{b}_3x_5 + \bar{b}_3b_9 + \bar{b}_3u_7 \\
&= b_8 + e_7 + c_7 + e_7 + \bar{w}_5 + b_8 + \bar{b}_3u_7, \\
(\bar{b}_3b_3)e_7 &= e_7 + b_8e_7 \\
&= 3e_7 + 2b_8 + c_7 + x_{19}.
\end{aligned}$$

Then $\bar{w}_5 + \bar{b}_3u_7 = e_7 + x_{19}$. Since the right side is real, we have that $w_5 \in Supp\{\bar{b}_3u_7\}$ by Substep 1, which implies that there exists a constituent t of u_7 such that $w_5 \in Supp\{\bar{b}_3t\}$. Then $t \in Supp\{b_3w_5\}$, so $t = y_7$ or v_8. It follows that $u_7 = y_7$. By (2.28) implies that

$$w_5 + \bar{w}_5 + b_8 + d_8 = \bar{w}_5 + \bar{b}_3y_7 = e_7 + x_{19},$$

a contradiction.

Subcase 4 Case 2 follows.

By Subcases 1, 2, 3, we have that e_7 and w_5 are not real, and f_{12} is reducible. Note that b_8c_7 is real, by the expression b_8c_7, we have that $b_3\bar{f}_{12}$ contains $\bar{w}_5$ and $\bar{e}_7$. Hence there exists a constituent r of f_{12} such that $b_3\bar{r}$ contains $\bar{w}_5$, which implies that $r \in Supp\{b_3w_5\}$. Therefore $r = y_7$ or v_8.

If $r = y_7$, then $b_3\bar{y}_7 = b_8 + d_8 + \bar{w}_5$ by (2.28). But (2.34) means that $b_3\bar{y}_7$ contains c_7, a contradiction.

If $r = v_8$, then there exists s_4 such that $f_{12} = v_8 + s_4$. Since $c_7 \in Supp\{b_3\bar{s}_4\}$ by (2.34), we have that $\bar{e}_7 \notin Supp\{b_3\bar{s}_4\}$. So $\bar{e}_7 \in Supp\{b_3\bar{v}_8\}$. Hence there exists a basis element a_5 of degree 5 such that

$$b_3\bar{v}_8 = \bar{w}_5 + \bar{e}_7 + c_7 + a_5, \tag{2.37}$$

$$b_3\bar{s}_4 = c_7 + c_5, \tag{2.38}$$

$$b_3c_7 = v_8 + s_4 + b_9. \tag{2.39}$$

Checking $b_3(b_5\bar{b}_5) = b_5(b_3\bar{b}_5)$, this implies that

$$b_5\bar{x}_5 + b_5\bar{y}_7 = 2b_3 + 2b_9 + x_5 + y_7 + b_3d_8. \tag{2.40}$$

We assert that $d_8 \in B$. In fact, if $d_8 \notin B$, then $d_8 = 2x_4$ or $x_4 + y_4$.

If $d_8 = 2x_4$, then $\bar{b}_3 y_7 = b_8 + 2x_4 + w_5$ by (2.28). Hence $(b_3 x_4, y_7) = 2$, which is impossible.

If $d_8 = x_4 + y_4$, then $\bar{b}_3 y_7 = b_8 + x_4 + y_4 + w_5$ by (2.28). Let $b_3 x_4 = y_7 + p_5$ and $b_3 y_4 = y_7 + q_5$. By (2.40), it holds that

$$b_5\bar{x}_5 + b_5\bar{y}_7 = 2b_3 + 2b_9 + x_5 + 3y_7 + p_5 + q_5.$$

But $(b_5\bar{x}_5, b_3) = (b_5\bar{x}_5, b_9) = 1$. So it is necessary to find elements from the right hand side of the above equation such that their sum having degree 13 yields $b_5\bar{x}_5$. This is impossible. Therefore $d_8 \in B$.

Since

$$\begin{aligned}
\bar{b}_3(b_3 w_5) &= \bar{b}_3 y_7 + \bar{b}_3 v_8 \\
&= b_8 + d_8 + w_5 + c_7 + w_5 + e_7 + \bar{a}_5, \\
b_3(\bar{b}_3 w_5) &= b_3\bar{b}_9 + b_3\bar{d}_6 \\
&= w_5 + c_7 + e_7 + b_8 + b_3\bar{d}_6,
\end{aligned}$$

then

$$b_3\bar{d}_6 = d_8 + w_5 + \bar{a}_5. \tag{2.41}$$

Now we assert that c_5 and e_5 are nonreal. Otherwise, $d_6, v_8, \in Supp\{b_3 a_5\}$ by (2.37) and (2.41). So $1 \in Supp\{b_3 a_5\}$, a contradiction. Hence c_5 and a_5 are nonreal, and $c_5 = \bar{a}_5$ by the expression b_8^2. Hence $b_3\bar{d}_6 = d_8 + w_5 + c_5$. Associated with (2.38), we have that $(b_3\bar{c}_5, d_6) = (b_3\bar{c}_5, s_4) = 1$, which implies that $(b_3\bar{c}_5, b_3\bar{c}_5) = 3$. Furthermore $(b_3 c_5, b_3 c_5) = 3$. Let

$$b_3 c_5 = v_8 + \delta_3 + \lambda_4.$$

Thus we may set $\bar{b}_3\delta_3 = c_5 + \Delta_4$. Then $\bar{\delta}_3\delta_3 = 1 + b_8$, and

$$\begin{aligned}
b_3(\delta_3\bar{\delta}_3) &= b_3 + b_3 b_8 \\
&= 2b_3 + b_9 + x_5 + y_7, \\
(b_3\bar{\delta}_3)\delta_3 &= \bar{c}_5\delta_3 + \bar{\Delta}_4\delta_3.
\end{aligned}$$

We have that

$$\bar{\Delta}_4\delta_3 = b_3 + b_9 \quad \text{and} \quad \bar{c}_5\delta_3 = b_3 + x_5 + y_7. \tag{2.42}$$

Furthermore

$$\begin{aligned}
\bar{b}_3(\bar{\Delta}_4\delta_3) &= \bar{b}_3 b_3 + \bar{b}_3 b_9 \\
&= 1 + 2b_8 + c_7 + \bar{w}_5 + \bar{e}_7, \\
(\bar{b}_3\delta_3)\bar{\Delta}_4 &= c_5\bar{\Delta}_4 + \Delta_4\bar{\Delta}_4.
\end{aligned}$$

It follows that

$$\Delta_4\bar{\Delta}_4 = 1 + b_8 + c_7 \quad \text{and} \quad c_5\bar{\Delta}_4 = b_8 + \bar{w}_5 + \bar{e}_7 = b_4\bar{b}_5.$$

Hence $(b_3c_5)\bar{\Delta}_4 = b_3(c_5\bar{\Delta}_4) = b_3(b_4\bar{b}_5) = b_4(b_3\bar{b}_5)$. But

$$b_4(b_3\bar{b}_5) = b_4\bar{b}_3 + b_4\bar{x}_5 + b_4\bar{y}_7 = b_3 + b_9 + b_4\bar{x}_5 + b_4\bar{y}_7,$$
$$(b_3c_5)\bar{\Delta}_4 = v_8\bar{\Delta}_4 + \delta_3\bar{\Delta}_4 + \lambda_4\bar{\Delta}_4.$$

So b_3 is a constituent of one of $v_8\bar{\Delta}_4$, $\delta_3\bar{\Delta}_4$ and $\lambda_4\bar{\Delta}_4$. Then one of v_8, δ_3 and λ_4 is a constituent of $b_3\Delta_4$. On the other hand $\delta_3 \in Supp\{b_3\Delta_4\}$ by (2.42). But $b_3\Delta_4$ contains at most two constituents, a contradiction. Subcase 4 follows, which concludes Step 2. Consequently the proposition follows. □

Proposition 2.3 *There exists no NITA such that* $(f(x), f(y)) = (6, 6)$.

Proof If $x_{12} = x_6 + y_6$, then

$$\bar{b}_3b_5 = b_3 + x_6 + y_6,$$
$$b_3b_8 = b_3 + b_9 + x_6 + y_6.$$

Assume that

$$\bar{b}_3x_6 = b_8 + h_{10}, \quad \text{for some } h_{10} \in N^*B,$$
$$\bar{b}_3y_6 = b_8 + i_{10}, \quad \text{for some } i_{10} \in N^*B,$$
$$b_3x_6 = b_5 + g_{13}, \quad \text{for some } g_{13} \in N^*B,$$
$$b_3y_6 = b_5 + f_{13}, \quad \text{for some } f_{13} \in N^*B.$$

Therefore,

$$b_8^2 = 1 + c_7 + 3b_8 + h_{10} + i_{10} + \bar{z}_{12}.$$

By the above equations and Lemma 2.3, one has that

$$(\bar{b}_3b_3)^2 = 2 + c_7 + 5b_8 + h_{10} + i_{10} + \bar{z}_{12},$$
$$\bar{b}_3^2b_3^2 = \bar{b}_4b_4 + \bar{b}_4b_5 + \bar{b}_5b_4 + \bar{b}_5b_5$$
$$= 2 + 5b_8 + c_7 + \bar{z}_{12} + z_{12} + d_8.$$

Hence

$$d_8 + z_{12} = h_{10} + i_{10}.$$

We assert that d_8 cannot be totally contained in one of h_{10} and i_{10}. Otherwise, without loss of generality, assume d_8 is contained in h_{10}. Then h_{10} contains an element of degree 2, a contradiction. Thus d_8 must be reducible and cannot be contained in one of h_{10} and i_{10}. For d_8, there are four possibilities:

$$d_8 = c_3 + d_5, \quad c_4 + d_4, \quad c_4 + \bar{c}_4, \quad \text{and} \quad 2c_4.$$

Step 1 There exists no NITA such that $d_8 = c_3 + d_5$.

If $d_8 = c_3 + d_5$, then $c_3 + d_5 + z_{12} = h_{10} + i_{10}$. Without loss of generality, let

$$h_{10} = c_3 + h_7, \qquad i_{10} = d_5 + i_5.$$

Then $z_{12} = h_7 + i_5$ and

$$\bar{b}_3 x_6 = c_3 + h_7 + b_8, \tag{2.43}$$

$$\bar{b}_3 y_6 = d_5 + i_5 + b_8, \tag{2.44}$$

$$\bar{b}_4 b_5 = b_8 + i_5 + h_7, \tag{2.45}$$

$$\bar{b}_3 b_9 = c_7 + b_8 + \bar{i}_5 + \bar{h}_7, \tag{2.46}$$

$$b_8^2 = 1 + c_7 + 3b_8 + c_3 + h_7 + d_5 + i_5 + \bar{h}_7 + \bar{i}_5. \tag{2.47}$$

Hence c_3 is real. By (2.45) and $(\bar{b}_4 b_5, \bar{b}_4 b_5) = 3$, we have that $h_7 \in B$. Set $b_3 c_3 = x_6 + u_3$, $u_3 \in B$ by Lemma 2.2. Thus $(b_3 c_3, b_3 c_3) = 2$ so that

$$c_3^2 = 1 + b_8.$$

Moreover,

$$\begin{aligned}(b_3 c_3) c_3 &= x_6 c_3 + u_3 c_3,\\ b_3 c_3^2 &= b_3^2 \bar{b}_3 = b_3 + b_3 b_8\\ &= 2b_3 + b_9 + x_6 + y_6.\end{aligned}$$

Hence

$$\begin{aligned}c_3 u_3 &= b_3 + x_6 \text{ or } b_3 + y_6,\\ c_3 x_6 &= b_3 + b_9 + y_6 \text{ or } b_3 + b_9 + x_6.\end{aligned}$$

If $c_3 u_3 = b_3 + x_6,\ c_3 x_6 = b_3 + b_9 + y_6$, then $c_3 \in Supp\{\bar{u}_3 x_6\}$. But

$$\begin{aligned}(b_3 \bar{b}_3)(c_3^2) &= 2 + 5b_8 + c_7 + c_3 + d_5 + h_7 + \bar{h}_7 + i_5 + \bar{i}_5,\\ (b_3 c_3)(\bar{b}_3 c_3) &= x_6 \bar{x}_6 + u_3 \bar{x}_6 + \bar{u}_3 x_6 + 1 + b_8.\end{aligned}$$

Note that $u_3 \bar{x}_6 + \bar{u}_3 x_6$ contains two c_3, and so we come to a contradiction. Therefore $c_3 u_3 = b_3 + y_6$, $c_3 x_6 = b_3 + b_9 + x_6$. Let $\bar{b}_3 u_3 = c_3 + w_6$. Then $u_3 \bar{u}_3 = 1 + b_8$ by Lemma 2.2 and

$$(c_3b_3)\bar{b}_3 = \bar{b}_3x_6 + \bar{b}_3u_3$$
$$= b_8 + c_3 + h_7 + c_3 + w_6.$$

Since $c_3(b_3\bar{b}_3)$ is real, it follows that h_7 and w_6 are real. Since

$$(\bar{b}_3u_3)c_3 = c_3^2 + c_3w_6$$
$$= 1 + b_8 + c_3w_6,$$
$$(\bar{b}_3c_3)u_3 = \bar{x}_6u_3 + \bar{u}_3u_3$$
$$= 1 + b_8 + \bar{x}_6u_3,$$
$$\bar{b}_3(c_3u_3) = \bar{b}_3b_3 + \bar{b}_3y_6$$
$$= 1 + b_8 + b_8 + d_5 + i_5,$$

we have that

$$c_3w_6 = \bar{x}_6u_3 = d_5 + i_5 + b_8.$$

Since c_3, w_6 are real, i_5 is also real by the above equation. By (2.43), (2.44) and (2.46), one has that

$$b_3i_5 = y_6 + b_9,$$
$$b_3h_7 = b_9 + x_6 + l_6, \quad \text{for some } l_6 \in N^*B.$$

Considering $c_3^2u_3\bar{u}_3$, we have that

$$(c_3u_3)(c_3\bar{u}_3) = b_3\bar{b}_3 + \bar{b}_3y_6 + b_3\bar{y}_6 + y_6\bar{y}_6$$
$$= 1 + b_8 + 2b_8 + 2d_5 + 2i_5 + y_6\bar{y}_6,$$
$$c_3^2(u_3\bar{u}_3) = 2 + 5b_8 + c_7 + c_3 + 2i_5 + 2h_7 + d_5.$$

Comparing the number of elements of degree 5, we come to a contradiction. Step 1 follows.

Step 2 There exists no NITA such that $d_8 = 2c_4$.

If $d_8 = 2c_4$, then $2c_4 + z_{12} = h_{10} + i_{10}$. There exist h_6 and i_6 such that $z_{12} = h_6 + i_6$. Since $(\bar{b}_4b_5, \bar{b}_4b_5) \le 4$, we obtain that $h_6, i_6 \in B$. Furthermore

$$\bar{b}_3x_6 = b_8 + c_4 + h_6, \qquad \bar{b}_3y_6 = b_8 + c_4 + i_6.$$

Thus $b_3c_4 = x_6 + y_6$. Then we can set

$$c_4^2 = 1 + b_8 + g_7, \quad g_7 \in N^*B.$$

Since $b_5\bar{b}_5 = 1 + 2b_8 + 2c_4$, $(c_4b_5, b_5) = 2$. Furthermore, $(c_4b_5, c_4b_5) \ge 5$. Hence

$$c_4^2 = 1 + b_8 + c_4 + e_3$$

and $(c_4b_5, c_4b_5) = 5$. There exists $t_{10} \in B$ such that

$$c_4b_5 = 2b_5 + t_{10}.$$

Since

$$\begin{aligned} b_3(b_5\bar{b}_5) &= b_3 + 2b_3b_8 + 2b_3c_4 \\ &= 3b_3 + 2b_9 + 4x_6 + 4y_6, \\ b_5(b_3\bar{b}_5) &= b_5(\bar{b}_3 + \bar{x}_6 + \bar{y}_6) \\ &= b_3 + x_6 + y_6 + \bar{x}_6b_5 + \bar{y}_6b_5, \end{aligned}$$

we have that $b_5\bar{x}_6 + b_5\bar{y}_6 = 2b_3 + 2b_9 + 3x_6 + 3y_6$. On the other hand,

$$\begin{aligned} &\bar{b}_3(c_4b_5) = 2\bar{b}_3b_5 + \bar{b}_3t_{10} = 2b_3 + 2x_6 + 2y_6 + \bar{b}_3t_{10}, \\ &(\bar{b}_3c_4)b_5 = \bar{x}_6b_5 + \bar{y}_6b_5 = 2b_3 + 2b_9 + 3x_6 + 3y_6. \end{aligned}$$

Hence $\bar{b}_3t_{10} = 2b_9 + x_6 + y_6$, from which it follows that $(b_3b_9, t_{10}) = 2$. Therefore $(b_3b_9, b_3b_9) \geq 5$. But $(\bar{b}_3b_9, \bar{b}_3b_9) = 4$ by Lemma 2.3 and h_6, $i_6 \in B$, a contradiction.

Step 3 There exists no NITA such that $d_8 = c_4 + d_4$.

If $d_8 = c_4 + d_4$, then there exist h_6, $i_6 \in N^*B$ such that $h_{10} = c_4 + h_6$, $i_{10} = d_4 + i_6$, $z_{12} = h_6 + i_6$ and

$$\bar{b}_3x_6 = c_4 + h_6 + b_8, \tag{2.48}$$

$$\bar{b}_3y_6 = d_4 + i_6 + b_8, \tag{2.49}$$

$$\bar{b}_4b_5 = b_8 + i_6 + h_6, \tag{2.50}$$

$$\bar{b}_3b_9 = c_7 + b_8 + \bar{i}_6 + \bar{h}_6, \tag{2.51}$$

$$b_8^2 = 1 + c_7 + 3b_8 + c_4 + h_6 + d_4 + i_6 + \bar{h}_6 + \bar{i}_6. \tag{2.52}$$

By (2.50) and $(\bar{b}_4b_5, \bar{b}_4b_5) = 3$, we have that $i_6, h_6 \in B$. Now we may set

$$b_3c_4 = x_6 + z_6, \quad z_6 \in B, \tag{2.53}$$

$$b_3d_4 = y_6 + p_6, \quad p_6 \in B. \tag{2.54}$$

Substep 1 h_6 and i_6 are nonreal.

If h_6 is real, then there exist e_3 and f_3 such that

$$\begin{aligned} \bar{b}_3h_6 &= \bar{b}_9 + \bar{x}_6 + e_3, \\ b_3e_3 &= f_3 + h_6. \end{aligned}$$

Thus $e_3\bar{e}_3 = 1 + b_8$ by Lemma 2.2. Since

$$b_3(e_3\bar{e}_3) = 2b_3 + b_9 + x_6 + y_6,$$
$$(b_3e_3)\bar{e}_3 = h_6\bar{e}_3 + f_3\bar{e}_3,$$

hence

$$f_3\bar{e}_3 = b_3 + x_6 \text{ or } b_3 + y_6,$$
$$h_6\bar{e}_3 = b_3 + b_9 + y_6 \text{ or } b_3 + b_9 + x_6.$$

Therefore

$$\begin{aligned}\bar{b}_3(f_3\bar{e}_3) &= \bar{b}_3b_3 + \bar{b}_3x_6 \text{ or } \bar{b}_3b_3 + \bar{b}_3y_6\\ &= 1 + 2b_8 + c_4 + h_6 \text{ or } 1 + 2b_8 + d_4 + i_6,\\ f_3(\bar{b}_3\bar{e}_3) &= f_3(h_6 + \bar{f}_3)\\ &= 1 + b_8 + f_3h_6.\end{aligned}$$

Thus

$$f_3h_6 = b_8 + c_4 + h_6 \text{ or } b_8 + d_4 + i_6.$$

On the other hand, one has the following equations:

$$\begin{aligned}(b_3\bar{b}_3)h_6 &= h_6 + h_6b_8,\\ (b_3h_6)\bar{b}_3 &= (\bar{e}_3 + x_6 + b_9)\bar{b}_3\\ &= \bar{b}_3b_9 + \bar{b}_3x_6 + \bar{b}_3\bar{e}_3\\ &= c_7 + 2b_8 + 3h_6 + \bar{i}_6 + c_4 + \bar{f}_3.\end{aligned}$$

Then

$$h_6b_8 = c_7 + c_4 + 2b_8 + 2h_6 + \bar{i}_6 + \bar{f}_3, \tag{2.55}$$

which implies that i_6 and f_3 are real.

Now let us calculate $(b_3\bar{b}_3)(e_3\bar{e}_3)$:

$$\begin{aligned}(b_3\bar{b}_3)(e_3\bar{e}_3) &= 2 + 5b_8 + c_7 + c_4 + d_4 + 2h_6 + 2i_6,\\ (b_3e_3)(\bar{b}_3\bar{e}_3) &= (h_6 + f_3)^2\\ &= h_6^2 + 2f_3h_6 + f_3^2\\ &= 1 + h_6^2 + 3b_8 + 2c_4 + 2h_6 \text{ or } 1 + h_6^2 + 3b_8 + 2d_4 + 2i_6.\end{aligned}$$

Comparing the number of elements of degree 4, we have that $c_4 = d_4$, a contradiction.

Symmetrically, we can prove that i_6 is nonreal.

Substep 2 There exists no NITA such that $d_8 = c_4 + d_4$ and h_6, i_6 are nonreal.

Since $(b_3\bar{b}_3)c_4 = c_4 + c_4b_8$ and $\bar{b}_3(b_3c_4) = \bar{b}_3x_6 + \bar{b}_3z_6 = c_4 + h_6 + b_8 + \bar{b}_3z_6$, we have that

$$c_4b_8 = b_8 + h_6 + \bar{b}_3z_6,$$

Since h_6 is nonreal and c_4b_8 is real, $\bar{h}_6 \in Supp\{\bar{b}_3z_6\}$. By (2.53), there exists $r_8 \in N^*B$ such that

$$\bar{b}_3z_6 = c_4 + \bar{h}_6 + r_8.$$

By (2.51), there exists α_3 such that

$$b_3\bar{h}_6 = b_9 + z_6 + \alpha_3.$$

Thus there exists γ_3 such that $b_3\bar{\alpha}_3 = h_6 + \gamma_3$. Hence $\alpha_3\bar{\alpha}_3 = 1 + b_8$ by Lemma 2.2.

Since

$$b_3(\alpha_3\bar{\alpha}_3) = 2b_3 + b_9 + x_6 + y_6,$$
$$(b_3\alpha_3)\bar{\alpha}_3 = h_6\bar{\alpha}_3 + \gamma_3\bar{\alpha}_3,$$

we have that

$$\gamma_3\bar{\alpha}_3 = b_3 + x_6 \text{ or } b_3 + y_6,$$
$$h_6\bar{\alpha}_3 = b_3 + b_9 + y_6 \text{ or } b_3 + b_9 + x_6.$$

Hence

$$\begin{aligned}\bar{b}_3(\gamma_3\bar{\alpha}_3) &= \bar{b}_3b_3 + \bar{b}_3x_6 \text{ or } \bar{b}_3b_3 + \bar{b}_3y_6\\ &= 1 + 2b_8 + c_4 + h_6 \text{ or } 1 + 2b_8 + d_4 + i_6,\\ \gamma_3(\bar{b}_3\bar{\alpha}_3) &= \gamma_3\bar{h}_6 + \gamma_3\bar{\gamma}_3\\ &= 1 + b_8 + \gamma_3\bar{h}_6.\end{aligned}$$

Therefore

$$\gamma_3\bar{h}_6 = b_8 + c_4 + h_6 \text{ or } b_8 + d_4 + i_6.$$

Since

$$\begin{aligned}(b_3\bar{b}_3)(\alpha_3\bar{\alpha}_3) &= 2 + 5b_8 + c_7 + c_4 + d_4 + 2h_6 + 2i_6,\\ (b_3\alpha_3)(\bar{b}_3\bar{\alpha}_3) &= (h_6 + \gamma_3)(\bar{h}_6 + \bar{\gamma}_3)\\ &= h_6\bar{h}_6 + \gamma_3\bar{h}_6 + \bar{\gamma}_3h_6 + \gamma_3\bar{\gamma}_3\\ &= 1 + 3b_8 + 2c_4 + h_6 + \bar{h}_6 + h_6\bar{h}_6\\ &\quad \text{or } 1 + 3b_8 + 2d_4 + i_6 + \bar{i}_6 + h_6\bar{h}_6,\end{aligned}$$

comparing the number of degree 4, we come to $c_4 = d_4$, a contradiction. So Substep 2 follows.

Step 4 There exists no NITA such that $d_8 = c_4 + \bar{c}_4$.

If $d_8 = c_4 + \bar{c}_4$, then $b_5\bar{b}_5 = 1 + 2b_8 + c_4 + \bar{c}_4$. And (2.48), (2.49), (2.50), (2.51), (2.52), (2.53) and (2.54) still hold with $d_4 = \bar{c}_4$. Thus

$$\begin{aligned}
\bar{b}_3(b_3c_4) &= \bar{b}_3x_6 + \bar{b}_3z_6,\\
&= c_4 + h_6 + b_8 + \bar{b}_3z_6,\\
b_3(\bar{b}_3c_4) &= b_3\bar{y}_6 + b_3\bar{p}_6\\
&= c_4 + \bar{i}_6 + b_8 + b_3\bar{p}_6.
\end{aligned}$$

Then

$$h_6 + \bar{b}_3z_6 = \bar{i}_6 + b_3\bar{p}_6. \tag{2.56}$$

Substep 1 There exists no NITA such that $h_6 = \bar{i}_6$.

If $h_6 = \bar{i}_6$, then it follows from (2.48) and (2.51) that there is $\alpha_3 \in B$ such that

$$\begin{aligned}
b_3h_6 &= b_9 + x_6 + \alpha_3,\\
b_3\bar{h}_6 &= b_9 + y_6 + \beta_3.
\end{aligned}$$

It is easy to see that $\alpha_3 \neq \beta_3$. Since $z_{12} = h_6 + \bar{h}_6$, this means by Lemma 2.3 that

$$b_5\bar{b}_9 = x_6 + y_6 + 3b_9 + \alpha_3 + \beta_3.$$

Thus $(b_5\bar{b}_9, b_5\bar{b}_9) = 13$.

By Lemma 2.3, we have that

$$b_9\bar{b}_9 = 1 + 3b_8 + c_4 + \bar{c}_4 + h_6 + \bar{h}_6 + \bar{b}_3f_{12}.$$

But $b_5\bar{b}_5 = 1 + 2b_8 + c_4 + \bar{c}_4$. Since $(b_9\bar{b}_9, b_8) = 3$, we have that

$$13 = (b_5\bar{b}_9, b_5\bar{b}_9) = 9 + (c_4, \bar{b}_3f_{12}) + (\bar{c}_4, \bar{b}_3f_{12}).$$

We conclude that $(c_4, \bar{b}_3f_{12}) = (\bar{c}_4, \bar{b}_3f_{12}) = 2$. Thus

$$(b_3c_4, f_{12}) = (b_3\bar{c}_4, f_{12}) = 2.$$

From (2.53) and (2.54), it follows that

$$(x_6 + z_6, f_{12}) = (y_6 + p_6, f_{12}) = 2.$$

Since $x_6 \neq y_6$ by $(b_3\bar{b}_3, b_5\bar{b}_5) = 3$ and $z_6 \neq x_6$, $y_6 \neq p_6$ by Lemma 2.2, it follows that $f_{12} = 2z_6$ and $z_6 = p_6$. Hence $b_3c_7 = b_9 + 2z_6$ by Lemma 2.3. Therefore $(b_3c_7, b_3c_7) = 5$ and $(\bar{b}_3z_6, c_7) = 2$. So $\bar{b}_3z_6 = 2c_7 + c_4$. But

$$b_8c_7 = b_8 + h_6 + \bar{h}_6 + 2\bar{b}_3z_6 = b_8 + h_6 + \bar{h}_6 + 4c_7 + 2c_4,$$

This is impossible for b_8c_7 is real and c_4 is not real.

Substep 2 There exists no NITA such that $h_6 \neq \bar{i}_6$.

Suppose that $h_6 \neq \bar{i}_6$. By (2.56), it follows that $(b_3\bar{p}_6, h_6) = (p_6, b_3\bar{h}_6) = (\bar{i}_6, \bar{b}_3z_6) = (b_3\bar{i}_6, z_6) = 1$. But $b_9 \in Supp\{b_3\bar{h}_6\}$ by (2.51) and $(c_4, b_6\bar{p}_6) = (c_4, \bar{b}_3z_6) = 1$ by (2.53) and (2.54). By (2.56), (2.51) and (2.54), there exist δ_3, z_3 and $v_8 \in N^*B$ such that

$$b_3\bar{h}_6 = b_9 + p_6 + \delta_3, \tag{2.57}$$

$$\bar{b}_3z_6 = \bar{i}_6 + c_4 + v_8, \tag{2.58}$$

$$b_3\bar{p}_6 = h_6 + c_4 + v_8. \tag{2.59}$$

Hence $\bar{b}_3\delta_3 = \bar{h}_6 + \gamma_3$, some $\gamma_3 \in B$. Therefore $(b_3\gamma_3, b_3\gamma_3) \geq 2$. It follows that

$$\delta_3\bar{\delta}_3 = \gamma_3\bar{\gamma}_3 = 1 + b_8$$

by Lemma 2.2.

Since

$$\begin{aligned}(b_3\bar{b}_3)c_4 &= c_4 + c_4b_8,\\ b_3(\bar{b}_3c_4) &= b_3\bar{y}_6 + b_3\bar{p}_6\\ &= c_4 + \bar{i}_6 + b_8 + h_6 + c_4 + v_8,\end{aligned}$$

we have that $c_4b_8 = c_4 + h_6 + \bar{i}_6 + v_8 + b_8$. Then $(c_4b_8, c_4b_8) \geq 5$. Let $c_4\bar{c}_4 = 1 + b_8 + r_7$, $r_{\in}N^*B$ and r_7 is real. By (2.52) and c_4 is not real, we have that $r_7 = c_7$ and then $(c_4b_8, c_4b_8) = 5$. Consequently $v_8 \in B$. By $b_3(\delta_3\bar{\delta}_3) = (b_3\bar{\delta}_3)\delta_3$, we have that

$$\begin{aligned}\delta_3\bar{\gamma}_3 &= b_3 + x_6 \text{ or } b_3 + y_6,\\ \delta_3\bar{h}_6 &= b_3 + b_9 + y_6 \text{ or } b_3 + b_9 + x_6.\end{aligned}$$

Without loss of generality, let $\delta_3\bar{\gamma}_3 = b_3 + x_6$. Then

$$\begin{aligned}1 + 2b_8 + b_8^2 = (\delta_3\bar{\delta}_3)(\gamma_3\bar{\gamma}_3) = (\delta_3\bar{\gamma}_3)(\bar{\delta}_3\gamma_3) &= b_3\bar{b}_3 + b_3\bar{x}_6 + \bar{b}_3x_6 + x_6\bar{x}_6\\ &= 1 + 3b_8 + c_4 + \bar{c}_4 + h_6 + \bar{h}_6 + x_6\bar{x}_6.\end{aligned}$$

It holds by (2.52) that

$$x_6\bar{x}_6 = 1 + 2b_8 + c_7 + i_6 + \bar{i}_6.$$

Furthermore $(b_5\bar{b}_5, x_6\bar{x}_6) = 5$. On the other hand, the following equations hold:

$$\begin{aligned}b_3(b_5\bar{b}_5) &= b_3 + 2b_3b_8 + b_3c_4 + b_3\bar{c}_4\\ &= 3b_3 + 2b_9 + 2x_6 + 2y_6 + x_6 + z_6 + y_6 + p_6,\end{aligned}$$

$$\begin{aligned}(b_3\bar{b}_5)b_5 &= \bar{b}_3b_5 + \bar{x}_6b_5 + \bar{y}_6b_5\\ &= b_3 + x_6 + y_6 + \bar{x}_6b_5 + \bar{y}_6b_5.\end{aligned}$$

Hence

$$\bar{x}_6b_5 + \bar{y}_6b_5 = 2b_3 + 2b_9 + 2x_6 + 2y_6 + z_6 + p_6. \tag{2.60}$$

It is proved that $(b_5\bar{b}_5, x_6\bar{x}_6) = 5$. By the above equation and $(b_5\bar{x}_5, b_3) = (b_5\bar{y}_5, b_3) = 1$, we can only have that

$$\begin{aligned}b_5\bar{x}_6 &= b_3 + b_9 + x_6 + y_6 + z_6 \text{ or } b_3 + b_9 + x_6 + y_6 + p_6,\\ b_5\bar{y}_6 &= b_3 + b_9 + x_6 + y_6 + p_6 \text{ or } b_3 + b_9 + x_6 + y_6 + z_6.\end{aligned}$$

Hence

$$\begin{aligned}\bar{b}_3(b_5\bar{x}_6) &= \bar{b}_3b_3 + \bar{b}_3b_9 + \bar{b}_3x_6 + \bar{b}_3y_6 + \bar{b}_3z_6\\ &\quad \text{or } \bar{b}_3b_3 + \bar{b}_3b_9 + \bar{b}_3x_6 + \bar{b}_3y_6 + \bar{b}_3p_6\\ &= 1 + b_8 + c_7 + b_8 + \bar{h}_6 + \bar{i}_6 + c_4 + h_6 + b_8\\ &\qquad + \bar{c}_4 + i_6 + b_8 + \bar{i}_6 + c_4 + v_8\\ &\quad \text{or } 1 + b_8 + c_7 + b_8 + \bar{h}_6 + \bar{i}_6 + c_4 + h_6 + b_8\\ &\qquad + \bar{c}_4 + i_6 + b_8 + \bar{h}_6 + \bar{c}_4 + \bar{v}_8,\\ (\bar{b}_3b_5)\bar{x}_6 &= b_3\bar{x}_6 + x_6\bar{x}_6 + y_6\bar{x}_6\\ &= \bar{c}_4 + \bar{h}_6 + b_8 + 1 + 2b_8 + c_7 + i_6 + \bar{i}_6 + y_6\bar{x}_6.\end{aligned}$$

Therefore

$$y_6\bar{x}_6 = 2c_4 + h_6 + b_8 + \bar{i}_6 + v_8 \text{ or } c_4 + \bar{c}_4 + h_6 + b_8 + \bar{h}_6 + \bar{v}_8.$$

If $y_6\bar{x}_6 = 2c_4 + h_6 + b_8 + \bar{i}_6 + v_8$, then $(c_4x_6, y_6) = 2$. Hence $(c_4x_6, c_4x_6) \geq 5$, a contradiction. Therefore

$$y_6\bar{x}_6 = c_4 + \bar{c}_4 + h_6 + b_8 + \bar{h}_6 + \bar{v}_8. \tag{2.61}$$

Thus $(\bar{c}_4x_6, y_6) = 1$.

Consider the following equations:

$$\begin{aligned}b_3(c_4\bar{c}_4) &= b_3 + b_3b_8 + b_3c_7\\ &= 2b_3 + b_9 + x_6 + y_6 + b_9 + f_{12},\\ (b_3c_4)\bar{c}_4 &= \bar{c}_4x_6 + \bar{c}_4z_6,\\ (b_3\bar{c}_4)c_4 &= c_4y_6 + c_4p_6.\end{aligned}$$

By the expression $x_6\bar{x}_6$, we have that $(x_6, \bar{c}_4x_6) = 0$. Consequently, $(x_6, \bar{c}_4z_6) = 1$ by the above equations. Since $b_3c_7 = b_9 + f_{12}$ and $L_1(B) = \{1\}$ and $L_2(B) = \emptyset$, so the constituents of f_{12} are of degrees larger than 3. Since $(\bar{c}_4x_6, b_3) = (\bar{c}_4z_6, b_3) = 1$ by (2.53), $(\bar{c}_4x_6, \bar{c}_4x_6) = 4$ and it is proved that $(\bar{c}_4x_6, y_6) = 1$, we have that

$$\bar{c}_4x_6 = b_3 + y_6 + b_9 + g_6, \quad g_6 \in B.$$

Consequently $\bar{c}_4z_6 = b_3 + x_6 + b_9 + f_6$, where $f_{12} = f_6 + g_6$ and $f_6 \in N^*B$.

Let $\bar{b}_3g_6 = c_7 + x_{11}$ and $\bar{b}_3f_6 = c_7 + y_{11}$, $x_{11}, y_{11} \in N^*(B)$. Then

$$\begin{aligned}
\bar{b}_3(\bar{c}_4x_6) &= \bar{b}_3b_3 + \bar{b}_3y_6 + \bar{b}_3b_9 + \bar{b}_3g_6 \\
&= 1 + b_8 + \bar{c}_4 + i_6 + b_8 + c_7 + b_8 + \bar{h}_6 + \bar{i}_6 + c_7 + x_{11}, \\
\bar{c}_4(\bar{b}_3x_6) &= \bar{c}_4c_4 + \bar{c}_4h_6 + \bar{c}_4b_8 \\
&= 1 + b_8 + c_7 + \bar{c}_4h_6 + \bar{c}_4 + \bar{h}_6 + i_6 + \bar{v}_8 + b_8, \\
\bar{b}_3(\bar{c}_4z_6) &= \bar{b}_3b_3 + \bar{b}_3x_6 + \bar{b}_3b_9 + \bar{b}_3f_6 \\
&= 1 + b_8 + c_4 + h_6 + b_8 + c_7 + b_8 + \bar{h}_6 + \bar{i}_6 + c_7 + y_{11}, \\
\bar{c}_4(\bar{b}_3z_6) &= \bar{c}_4c_4 + \bar{c}_4\bar{i}_6 + \bar{c}_4v_8 \\
&= 1 + b_8 + c_7 + \bar{c}_4\bar{i}_6 + \bar{c}_4v_8.
\end{aligned}$$

Hence

$$\bar{c}_4h_6 + \bar{v}_8 = b_8 + \bar{i}_6 + c_7 + x_{11}, \tag{2.62}$$

$$\bar{c}_4\bar{i}_6 + \bar{c}_4v_8 = c_4 + h_6 + 2b_8 + \bar{i} + \bar{h}_6 + c_7 + y_{11}. \tag{2.63}$$

We assert that $v_8 \neq b_8$. Otherwise, if $v_8 = b_8$, then $(c_4b_8, b_8) = 2$. So $(b_8^2, c_4) = 2$, a contradiction by (2.52). Hence $\bar{v}_8 \in Supp\{x_{11}\}$ by (2.62). Let $x_{11} = \bar{v}_8 + x_3$, then it follows by (2.62) that

$$\bar{c}_4h_6 = b_8 + \bar{i}_6 + c_7 + x_3.$$

By (2.63), we have that $\bar{h}_6 \in Supp\{\bar{c}_4\bar{i}_6 + \bar{c}_4v_8\}$. So $(\bar{c}_4h_6, i_6) \geq 1$ or $(\bar{c}_4h_6, \bar{v}_8) \geq 1$. The latter case will lead to $v_8 = b_8$, a contradiction. Hence $(\bar{c}_4h_6, i_6) \geq 1$. Thus $i_6 = \bar{i}_6$ and $(\bar{c}_4h_6, i_6) = 1$. Furthermore,

$$x_6\bar{x}_6 = 1 + 2b_8 + c_7 + 2i_6.$$

And it follows by (2.49) and (2.51) that

$$b_3i_6 = b_9 + y_6 + z_3, \quad \text{some } z_3 \in B.$$

Thus

$$\begin{aligned}
b_3(x_6\bar{x}_6) &= b_3 + 2b_3b_8 + b_3c_7 + 2b_3i_6 \\
&= 3b_3 + 2b_9 + 2x_6 + 2y_6 + b_9 + g_6 + f_6 + 2b_9 + 2y_6 + 2z_3,
\end{aligned}$$

$$x_6(b_3\bar{x}_6) = x_6\bar{c}_4 + x_6\bar{h}_6 + x_6b_8$$
$$= b_3 + y_6 + b_9 + g_6 + x_6\bar{h}_6 + x_6b_8.$$

Then $x_6\bar{h}_6 + x_6b_8 = 2b_3 + 2b_9 + 2x_6 + 3y_6 + f_6 + 2b_9 + 2z_3$.

By the expression $y_6\bar{x}_6$ and $v_8 \neq b_8$, we have that $(b_8x_6, y_6) = 1$. Hence $(x_6\bar{h}_6, y_6) = 2$ and so $(\bar{x}_6y_6, \bar{h}_6) = 2$. Again by the expression $y_6\bar{x}_6$, we have that $h_6 = \bar{h}_6$. Now (2.48) implies that $x_6 \in Supp\{b_3h_6\}$. Then $p_6 = x_6$ by (2.57). Consequently $b_3\bar{c}_4 = x_6 + y_6$, from which we have $(bc_4, \bar{b}_3x_6) = 1$. Thus $c_4 = \bar{c}_4$ by (2.48). Step 4 follows. This completes the proof of Lemma 2.6. □

Therefore $(b_5\bar{b}_5, b_8) = 1$ and $(b_3\bar{b}_3, b_5\bar{b}_5) = 2$. Consequently by Lemma 2.3 we have $x_{12} \in B$ and $(b_3b_8, b_3b_8) = 3$. Thus Theorem 2.2 holds. Furthermore, in Lemma 2.3 we proved that $c_7 \in B$.

2.4 General Information on NITA Generated by b_3 and Satisfying $b_3^2 = \bar{b}_3 + b_6$ and $b_3^2 = c_3 + b_6$

Since the structure of NITA generated by b_3 and satisfying $b_3^2 = \bar{b}_3 + b_6$ and $b_3^2 = c_3 + b_6$ are related, we investigate them simultaneously. Some results are written in the same proposition. The purpose of this section is to prove (3) and (4) of the Main Theorem 1. The following lemmas will be useful during our investigation.

Lemma 2.7

(1) *If* $b_3^2 = \bar{b}_3 + b_6$, *then* $b_3b_8 = b_6 + \bar{b}_3b_6$;
(2) *If* $b_3^2 = c_3 + b_6$, *then there exists an element* $x_6 \in B$ *such that* $\bar{b}_3c_3 = b_3 + x_6$ *and* $c_3\bar{c}_3 = 1 + b_8$. *Furthermore,* $b_3b_8 = x_6 + \bar{b}_3b_6$ *and* $c_3b_8 = b_6 + b_3x_6$.

Proof (1) follows from $(b_3\bar{b}_3)b_3 = b_3^2\bar{b}_3$.

(2) Since $b_3 \in Supp\{\bar{b}_3c_3\}$, $(b_3\bar{b}_3, c_3\bar{c}_3) \geq 2$. But $(b_3\bar{b}_3, c_3\bar{c}_3) \leq 2$, so $(b_3\bar{b}_3, c_3\bar{c}_3) = 2$ and

$$c_3\bar{c}_3 = 1 + b_8, \qquad \bar{b}_3c_3 = b_3 + x_6, \quad x_6 \in B.$$

Checking $(b_3\bar{b}_3)b_3 = b_3^2\bar{b}_3$ and $c_3(c_3\bar{c}_3) = b_3(\bar{b}_3c_3)$, one has (2). □

Lemma 2.8 *If* $(b_3b_8, b_3b_8) \geq 4$, *then any constituent of* b_3b_8 *and* $\bar{b}_3b_6$ *is either* b_3 *or of degree* > 3 *and* $(b_3b_8, b_3b_8) = 4, 5$.

Proof Since $(b_3b_8, b_3) = 1$, the degree of the sum of the remaining constituents of b_3b_8 must be 21.

On the other hand, any constituent y_n of b_3b_8 must have degree ≥ 4. Otherwise, $n = 3$ and $b_3\bar{y}_3 = 1 + b_8$, which implies that $b_3 = y_3$, a contradiction. □

Theorem 2.3 *There exists no NITA generated by b_3 and satisfying $b_3^2 = \bar{b}_3 + b_6$ or $b_3^2 = c_3 + b_6$ and $(b_3b_8, b_3b_8) = 2$.*

Proof Since $b_3\bar{b}_3 = 1 + b_8$ and $(b_3b_8, b_3b_8) = 2$, we may assume that $b_3b_8 = b_3 + b_{21}$, where $b_{21} \in B$. By Lemma 2.7, one has that

$$b_3 + b_{21} = b_6 + \bar{b}_3b_6 \text{ or } x_6 + \bar{b}_3b_6,$$

which is impossible. □

Theorem 2.4 *There is no NITA generated by b_3 and satisfying $b_3^2 = \bar{b}_3 + b_6$ or $b_3^2 = c_3 + b_6$ and $(b_3b_8, b_3b_8) = 5$.*

Proof (1) There is no NITA satisfying $b_3^2 = \bar{b}_3 + b_6$ and $(b_3b_8, b_3b_8) = 5$.

By $b_3^2\bar{b}_3 = (b_3\bar{b}_3)b_3$, we have that $b_3b_8 = b_6 + \bar{b}_3b_6$, which implies that $b_8 \in Supp\{\bar{b}_3b_6\}$. But $b_3 \in Supp\{\bar{b}_3b_6\}$. We need a sum with degree 7 of two constituents to make up $\bar{b}_3b_6$, which is impossible by Lemma 2.7.

(2) There is no NITA satisfying $b_3^2 = c_3 + b_6$ and $(b_3b_8, b_3b_8) = 5$.

By Lemma 2.7 and $(b_3b_8, b_3b_8) = 5$, we may assume that

$$\begin{aligned}
&(A) \quad b_3b_8 = b_3 + x_6 + x_4 + y_4 + z_7,\\
&(B) \quad b_3b_8 = b_3 + x_6 + x_4 + y_5 + z_6,\\
&(C) \quad b_3b_8 = b_3 + x_6 + x_5 + y_5 + z_5.
\end{aligned}$$

Since

$$\begin{aligned}
\bar{b}_3(\bar{b}_3c_3) &= b_3\bar{b}_3 + x_6\bar{b}_3,\\
\bar{b}_3^2c_3 &= c_3\bar{c}_3 + c_3\bar{b}_6\\
&= 1 + b_8 + c_3\bar{b}_6,
\end{aligned}$$

we have that $\bar{b}_3x_6 = c_3\bar{b}_6$.

Since $(c_3b_8, c_3b_8) = (b_3b_8, b_3b_8) = 5$ by Lemma 2.7 and $c_3b_8 = b_6 + b_3x_6$ by Lemma 2.7, we have $(b_3x_6, b_3x_6) = 4$. Furthermore, $(\bar{b}_3x_6, \bar{b}_3x_6) = 4$. But $(\bar{b}_3x_6, b_8) = 1$. We may assume

$$c_3\bar{b}_6 = \bar{b}_3x_6 = b_8 + m_3 + n_3 + v_4,$$

where $m_3, n_3, v_4 \in B$ and m_3, n_3, v_4 are distinct. We may assume

$$\begin{aligned}
c_3\bar{m}_3 &= b_6 + m_3^*, \quad m_3^* \in B,\\
c_3\bar{n}_3 &= b_6 + n_3^*.
\end{aligned}$$

Hence $(c_3\bar{m}_3, c_3\bar{m}_3) = 2$. So $m_3\bar{m}_3 = 1 + b_8$. Moreover,

$$(c_3\bar{c}_3)\bar{m}_3 = \bar{m}_3 + \bar{m}_3b_8,$$

$$\begin{aligned}(c_3\bar{m}_3)\bar{c}_3 &= \bar{c}_3 b_6 + \bar{c}_3 m_3^* \\ &= b_8 + \bar{m}_3 + \bar{n}_3 + \bar{v}_4 + \bar{c}_3 m_3^*.\end{aligned}$$

Then

$$\bar{m}_3 b_8 = b_8 + \bar{n}_3 + \bar{v}_4 + \bar{c}_3 m_3^*,$$

which implies that $m_3\bar{n}_3 = 1 + b_8$. So $m_3 = n_3$, a contradiction. □

Theorem 2.5 *There exists no NITA generated by b_3 and satisfying $b_3^2 = \bar{b}_3 + b_6$, $(b_3b_8, b_3b_8) = 4$.*

Proof Since $b_3b_8 = b_6 + \bar{b}_3b_6$ by Lemma 2.7 and by assumption that $(b_3b_8, b_3b_8) = 4$, then $b_8 \in Supp\{\bar{b}_3b_6\}$ and $(\bar{b}_3b_6, \bar{b}_3b_6) = 3$. But $b_3 \in Supp\{\bar{b}_3b_6\}$. Thus there exists $p_7 \in B$ such that

$$\bar{b}_3b_6 = b_3 + b_8 + p_7, \qquad b_3b_8 = b_3 + b_6 + b_8 + p_7.$$

Hence

$$\begin{aligned}(b_3\bar{b}_3)b_8 &= b_8 + b_8^2, \\ \bar{b}_3(b_3b_8) &= \bar{b}_3b_3 + \bar{b}_3b_6 + \bar{b}_3b_8 + \bar{b}_3p_7 \\ &= 1 + b_8 + b_3 + b_8 + p_7 + \bar{b}_3 + \bar{b}_6 + \bar{b}_8 + \bar{p}_7 + \bar{b}_3p_7.\end{aligned}$$

Consequently, $b_8^2 = 1 + b_3 + \bar{b}_3 + p_7 + \bar{p}_7 + 2b_8 + \bar{b}_6 + \bar{b}_3p_7$. On the other hand,

$$\begin{aligned}(b_3\bar{b}_3)^2 &= 1 + 2b_8 + b_8^2, \\ b_3^2\bar{b}_3^2 &= b_3\bar{b}_3 + \bar{b}_3b_6 + b_3\bar{b}_6 + b_6\bar{b}_6 \\ &= 1 + b_8 + b_3 + b_8 + p_7 + \bar{b}_3 + b_8 + \bar{p}_3 + b_6\bar{b}_6.\end{aligned}$$

We have that $b_8^2 = b_3 + \bar{b}_3 + p_7 + \bar{p}_7 + b_8 + b_6\bar{b}_6$. Hence $b_6\bar{b}_6 = 1 + \bar{b}_6 + b_8 + \bar{b}_3p_7$.
 Since

$$\begin{aligned}\bar{b}_3^2b_6 &= b_3b_6 + \bar{b}_6b_6 \\ &= b_3b_6 + 1 + b_8 + \bar{b}_6 + \bar{b}_3p_7, \\ \bar{b}_3(\bar{b}_3b_6) &= \bar{b}_3b_3 + \bar{b}_3b_8 + \bar{b}_3p_7 \\ &= 1 + b_8 + \bar{b}_3 + \bar{b}_6 + b_8 + \bar{p}_7 + \bar{b}_3p_7,\end{aligned}$$

it holds that $b_3b_6 = \bar{b}_3 + \bar{p}_7 + b_8$. Now the following equations follow:

$$\begin{aligned}b_3(\bar{b}_3b_6) &= b_3^2 + b_3b_8 + b_3p_7 \\ &= \bar{b}_3 + b_6 + b_3 + b_6 + b_8 + p_7 + b_3p_7, \\ (b_3b_6)\bar{b}_3 &= \bar{b}_3^2 + \bar{b}_3\bar{p}_7 + b_8\bar{b}_3 \\ &= b_3 + \bar{b}_6 + \bar{b}_3 + \bar{b}_6 + b_8 + \bar{p}_7 + \bar{b}_3\bar{p}_7.\end{aligned}$$

Then $2b_6 + p_7 + b_3 p_7 = 2\bar{b}_6 + \bar{p}_7 + \bar{b}_3 \bar{p}_7$. Since b_6 is nonreal, we have that $(b_3 p_7, \bar{b}_6) = 2$, a contradiction to the expression $\bar{b}_3 b_6$. The theorem follows. □

Theorem 2.6 *There exists no NITA satisfying* $b_3^2 = c_3 + b_6$, $(b_3 b_8, b_3 b_8) = 4$ *and* $c_3^2 = r_4 + s_5$.

Proof Let us investigate Sub-NITA generated by c_2. By assumptions, we have that $(b_3\bar{b}_3, b_3\bar{b}_3) = (c_3\bar{c}_3, c_3\bar{c}_3)$, hence $b_3\bar{b}_3 = c_3\bar{c}_3$. Therefore $(c_3\bar{c}_3, b_8^2) = (b_3\bar{b}_3, b_8^2) = 4$, thus $(c_3 b_8, c_3 b_8) = 4$, a contradiction to Theorem 2.2. □

2.5 NITA Generated by b_3 Satisfying $b_3^2 = \bar{b}_3 + b_6$ and b_6 Nonreal and $b_{10} \in B$ is Real

Now we discuss NITA satisfying the following hypothesis.

Hypothesis 2.1 *Let* (A, B) *be a NITA generated by a nonreal element* $b_3 \in B$ *satisfies* $b_3\bar{b}_3 = 1 + c_8$ *and* $b_3^2 = \bar{b}_3 + b_6$, *where* $b_6 \in B$ *is nonreal. Here we still assume that* $L(B) = 1$ *and* $L_2(B) = \emptyset$. *In this section, we use the symbols* b_i, c_i, d_i *to denote elements of* B *with degree* i, *where* $i \geq 2$.

By the associative law and Hypothesis 2.1, we have $b_3(b_3\bar{b}_3) = (b_3^2)\bar{b}_3$, $b_3 + c_8 b_3 = \bar{b}_3^2 + b_6\bar{b}_3$, so

$$c_8 b_3 = \bar{b}_6 + b_6\bar{b}_3.$$

Hence $(b_3 b_6, c_8) = (b_3 c_8, \bar{b}_6) = 1$, so

$$b_3 b_6 = c_8 + b_{10},$$

therefore $b_{10} \in B$ since $(b_3 c_8, b_3 c_8) = 3$.

Now we shall deal with the case $b_{10} = \bar{b}_{10} \in B$.

Lemma 2.9 *Let* (A, B) *satisfy Hypothesis 2.1 and* $\bar{b}_{10} = b_{10} \in B$, *then*

1) $b_3 b_6 = c_8 + b_{10}$;
2) $\bar{b}_3 b_6 = b_3 + b_{15}$, *where* $b_{15} \in B$;
3) $b_3 c_8 = b_3 + \bar{b}_6 + b_{15}$;
4) $b_3 b_{10} = b_{15} + \bar{b}_6 + \bar{c}_9$, $c_9 \in B$;
5) $b_3 b_{15} = 2\bar{b}_{15} + b_6 + c_9$;
6) $b_6^2 = 2\bar{b}_6 + \bar{c}_9 + b_{15}$;
7) $b_6 c_8 = \bar{b}_3 + 2\bar{b}_{15} + b_6 + c_9$;
8) $\bar{b}_6 b_6 = 1 + c_8 + x + y + z + w$, *where* $x, y, z, w \in B$, $|x + y + z + w| = 27$, *and the degrees of* x, y, z *and* $w \geq 5$. *Moreover,* $\overline{x + y + z + w} = x + y + z + w$;
9) $c_8^2 = 1 + 2c_8 + 2b_{10} + x + y + z + w$;

10) $b_3\bar{b}_{15} = b_{10} + c_8 + x + y + z + w$;
11) $(\bar{c}_9 b_3, c_9) = 1$ *or* $(b_6 c_9, \bar{c}_9) \geq 1$.

Proof We have seen that

$$c_8 b_3 = \bar{b}_6 + b_6\bar{b}_3 \tag{1}$$

and

$$b_3 b_6 = c_8 + b_{10}, \tag{2}$$

where b_{10} is real. So we obtain $(\bar{b}_3 b_6, \bar{b}_3 b_6) = 2$ and consequently we obtain

$$\bar{b}_3 b_6 = b_3 + b_{15}, \tag{3}$$

where $b_{15} \in B$. Hence by (1)

$$c_8 b_3 = b_3 + \bar{b}_6 + b_{15}. \tag{4}$$

By the associative law and (2), (3), we have $b_3(\bar{b}_3 b_6) = (b_3 b_6)\bar{b}_3$ and $b_3^2 + b_{15}b_3 = c_8\bar{b}_3 + b_{10}\bar{b}_3$, so by (4) and Hypothesis 2.1 we have

$$b_{15}b_3 = \bar{b}_{15} + \bar{b}_3 b_{10}. \tag{5}$$

If $x \in B$ and $(b_{15}b_3, x) = (b_{15}, \bar{b}_3 x)$, so $|x| \geq 5$, $\forall x \in B$. So every constituent of $\bar{b}_3 b_{10}$ has degrees ≥ 5. Since b_{10} is real, the constituents of $b_3 b_{10}$ also have degrees ≥ 5, so by the associative law and Hypothesis 2.1 and (4), (2):

$$\begin{aligned}
(\bar{b}_3 b_3)c_8 &= (b_3 c_8)\bar{b}_3, \\
c_8 + c_8^2 &= b_3\bar{b}_3 + \bar{b}_6\bar{b}_3 + b_{15}\bar{b}_3, \\
c_8^2 &= 1 + c_8 + b_{10} + \bar{b}_3 b_{15}.
\end{aligned} \tag{6}$$

By the associative law and Hypothesis 2.1 and (2):

$$\begin{aligned}
(b_3\bar{b}_3)(b_3\bar{b}_3) &= b_3^2\bar{b}_3^2, \\
1 + 2c_8 + c_8^2 &= \bar{b}_3 b_3 + \bar{b}_3\bar{b}_6 + b_6 b_3 + b_6\bar{b}_6, \\
c_8^2 &= 2b_{10} + c_8 + b_6\bar{b}_6;
\end{aligned} \tag{7}$$

so we get

$$1 = (\bar{b}_3 b_{15}, b_{10}) = (b_{15}, b_3 b_{10}).$$

By (2),

$$(b_3 b_{10}, \bar{b}_6) = (b_3 b_6, b_{10}) = 1.$$

$b_3 b_{10}$ have constituents of degree ≥ 5 as we have proved. Therefore

$$b_3 b_{10} = b_{15} + \bar{b}_6 + \bar{c}_9 \tag{8}$$

where $c_9 \in B$. So by (5),

$$b_3 b_{15} = 2\bar{b}_{15} + b_6 + c_9. \tag{9}$$

By the associative law and Hypothesis 2.1 and (2), (3), (4):

$$(b_3^2)b_6 = (b_3 b_6)b_3,$$
$$\bar{b}_3 b_6 + b_6^2 = c_8 b_3 + b_{10} b_3,$$

consequently

$$b_6^2 = 2\bar{b}_6 + b_{15} + \bar{c}_9.$$

Now

$$b_3(\bar{b}_3 b_6) = (b_3 \bar{b}_3) b_6,$$
$$b_3^2 + b_{15} b_3 = b_6 + b_6 b_8.$$

Thus

$$c_8 b_6 = \bar{b}_3 + 2\bar{b}_{15} + b_6 + c_9.$$

So

$$(\bar{b}_6 b_6, c_8) = (b_6, b_6 c_8) = 1.$$

Since

$$(b_6^2, b_6^2) = 6 = (\bar{b}_6 b_6, \bar{b}_6 b_6),$$

hence

$$b_6 \bar{b}_6 = 1 + c_8 + x + y + z + w$$

where $x, y, z, w \in B$. Therefore by (7), (6)

$$c_8^2 = 1 + 2c_8 + 2b_{10} + x + y + z + w,$$

and

$$b_3 \bar{b}_5 = b_{10} + c_8 + x + y + z + w.$$

So $(b_3 \bar{b}_{15}, m) = (b_3 \bar{m}, b_{15})$, so $|m| \geq 5$, hence

$$|x|, |y|, |z|, |w| \geq 5.$$

By (8),

$$(b_3 c_9, b_{10}) = (b_3 b_{10}, \bar{c}_9) = 1,$$

hence

$$b_3 c_9 = b_{10} + \alpha,$$

where $\alpha \in N^*B$. By the associative law and Hypothesis 2.1:

$$(b_3^2)c_9 = (b_3c_9)b_3,$$

$$c_9\bar{b}_3 + c_9b_6 = b_{10}b_3 + \alpha b_3 = b_{15} + \bar{b}_6 + \bar{c}_9 + \alpha b_3;$$

hence $(c_9\bar{b}_3, \bar{c}_9) \geq 1$, or $(b_6c_9, \bar{c}_9) \geq 1$. By (9) we conclude that either $(c_9\bar{b}_3, \bar{c}_9) = 1$ or $(b_6c_9, \bar{c}_9) \geq 1$. □

We start to investigate the case $(\bar{c}_9b_3, c_9) = 1$ of Lemma 2.9.

Lemma 2.10 *Let (A, B) satisfy Hypothesis 2.1. Assume that $\bar{b}_{10} = b_{10} \in B$ and $(\bar{c}_9b_3, c_9) = 1$. Then we have the following*:

1) $b_3\bar{c}_9 = \bar{b}_{15} + c_9 + c_3, c_3, c_9 \in B$;
2) $b_6b_{10} = 3\bar{b}_{15} + \bar{b}_3 + c_3 + c_9, b_{15} \in B$;
3) $c_8b_{15} = 5b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 3\bar{c}_9$;
4) $b_3\bar{c}_3 = c_9$;
5) $b_3c_3 = b_9, \bar{b}_9 = b_9 \in B$;
6) $b_3c_9 = b_9 + b_{10} + b_8, b_9 = \bar{b}_9, \bar{b}_8 = b_8 \in B$;
7) $\bar{c}_3b_6 = b_{10} + b_8$;
8) $b_6\bar{b}_{15} = 4b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
9) $c_3\bar{c}_3 = 1 + b_8$;
10) $\bar{b}_6b_6 = 1 + c_8 + b_9 + b_8 + c_5 + b_5, \overline{c_5 + b_5} = c_5 + b_5$;
11) $c_8^2 = 1 + 2c_8 + 2b_{10} + b_9 + b_8 + b_5 + c_5$;
12) $b_3\bar{b}_{15} = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5$;
13) $b_6b_{15} = 2c_8 + 3b_{10} + 2b_9 + 2b_8 + b_5 + c_5$;
14) $c_8b_{10} = 2c_8 + 2b_{10} + 2b_9 + 2b_8 + b_5 + c_5$;
15) $b_6\bar{c}_9 = b_{10} + c_8 + 2b_9 + b_8 + b_5 + c_5$;
16) $b_{10}^2 = 1 + 2c_8 + 2b_{10} + 3b_9 + 2b_8 + 2b_5 + 2c_5$.

Proof By 5) in Lemma 2.9, $(\bar{b}_3c_9, b_{15}) = (c_9, b_3b_{15}) = 1$. So by the assumption in the lemma, $(b_3\bar{c}_9, c_9) = 1$, hence

$$b_3\bar{c}_9 = \bar{b}_{15} + c_9 + c_3, \tag{1}$$

where $c_3 \in B$. By the associative law and Hypothesis 2.1 and 4) in Lemma 2.9

$$(b_3^2)b_{10} = (b_3b_{10})b_3,$$

$$\bar{b}_3b_{10} + b_6b_{10} = b_{15}b_3 + \bar{b}_6b_3 + \bar{c}_9b_3;$$

by 2), 4), 5) in Lemma 2.9 and (1),

$$b_6b_{10} = 3\bar{b}_{15} + \bar{b}_3 + c_3 + c_9. \tag{2}$$

By the associative law and Hypothesis 2.1 and 5) in Lemma 2.9:

$$(\bar{b}_3 b_3) b_{15} = (b_3 b_{15}) \bar{b}_3,$$
$$c_8 b_{15} + b_{15} = 2\bar{b}_{15}\bar{b}_3 + \bar{b}_3 b_6 + c_9 \bar{b}_3;$$

by (1) and 2), 5) in Lemma 2.9,

$$c_8 b_{15} = 5b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 3\bar{c}_9. \tag{3}$$

By (1),

$$(\bar{c}_3 b_3, c_9) = (b_3 \bar{c}_9, c_3) = 1,$$

hence

$$\bar{c}_3 b_3 = c_9. \tag{3'}$$

So

$$(c_3 b_3, c_3 b_3) = (b_3 \bar{c}_9, c_3) = 1,$$

hence

$$c_3 b_3 = b_9. \tag{4}$$

By the associative law and Hypothesis 2.1 and (3)

$$\bar{c}_3 (b_3^2) = (\bar{c}_3 b_3) b_3,$$
$$\bar{c}_3 \bar{b}_3 + \bar{c}_3 b_6 = c_9 b_3,$$

hence

$$c_9 b_3 = \bar{b}_9 + \bar{c}_3 b_6. \tag{5}$$

By (1) and (4) in Lemma 2.9, $(c_9 b_3, b_{10}) = (b_3 b_{10}, \bar{c}_9) = 1, (b_3 c_9, b_3 c_9) = (b_3 \bar{c}_9, b_3 \bar{c}_9) = 3$, hence

$$c_9 b_3 = \bar{b}_9 + b_{10} + b_8, \tag{6}$$

where $b_8 \in B$. So

$$\bar{c}_3 b_6 = b_{10} + b_8. \tag{7}$$

By the associative law and Hypothesis 2.1 and 4) in Lemma 2.9:

$$(\bar{b}_3 b_3) b_{10} = (b_3 b_{10}) \bar{b}_3,$$
$$b_{10} + c_8 b_{10} = b_{15}\bar{b}_3 + \bar{b}_6 \bar{b}_3 + \bar{c}_9 \bar{b}_3,$$

by 1), 10) in Lemma 2.9 and (6)

$$c_8 b_{10} = c_8 + \bar{x} + \bar{y} + \bar{z} + \bar{w} + \bar{b}_8. \tag{8}$$

b_{10}, c_8 are reals, so if $\bar{b}_8 \neq b_8$, then $(c_8 b_{10}, b_8) \neq 0$, and without loss of generality $\bar{x} = b_8$, so $x = b_8$ and by 8) in Lemma 2.9 and without loss of generality $y = b_8$, so by (8) $(c_8b_{10}, \bar{b}_8) = 2$, hence $(c_8b_{10}, b_8) = 2$, so without loss of generality $\bar{z} = b_8$, hence $z = \bar{b}_8$ and by 8) in Lemma 2.9 without loss of generality $w = b_8$ then $(c_8b_{10}, b_8) = 3$, a contradiction. Hence $b_8 = \bar{b}_8$. In the same way $b_9 = \bar{b}_9$.

By the associative law and 2), 6) in Lemma 2.9:

$$(b_6^2)\bar{b}_3 = (b_6\bar{b}_3)b_6,$$

$$2\bar{b}_6\bar{b}_3 + \bar{c}_9\bar{b}_3 + b_{15}\bar{b}_3 = b_3b_6 + b_{15}b_6,$$

and by Lemma 2.9 and (6)

$$b_6b_{15} = 2c_8 + 3b_{10} + b_8 + b_9 + x + y + w + z. \tag{9}$$

By the associative law and the Hypothesis 2.1 and 4) in Lemma 2.9:

$$b_{10}(b_3\bar{b}_3) = (\bar{b}_3b_{10})b_3,$$

$$b_{10} + c_8b_{10} = b_3\bar{b}_{15} + b_3b_6 + b_3c_9;$$

by (6) and 1), 10) in Lemma 2.9

$$c_8b_{10} = 2c_8 + 2b_{10} + x + y + z + w + b_8 + b_9. \tag{10}$$

By the associative law and 2), 3), 6) in Lemma 2.9 and (3):

$$c_8(\bar{b}_3b_6) = (\bar{b}_3c_8)b_6,$$

$$b_3c_8 + b_{15}c_8 = \bar{b}_3b_6 + b_6^2 + \bar{b}_{15}b_6,$$

$$b_6\bar{b}_{15} = 4b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9.$$

By the associative law and (1), (4), (6) and Hypothesis 2.1:

$$(b_3^2)\bar{c}_9 = (b_3\bar{c}_9)b_3,$$

$$\bar{b}_3\bar{c}_9 + b_6\bar{c}_9 = \bar{b}_{15}b_3 + c_9b_3 + c_3b_3,$$

$$b_6\bar{c}_9 = b_{10} + c_8 + x + y + z + w + b_9. \tag{11}$$

By the associative law and 1) in Lemma 2.9 and (2):

$$(b_3b_6)b_{10} = (b_6b_{10})b_3,$$

$$c_8b_{10} + b_{10}^2 = 3\bar{b}_{15}b_3 + \bar{b}_3b_3 + c_3b_3 + c_9b_3,$$

by (4), (6), (10) and Hypothesis 2.1

$$b_{10}^2 = 1 + 2c_8 + b_9 + 2b_{10} + 2x + 2y + 2z + 2w. \tag{12}$$

By (2), $(b_6b_{10}, b_6b_{10}) = 12$. But by 8) in Lemma 2.9 and (12) $(b_6\bar{b}_6, b_{10}^2) = 10$, so we obtain, without loss of generality,

$$x = b_9. \tag{13}$$

By (4) and that $b_9 = \bar{b}_9$, $(c_3b_3, \bar{c}_3\bar{b}_3) = (b_3^2, \bar{c}_3^2) = 1$, but by (3′) $(c_3^2, b_3) = (c_3\bar{b}_3, \bar{c}_3) = 0$, so by Hypothesis 2.1

$$(\bar{c}_3^2, b_6) = 1. \tag{14}$$

Hence

$$(c_3\bar{c}_3, c_3\bar{c}_3) = 2,$$

so

$$c_3\bar{c}_3 = 1 + t_8. \tag{15}$$

By (4)

$$(c_3b_3, c_3b_3) = (c_3\bar{c}_3, b_3\bar{b}_3) = 1,$$

therefore by hypothesis $t_8 \neq c_8$. By (12), (13) and 9) in Lemma 2.9

$$(b_{10}^2, c_8^2) = 18 = (c_8b_{10}, c_8b_{10}).$$

So by (8) we conclude that, without loss of generality,

$$y = b_8. \tag{16}$$

So by (13), (16) and 8) in Lemma 2.9, we obtain $|z| = |w| = 5$, set $z = b_5$ and $w = c_5$. By (7),

$$(\bar{c}_3b_6, \bar{c}_3b_6) = (\bar{c}_3c_3, \bar{b}_6b_6) = 2,$$

so by (15) and 8) in Lemma 2.9 and by (16) $t_8 = b_8$. So $c_3\bar{c}_3 = 1 + b_8$. By 8), 9), 10) in Lemma 2.9 and (10), (11), (12), (13), (16), (9)

$$\bar{b}_6b_6 = 1 + c_8 + b_9 + b_8 + c_5 + b_5,$$

$$c_8^2 = 1 + 2c_8 + 2b_{10} + b_9 + b_8 + b_5 + c_5,$$

$$b_3\bar{b}_{15} = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5,$$

$$b_6b_{15} = 2c_8 + 3b_{10} + 2b_9 + 2b_8 + b_5 + c_5,$$

$$c_8b_{10} = 2c_8 + 2b_{10} + 2b_9 + 2b_8 + b_5 + c_5,$$

$$b_6\bar{c}_9 = b_{10} + c_8 + 2b_9 + b_8 + b_5 + c_5,$$

$$b_{10}^2 = 1 + 2c_8 + 2b_{10} + 3b_9 + 2b_8 + 2b_5 + 2c_5.$$

□

Lemma 2.11 *Let (A, B) satisfy Hypothesis 2.1. Assume that $\bar{b}_{10} = b_{10} \in B$ and $(\bar{c}_9b_3, c_9) = 1$. Then we obtain the following equations*:

1) $b_3b_9 = b_{15} + \bar{c}_9 + \bar{c}_3$;
2) $c_3c_8 = c_9 + \bar{b}_{15}$;

3) $b_3b_8 = b_{15} + \bar{c}_9$;
4) $b_3c_5 = b_{15}$;
5) $b_3b_5 = b_{15}$;
6) $b_6b_5 = \bar{b}_{15} + b_6 + c_9$;
7) $b_6c_9 = 2b_{15} + 2\bar{c}_9 + \bar{b}_6$;
8) $c_3b_6 = \bar{c}_3 + b_{15}$;
9) $b_6b_9 = 2\bar{b}_{15} + b_6 + 2c_9$;
10) $b_6c_5 = \bar{b}_{15} + b_6 + c_9$;
11) $b_6b_8 = 2\bar{b}_{15} + b_6 + c_3 + c_9$;
12) $c_3b_{10} = \bar{b}_{15} + b_6 + c_9$;
13) $c_3c_9 = b_{15} + \bar{c}_9 + \bar{b}_3$;
14) $c_3b_8 = c_3 + b_6 + \bar{b}_{15}$;
15) $c_3b_9 = \bar{b}_{15} + \bar{b}_3 + c_9$;
16) $c_3\bar{b}_{15} = 2b_{15} + \bar{b}_6 + \bar{c}_9$;
17) $b_8\bar{b}_{15} = 5\bar{b}_{15} + \bar{b}_3 + c_3 + 2b_6 + 3c_9$;
18) $b_8c_9 = 3\bar{b}_{15} + 2c_9 + b_6 + \bar{b}_3$.

Proof By 5), 6), 12), in Lemma 2.10

$$(b_3b_9, \bar{c}_3) = (b_3c_3, b_9) = 1,$$
$$(b_3b_9, \bar{c}_9) = (b_3c_9, b_9) = 1,$$
$$(b_3b_9, b_{15}) = (b_3\bar{b}_{15}, b_9) = 1,$$

then

$$b_3b_9 = b_{15} + \bar{c}_9 + \bar{c}_3. \tag{1}$$

By the associative law and 1), 4) in Lemma 2.10 and Hypothesis 2.1,

$$(\bar{b}_3c_3)b_3 = (\bar{b}_3b_3)c_3,$$
$$\bar{c}_9b_3 = c_3 + c_3c_8,$$

so

$$c_3c_8 = c_9 + \bar{b}_{15}. \tag{2}$$

By 6), 12) in Lemma 2.10,

$$(b_3b_8, b_{15}) = (b_3\bar{b}_{15}, b_8) = 1,$$
$$(b_3b_8, \bar{c}_9) = (b_3c_9, b_8) = 1,$$

then

$$b_3b_8 = b_{15} + \bar{c}_9. \tag{3}$$

By the associative law and 2) in Lemma 2.9 and 10) in Lemma 2.10:

$$(b_6\bar{b}_6)b_3 = (b_3\bar{b}_6)b_6,$$
$$b_3 + c_8b_3 + b_9b_3 + b_8b_3 + c_5b_3 + b_5b_3 = \bar{b}_3b_6 + \bar{b}_{15}b_6,$$

by 2), 3) in Lemma 2.9 and (3), (1) and 8) in Lemma 2.10:

$$c_5 b_3 + b_5 b_3 = 2b_{15},$$

so

$$c_5 b_3 = b_{15}, \tag{4}$$

$$b_5 b_3 = b_{15}. \tag{5}$$

By the associative law and Hypothesis 2.1 and (5) and 5) in Lemma 2.9

$$(b_3^2)b_5 = (b_3 b_5)b_3 = b_3 b_{15},$$

$$\bar{b}_3 b_5 + b_6 b_5 = 2\bar{b}_{15} + b_6 + c_9,$$

by (5)

$$(\bar{b}_3 b_5, \bar{b}_3 b_5) = (b_5 b_3, b_5 b_3) = 1.$$

So $\bar{b}_3 b_5 = \bar{b}_{15}$, therefore

$$b_6 b_5 = \bar{b}_{15} + b_6 + c_9. \tag{6}$$

By the associative law and 4), 7) in Lemma 2.10:

$$(b_3 \bar{c}_3)b_6 = (\bar{c}_3 b_6)b_3,$$

$$c_9 b_6 = b_{10} b_3 + b_8 b_3,$$

by 4) in Lemma 2.9 and by (3)

$$b_6 c_9 = 2b_{15} + 2\bar{c}_9 + \bar{b}_6. \tag{7}$$

By the associative law and the Hypothesis 2.1 and 5) in Lemma 2.10:

$$(b_3^2)c_3 = (b_3 c_3)b_3, \qquad \bar{b}_3 c_3 + b_6 c_3 = b_9 b_3,$$

so by (1) and 4) in Lemma 2.10

$$b_6 c_3 = \bar{c}_3 + b_{15}. \tag{8}$$

10), 13), 15) in Lemma 2.10,

$$(b_6 b_9, \bar{b}_{15}) = (b_6 b_{15}, b_9) = 2,$$

$$(b_6 b_9, b_6) = (b_9, \bar{b}_6 b_6) = 1,$$

$$(b_6 b_9, c_9) = (b_6 \bar{c}_9, b_9) = 2,$$

so

$$b_6 b_9 = 2\bar{b}_{15} + b_6 + 2c_9. \tag{9}$$

By the associative law and Hypothesis 2.1 and (4):

$$(b_3^2)c_5 = (b_3c_5)b_3, \qquad \bar{b}_3c_5 + b_6c_5 = b_{15}b_3,$$

by 5) in Lemma 2.9

$$\bar{b}_3c_5 + b_6c_5 = 2\bar{b}_{15} + b_6 + c_9,$$
$$(c_5\bar{b}_3, c_5\bar{b}_3) = (c_5b_3, c_5b_3) = 1,$$

so $\bar{b}_3c_5 = \bar{b}_{15}$, hence by 7), 10), 13), and 15) in Lemma 2.10

$$b_6c_5 = \bar{b}_{15} + b_6 + c_9, \tag{10}$$
$$(b_6b_8, c_3) = (b_6\bar{c}_3, b_8) = 1, \qquad (b_6b_8, b_6) = (b_8, \bar{b}_6b_6) = 1,$$
$$(b_6b_8, c_9) = (b_8, \bar{b}_6c_9) = 1, \qquad (b_6b_8, \bar{b}_{15}) = (b_6b_{15}, b_8) = 2,$$

so

$$b_6b_8 = 2\bar{b}_{15} + b_6 + c_3 + c_9. \tag{11}$$

By the associative law and 1) in Lemma 2.9 and (8)

$$c_3(b_3b_6) = (c_3b_6)b_3, \qquad c_3c_8 + c_3b_{10} = \bar{c}_3b_3 + b_{15}b_3,$$

by (2), 4) in Lemma 2.10 and 5) in Lemma 2.9

$$c_3b_{10} = \bar{b}_{15} + b_6 + c_9. \tag{12}$$

By the associative law and 4), 9) in Lemma 2.10:

$$(c_3\bar{c}_3)b_3 = (\bar{c}_3b_3)c_3, \qquad b_3 + b_8b_3 = c_9c_3,$$

by (3)

$$c_3c_9 = b_{15} + \bar{c}_9 + b_3. \tag{13}$$

By the associative law and 6) in Lemma 2.10 and (13)

$$(b_3c_9)c_3 = (c_3c_9)b_3,$$
$$b_9c_3 + b_{10}c_3 + b_8c_3 = b_{15}b_3 + \bar{c}_9b_3 + b_3^2;$$

by (12) and 5) in Lemma 2.9 and 1) in Lemma 2.10 and Hypothesis 2.1

$$b_9c_3 + b_8c_3 + b_{15} + b_6 + c_9 = 2\bar{b}_{15}b_6 + c_9 + \bar{b}_{15} + c_9 + c_3 + \bar{b}_3 + b_6,$$
$$b_9c_3 + b_8c_3 = 2\bar{b}_{15} + c_9 + c_3 + \bar{b}_3 + b_6;$$

by 5), 7), 9) in Lemma 2.10

$$(b_8c_3, \bar{b}_3) = (c_3b_3, b_8) = 0,$$
$$(b_8c_3, c_3) = (b_8, \bar{c}_3c_3) = 1,$$
$$(b_8c_3, b_6) = (b_8, \bar{c}_3b_6) = 1.$$

So

$$b_8c_3 = c_3 + b_6 + \bar{b}_{15}, \tag{14}$$

hence

$$b_9c_3 = \bar{b}_{15} + \bar{b}_3 + c_9. \tag{15}$$

By the associative law and 2) in Lemma 2.9 and 7) in Lemma 2.10:

$$c_3(b_3\bar{b}_6) = (c_3\bar{b}_6)b_3, \qquad c_3\bar{b}_3 + c_3\bar{b}_{15} = b_{10}b_3 + b_8b_3,$$

by 4) in Lemma 2.10 and 4) in Lemma 2.9 and (3):

$$c_3\bar{b}_{15} = 2b_{15} + \bar{b}_6 + \bar{c}_9. \tag{16}$$

By the associative law and 9) in Lemma 2.10:

$$(c_3\bar{b}_{15})\bar{c}_3 = (c_3\bar{c}_3)\bar{b}_{15}, \qquad 2b_{15}\bar{c}_3 + \bar{b}_6\bar{c}_3 + \bar{c}_9\bar{c}_3 = \bar{b}_{15} + b_8\bar{b}_{15},$$

by (16) and (8), (13)

$$b_8\bar{b}_{15} = 5\bar{b}_{15} + \bar{b}_3 + c_3 + 2b_6 + 3c_9. \tag{17}$$

By the associative law and (13) and 9) in Lemma 2.10:

$$(c_3\bar{c}_3)c_9 = (c_3c_9)\bar{c}_3, \qquad c_9 + b_8c_9 = b_{15}\bar{c}_3 + \bar{c}_9\bar{c}_3 + b_3\bar{c}_3,$$

by (13), (16) and 4) in Lemma 2.10:

$$c_9 + b_8c_9 = 2\bar{b}_{15} + b_6 + c_9 + \bar{b}_{15} + c_9 + \bar{b}_3 + c_9,$$

then

$$b_8c_9 = 3\bar{b}_{15} + 2c_9 + b_6 + \bar{b}_3. \tag{18}$$

□

Lemma 2.12 *Let* (A, B) *satisfy Hypothesis* 2.1. *Assume that* $\bar{b}_{10} = b_{10} \in B$ *and* $(\bar{c}_9b_3, c_9) = 1$. *Then we obtain the following equations*:

1) $c_3\bar{c}_9 = b_{10} + b_9 + c_8$;
2) $c_3b_{15} = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10}$;
3) $c_3^2 = \bar{c}_3 + \bar{b}_6$;
4) $\bar{b}_{15}b_{15} = 1 + 3b_5 + 3c_5 + 5c_8 + 5b_8 + 6b_6 + 6b_{10}$;
5) $b_{15}^2 = 9\bar{b}_{15} + 4b_6 + 2\bar{b}_3 + 2c_3 + 6c_9$;

6) $b_{10}b_{15} = 6b_{15} + 3\bar{b}_6 + b_3 + \bar{c}_3 + 4\bar{c}_9$;
7) $c_9b_{15} = 2b_5 + 2c_5 + 3c_8 + 3b_8 + 3b_9 + 4b_{10}$;
8) $c_9\bar{b}_{15} = 6b_{15} + 2\bar{b}_6 + b_3 + \bar{c}_3 + 3\bar{c}_9$;
9) $c_9^2 = 3b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 2\bar{c}_9$;
10) $c_9\bar{c}_9 = 1 + 2c_8 + 2b_9 + 2b_{10} + c_5 + b_5 + 2b_8$;
11) $c_8c_9 = 3\bar{b}_{15} + b_6 + c_3 + 2c_9$;
12) $b_9c_9 = \bar{b}_3 + 2c_9 + 3\bar{b}_{15} + c_3 + 2b_6$;
13) $c_9b_{10} = \bar{b}_3 + 2c_9 + 3\bar{b}_{15} + c_3 + 2b_6$;
14) $b_5c_9 = 2\bar{b}_{15} + b_6 + c_9$;
15) $c_5c_9 = 2\bar{b}_{15} + b_6 + c_9$;
16) $b_5 = \bar{b}_5, c_5 = \bar{c}_5$;
17) $c_3b_5 = \bar{b}_{15}$;
18) $c_3c_5 = \bar{b}_{15}$.

Proof By 2), 12), 15) in Lemma 2.11, $(c_3\bar{c}_9, b_{10}) = (c_3b_{10}, c_9) = 1$, $(c_3\bar{c}_9, b_9) = (c_3b_9, c_9) = 1$, $(c_3\bar{c}_9, c_8) = (c_3c_8, c_9) = 1$, then

$$c_3\bar{c}_9 = b_{10} + b_9 + c_8. \tag{1}$$

By the associative law and 3) and 14) in Lemma 2.11:

$$(b_3b_8)c_3 = (c_3b_8)b_3, \qquad b_{15}c_3 + \bar{c}_9c_3 = c_3b_3 + b_6b_3 + \bar{b}_{15}b_3,$$

by 5), 12) in Lemma 2.10 and (1) and 1) in Lemma 2.9

$$c_3b_{15} = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10}. \tag{2}$$

By the associative law and (1) and 1) Lemma 2.10:

$$(b_3\bar{c}_9)c_3 = (c_3\bar{c}_9)b_3, \qquad \bar{b}_{15}c_3 + c_9c_3 + c_3^2 = b_{10}b_3 + b_9b_3 + c_8b_3;$$

by 1), 13), 16) in Lemma 2.11 and 3), 4) in Lemma 2.9

$$c_3^2 = \bar{c}_3 + \bar{b}_6. \tag{3}$$

By the associative law and 2) in Lemma 2.9 and 8) in Lemma 2.10:

$$(b_3\bar{b}_6)b_{15} = (\bar{b}_6b_{15})b_3,$$
$$\bar{b}_3b_{15} + \bar{b}_{15}b_{15} = 4\bar{b}_{15}b_3 + b_6b_3 + \bar{b}_3b_3 + c_3b_3 + 2c_9b_3.$$

By 6), 5), 12) in Lemma 2.10 and 1) in Lemma 2.9 and Hypothesis 2.1, we obtain that

$$\bar{b}_{15}b_{15} = 1 + 3b_5 + 3c_5 + 5c_8 + 5b_8 + 6b_6 + 6b_{10}. \tag{4}$$

By the associative law and 2) in Lemma 2.9 and 13) in Lemma 2.10:

$$(b_3\bar{b}_6)\bar{b}_{15} = b_3(\bar{b}_6\bar{b}_{15}),$$
$$\bar{b}_3\bar{b}_{15} + \bar{b}_{15}^2 = 2c_8b_3 + 3b_{10}b_3 + 2b_9b_3 + 2b_8b_3 + b_5b_3 + c_5b_3;$$

by 3), 4) in Lemma 2.9 and 1, 3), 4), 5) in Lemma 2.11

$$b_{15}^2 = 9\bar{b}_{15} + 4b_6 + 2\bar{b}_3 + 3c_3 + 6c_9. \tag{5}$$

By 4), 12), 13) in Lemma 2.10 and 12) in Lemma 2.11 and 5) in Lemma 2.9:

$$(b_{10}b_{15}, b_3) = (b_{10}, \bar{b}_{15}b_{15}) = 6, \qquad (b_{10}b_{15}, \bar{b}_6) = (b_{15}b_6, b_{10}) = 3,$$
$$(b_{10}b_{15}, b_3) = (\bar{b}_{15}b_3, b_{10}) = 1, \qquad (b_{10}b_{15}, \bar{c}_3) = (b_{10}c_3, \bar{b}_{15}) = 1,$$
$$(b_{10}b_{15}, \bar{b}_3c_3) = (b_{15}b_3, b_{10}c_3) = 4,$$

so $(b_{10}b_{15}, \bar{c}_9) = 4$. Hence

$$b_{10}b_{15} = 6b_{15} + 3\bar{b}_6 + b_3 + \bar{c}_3 + 4\bar{c}_9. \tag{6}$$

By the associative law and 4) in Lemma 2.10 and 16) in Lemma 2.11:

$$(b_3\bar{c}_3)b_{15} = b_3(\bar{c}_3b_{15}), \qquad c_9b_{15} = 2\bar{b}_{15}b_3 + b_3b_6 + b_3c_9;$$

by 6), 12) in Lemma 2.10 and 1) in Lemma 2.9:

$$c_9b_{15} = 2b_5 + 2c_5 + 3c_8 + 3b_8 + 3b_9 + 4b_{10}. \tag{7}$$

By 4) and Lemma 2.10 and (2) we get:

$$(b_3\bar{c}_3)\bar{b}_{15} = (\bar{c}_3\bar{b}_{15})b_3,$$
$$c_9\bar{b}_{15} = b_5b_3 + c_5b_3 + c_8b_3 + b_8b_3 + b_9b_3 + b_{10}b_3;$$

by 1), 3), 4), 5) in Lemma 2.11 and 3), 4) in Lemma 2.9,

$$c_9\bar{b}_{15} = 6b_{15} + 2\bar{b}_6 + b_3 + \bar{c}_3 + 3\bar{c}_9. \tag{8}$$

By the associative law and 4) in Lemma 2.10 and (1):

$$(b_3\bar{c}_3)c_9 = b_3(\bar{c}_3c_9), \qquad c_9^2 = b_{10}b_3 + b_9b_3 + c_8b_3;$$

by 3), 4) in Lemma 2.9 and 1) in Lemma 2.11

$$c_9^2 = 3b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 2\bar{c}_9. \tag{9}$$

By the associative law and 4) in Lemma 2.10 and 13) in Lemma 2.11:

$$(b_3\bar{c}_3)\bar{c}_9 = b_3(\bar{c}_3\bar{c}_9), \qquad c_9\bar{c}_9 = b_3\bar{b}_{15} + b_3c_9 + b_3\bar{b}_3;$$

by 6), 12) in Lemma 2.10 and Hypothesis 2.1:

$$c_9\bar{c}_9 = 1 + 2c_8 + 2b_9 + 2b_{10} + c_5 + b_5 + 2b_8. \tag{10}$$

By the associative law and 4) in Lemma 2.10 and 2) in Lemma 2.11:

$$(b_3\bar{c}_3)c_8 = (\bar{c}_3c_8)b_3, \qquad c_9c_8 = \bar{c}_9b_3 + b_{15}b_3;$$

by 1) in Lemma 2.10 and 5) in Lemma 2.9:

$$c_9c_8 = 3\bar{b}_{15} + b_6 + c_3 + 2c_9. \tag{11}$$

By the associative law and 5) in Lemma 2.10 and 13) in Lemma 2.11:

$$(b_3c_3)c_9 = (c_3c_9)b_3, \qquad b_9c_9 = b_{15}b_3 + \bar{c}_9b_3 + b_3^2;$$

by 5) in Lemma 2.9 and Hypothesis 2.1 and 1) in Lemma 2.10

$$b_9c_9 = \bar{b}_3 + 2c_9 + 3\bar{b}_{15} + c_3 + 2b_6. \tag{12}$$

By the associative law and 4) in Lemma 2.10 and 12) in Lemma 2.11:

$$(b_3\bar{c}_3)b_{10} = (\bar{c}_3b_{10})b_3, \qquad c_9b_{10} = b_{15}b_3 + \bar{b}_6b_3 + \bar{c}_9b_3;$$

by 2) and 5) in Lemma 2.9 and 1) in Lemma 2.10

$$c_9b_{10} = \bar{b}_3 + 2c_9 + 4\bar{b}_{15} + c_3 + b_6. \tag{13}$$

By (2),

$$(\bar{c}_3b_5, b_{15}) = (b_5, c_3b_{15}) = 1, \qquad (\bar{c}_3c_5, b_{15}) = (c_5, c_3b_{15}) = 1.$$

So

$$\bar{c}_3b_5 = b_{15}, \tag{14}$$

$$\bar{c}_3c_5 = b_{15}. \tag{15}$$

Hence by 4), 5) in Lemma 2.10 and the associative law:

$$(\bar{c}_3b_5)b_3 = (b_3\bar{c}_5)b_5, \qquad (\bar{c}_3c_5)b_3 = (b_3\bar{c}_3)c_5,$$

$$b_{15}b_3 = c_9b_5, \qquad b_{15}b_3 = c_9c_5.$$

By 5) in Lemma 2.9 we get:

$$b_5c_9 = 2\bar{b}_{15} + b_6 + c_9, \tag{16}$$

$$c_5c_9 = 2\bar{b}_{15} + b_6 + c_9. \tag{17}$$

By the associative law and 12) in Lemma 2.10 and 5) in Lemma 2.11:

$$(b_3\bar{b}_{15})b_5 = (b_3b_5)\bar{b}_{15} = b_{15}\bar{b}_{15},$$

$$b_{10}b_5 + c_8b_5 + b_9b_5 + b_8b_5 + b_5^2 + c_5b_5 = b_{15}\bar{b}_{15},$$

so b_5 is real and so c_5 is real. So by (14), (15) $c_3b_5 = \bar{b}_{15}, c_3b_5 = b_{15}$. □

Lemma 2.13 *Let (A, B) satisfy Hypothesis 2.1. Assume that $\bar{b}_{10} = b_{10} \in B$ and $(\bar{c}_9b_3, c_9) = 1$. Then we obtain the following equations*:

1) $b_5c_8 = c_5 + c_8 + b_8 + b_9 + b_{10}$;
2) $b_5b_8 = c_5 + c_8 + b_8 + b_9 + b_{10}$;
3) $b_5b_9 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5$;
4) $b_5b_{10} = c_8 + b_8 + b_9 + b_5 + 2b_{10}$;
5) $b_5^2 = 1 + b_{10} + b_9 + b_5$;
6) $c_5b_5 = b_8 + c_8 + b_9$;
7) $c_8b_8 = b_5 + c_5 + c_8 + b_8 + 2b_9 + 2b_{10}$;
8) $c_8b_9 = 2b_8 + 2b_9 + 2b_{10} + b_5 + c_5 + c_8$;
9) $b_9^2 = 1 + b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}$;
10) $b_8^2 = 1 + b_5 + c_5 + c_8 + 2b_8 + b_9 + 2b_{10}$;
11) $b_8b_9 = b_5 + c_5 + 2c_8 + b_8 + 2b_9 + 2b_{10}$;
12) $b_8b_{10} = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}$;
13) $c_5c_8 = b_5 + c_8 + b_8 + b_9 + b_{10}$;
14) $c_5b_8 = b_5 + c_8 + b_8 + b_9 + b_{10}$;
15) $c_5b_9 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5$;
16) $c_5b_{10} = b_8 + c_8 + b_9 + c_5 + 2b_{10}$;
17) $c_5^2 = 1 + b_9 + b_{10} + c_5$;
18) $b_5b_{15} = 3b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
19) $c_5b_{15} = 3b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
20) $b_9b_{15} = 6b_{15} + 3\bar{c}_9 + b_3 + 2\bar{b}_6 + \bar{c}_3$;
21) $b_9b_{10} = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 3b_{10}$.

Proof By the associative law and Hypothesis 2.1 and 12) in Lemma 2.10:

$$b_5(b_3\bar{b}_3) = (\bar{b}_3b_5)b_3 = \bar{b}_{15}b_3,$$

$$b_5 + b_5c_8 = \bar{b}_{15}b_3 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5.$$

Thus

$$c_8b_5 = c_5 + c_8 + b_8 + b_9 + b_{10}. \tag{1}$$

By the associative law and 9) in Lemma 2.10 and 18) in Lemma 2.12:

$$(c_3\bar{c}_3)b_5 = (\bar{c}_3b_5)c_3, \qquad b_5 + b_5b_8 = b_{15}c_3;$$

by 2) in Lemma 2.12

$$b_5b_8 = c_5 + c_8 + b_8 + b_9 + b_{10}. \tag{2}$$

By the associative law and 5), 12) in Lemma 2.10 and 18) in Lemma 2.12:

$$(b_3c_3)b_5 = (c_3b_5)b_3,$$
$$b_9b_5 = \bar{b}_{15}b_3 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5. \tag{3}$$

By the associative law and 1) in Lemma 2.9 and 5) in Lemma 2.11:

$$(b_3b_6)b_5 = (b_3b_5)b_6, \qquad c_8b_5 + b_{10}b_5 = b_{15}b_6;$$

by (1) and 13) in Lemma 2.10:

$$b_5b_{10} = c_8 + b_8 + b_9 + b_5 + 2b_{10}. \tag{4}$$

By the associative law and 12) in Lemma 2.10 and 5) in Lemma 2.11:

$$(b_3\bar{b}_{15})b_5 = (b_3b_5)\bar{b}_{15},$$
$$b_{10}b_5 + c_8b_5 + b_9b_5 + b_8b_5 + b_5^2 + c_5b_5 = b_{15}\bar{b}_{15};$$

so by 4) in Lemma 2.12 and (1), (2), (3), (4),

$$b_5^2 + c_5b_5 = 1 + b_5 + 2b_9 + b_{10} + b_8 + c_8,$$

and by (2), (4), (3), (1),

$$(b_5^2, b_{10}) = (b_5, b_5b_{10}) = 1, \qquad (b_5^2, b_9) = (b_5, b_5b_9) = 1.$$

Therefore

$$b_5^2 = 1 + b_{10} + b_9 + b_5, \tag{5}$$
$$c_5b_5 = b_8 + c_8 + b_9. \tag{6}$$

By the associative law and Hypothesis 2.1 and 3) in Lemma 2.11:

$$(b_3\bar{b}_3)b_8 = (\bar{b}_3b_8)b_3, \qquad b_8 + c_8b_8 = \bar{b}_{15}b_3 + c_9b_3;$$

by 6), 12) in Lemma 2.10, we obtain

$$c_8b_8 = b_5 + c_5 + c_8 + b_8 + 2b_9 + 2b_{10}. \tag{7}$$

By the associative law and Hypothesis 2.1 and 1) in Lemma 2.11:

$$(b_3\bar{b}_3)b_9 = (\bar{b}_3b_9)b_3, \qquad b_9 + c_8b_9 = \bar{b}_{15}b_3 + c_9b_3 + c_3b_3;$$

by 5), 6), 12) in Lemma 2.10 we obtain

$$b_9 + c_8b_9 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5 + b_9 + b_{10} + b_8 + b_9.$$

By 5) in Lemma 2.10 and 15) in Lemma 2.11 we obtain

$$c_8 b_9 = 2b_8 + 2b_9 + 2b_{10} + b_5 + c_5 + c_8, \tag{8}$$
$$(b_3 c_3) b_9 = (c_3 b_9) b_3, \qquad b_9^2 = \bar{b}_{15} b_3 + \bar{b}_3 b_3 + c_9 b_3;$$

by 12) in Lemma 2.10 and Hypothesis 2.1 and 6) in Lemma 2.10:

$$b_9^2 = 1 + b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}. \tag{9}$$

By the associative law and 9) in Lemma 2.10 and 14) in Lemma 2.11:

$$(c_3 \bar{c}_3) b_8 = c_3 (\bar{c}_3 b_8), \qquad b_8 + b_8^2 = c_3 \bar{c}_3 + c_3 \bar{b}_6 + c_3 b_{15};$$

by 7), 9) in Lemma 2.10 and 2) in Lemma 2.12:

$$b_8^2 = 1 + b_5 + c_5 + c_8 + 2b_8 + b_9 + 2b_{10}. \tag{10}$$

By the associative law and 9) in Lemma 2.10 and 15) in Lemma 2.11:

$$(c_3 \bar{c}_3) b_9 = (\bar{c}_3 b_9) c_3, \qquad b_9 + b_8 b_9 = b_{15} c_3 + b_3 c_3 + \bar{c}_9 c_3;$$

by 1), 2) in Lemma 2.12 and 5) in Lemma 2.10:

$$b_9 + b_8 b_9 = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10} + b_9 + b_{10} + b_9 + c_8,$$
$$b_8 b_9 = b_5 + c_5 + 2c_8 + b_8 + 2b_9 + 2b_{10}. \tag{11}$$

By the associative law and 9) in Lemma 2.10, 12) in Lemma 2.11:

$$(c_3 \bar{c}_3) b_{10} = (\bar{c}_3 b_{10}) c_3, \qquad b_{10} + b_8 b_{10} = b_{15} c_3 + \bar{b}_6 c_3 + \bar{c}_9 c_3,$$

so by 1), 2) in Lemma 2.12 and 7) in Lemma 2.10:

$$b_8 b_{10} = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}. \tag{12}$$

By the associative law and Hypothesis 2.1 and 4) in Lemma 2.11, 12) in Lemma 2.10:

$$(b_3 \bar{b}_3) c_5 = (\bar{b}_3 c_5) b_3,$$
$$c_5 + c_5 c_8 = \bar{b}_{15} b_3 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5,$$
$$c_5 c_8 = b_{10} + c_8 + b_9 + b_8 + b_5. \tag{13}$$

By the associative law and 9) in Lemma 2.10 and 2), 18) in Lemma 2.12:

$$(c_3 \bar{c}_3) c_5 = (\bar{c}_3 c_5) c_3,$$
$$c_5 + c_5 b_8 = b_{15} c_3 = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10},$$
$$c_5 b_8 = b_5 + c_8 + b_8 + b_9 + b_{10}. \tag{14}$$

By the associative law and 5), 12) in Lemma 2.10, 18) in Lemma 2.12:

$$(b_3c_3)c_5 = (c_3c_5)b_3,$$
$$b_9c_5 = \bar{b}_{15}b_3 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5. \tag{15}$$

By the associative law and 1) in Lemma 2.9 and 10) in Lemma 2.11:

$$(b_3b_6)c_5 = (b_6c_5)b_3,$$
$$c_8c_5 + b_{10}c_5 = \bar{b}_{15}b_3 + b_3b_6 + b_3c_9.$$

By 6), 12) in Lemma 2.10 and (13) and 1) in Lemma 2.9 we get:

$$b_{10} + c_8 + b_9 + b_8 + b_5 + b_{10}c_5 = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5 + c_8 + b_{10}$$
$$+ b_9 + b_{10} + b,$$
$$b_{10}c_5 = c_5 + c_8 + 2b_{10} + b_9 + b_8. \tag{16}$$

By the associative law and 10) in Lemma 2.10, 10) in Lemma 2.11:

$$(b_6\bar{b}_6)c_5 = \bar{b}_6(b_6c_5),$$
$$c_5 + c_5c_8 + b_9c_5 + b_8c_5 + c_5^2 + b_5c_5 = \bar{b}_6\bar{b}_{15} + \bar{b}_6b_6;$$

by (13), (15), (14), (6) and 10), 13) in Lemma 2.10:

$$c_5^2 = 1 + b_9 + b_{10} + c_5. \tag{17}$$

By the associative law and 18) in Lemma 2.12 and (5):

$$c_3b_5^2 = (c_3b_5)b_5,$$
$$c_3 + b_{10}c_3 + b_9c_3 + b_5c_3 = \bar{b}_{15}b_5;$$

so by 12), 15) in Lemma 2.11 and 17) in Lemma 2.12:

$$b_5\bar{b}_{15} = 3\bar{b}_{15} + b_6 + \bar{b}_3 + c_3 + 2c_9. \tag{18}$$

By the associative law and 18) in Lemma 2.12 and (17):

$$(c_5^2)c_3 = (c_5c_3)c_5, \qquad c_3 + b_9c_3 + b_{10}c_3 + c_5c_3 = \bar{b}_{15}c_5;$$

so by 12), 15) in Lemma 2.11 and 18) in Lemma 2.12:

$$\bar{b}_{15}c_5 = 3\bar{b}_{15} + b_6 + \bar{b}_3 + c_3 + 2c_9. \tag{19}$$

By the associative law and 4) in Lemma 2.11 and (15):

$$(b_3c_5)b_9 = b_3(c_5b_9),$$
$$b_{15}b_9 = b_3b_8 + c_8b_3 + b_5b_3 + c_5b_3 + b_9b_3 + b_{10}b_3;$$

by 3), 4) in Lemma 2.9 and 1), 3), 4), 5) in Lemma 2.11:

$$b_{15}b_9 = b_{15} + \bar{c}_9 + b_3 + \bar{b}_6 + b_{15} + 2b_{15} + 2b_{15} + 2\bar{c}_9 + \bar{c}_3 + \bar{b}_6,$$
$$b_{15}b_9 = 6b_{15} + 3\bar{c}_9 + b_3 + 2\bar{b}_6 + \bar{c}_3.$$

By the associative law and 5) in Lemma 2.10 and 4) in Lemma 2.9:

$$b_{10}(b_3c_3) = (b_{10}b_3)c_3,$$
$$b_{10}b_9 = b_{15}c_3 + \bar{b}_6c_3 + \bar{c}_9c_3,$$
$$b_{10}b_9 = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10} + b_8 + c_8 + b_9 + b_{10},$$
$$b_{10}b_9 = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 3b_{10}.$$

□

This completes the investigation of 11) in Lemma 2.9, subcase $(\bar{c}_9b_3, c_9) = 1$. Thus we have the remaining case that $(b_3\bar{c}_9, c_9) = 0$. In the remainder of this section, we deal with the investigation of 11) in Lemma 2.9, subcase $(b_6c_9, \bar{c}_9) \geq 1$.

Lemma 2.14 *Let* (A, B) *satisfy Hypothesis* 2.1. *Assume that* $\bar{b}_{10} = b_{10} \in B$ *and* $(b_6c_9, \bar{c}_9) \geq 1$. *Then we obtain*:

1) $b_3c_9 = b_{10} + \bar{x} + s$, $s \in B$, $|\bar{x} + s| = 17$, $|\bar{s}| = |s|$;
2) $(c_8b_{15}, \bar{c}_9) = (c_8c_9, \bar{b}_{15}) = 2$;
3) $\bar{b}_6\bar{c}_9 = 3c_9 + \bar{b}_{15} + b_6 + t_6$, $t_6 \in N^*B$;
4) $c_8b_{10} = 2c_8 + 2b_{10} + 2x + y + z + w + s$;
5) $b_6b_{15} = 3b_{10} + 2c_8 + x + y + z + w + \bar{x} + s$;
6) $\bar{b}_3c_9 = b_{15} + \varepsilon + \theta$, $\varepsilon, \theta \in B$, $|\varepsilon + \theta| = 12$;
7) $b_6b_{10} = \bar{b}_3 + 3\bar{b}_{15} + \bar{\varepsilon} + \bar{\theta}$;
8) $c_8b_{15} = 5b_{15} + 2\bar{b}_6 + 2\bar{c}_9 + b_3 + \varepsilon + \theta$.

Proof By the associative law and Hypothesis 2.1 and 10) in Lemma 2.9:

$$(b_3^2)\bar{b}_{15} = (b_3\bar{b}_{15})b_3,$$
$$\bar{b}_3\bar{b}_{15} + b_6\bar{b}_{15} = b_{10}b_3 + c_8b_3 + b_3(x + y + z + w).$$

So by 5), 4), 3) in Lemma 2.9,

$$b_6\bar{b}_{15} = b_3 + \bar{b}_6 + b_3(x + y + z + w). \tag{1}$$

By the associative law and Hypothesis 2.1 and 4) in Lemma 2.9:

$$(b_3^2)b_{10} = (b_3b_{10})b_3,$$
$$\bar{b}_3b_{10} + b_6b_{10} = b_{15}b_3 + \bar{b}_6b_3 + \bar{c}_9b_3;$$

by 4), 5), 2) in Lemma 2.9:

$$b_6b_{10} = \bar{b}_3 + 2\bar{b}_{15} + b_3\bar{c}_9. \tag{2}$$

By the associative law and (2) and 4) in Lemma 2.9:

$$(b_3 b_{10})\bar{b}_6 = (\bar{b}_6 b_{10}) b_3,$$
$$\bar{b}_6^2 + b_{15}\bar{b}_6 + \bar{b}_6\bar{c}_9 = b_3^2 + 2b_{15}b_3 + (\bar{b}_3 b_3)c_9.$$

So by 5), 6) in Lemma 2.9 and Hypothesis 2.1:

$$\begin{aligned} &2b_6 + c_9 + \bar{b}_{15} + \bar{b}_3 + b_6 + \bar{b}_3(x+y+z+w) + \bar{b}_6\bar{c}_9 \\ &\quad = \bar{b}_3 + b_6 + 4\bar{b}_{15} + 2b_6 + 2c_9 + c_9 + c_8c_9, \\ &\bar{b}_6\bar{c}_9 + \bar{b}_3(x+y+z+w) = 3\bar{b}_{15} + 2c_9 + c_8c_9. \end{aligned} \tag{3}$$

By the associative law and 6), 7) in Lemma 2.9:

$$(b_6^2)c_8 = (b_6c_8)b_6,$$
$$2\bar{b}_6c_8 + \bar{c}_9c_8 + b_{15}c_8 = \bar{b}_3b_6 + 2\bar{b}_{15}b_6 + b_6^2 + c_9b_6;$$

by 7), 2), 6) in Lemma 2.9 and (3), (1)

$$c_8b_{15} = b_3 + 2\bar{b}_6 + \bar{c}_9 + b_{15} + b_3(x+y+z+w). \tag{4}$$

By the associative law and Hypothesis 2.1:

$$(b_3\bar{b}_3)c_9 = (\bar{b}_3c_9)b_3, \qquad c_9 + c_9c_8 = (\bar{b}_3c_9)b_3,$$

but by 5) in Lemma 2.9,

$$(\bar{b}_3c_9, b_{15}) = (c_9, b_3b_{15}) = 1.$$

So

$$c_9 + c_9c_8 = b_{15}b_3 + \alpha_1 b_3;$$

hence by 5) in Lemma 2.9 we obtain that:

$$c_9c_8 = 2\bar{b}_{15} + b_6 + \alpha_1 b_3. \tag{5}$$

Hence

$$(c_8c_9, \bar{b}_{15}) \geq 2. \tag{6}$$

By 10) in Lemma 2.9,

$$(\bar{b}_3x, \bar{b}_{15}) = 1, \qquad (\bar{b}_3y, \bar{b}_{15}) = 1, \qquad (\bar{b}_3z, \bar{b}_{15}) = 1, \qquad (\bar{b}_3w, \bar{b}_{15}) = 1.$$

So by (3),

$$(\bar{b}_6\bar{c}_9, \bar{b}_{15}) \geq 1. \tag{7}$$

By (6), $(c_8 b_{15}, \bar{c}_9) \geq 2$, so by (4),

$$(b_3(x + y + z + w), \bar{c}_9) \neq 0.$$

Hence one of $b_3 x$, $b_3 y$, $b_3 z$, and $b_3 w$ contains $\bar{c}_9$; thus, without loss of generality, we may assume that

$$(b_3 x, \bar{c}_9) \neq 0. \tag{8}$$

By the associative law and Hypothesis 2.1 and 4) in Lemma 2.9:

$$(b_3 \bar{b}_3) b_{10} = (b_3 b_{10}) \bar{b}_3,$$
$$b_{10} + c_8 b_{10} = b_{15} \bar{b}_3 + \bar{b}_6 \bar{b}_3 + \bar{c}_9 \bar{b}_3;$$

by 10), 1) in Lemma 2.9

$$c_8 b_{10} = 2c_8 + b_{10} + x + y + z + w + \bar{b}_3 \bar{c}_9. \tag{9}$$

By (8) and 4) in Lemma 2.9, $(b_3 c_9, b_{10}) = (b_3 b_{10}, \bar{c}_9) = 1$, $(b_3 c_9, \bar{x}) \neq 0$; so

$$b_3 c_9 = b_{10} + \bar{x} + \alpha, \quad \alpha \in N^* B. \tag{10}$$

Then $(b_3 x, \bar{c}_9) = (b_3 c_9, \bar{x}) \neq 0$. By (9) since c_8, b_{10}, $x + y + z + w$ are reals then $b_3 c_9$ is real. By 10) in Lemma 2.9, $(b_3 x, b_{15}) = 1$; therefore

$$b_3 x = \bar{c}_9 + b_{15} + \beta, \tag{11}$$

where $\beta \in N^* B$. So $|x| \geq 8$. Hence by (10)

$$3 \leq (b_3 c_9, b_3 c_9) \leq 5.$$

Thus

$$3 \leq (b_3 \bar{b}_3, c_9 \bar{c}_9) \leq 5.$$

So by Hypothesis $2 \leq (c_9 \bar{c}_9, c_8) \leq 4$. Thus

$$2 \leq (c_9, c_9 c_8) \leq 4. \tag{12}$$

If $(c_8 b_{15}, \bar{c}_9) \geq 4$ then by (4), we have that $(b_3(x + y + z + w), \bar{c}_9) \geq 3$. Since $1 + (c_8, b_{15} \bar{b}_{15}) = (1 + c_8, b_{15} \bar{b}_{15}) = (b_3 \bar{b}_3, b_{15} \bar{b}_{15}) = (b_3 b_{15}, b_3 b_{15}) = 6$ by 10) in Lemma 2.9, we have that $(c_8 b_{15}, b_{15}) = 5$. Hence we obtain, without loss of generality, by (11) that

$$b_3 y = \bar{c}_9 + b_{15} + \beta_1, \qquad b_3 z = \bar{c}_9 + b_{15} + \beta_2.$$

So the degrees of x, y and z are all larger or equal to 8, but $|x + y + z + w| = 27$; hence $|x| = |y| = |z| = 8$ by (11) and $(b_3 c_9, x) = (b_3 c_9, y) = (b_3 c_9, z) = 1$, a contradiction to (10). If $(c_8 b_{15}, \bar{c}_9) = 3$. Then by (5) and (12)

$$c_8 c_9 = 3\bar{b}_{15} + b_6 + 2c_9 + c_3, \tag{13}$$

where $c_3 \in B$. By (4), $(b_3(x+y+z+w), \bar{c}_9) = 2$, so by (3) $(\bar{b}_6\bar{c}_9, c_9) = 2$ and by 10), 6) in Lemma 2.9, $(\bar{b}_6\bar{c}_9, b_6) = 1$, $(\bar{b}_3(x+y+z+w), \bar{b}_{15}) = 4$, so by (3), (13) $(\bar{b}_6\bar{c}_9, \bar{b}_{15}) = 2$. Hence

$$b_6c_9 = \bar{b}_6 + 2\bar{c}_9 + 2b_{15}. \tag{14}$$

So by (3),

$$\bar{b}_3(x+y+z+w) = 4\bar{b}_{15} + 2c_9 + c_3.$$

Then we obtain, without loss of generality, that the only possibility is

$$x = b_8, \qquad y = b_9, \qquad z = b_5, \qquad w = c_5,$$

when b_8, b_9, b_5, c_5 are reals. So

$$\bar{b}_3b_8 = \bar{b}_{15} + c_9, \tag{15}$$

$$\bar{b}_3b_9 = \bar{b}_{15} + c_9 + c_3, \tag{16}$$

$$\bar{b}_3b_5 = \bar{b}_{15}, \tag{17}$$

$$\bar{b}_3c_5 = \bar{b}_{15}. \tag{18}$$

So $(b_3c_9, b_8) = 1$, and by 4) in Lemma 2.9, $(b_3c_9, b_{10}) = 1$. Thus

$$b_3c_9 = b_{10} + b_8 + b_9. \tag{19}$$

So by the associative law and Hypothesis 2.1

$$(b_3^2)c_9 = (b_3c_9)b_3,$$

$$c_9\bar{b}_3 + c_9b_6 = b_{10}b_3 + b_8b_3 + b_9b_3.$$

Therefore by (15), (16), (14) $c_9\bar{b}_3 = b_{15} + \bar{c}_9 + \bar{c}_3$ which contradicts the assumption in the lemma, we conclude that $(c_8b_{15}, \bar{c}_9) \le 2$ and by (8) and (4),

$$(c_8b_{15}, \bar{c}_9) = 2. \tag{20}$$

By (4)

$$(b_3(x+y+z+w), c_9) = 1, \tag{21}$$

and by (3), (12) we obtain $3 \le (\bar{b}_6\bar{c}_9, c_9) \le 5$, by (20) $(c_8c_9, \bar{b}_{15}) = 2$. So by (3) and that $(\bar{b}_3(x+y+z+w), \bar{b}_{15}) = 4$ we get $(\bar{b}_6\bar{c}_9, \bar{b}_{15}) = 1$. And by 6) in Lemma 2.9, $(\bar{b}_6\bar{c}_9, b_6) = 1$. So $(\bar{b}_6\bar{c}_9, c_9) = 3$. Then by (3), (21), $(c_8c_9, c_9) = 2$. So $(c_8, \bar{c}_9c_9) = 2$, and by Hypothesis 2.1 $(\bar{c}_9c_9, \bar{b}_3b_3) = 3$. Thus $(b_3c_9, b_3c_9) = 3$. So by 4) in Lemma 2.9 and (10), we obtain that

$$b_3c_9 = b_{10} + \bar{x} + s, \tag{22}$$

$s \in B$. Also

$$\bar{b}_6\bar{c}_9 = 3c_9 + \bar{b}_{15} + b_6 + t_6, \tag{23}$$

where $t_6 \in N^*B$, $t_6 \neq b_6, \bar{b}_6$. By (9) and (22),

$$c_8 b_{10} = 2c_8 + 2b_{10} + 2x + y + z + w + \bar{S}. \tag{24}$$

By the associative law and 5) in Lemma 2.9:

$$(b_3^2)b_{15} = (b_3 b_{15})b_3,$$
$$\bar{b}_3 b_{15} + b_6 b_{15} = 2\bar{b}_{15} b_3 + b_6 b_3 + c_9 b_3;$$

by (22) and 1), 10) in Lemma 2.9:

$$b_6 b_{15} = 3b_{10} + 2c_8 + x + y + z + w + \bar{x} + S. \tag{25}$$

By the associative law and Hypothesis 2.1 and 5) in Lemma 2.9:
Since $b_{15}(b_3\bar{b}_3) = \bar{b}_3(b_{15}b_3)$, then

$$c_8 b_{15} = 4b_{15} + 2\bar{b}_6 + 2\bar{c}_9 + b_3 + \bar{b}_3 c_9. \tag{26}$$

By (22) and 5) in Lemma 2.9:

$$\bar{b}_3 c_9 = b_{15} + \varepsilon + \theta, \tag{27}$$

where $\varepsilon, \theta \in B$, $|\varepsilon + \theta| = 12$, so by (26)

$$c_8 b_{15} = 5b_{15} + 2\bar{b}_6 + 2\bar{c}_9 + b_3 + \varepsilon + \theta.$$

By the associative law and Hypothesis 2.1 and 4) in Lemma 2.9:

$$(b_3^2)b_{10} = b_3(b_3 b_{10}),$$
$$b_{10}\bar{b}_3 + b_{10}b_6 = b_{15}b_3 + \bar{b}_6 b_3 + \bar{c}_9 b_3;$$

so by (27) and 5), 2), 4) in Lemma 2.9:

$$b_6 b_{10} = \bar{b}_3 + 3\bar{b}_{15} + \bar{\varepsilon} + \bar{\theta}.$$

□

Lemma 2.15 *Let (A, B) satisfy Hypothesis 2.1. Assume that $\bar{b}_{10} = b_{10} \in B$ and $(b_3\bar{c}_9, c_9) = 0$. Then $(b_6 c_9, \bar{c}_9) \geq 1$ is impossible.*

Proof We assume that our assumption is possible. By Hypothesis 2.1 and 10) in Lemma 2.9 and the associative law:

$$(b_3\bar{b}_3)b_{15} = b_3(\bar{b}_3 b_{15}),$$
$$b_{15} + c_8 b_{15} = b_3 b_{10} + b_3 c_8 + b_3\bar{x} + b_3\bar{y} + b_3\bar{z} + b_3\bar{w}.$$

By 3), 4) in Lemma 2.9, 8) in Lemma 2.14:

$$\begin{aligned} &b_{15} + 5b_{15} + 2\bar{b}_6 + 2\bar{c}_9 + b_3 + \varepsilon + \theta \\ &\quad = b_{15} + \bar{b}_6 + \bar{c}_9 + b_3 + \bar{b}_6 + b_{15} + b_3\bar{x} + b_3\bar{y} + b_3\bar{z} + b_3\bar{w}, \\ &\qquad 4b_{15} + \bar{c}_9 + \varepsilon + \theta = b_3\bar{x} + b_3\bar{y} + b_3\bar{z} + b_3\bar{w}. \end{aligned} \tag{1}$$

By 10) in Lemma 2.9, and 1) in Lemma 2.14:

$$(b_3\bar{x}, b_{15}) = (\bar{b}_{15}b_3, x) = 1, \qquad (b_3\bar{y}, b_{15}) = (b_3\bar{b}_{15}, y) = 1,$$
$$(b_3\bar{z}, b_{15}) = (\bar{b}_{15}b_3, z) = 1, \qquad (b_3\bar{w}, b_{15}) = (b_3\bar{b}_{15}, w) = 1,$$

$(b_3\bar{x}, \bar{c}_9) = (b_3c_9, x) = 1$, so $x = \bar{x}$. If otherwise two of x, y, z, w have degrees ≥ 8, this is impossible. Then

$$b_3x = \bar{c}_9 + b_{15}, \tag{2}$$

and then $x = x_8$. So $|\bar{y} + \bar{z} + \bar{w}| = 19$, and also we get $5 \leq |\bar{y}|, |\bar{z}|, |\bar{w}| \leq 9$ and also, without loss of generality, $|\bar{y}| = 5$, hence $\bar{y} = \bar{y}_5$, so

$$b_3\bar{y}_5 = b_{15}. \tag{3}$$

So

$$|\bar{z} + \bar{w}| = 14. \tag{4}$$

By 1) in Lemma 2.14 and the fact that $x = x_8$, we obtain that

$$b_3c_9 = b_{10} + x_8 + s_9. \tag{5}$$

So by the associative law and Hypothesis 2.1 and (5):

$$(b_3^2)c_9 = b_3(b_3c_9),$$
$$c_9\bar{b}_3 + b_6c_9 = b_3b_{10} + b_3x_8 + s_9b_3.$$

Then by (2) and (6), 3) in Lemma 2.14, 4) in Lemma 2.9:

$$\begin{aligned} b_{15} + \varepsilon + \theta + 3\bar{c}_9 + b_{15} + \bar{b}_6 + \bar{t}_6 &= b_{15} + \bar{b}_6 + \bar{c}_9 + b_{15} + \bar{c}_9 + s_9b_3, \\ s_9b_3 &= \varepsilon + \theta + \bar{c}_9 + \bar{t}_6. \end{aligned} \tag{6}$$

If $|z| = 5$, then by (4) $|w| = 9$, hence $z = \bar{z}_5$, $w = \bar{w}_9$. So

$$b_3z_5 = b_{15},$$
$$b_3w_9 = b_{15} + \varepsilon + \theta. \tag{7}$$

So by (6)

$$(\bar{b}_3\varepsilon, s_9) = (\varepsilon, b_3 s_9) = 1, \qquad (\bar{b}_3\theta, s_9) = (\theta, b_3 s_9) = 1,$$
$$(\bar{b}_3\varepsilon, w_9) = (\varepsilon, b_3 w_9) = 1, \qquad (\bar{b}_3\theta, w_9) = (\theta, b_3 w_9) = 1.$$

Hence

$$\bar{b}_3\varepsilon = s_9 + w_9, \tag{8}$$
$$\bar{b}_3\theta = s_9 + w_9, \tag{9}$$

and hence $\varepsilon = \varepsilon_6$, $\theta = \theta_6$, by 4) in the Lemma 2.14 $s_9 = \bar{s}_9$. So by the associative law and Hypothesis 2.1:

$$\bar{\varepsilon}_6(b_3^2) = b_3(\bar{\varepsilon}_6 b_3),$$
$$\bar{\varepsilon}_6\bar{b}_3 + \bar{\varepsilon}_6 b_6 = s_9 b_3 + w_9 b_3;$$

so by (6), (7)

$$\bar{\varepsilon}_6\bar{b}_3 + \bar{\varepsilon}_6 b_6 = \varepsilon_6 + \theta_6 + \bar{c}_9 + \bar{t}_6 + b_{15} + \varepsilon_6 + \theta_6,$$
$$\bar{\varepsilon}_6\bar{b}_3 + \bar{\varepsilon}_6 b_6 = 2\varepsilon_6 + 2\theta_6 + \bar{c}_9 + \bar{t}_6 + b_{15}.$$

By 6) in Lemma 2.14 $(\bar{\varepsilon}_6\bar{b}_3, \bar{c}_9) = (\varepsilon_6, \bar{b}_3 c_9) = 1$ and $(\bar{\varepsilon}_6\bar{b}_3, \delta) = 1$, where $\delta = \varepsilon_6$ or θ_6, so $\bar{\varepsilon}_6\bar{b}_3 = \delta + \bar{c}_9 + m_3$, so $\bar{t}_6 = m_3 + k_3$, and $(b_3 m_3, \bar{\varepsilon}_6) = (\bar{b}_3\bar{\varepsilon}_6, m_3) = 1$. Therefore

$$(b_3 m_3, b_3 m_3) = (b_3\bar{b}_3, m_3\bar{m}_3) > 1.$$

In (6) we get

$$(\bar{b}_3 m_3, s_9) = (m_3, b_3 s_9) = 1,$$

then

$$(\bar{b}_3 m_3, \bar{b}_3 m_3) = (\bar{b}_3 b_3, \bar{m}_3 m_3) = 1,$$

a contradiction.

If $|\bar{z}| = 6$ then by (4) $|\bar{w}| = 8$, hence $z = \bar{z}$, $w = \bar{w}$. So

$$b_3 z_6 = b_{15} + \varepsilon_3, b_3 w_8 = b_{15} + \theta_9.$$

So $(\varepsilon_3\bar{b}_3, z_6) = (\varepsilon_3, b_3 z_6) = 1$, hence $(\varepsilon_3\bar{b}_3, \varepsilon_3\bar{b}_3) > 1$. By (6), $(s_9 b_3, \varepsilon_3) = (s_9, \bar{b}_3 z_6) = 1$, hence $\bar{b}_3\varepsilon_3 = s_9$, so $(\varepsilon_3\bar{b}_3, \varepsilon_3\bar{b}_3) = 1$, a contradiction. If $|\bar{z}| = 7$ then by (4) $|\bar{w}| = 7$, so

$$z = \bar{z}, \qquad w = \bar{w} \quad \text{or} \quad \bar{z} = w,$$
$$b_3\bar{z}_7 = b_{15} + \varepsilon_6, \tag{10}$$
$$b_3\bar{w}_7 = b_{15} + \theta_6. \tag{11}$$

So $(\bar{b}_3\varepsilon_6, \bar{z}_7) = (\varepsilon_6, b_3\bar{z}_7) = 1$. By (6) $(\bar{b}_3\varepsilon_6, s_9) = (\varepsilon_6, b_3 s_9) = 1$, hence

$$\bar{b}_3\varepsilon_6 = \bar{z}_7 + s_9 + m_2,$$

a contradiction. □

Consequently, 11) in Lemma 2.9 subcase $(b_6c_9, \bar{c}_9) \geq 1$ is impossible and we have only the case $(b_3\bar{c}_9, c_9) = 1$. We can now state Theorem 2.7.

Theorem 2.7 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ satisfying $b_3\bar{b}_3 = 1 + c_8$ and $b_3^2 = \bar{b}_3 + b_6$, where $b_6 \in B$ is nonreal and satisfying Hypothesis 2.1. Then $b_3b_6 = c_8 + b_{10}$ and if $b_{10} = \bar{b}_{10}$ then $(A, B) \cong (CH(3A_6), Irr(3A_6))$. (A, B) is a Table Algebra of dimension 17: $B = \{1, b_3, \bar{b}_3, c_3, \bar{c}_3, b_6, \bar{b}_6, c_9, \bar{c}_9, b_{15}, \bar{b}_{15}, b_5, c_5, b_8, c_8, b_9, b_{10}\}$ and B has an increasing series of table subsets $\{b_1\} \subseteq \{b_1, b_5, c_5, b_8, c_8, b_9, b_{10}\} \subseteq B$ defined by*

1) $b_3\bar{b}_3 = 1 + c_8$;
2) $b_3^2 = \bar{b}_3 + b_6$;
3) $b_3c_3 = b_9$;
4) $b_3\bar{c}_3 = c_9$;
5) $b_3b_6 = c_8 + b_{10}$;
6) $b_3\bar{b}_6 = \bar{b}_3 + \bar{b}_{15}$;
7) $b_3c_9 = b_9 + b_{10} + b_8$;
8) $b_3\bar{c}_9 = \bar{b}_{15} + c_9 + c_3$;
9) $b_3b_{15} = 2\bar{b}_{15} + b_6 + c_9$;
10) $b_3\bar{b}_{15} = b_{10} + c_8 + b_9 + b_8 + b_5 + c_5$;
11) $b_3b_5 = b_{15}$;
12) $b_3c_5 = b_{15}$;
13) $b_3b_8 = b_{15} + \bar{c}_9$;
14) $b_3c_8 = b_3 + \bar{b}_6 + b_{15}$;
15) $b_3b_9 = b_{15} + \bar{c}_9 + \bar{c}_3$;
16) $b_3b_{10} = b_{15} + \bar{b}_6 + \bar{c}_9$;
18) $c_3^2 = \bar{c}_3 + \bar{b}_6$;
19) $c_3b_6 = \bar{c}_3 + b_{15}$;
20) $c_3\bar{b}_6 = b_{10} + b_8$;
21) $c_3c_9 = b_3 + b_{15} + \bar{c}_9$;
22) $c_3\bar{c}_9 = c_8 + b_9 + b_{10}$;
23) $c_3b_{15} = b_5 + c_5 + c_8 + b_8 + b_9 + b_{10}$;
24) $c_3\bar{b}_{15} = 2b_{15} + \bar{b}_6 + \bar{c}_9$;
25) $c_3b_5 = \bar{b}_{15}$;
26) $c_3c_5 = \bar{b}_{15}$;
27) $c_3b_8 = c_3 + b_6 + \bar{b}_{15}$;
28) $c_3c_8 = c_9 + \bar{b}_{15}$;
29) $c_3b_9 = b_{15} + \bar{b}_3 + c_9$;
30) $c_3b_{10} = \bar{b}_{15} + b_6 + c_9$;
31) $b_6\bar{b}_6 = 1 + c_8 + b_9 + b_8 + c_5 + b_5$;
32) $b_6c_9 = 2b_{15} + 2\bar{c}_9 + \bar{b}_6$;
33) $b_6\bar{c}_9 = b_{10} + c_8 + 2b_9 + b_8 + b_5 + c_5$;
34) $b_6b_{15} = 2c_8 + 3b_{10} + 2b_9 + 2b_8 + b_5 + c_5$;
35) $b_6\bar{b}_{15} = 4b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
36) $b_6b_5 = \bar{b}_{15} + b_6 + c_9$;
37) $b_6c_5 = \bar{b}_{15} + b_6 + c_9$;

38) $b_6 b_8 = 2\bar{b}_{15} + b_6 + c_3 + c_9$;
39) $b_6 c_8 = \bar{b}_3 + 2\bar{b}_{15} + b_6 + c_9$;
40) $b_6 b_9 = b_6 + 2c_9 + 2\bar{b}_{15}$;
41) $b_6 b_{10} = 3\bar{b}_{15} + \bar{b}_3 + c_3 + c_9$;
42) $b_6^2 = 2\bar{b}_6 + \bar{c}_9 + b_{15}$;
43) $c_9 \bar{c}_9 = 1 + 2c_8 + 2b_9 + 2b_{10} + c_5 + b_5 + 2b_8$;
44) $c_9 b_{15} = 2b_5 + 2c_5 + 3c_8 + 3b_8 + 3b_9 + 4b_{10}$;
45) $c_9 \bar{b}_{15} = 6b_{15} + 2\bar{b}_6 + b_3 + \bar{c}_3 + 3\bar{c}_9$;
46) $c_9 b_5 = 2\bar{b}_{15} + b_6 + c_9$;
47) $c_9 c_5 = 2\bar{b}_{15} + b_6 + c_9$;
48) $c_9 b_8 = 3\bar{b}_{15} + \bar{b}_3 + b_6 + 2c_9$;
49) $c_9 c_8 = 3\bar{b}_{15} + c_3 + b_6 + 2c_9$;
50) $c_9 b_9 = \bar{b}_3 + 2c_9 + 3\bar{b}_{15} + c_3 + 2b_6$;
51) $c_9 b_{10} = \bar{b}_3 + 2c_9 + 4\bar{b}_{15} + c_3 + b_6$;
52) $c_9^2 = 3b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 2\bar{c}_9$;
53) $b_{15}\bar{b}_{15} = 1 + 3b_5 + 3c_5 + 5c_8 + 5b_8 + 6b_9 + 6b_{10}$;
54) $b_{15}^2 = 9\bar{b}_{15} + 4b_6 + 2\bar{b}_3 + 2c_3 + 6c_9$;
55) $b_{15} b_5 = 3b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
56) $b_{15} c_5 = 3b_{15} + \bar{b}_6 + b_3 + \bar{c}_3 + 2\bar{c}_9$;
57) $b_{15} b_8 = 5b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 3\bar{c}_9$;
58) $b_{15} c_8 = 5b_{15} + b_3 + \bar{c}_3 + 2\bar{b}_6 + 3\bar{c}_9$;
59) $b_{15} b_9 = 6b_{15} + 3\bar{c}_9 + b_3 + 2\bar{b}_6 + \bar{c}_3$;
60) $b_{15} b_{10} = 6b_{15} + 3\bar{b}_6 + +b_3 + \bar{c}_3 + 4\bar{c}_9$;
61) $b_5^2 = 1 + b_{10} + b_9 + b_5$;
62) $b_5 c_5 = b_8 + c_8 + b_9$;
63) $b_5 b_8 = c_5 + c_8 + b_8 + b_9 + b_{10}$;
64) $b_5 c_8 = c_5 + b_8 + b_9 + b_{10} + c_8$;
65) $b_5 b_9 = c_8 + c_5 + b_8 + b_5 + b_9 + b_{10}$;
66) $b_5 b_{10} = c_8 + b_8 + b_9 + b_5 + 2b_{10}$;
67) $c_5^2 = 1 + b_9 + b_{10} + c_5$;
68) $c_5 b_8 = b_5 + c_8 + b_8 + b_9 + b_{10}$;
69) $c_5 c_8 = b_5 + c_8 + b_8 + b_9 + b_{10}$;
70) $c_5 b_9 = b_8 + c_8 + b_5 + c_5 + b_9 + b_{10}$;
71) $c_5 b_{10} = b_8 + c_8 + b_9 + c_5 + 2b_{10}$;
72) $b_8^2 = 1 + b_5 + c_5 + c_8 + 2b_8 + b_9 + 2b_{10}$;
73) $b_8 c_8 = b_5 + c_5 + c_8 + b_8 + 2b_9 + 2b_{10}$;
74) $b_8 b_9 = b_5 + c_5 + 2c_8 + b_8 + 2b_9 + 2b_{10}$;
75) $b_8 b_{10} = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}$;
76) $c_8^2 = 1 + 2c_8 + 2b_{10} + b_9 + b_8 + b_5 + c_5$;
77) $c_8 b_9 = b_5 + c_5 + c_8 + 2b_8 + 2b_9 + 2b_{10}$;
78) $c_8 b_{10} = 2c_8 + 2b_{10} + 2b_9 + 2b_8 + b_5 + c_5$;
79) $b_9 b_{10} = b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 3b_{10}$;
80) $b_9^2 = 1 + b_5 + c_5 + 2c_8 + 2b_8 + 2b_9 + 2b_{10}$;
81) $b_{10}^2 = 1 + 2c_8 + 2b_{10} + 3b_9 + 2b_8 + 2b_5 + 2c_5$.

Proof The theorem follows by Lemmas 2.9–2.15. □

In this section, we do not have information about what NITA satisfying b_{10} nonreal. But we conjecture:

Conjecture 2.1 *There exists no NITA satisfying Hypothesis* 2.1 *and* $b_3b_6 = c_8 + b_{10}$, *where* b_{10} *is nonreal.*

2.6 NITA Generated by b_3 Satisfying $b_3^2 = c_3 + b_6$, $c_3 \neq b_3, \bar{b}_3$, b_6 Non-real, $(b_3b_8, b_3b_8) = 4$ and $c_3^2 = r_3 + s_6$

Now we discuss NITA satisfying the following hypothesis.

Hypothesis 2.2 *Let* (A, B) *be a NITA generated by a nonreal element* $b_3 \in B$ *satisfying* $b_3^2 = c_3 + b_6$, $c_3 \neq b_3, \bar{b}_3$, b_6 *nonreal*, $(b_3b_8, b_3b_8) = 4$ *and* $c_3^2 = r_3 + s_6$. *Here we still assume that* $L(B) = 1$ *and* $L_2(B) = \emptyset$.

In this section, we are not able to classify NITAs satisfying Hypothesis 2.2, but we have the following theorem.

Theorem 2.8 *There exists no NITA satisfying Hypothesis* 2.2 *and* c_3 *generates a sub-NITA isomorphic to one of the four known NITAs in the Main Theorem* 1.

Proof By assumptions we have that (1) $b_3\bar{b}_3 = 1 + b_8 = c_3\bar{c}_3$; (2) $b_3^2 = c_3 + b_6$; (3) $(b_3b_8, b_3b_8) = 4$ and (4) $c_3^2 = r_3 + s_6$.

Step 1 There exists no NITA satisfying hypotheses (1) to (4) and containing a NITA strictly isomorphic to $(Ch(PSL(2, 7)), Irr(PSL(2, 7)))$.

By [CA], $(Ch(PSL(2, 7)), Irr(PSL(2, 7)))$ the sub-NITA generated by c_3 which is of dimension 6 and has base elements: $1, c_3, \bar{c}_3, s_6, b_7, b_8$. Moreover, we have the following equations:

(a1) $c_3b_8 = c_3 + s_6 + b_7 + b_8$,
(a2) $s_6^2 = 1 + 2s_6 + b_7 + b_8$,
(a3) $c_3s_6 = \bar{c}_3 + b_7 + b_8$.

It follows by (1) that $(b_3\bar{c}_3, b_3\bar{c}_3) = (b_3\bar{b}_3, c_3\bar{c}_3) = 2$, hence we obtain equation (a4): $b_3\bar{c}_3 = \bar{b}_3 + t_6$, where $t_6 \in B$. By the associative law and equations (a1) and (a4), we have

$$(b_3\bar{b}_3)\bar{c}_3 = \bar{c}_3 + \bar{c}_3b_8 = \bar{c}_3 + \bar{c}_3 + s_6 + b_7 + b_8,$$

$$(b_3\bar{c}_3)\bar{b}_3 = \bar{b}_3^2 + \bar{b}_3t_6 = \bar{c}_3 + \bar{b}_6 + \bar{b}_3t_6.$$

Then we get $\bar{b}_6 = s_6$, so b_6 is real. Moreover we have equation (a5): $\bar{b}_3t_6 = \bar{c}_3 + s_6 + b_7 + b_8$. By $(b_3\bar{b}_3)b_3 = b_3^2\bar{b}_3$, we arrive at $b_3b_8 = \bar{t}_6 + \bar{b}_3s_6$. Now by

(3) we obtain equation (a6): $\bar{b}_3 s_6 = b_3 + x + y$, where distinct b_3, x and y such that $x + y$ is of degree 15. Hence we have equation (a7): $b_3 b_8 = b_3 + \bar{t}_6 + x + y$. Therefore by (a2), (a3), (a5) and (a6)

$$b_3^2 s_6 = (c_3 + s_6)s_6 = c_3 s_6 + s_6 s_6 = \bar{c}_3 + b_7 + b_8 + 1 + 2s_6 + b_7 + b_8,$$

$$b_3(b_3 s_6) = b_3(\bar{b}_3 + \bar{x} + \bar{y}) = 1 + b_8 + b_3\bar{x} + b_3\bar{y}.$$

So $b_3\bar{x} + b_3\bar{y} = \bar{c}_3 + 2s_6 + 2b_7 + b_8$, but by (a7), the left hand side of this equation contains two b_8, and the right side only one, a contradiction.

Step 2 There exists no NITA satisfying hypotheses (1) to (4) and containing a sub-NITA generated by c_3 which is a NITA of dimension 17, 32 or 22.

By [CA], we see that for the NITAs of dimensions 17, 32 and 22, one can find new base elements: b_5, c_5, s_6, x_8, b_9 and b_{10} such that the following equations hold:

(a1) $c_3 b_8 = c_3 + \bar{s}_6 + b_{15}$,
(a2) $s_6\bar{s}_6 = 1 + b_5 + c_5 + b_8 + x_8 + b_9$,
(a3) $c_3 s_6 = b_8 + b_{10}$.

Suppose $b_3\bar{c}_3 = \bar{b}_3 + t_6$, where $t_6 \in B$. Then by hypothesis and equation (a1) it follows that $(b_3\bar{b}_3)\bar{c}_3 = \bar{c}_3 + \bar{c}_3 b_8 = \bar{c}_3 + \bar{c}_3 + s_6 + \bar{b}_{15}$, $(b_3\bar{c}_3)\bar{b}_3 = \bar{b}_3^2 + \bar{b}_3 t_6 = \bar{c}_3 + \bar{b}_6 + \bar{b}_3 t_6$. We get $s_6 = \bar{b}_6$ and equation (a4): $\bar{b}_3 t_6 = \bar{c}_3 + \bar{b}_{15}$. Hence $b_3 + b_3 b_8 = (b_3\bar{b}_3)b_3 = b_3^2\bar{b}_3 = \bar{b}_3 c_3 + \bar{b}_3\bar{s}_6$, and we obtain the equation (a5): $b_3 b_8 = \bar{t}_6 + \bar{b}_3\bar{s}_6$. Now by (3) we obtain equation (a6): $\bar{b}_3\bar{s}_6 = b_3 + x + y$, where distinct b_3, x and y such that $x + y$ is of degree 15. Hence we have equation (a7): $b_3 b_8 = b_3 + \bar{t}_6 + x + y$. Therefore

$$b_3^2 s_6 = (c_3 + \bar{s}_6)s_6 = c_3 s_6 + s_6\bar{s}_6 = b_8 + b_{10} + 1 + b_5 + c_5 + b_8 + x_8 + b_9,$$

$$b_3(b_3 s_6) = b_3(\bar{b}_3 + \bar{x} + \bar{y}) = 1 + b_8 + b_3\bar{x} + b_3\bar{y}.$$

So $b_3\bar{x} + b_3\bar{y} = 1 + b_5 + c_5 + b_8 + x_8 + b_9$, which implies $x_8 = b_8$ by (a7), a contradiction. This concludes the proof. □

2.7 Structure of NITA Generated by b_3 and Satisfying $b_3^2 = c_3 + b_6$, $c_3 \neq b_3, \bar{b}_3$, $(b_3 b_8, b_3 b_8) = 3$ and c_3 Non-real

In this section we classify NITA generated by b_3 and satisfying $b_3^2 = c_3 + b_6$, $c_3 \neq b_3, \bar{b}_3$ and c_3 nonreal and $(b_3 b_8, b_3 b_8) = 3$. First, we state Hypothesis 2.3.

Hypothesis 2.3 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ such that $b_3\bar{b}_3 = 1 + b_8$ and $b_3^2 = c_3 + b_6$, $c_3, b_6 \in B$, $c_3 \neq \bar{b}_3$, $c_3 \neq \bar{c}_3$ and $(b_3 b_8, b_3 b_8) = 3$.*

Lemma 2.16 *Let (A, B) be a NITA satisfying Hypothesis 2.3. Then $c_3\bar{c}_3 = b_3\bar{b}_3 = 1 + b_8$ and $b_3b_8 = b_3 + x_6 + y_{15}$ and $c_3b_8 = c_3 + y_6 + z_{15}$, where $b_8, x_6, y_{15} \in B$, and there exist $r_3, s_6, u_3, v_6, w_3, z_6 \in B$ such that*

$$\begin{aligned}
b_3^2 &= r_3 + s_6, & \bar{b}_3r_3 &= b_3 + x_6, & \bar{b}_3s_6 &= b_3 + y_{15},\\
b_3c_3 &= u_3 + v_6, & \bar{b}_3u_3 &= c_3 + y_6, & \bar{b}_3v_6 &= c_3 + z_{15},\\
\bar{b}_3c_3 &= w_3 + z_6, & b_3w_3 &= c_3 + y_6, & b_3z_6 &= c_3 + z_{15}.
\end{aligned}$$

Proof It is obvious that $(b_3^2, b_3^2) = 2$. So $b_3^2 = r_3 + s_6$ or $r_4 + s_5$. Hence

$$\begin{aligned}
b_3^2\bar{b}_3 &= \bar{b}_3r_3 + \bar{b}_3s_6 \text{ or } \bar{b}_3r_4 + \bar{b}_3s_6,\\
(b_3\bar{b}_3)b_3 &= b_3 + b_3b_8\\
&= 2b_3 + x_6 + y_{15}.
\end{aligned}$$

Since $b_3 \in Supp\{\bar{b}_3r_3\}$ and $Supp\{\bar{b}_3s_6\}$ or $b_3 \in Supp\{\bar{b}_3r_4\}$ and $Supp\{\bar{b}_3s_5\}$, the first part of the lemma follows.

Since $b_3\bar{b}_3 = c_3\bar{c}_3 = 1 + b_8$, it follows that $b_3c_3 = u_3 + v_6$ or $u_4 + v_5$ and $\bar{b}_3c_3 = w_3 + z_6$ or $w_4 + z_5$. The second and third parts of the lemma follow from $(b_3\bar{b}_3)c_3 = (b_3c_3)\bar{b}_3 = (\bar{b}_3c_3)b_3$. □

Lemma 2.17 *Let (A, B) satisfy Hypothesis 2.3. Then there are nonreal base elements x_6, y_{15}, x_{15} and real basis elements $r_3, s_6, d_9, x_{10}, t_{15}$ such that the following equations hold*:

$$\begin{aligned}
b_3\bar{b}_3 &= 1 + b_8, & b_3^2 &= c_3 + b_6,\\
b_3\bar{c}_3 &= \bar{b}_3 + \bar{x}_6, & b_3x_6 &= c_3 + y_{15},\\
b_3b_8 &= b_3 + x_6 + x_{15}, & b_3\bar{x}_6 &= b_8 + x_{10},\\
b_3\bar{b}_6 &= \bar{b}_3 + \bar{x}_{15}, & b_3c_6 &= \bar{c}_3 + \bar{y}_{15},\\
b_3c_3 &= r_3 + s_6, & b_3r_3 &= \bar{c}_3 + \bar{b}_6,\\
b_3s_6 &= \bar{c}_3 + \bar{y}_{15}, & b_3x_{10} &= x_6 + x_{15} + b_9,\\
b_3\bar{y}_{15} &= \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9, & b_3y_{15} &= s_6 + 2t_{15} + d_9,\\
b_3b_6 &= r_3 + t_{15}, & b_3x_{15} &= b_6 + 2y_{15} + c_9,\\
b_3t_{15} &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}, & &
\end{aligned}$$

$$\begin{aligned}
c_3\bar{c}_3 &= 1 + b_8, & c_3r_3 &= \bar{b}_3 + \bar{x}_6,\\
c_3s_6 &= \bar{b}_3 + \bar{x}_{15}, & c_3^2 &= \bar{c}_3 + \bar{b}_6,\\
c_3b_8 &= c_3 + b_6 + y_{15}, & c_3\bar{x}_6 &= b_3 + x_{15},\\
c_3x_6 &= r_3 + t_{15}, & c_3x_{10} &= b_6 + y_{15} + c_9,\\
c_3y_{15} &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}, & c_3b_6 &= \bar{c}_3 + \bar{y}_{15},\\
c_3t_{15} &= \bar{x}_6 + \bar{b}_9 + 2\bar{x}_{15}, & c_3x_{15} &= 2t_{15} + s_6 + d_9,\\
c_3\bar{x}_{15} &= 2x_{15} + x_6 + b_9, & c_3\bar{b}_6 &= b_8 + x_{10},
\end{aligned}$$

$$r_3\bar{c}_3 = c_3 + x_6, \qquad r_3\bar{b}_3 = c_3 + b_6,$$
$$r_3\bar{x}_6 = c_3 + y_{15}, \qquad r_3 y_{15} = \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9,$$
$$r_3 b_6 = \bar{x}_{15} + \bar{b}_3, \qquad r_3 b_8 = s_6 + r_3 + t_{15},$$
$$r_3\bar{x}_{15} = b_6 + c_9 + 2y_{15},$$
$$r_3 x_{10} = s_6 + t_{15} + d_9,$$

$$x_6^2 = 2b_6 + y_{15} + c_9, \qquad x_6 b_8 = b_3 + x_6 + 2x_{15} + b_9,$$

$$b_6^2 = 2\bar{b}_6 + \bar{y}_{15} + \bar{c}_9, \qquad b_6\bar{x}_6 = 2x_6 + b_9 + x_{15},$$
$$b_6 x_6 = 2s_6 + t_{15} + d_9, \qquad b_6 b_8 = c_3 + b_6 + c_9 + 2y_{15},$$

$$s_6\bar{b}_6 = 2x_6 + x_{15} + b_9, \qquad s_6 b_8 = r_3 + s_6 + 2t_{15} + d_9,$$
$$s_6\bar{c}_3 = b_3 + x_{15},$$

$$b_8 y_{15} = c_3 + 2b_6 + 2c_9 + 4y_{15} + \bar{b}_3 d_9.$$

Proof By Hypothesis 2.3 $(b_3b_8, b_3b_8) = 3$. Then the following equations hold by Lemma 2.7:

$$b_3\bar{b}_3 = 1 + b_8, \qquad c_3\bar{c}_3 = 1 + b_8,$$
$$b_3^2 = c_3 + b_6, \qquad \bar{b}_3 c_3 = b_3 + x_6.$$

Let $b_3 x_6 = c_3 + y_{15}$, then

$$(b_3\bar{b}_3)c_3 = c_3 + c_3 b_8,$$
$$(\bar{b}_3 c_3)b_3 = b_3^2 + b_3 x_6$$
$$= c_3 + b_6 + c_3 + y_{15}.$$

Hence

$$c_3 b_8 = c_3 + b_6 + y_{15}.$$

Therefore $y_{15} \in B$ by Lemma 2.7.

Let $c_3\bar{x}_6 = b_3 + x_{15}$, $x_{15} \in N^*B$. Since $(c_3\bar{c}_3)b_3 = (b_3\bar{c}_3)c_3$, it follows that $x_{15} \in B$ and

$$b_3 b_8 = b_3 + x_6 + x_{15}$$

by $b_3^2\bar{b}_3 = (b_3\bar{b}_3)b_3$. Since $y_{15} \in B$, so $(\bar{b}_3 x_6, \bar{b}_3 x_6) = (b_3 x_6, b_3 x_6) = 2$. Thus

$$\bar{b}_3 x_6 = b_8 + x_{10}, \quad x_{10} \in B.$$

Since $b_3^2\bar{c}_3 = (b_3\bar{c}_3)b_3$ and $b_3^2\bar{b}_3 = (b_3\bar{b}_3)b_3$, one has that

$$c_3\bar{b}_6 = b_8 + x_{10}, \qquad \bar{b}_3 b_6 = b_3 + x_{15}.$$

By Lemma 2.16, there exist $w_3, z_6, r_3, s_6 \in B$ such that

$$c_3^2 = w_3 + z_6, \qquad w_3\bar{c}_3 = c_3 + b_6, \qquad z_6\bar{c}_3 = c_3 + y_{15},$$
$$b_3c_3 = r_3 + s_6, \qquad b_3r_3 = w_3 + z_6, \qquad b_3s_6 = w_3 + z_{15}.$$

By Lemma 2.16 again, it follows that

$$r_3\bar{c}_3 = b_3 + x_6, \qquad \bar{c}_3s_6 = b_3 + x_{15},$$
$$r_3\bar{b}_3 = c_3 + b_6, \qquad \bar{b}_3s_6 = c_3 + y_{15}.$$

Thus $(\bar{c}_3r_3, \bar{c}_3r_3) = 2$, which implies that $r_3\bar{r}_3 = 1 + b_8$.

Since

$$\begin{aligned}(b_3r_3)\bar{c}_3 &= w_3\bar{c}_3 + z_6\bar{c}_3\\ &= 2c_3 + b_6 + y_{15},\\ (b_3\bar{c}_3)r_3 &= r_3\bar{b}_3 + r_3\bar{x}_6\\ &= c_3 + b_6 + r_3\bar{x}_6,\end{aligned}$$

which implies that $r_3\bar{x}_6 = c_3 + y_{15}$.

We assert that x_{10} is real. In fact,

$$\begin{aligned}\bar{b}_3(r_3\bar{c}_3) &= b_3\bar{b}_3 + x_6\bar{b}_3\\ &= 1 + b_8 + b_8 + x_{10},\\ (\bar{b}_3r_3)\bar{c}_3 &= c_3\bar{c}_3 + b_6\bar{c}_3\\ &= 1 + b_8 + b_8 + \bar{x}_{10},\end{aligned}$$

so it follows that x_{10} is real.

Since

$$\begin{aligned}\bar{b}_3(w_3\bar{c}_3) &= c_3\bar{b}_3 + b_6\bar{b}_3\\ &= b_3 + x_6 + b_3 + x_{15},\\ (\bar{b}_3\bar{c}_3)w_3 &= \bar{r}_3w_3 + \bar{s}_6w_3,\end{aligned}$$

we have that

$$\bar{r}_3w_3 = b_3 + x_6, \qquad w_3\bar{s}_6 = b_3 + x_{15}.$$

Consequently $\bar{b}_3w_3 = r_3 + s_6$.

Now we have that $(b_3\bar{b}_3)w_3 = (\bar{b}_3w_3)b_3$ and

$$w_3b_8 = w_3 + z_6 + z_{15}.$$

Then $w_3\bar{w}_3 = 1 + b_8$.

Since

$$
\begin{aligned}
(b_3\bar{b}_3)^2 &= 1 + 2b_8 + b_8^2, \\
b_3^2\bar{b}_3^2 &= (c_3 + b_6)(\bar{c}_3 + \bar{b}_6) \\
&= 1 + b_8 + c_3\bar{b}_6 + \bar{c}_3 b_6 + b_6\bar{b}_6 \\
&= 1 + 3b_8 + 2x_{10} + b_6\bar{b}_6,
\end{aligned}
$$

so

$$b_8^2 = 2x_{10} + b_8 + b_6\bar{b}_6.$$

By $b_3^2\bar{b}_6 = (b_3\bar{b}_6)b_3$, we have that $x_{10} + b_6\bar{b}_6 = 1 + b_3\bar{x}_{15}$. Then $(b_3x_{10}, x_{15}) = 1$. But $(b_3x_{10}, x_6) = 1$. There exists $b_9 \in B$ such that

$$b_3x_{10} = x_6 + x_{15} + b_9, \quad b_9 \in N^*B.$$

It is easy to see that b_3x_{10} cannot have constituents of degree 3 and 4 by $L_1(B) = \{1\}$ and $L_2(B) = \emptyset$. Thus $b_9 \in B$.

Since

$$
\begin{aligned}
(\bar{b}_3x_6)b_3 &= b_3b_8 + b_3x_{10} \\
&= b_3 + x_6 + x_{15} + x_6 + x_{15} + b_9, \\
(b_3\bar{b}_3)x_6 &= x_6 + x_6b_8,
\end{aligned}
$$

we have

$$x_6b_8 = b_3 + x_6 + 2x_{15} + b_9.$$

Multiplying both sides of the equation $x_{10} + b_6\bar{b}_6 = 1 + b_3\bar{x}_{15}$ by b_3, we have that

$$b_3x_{10} + (b_3\bar{b}_6)b_6 = b_3 + b_3^2\bar{x}_{15},$$

so

$$x_6 + x_{15} + b_9 + b_6\bar{b}_3 + b_6\bar{x}_{15} = c_3\bar{x}_{15} + b_6\bar{x}_{15}.$$

Hence

$$c_3\bar{x}_{15} = 2x_{15} + x_6 + b_9.$$

Now checking the associative law of $(b_3\bar{c}_3)b_6 = (\bar{c}_3b_6)b_3$, $(r_3\bar{r}_3)x_6 = (\bar{r}_3x_6)r_3$ and $(r_3\bar{b}_3)\bar{x}_6 = (r_3\bar{x}_6)\bar{b}_3$, we obtain

$$
\begin{aligned}
b_6\bar{x}_6 &= 2x_6 + b_9 + x_{15}, \\
r_3\bar{y}_{15} &= x_6 + 2x_{15} + b_9, \\
\bar{b}_3y_{15} &= x_6 + 2x_{15} + b_9.
\end{aligned}
$$

Hence

$$c_3\bar{y}_{15} + b_6\bar{y}_{15} = b_3^2\bar{y}_{15} = b_3(b_3\bar{y}_{15}) = b_3\bar{x}_6 + 2b_3\bar{x}_{15} + b_3\bar{b}_9.$$

Since $(b_3\bar{x}_6, b_8) = (b_3\bar{x}_6, x_{10}) = (b_3\bar{x}_{15}, b_8) = (b_3\bar{x}_{15}, x_{10}) = (b_3\bar{b}_9, x_{10}) = 1$ and $(b_3\bar{b}_9, b_8) = 0$, we have that

$$(c_3\bar{y}_{15} + b_6\bar{y}_{15}, b_8) = 3 \quad \text{and} \quad (c_3\bar{y}_{15} + b_6\bar{y}_{15}, x_{10}) = 4.$$

From the degrees, we know that $(c_3\bar{y}_{15}, x_{10}) \leq 1$. If $(c_3\bar{y}_{15}, x_{10}) = 0$, then $(b_6\bar{y}_{15}, x_{10}) = 4$, so $b_6x_{10} = 4y_{15}$. But $(b_6x_{10}, c_3) = 1$, a contradiction. Hence $(c_3\bar{y}_{15}, x_{10}) = 1$ and $(b_6x_{10}, y_{15}) = (b_6\bar{y}_{15}, x_{10}) = 3$.

Since $(c_3x_{10}, c_3x_{10}) = (b_3x_{10}, b_3x_{10}) = 3$ and $(c_3x_{10}, b_6) = 1$, there exists $c_9 \in B$ such that

$$c_3x_{10} = b_6 + y_{15} + c_9.$$

Then

$$\begin{aligned}
(\bar{b}_3c_3)x_6 &= b_3x_6 + x_6^2 \\
&= c_3 + y_{15} + x_6^2, \\
(\bar{b}_3x_6)c_3 &= c_3b_8 + c_3x_{10} \\
&= c_3 + b_6 + y_{15} + b_6 + y_{15} + c_9, \\
(c_3\bar{c}_3)b_6 &= b_6 + b_6b_8, \\
(\bar{c}_3b_6)c_3 &= b_8c_3 + x_{10}c_3 \\
&= c_3 + b_6 + y_{15} + y_{15} + b_6 + c_9, \\
(c_3b_6)\bar{c}_3 &= c_3 + b_6 + \bar{c}_3z_{15},
\end{aligned}$$

which implies that

$$\begin{aligned}
x_6^2 &= 2b_6 + y_{15} + c_9, \\
b_6b_8 &= c_3 + b_6 + c_9 + 2y_{15}, \\
c_3\bar{z}_{15} &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}.
\end{aligned}$$

Since

$$\begin{aligned}
(b_3\bar{c}_3)^2 &= (\bar{b}_3 + \bar{x}_6)^2 \\
&= \bar{b}_3^2 + 2\bar{b}_3\bar{x}_6 + \bar{x}_6^2 \\
&= 2\bar{c}_3 + 3\bar{b}_6 + 3\bar{y}_{15} + \bar{c}_9, \\
b_3^2\bar{c}_3^2 &= c_3\bar{w}_3 + c_3\bar{z}_6 + \bar{w}_3b_6 + b_6\bar{z}_6 \\
&= \bar{c}_3 + \bar{b}_6 + \bar{c}_3 + \bar{y}_{15} + b_6\bar{w}_3 + b_6\bar{z}_6,
\end{aligned}$$

but $(\bar{w}_3 b_6, \bar{c}_3) = 1$, we have that

$$w_3\bar{b}_6 = c_3 + y_{15},$$
$$b_6\bar{z}_6 = 2\bar{b}_6 + \bar{y}_{15} + \bar{c}_9.$$

Since

$$\begin{aligned}(b_3\bar{r}_3)\bar{b}_6 &= \bar{c}_3\bar{b}_6 + \bar{b}_6^2\\ &= \bar{w}_3 + \bar{z}_{15} + \bar{b}_6^2,\\ (b_3\bar{b}_6)\bar{r}_3 &= \bar{b}_3\bar{r}_3 + \bar{x}_{15}\bar{r}_3\\ &= \bar{w}_3 + \bar{z}_6 + \bar{r}_3\bar{x}_{15},\end{aligned}$$

and $(\bar{b}_6^2, \bar{z}_6) = 2$, we have that $(\bar{r}_3\bar{x}_{15}, \bar{z}_6) = 1$. Hence $(r_3\bar{z}_6, \bar{x}_{15}) = 1$. But $(r_3\bar{z}_6, \bar{b}_3) = 1$ by $b_3r_3 = w_3 + z_6$. Thus

$$\bar{r}_3 z_6 = b_3 + x_{15}.$$

From the following equations

$$\begin{aligned}\bar{b}_3(w_3\bar{b}_6) &= \bar{b}_3(c_3 + y_{15})\\ &= b_3 + x_6 + x_6 + 2x_{15} + b_9,\\ (\bar{b}_3 w_3)\bar{b}_6 &= r_3\bar{b}_6 + s_6\bar{b}_6,\end{aligned}$$

and $(r_3\bar{b}_6, r_3\bar{b}_6) = (b_3\bar{b}_6, b_3\bar{b}_6) = 2$, $(r_3\bar{b}_6, b_3) = 1$, we obtain

$$r_3\bar{b}_6 = b_3 + x_{15}, \qquad s_6\bar{b}_6 = 2x_6 + x_{15} + b_9.$$

By $(b_3c_3)\bar{r}_3 = (c_3\bar{r}_3)b_3$ and $\bar{c}_3(w_3b_8) = (\bar{c}_3b_8)w_3$, we have that

$$\bar{r}_3 s_6 = b_8 + x_{10}, \qquad w_3\bar{y}_{15} = b_6 + c_9 + 2y_{15}.$$

Since $(\bar{r}_3b_8, \bar{r}_3b_8) = (r_3\bar{r}_3, b_8^2) = (b_3\bar{b}_3, b_8^2) = (b_3b_8, b_3b_8) = 3$ by Lemma 2.16 and $(\bar{r}_3b_8, \bar{s}_6) = (\bar{r}_3b_8, \bar{r}_3) = 1$, there exists $t_{15} \in B$ such that

$$\bar{r}_3 b_8 = \bar{s}_6 + \bar{r}_3 + t_{15}.$$

Since

$$\begin{aligned}(w_3b_8)\bar{r}_3 &= w_3\bar{r}_3 + z_6\bar{r}_3 + z_{15}\bar{r}_3\\ &= b_3 + x_6 + b_3 + x_{15} + \bar{r}_3 z_{15},\\ (\bar{r}_3 w_3)b_8 &= b_3b_8 + x_6b_8\\ &= b_3 + x_6 + x_{15} + b_3 + x_6 + 2x_{15} + b_9,\\ (\bar{r}_3b_8)w_3 &= \bar{s}_6w_3 + \bar{r}_3w_3 + t_{15}w_3\\ &= b_3 + x_{15} + b_3 + x_6 + w_3t_{15},\end{aligned}$$

we have that

$$r_3\bar{z}_{15} = \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9,$$
$$w_3 t_{15} = x_6 + b_9 + 2x_{15}.$$

Since

$$\begin{aligned}
(\bar{b}_3 r_3)b_8 &= c_3 b_8 + b_8 b_6 \\
&= c_3 + b_6 + y_{15} + 2y_{15} + b_6 + c_3 + c_9, \\
(\bar{b}_3 b_8)r_3 &= r_3\bar{b}_3 + r_3\bar{x}_6 + r_3\bar{x}_{15} \\
&= c_3 + b_6 + c_3 + y_{15} + r_3\bar{x}_{15}, \\
(r_3 b_8)\bar{b}_3 &= r_3\bar{b}_3 + s_6\bar{b}_3 + \bar{t}_{15}\bar{b}_3 \\
&= c_3 + b_6 + c_3 + y_{15} + \bar{b}_3\bar{t}_{15},
\end{aligned}$$

we have that $w_3 = \bar{c}_3, z_6 = \bar{b}_6, z_{15} = \bar{y}_{15}$ and

$$r_3\bar{x}_{15} = b_6 + c_9 + 2y_{15},$$
$$b_3 t_{15} = \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}.$$

Furthermore, $b_3 r_3 = \bar{c}_3 + \bar{b}_6$, and we have that $(b_3 c_3, \bar{r}_3) = 1$. But $(b_3 c_3, r_3) = 1$; hence $r_3 = \bar{r}_3$. Moreover $\bar{r}_3 + \bar{s}_6 = b_3\bar{w}_3 = b_3 c_3 = r_3 + s_6$, which implies that $s_6 = \bar{s}_6$. Therefore t_{15} is real by the expression $r_3 b_8$, and

$$c_3 t_{15} = \bar{x}_6 + \bar{c}_9 + 2\bar{x}_{15}.$$

Now we have that $t_{15} \in Supp\{c_3 x_6\}$. Thus

$$c_3 x_6 = r_3 + t_{15}.$$

Since

$$\begin{aligned}
(c_3 r_3)\bar{b}_6 &= \bar{b}_3\bar{b}_6 + \bar{x}_6\bar{b}_6, \\
(r_3\bar{b}_6)c_3 &= b_3 c_3 + c_3 x_{15} \\
&= r_3 + s_6 + c_3 x_{15}, \\
(c_3\bar{b}_6)r_3 &= r_3 b_8 + r_3 x_{10} \\
&= r_3 + s_6 + t_{15} + r_3 x_{10}
\end{aligned}$$

and $(r_3 x_{10}, s_6) = 1$, we have $s_6 \in Supp\{c_3 x_{15}\}$. But $(c_3 x_{15}, t_{15}) = (c_3 t_{15}, \bar{x}_{15}) = 2$. Let

$$c_3 x_{15} = 2t_{15} + s_6 + d_9, \quad d_9 \in N^* B.$$

Then $r_3 x_{10} = s_6 + t_{15} + d_9$ and by $(c_3 r_3)\bar{b}_6 = (c_3\bar{b}_6)r_3$ we obtain that

$$\bar{b}_3\bar{b}_6 + \bar{x}_6\bar{b}_6 = r_3 + s_6 + t_{15} + s_6 + t_{15} + d_9.$$

Furthermore d_9 is real for r_3 and x_{10} are real, and

$$b_3b_6 = r_3 + t_{15},$$
$$x_6b_6 = 2s_6 + t_{15} + d_9.$$

Since

$$\begin{aligned}(b_3\bar{b}_3)b_6 &= b_6 + b_6b_8\\ &= b_6 + c_3 + b_6 + c_9 + 2y_{15},\\ b_3(\bar{b}_3b_6) &= b_3^2 + b_3x_{15}\\ &= c_3 + b_6 + b_3x_{15},\\ (b_3b_6)\bar{b}_3 &= r_3\bar{b}_3 + t_{15}\bar{b}_3\\ &= c_3 + b_6 + \bar{b}_3t_{15},\end{aligned}$$

we have that

$$b_3x_{15} = b_6 + 2y_{15} + c_9,$$
$$b_3t_{15} = \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}.$$

Since

$$\begin{aligned}b_3^2x_6 &= c_3x_6 + b_6x_6\\ &= r_3 + t_{15} + 2s_6 + t_{15} + d_9,\\ (b_3x_6)b_3 &= b_3c_3 + b_3y_{15}\\ &= r_3 + s_6 + b_3y_{15},\end{aligned}$$

we have that

$$b_3y_{15} = 2t_{15} + s_6 + d_9.$$

Since $(b_3y_{15}, b_3y_{15}) = (b_3\bar{y}_{15}, b_3\bar{y}_{15}) = 6$, we have that $d_9 \in B$.
Checking the associative law of $(b_3\bar{b}_3)s_6 = (b_3s_6)\bar{b}_3$, we have

$$s_6b_8 = r_3 + s_6 + 2t_{15} + d_9.$$

Since

$$\begin{aligned}(b_3\bar{b}_3)y_{15} &= y_{15} + b_8y_{15},\\ (b_3y_{15})\bar{b}_3 &= \bar{b}_3s_6 + 2\bar{b}_3t_{15} + \bar{b}_3d_9\\ &= c_3 + y_{15} + 2b_6 + 2c_9 + 4y_{15} + \bar{b}_3d_9,\end{aligned}$$

it follows that $b_8y_{15} = c_3 + 2b_6 + 2c_9 + 4y_{15} + \bar{b}_3d_9$.
This completes the proof of the lemma. □

Remark It always follows that $(c_3c_9, \bar{y}_{15}) = 1$ by Lemma 2.17. So $(c_3c_9, c_3c_9) \geq 2$. In the following, we shall investigate the expression c_3c_9.

Lemma 2.18 *There exist no $x_4, y_8 \in B$ such that $c_3c_9 = x_4 + y_8 + \bar{y}_{15}$.*

Proof If there exist no $x_4, y_8 \in B$ such that

$$c_3c_9 = x_4 + y_8 + \bar{y}_{15},$$

then there exists $z_3 \in B$ such that $c_3\bar{x}_4 = \bar{c}_9 + z_3$. Furthermore, there is $y_5 \in B$ such that $c_3\bar{z}_3 = x_4 + y_5$. Thus $z_3\bar{z}_3 = 1 + b_8$. By Lemma 2.16 we have that

$$\begin{aligned} c_3(z_3\bar{z}_3) &= c_3 + c_3b_8 \\ &= 2c_3 + b_6 + y_{15}, \\ z_3(c_3\bar{z}_3) &= z_3x_4 + z_3y_5. \end{aligned}$$

It must follow that $z_3x_4 = 2c_3 + b_6$, which implies that $(z_3\bar{c}_3, x_4) = 2$, a contradiction. $\square$

Lemma 2.19 *There exist no $x_5, y_7 \in B$ such that $c_3c_9 = x_5 + y_7 + \bar{y}_{15}$.*

Proof If some $x_5, y_7 \in B$ such that $c_3c_9 = x_5 + y_7 + \bar{y}_{15}$. Then $(c_3\bar{x}_5, \bar{c}_9) = 1$. There are three possibilities:

$$\begin{aligned} &\text{(I)} && c_3\bar{x}_5 = \bar{c}_9 + 2x_3, \\ &\text{(II)} && c_3\bar{x}_5 = \bar{c}_9 + x_3 + y_3, \\ &\text{(III)} && c_3\bar{x}_5 = \bar{c}_9 + y_6. \end{aligned}$$

It is easy to see that (I) will lead to a contradiction.

If (II) follows, then $c_3\bar{x}_5 = \bar{c}_9 + x_3 + y_3$. Set $c_3\bar{x}_3 = x_5 + z_4$, $c_3\bar{y}_3 = x_5 + t_4$, where $z_4, t_4 \in B$. Hence

$$\begin{aligned} \bar{c}_3(c_3\bar{x}_3) &= \bar{c}_3x_5 + \bar{c}_3z_4 \\ &= c_9 + \bar{x}_3 + \bar{y}_3 + \bar{c}_3z_4, \\ (c_3\bar{c}_3)\bar{x}_3 &= \bar{x}_3 + \bar{x}_3b_8. \end{aligned}$$

Therefore $(x_3\bar{y}_3, b_8) = 1$, which implies that $\bar{y}_3x_3 = 1 + b_8$. So $y_3 = x_3$, a contradiction.

If (III) follows, then

$$\begin{aligned} \bar{c}_3(c_3\bar{x}_5) &= \bar{c}_3\bar{c}_9 + \bar{c}_3y_6 \\ &= y_{15} + \bar{x}_5 + \bar{y}_7 + \bar{c}_3y_6, \\ (c_3\bar{c}_3)\bar{x}_5 &= \bar{x}_5 + \bar{x}_5b_8. \end{aligned}$$

Hence $\bar{x}_5 b_8 = y_{15} + \bar{y}_7 + \bar{c}_3 y_6$. Therefore $(b_8 y_{15}, \bar{x}_5) = 1$. By Lemma 2.17, we have that $(\bar{x}_5, \bar{b}_3 d_9) = 1$. Since $(\bar{b}_3 d_9, y_{15}) = 1$, we may set

$$\bar{b}_3 d_9 = \bar{x}_5 + y_{15} + e_7, \quad e_7 \in N^* B.$$

If $e_7 = m_3 + n_4$, then $b_3 n_4 = d_9 + p_3$, some $p_3 \in B$. Let $\bar{b}_3 p_3 = n_4 + z_5$, some $z_5 \in B$. Then $p_3 \bar{p}_3 = 1 + b_8$. Furthermore,

$$\begin{aligned} b_3(p_3\bar{p}_3) &= b_3 + b_3 b_8 \\ &= 2b_3 + x_6 + x_{15}, \\ (b_3\bar{p}_3)p_3 &= \bar{n}_4 p_3 + \bar{z}_5 p_3. \end{aligned}$$

Note that $b_3 \in Supp\{\bar{n}_4 p_3\}$ and $b_3 \in Supp\{\bar{z}_5 p_3\}$, we come to a contradiction.

If $e_7 \in B$, then $\bar{b}_3 d_9 = \bar{x}_5 + y_{15} + e_7$. Let $b_3 \bar{x}_5 = d_9 + u_6$.

We assert that $u_6 \in B$. Otherwise, let $u_6 = r_3 + s_3$, where $r_3, s_3 \in B$, then

$$b_3 \bar{r}_3 = x_5 + u_4.$$

Hence $r_3 \bar{r}_3 = 1 + b_8$. Moreover,

$$\begin{aligned} b_3(r_3\bar{r}_3) &= b_3 + b_3 b_8 \\ &= 2b_3 + x_6 + x_{15}, \\ r_3(b_3\bar{r}_3) &= r_3 x_5 + r_3 u_4, \end{aligned}$$

which is impossible for $b_3 \in Supp\{r_3 x_5\}$ and $b_3 \in Supp\{r_3 u_4\}$.

Now we have that

$$\begin{aligned} \bar{b}_3(c_3\bar{x}_5) &= \bar{b}_3\bar{c}_9 + \bar{b}_3 y_6, \\ (\bar{b}_3 c_3)\bar{x}_5 &= b_3\bar{x}_5 + x_6\bar{x}_5 \\ &= d_9 + u_6 + x_6\bar{x}_5. \end{aligned}$$

If $d_9, u_6 \in Supp\{\bar{b}_3 y_6\}$, then $\bar{b}_3 y_6 = d_9 + u_6 + t_3$. Let $b_3 t_3 = y_6 + p_3$. Hence $t_3 \bar{t}_3 = 1 + b_8$. Moreover,

$$\begin{aligned} b_3(t_3\bar{t}_3) &= 2b_3 + x_6 + x_{15}, \\ \bar{t}_3(b_3 t_3) &= \bar{t}_3 y_6 + \bar{t}_3 p_3. \end{aligned}$$

So $\bar{t}_3 y_6 = b_3 + x_{15}$. But $3 = (\bar{b}_3 y_6, \bar{b}_3 y_6) = (\bar{t}_3 y_6, \bar{t}_3 y_6) = 2$ by Lemma 2.1, a contradiction.

If $u_6 \in Supp\{\bar{b}_3 y_6\}$, $d_9 \in Supp\{\bar{b}_3 \bar{c}_9\}$ or $u_6 \in Supp\{\bar{b}_3 \bar{c}_9\}$, $d_9 \in Supp\{\bar{b}_3 y_6\}$.

If the former one follows, then there exists a q_3 such that

$$b_3 c_9 = t_{15} + d_9 + q_3.$$

Hence $c_9 \in Supp\{\bar{b}_3 d_9\}$, a contradiction.

If $u_6 \in Supp\{\bar{b}_3\bar{c}_9\}$, $d_9 \in Supp\{\bar{b}_3 y_6\}$, then

$$b_3 d_9 = \bar{y}_{15} + y_6 + r_6,$$

where $r_6 \in N^* B$. But d_9 is real, $\bar{b}_3 d_9 = \bar{x}_5 + y_{15} + e_7$, a contradiction. This completes the proof. □

Lemma 2.20 *There exist no m_6 and n_6 such that $c_3 c_9 = \bar{y}_{15} + m_6 + n_6$.*

Proof If there exist m_6 and n_6 such that $c_3 c_9 = \bar{y}_{15} + m_6 + n_6$. Then $m_6 \neq b_6, \bar{b}_6$, $n_6 \neq b_6, \bar{b}_6$ by Lemma 2.17 and $c_3 \bar{m}_6$ has three possibilities:

$$\begin{array}{rll} \text{(I)} & c_3\bar{m}_6 = \bar{c}_9 + x_3 + y_3 + z_3, & x_3, y_3, z_3 \text{ distinct} \\ \text{(II)} & c_3\bar{m}_6 = \bar{c}_9 + x_3 + y_6, & x_3 \text{ and } y_6 \in B, \\ \text{(III)} & c_3\bar{m}_6 = \bar{c}_9 + x_4 + y_5, & x_4 \text{ and } y_5 \in B. \end{array}$$

Suppose (I) follows. We may assume that $c_3\bar{x}_3 = m_6 + r_3$, where $r_3 \in B$. Then

$$\bar{x}_3 + \bar{x}_3 b_8 = \bar{x}_3(c_3\bar{c}_3) = (\bar{x}_3 c_3)\bar{c}_3 = m_6\bar{c}_3 + r_3\bar{c}_3 = c_9 + \bar{x}_3 + \bar{y}_3 + \bar{z}_3 + r_3\bar{c}_3.$$

Hence $\bar{y}_3 \in Supp\{\bar{x}_3 b_8\}$, which implies that $x_3 = y_3$, a contradiction.

If (II) follows, then $(c_3\bar{x}_3, c_3\bar{x}_3) = 2$, from which $x_3\bar{x}_3 = 1 + b_8$ follows. Also we may set $c_3\bar{m}_3 = x_3 + t_6$, where $t_6 \in B$. Hence

$$\begin{aligned} (c_3\bar{c}_3)\bar{x}_3 &= \bar{x}_3 + \bar{x}_3 b_8, \\ \bar{c}_3(c_3\bar{x}_3) &= \bar{c}_3 m_6 + \bar{c}_3 m_3 \\ &= \bar{x}_4 + \bar{y}_5 + c_9 + \bar{x}_3 + \bar{t}_6. \end{aligned}$$

Therefore $x_3 b_8 = x_4 + y_5 + \bar{c}_9 + t_6$. We have that $(x_3 b_8, x_3 b_8) = 4$. But by Lemma 2.8, $(x_3 b_8, x_3 b_8) = (b_3 b_8, b_3 b_8) = 3$, a contradiction.

If (III) follows, set $c_3\bar{x}_4 = m_6 + p_6$, where $p_6 \in B$. By $\bar{c}_3(c_3\bar{m}_6) = (c_3\bar{c}_3)\bar{m}_6$, one has that

$$\bar{m}_6 b_8 = y_{15} + \bar{m}_6 + \bar{n}_6 + \bar{p}_6 + \bar{c}_3 y_5.$$

Hence $(b_8 y_{15}, \bar{m}_6) = 1$. By the expression $b_8 y_{15}$ in Lemma 2.17, $(\bar{m}_6, \bar{b}_3 d_9) = 1$. But $(y_{15}, \bar{b}_3 d_9) = 1$. There exists $q_6 \in N^* B$ such that

$$\bar{b}_3 d_9 = y_{15} + \bar{m}_6 + q_6. \tag{2.64}$$

If $q_6 \notin B$, then $q = 2z_3$ or $q_6 = w_3 + u_3$. If the first one holds, then $(\bar{b}_3 d_9, z_3) = 2$ and $(b_3 z_3, d_9) = 2$, which is impossible. So the second one follows. Consequently $b_3 w_3 = d_9$. Hence

$$w_3 + w_3 b_8 = w_3(b_3\bar{b}_3) = \bar{b}_3(b_3 w_3) = \bar{b}_3 d_9 = y_{15} + \bar{m}_6 + w_3 + u_3.$$

We can see that $w_3\bar{u}_3 = 1 + b_8$. Then $w_3 = u_3$, a contradiction. Hence $p_6 \in B$. It follows by Lemma 2.17 and (2.64) that

$$b_8 y_{15} = 5y_{15} + 2c_9 + 2b_6 + c_3 + \bar{m}_6 + q_6.$$

Since

$$\begin{aligned} c_3(\bar{b}_3 d_9) &= c_3 y_{15} + c_3\bar{m}_6 + c_3 q_6 \\ &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15} + x_4 + y_5 + \bar{c}_9 + c_3 q_6, \\ (c_3\bar{b}_3)d_9 &= b_3 d_9 + x_6 d_9 \\ &= \bar{y}_{15} + m_6 + \bar{q}_6 + x_6 d_9, \end{aligned}$$

we have that $(c_3 q_6, m_6) \geq 1$. Hence $(c_3\bar{m}_6, \bar{q}_6) \geq 1$, a contradiction to our assumption. This completes the proof. $\square$

Lemma 2.21 *There exists no NITA such that* $(c_3c_9, c_3c_9) \geq 4$.

Proof The proof is given in three steps.

Step 1 The following equations hold:

$$\begin{aligned} &\text{(i)} \quad c_3c_9 = \bar{y}_{15} + y_3 + z_3 + w_3 + u_3, \\ &\text{(ii)} \quad c_3c_9 = \bar{y}_{15} + y_3 + y_3 + y_6, \\ &\text{(iii)} \quad c_3c_9 = \bar{y}_{15} + y_3 + y_4 + y_5, \\ &\text{(iv)} \quad c_3c_9 = \bar{y}_{15} + y_4 + z_4 + w_4. \end{aligned}$$

For any $z \in Supp\{c_3c_9\} \setminus \{\bar{y}_{15}\}$, it is sufficient to prove that $(c_3c_9, z) = 1$. Otherwise $(c_3c_9, z) \geq 2$. Since $c_3c_9 - \bar{y}_{15}$ is of degree 12, one has that $(c_3c_9, z) = 2$ and z is of degree 6. So $c_3c_9 = \bar{y}_{15} + 2z_6$, $c_3\bar{z}_6 = 2\bar{c}_9$. Hence

$$\bar{z}_6 + \bar{z}_6 b_8 = (c_3\bar{c}_3)\bar{z}_6 = \bar{c}_3(c_3\bar{z}_6) = 2\bar{c}_3\bar{c}_9 = 2y_{15} + 4\bar{z}_6.$$

Thus $\bar{z}_6 b_8 = 2y_{15} + 3\bar{z}_6$. Consequently $(b_8 y_{15}, \bar{z}_6) = 2$. By the expression $b_6 b_8$ in Lemma 2.17, $z_6 \neq b_6, \bar{b}_6$ and the expression $b_8 y_{15}$ in Lemma 2.17, we have that

$$\begin{aligned} \bar{b}_3 d_9 &= y_{15} + 2\bar{z}_6, \\ b_8 y_{15} &= 5y_{15} + 2c_9 + 2b_6 + c_3 + 2\bar{z}_6. \end{aligned}$$

Therefore $b_3\bar{z}_6 = 2d_9$.

Since

$$\begin{aligned} (b_3\bar{b}_3)d_9 &= d_9 + b_8 d_9, \\ b_3(\bar{b}_3 d_9) &= b_3 y_{15} + 2b_3\bar{z}_6 \\ &= s_6 + 2t_{15} + 5d_9, \end{aligned}$$

it follows that $b_8d_9 = s_6 + 2t_{15} + 4d_9$. Consequently,

$$\begin{aligned} b_8(b_3\bar{z}_6) &= 2b_8b_9 \\ &= 2s_6 + 4t_{15} + 8d_9, \\ (b_3b_8)\bar{z}_6 &= b_3\bar{z}_6 + x_6\bar{z}_6 + x_{15}\bar{z}_6 \\ &= 2d_9 + x_6\bar{z}_6 + x_{15}\bar{z}_6. \end{aligned}$$

Then $x_6\bar{z}_6 + x_{15}\bar{z}_6 = 2s_6 + 4t_{15} + 6d_9$. We have exactly three possibilities:

$$\begin{array}{rll} \text{(I)} & x_6\bar{z}_6 = 4d_9, & x_{15}\bar{z}_6 = 2s_6 + 4t_{15} + 2d_9, \\ \text{(II)} & x_6\bar{z}_6 = s_6 + 2t_{15}, & x_{15}\bar{z}_6 = s_6 + 2t_{15} + 6d_9, \\ \text{(III)} & x_6\bar{z}_6 = 2s_6 + t_{15} + d_9, & x_{15}\bar{z}_6 = 3t_{15} + 5d_9. \end{array}$$

The first case implies that

$$\begin{aligned} \bar{b}_3(x_6\bar{z}_6) &= 4\bar{b}_3d_9 \\ &= 4y_{15} + 8\bar{z}_6, \\ (\bar{b}_3x_6)\bar{z}_6 &= b_8\bar{z}_6 + x_{10}\bar{z}_6 \\ &= 2y_{15} + 3\bar{z}_6 + x_{10}\bar{z}_6. \end{aligned}$$

So $x_{10}\bar{z}_6 = 2y_{15} + 5\bar{z}_6$. Thus $(z_6\bar{z}_6, x_{10}) = 5$, a contradiction.

The second and the third cases mean $(z_6d_9, x_{15}) = 6$ and $(z_6d_9, x_{15}) = 5$, respectively. This is impossible.

Step 2 No NITA satisfies either (i), (ii) or (iii).

It is sufficient to show that $y_3 \in Supp\{c_3c_9\}$. If $y_3 \in Supp\{c_3c_9\}$, then $\bar{c}_3y_3 = c_9$. Hence $(\bar{b}_3y_3, \bar{b}_3y_3) = (\bar{c}_3y_3, \bar{c}_3y_3) = 1$ by Lemma 2.1. Let $\bar{b}_3y_3 = g_9$, $g_9 \in B$. Then

$$b_3c_9 = b_3(\bar{c}_3y_3) = (b_3\bar{c}_3)y_3 = \bar{b}_3y_3 + \bar{x}_6y_3 = g_9 + \bar{x}_6y_3.$$

Since $(t_{15}, b_3c_9) = 1$, we have that $b_3c_9 = g_9 + t_{15} + \alpha_3$, some $\alpha_3 \in B$. Therefore $(b_3c_9, b_3c_9) = 3$. But $(b_3c_9, b_3c_9) = (c_3c_9, c_3c_9)$ by Lemma 2.1, which is impossible for $(c_3c_9, c_3c_9) \geq 4$.

Step 3 No NITA satisfies (iv).

If (iv) holds, then there exists m_3 such that $\bar{c}_3y_4 = c_9 + m_3$. Thus $c_3m_3 = y_4 + k_5$, some $k_5 \in B$. So $(c_3m_3, c_3m_3) = 2$, which implies that $m_3\bar{m}_3 = 1 + b_8$. Hence

$$\bar{m}_3y_4 + \bar{m}_3k_5 = c_3(m_3\bar{m}_3) = c_3 + c_3b_8 = (c_3\bar{c}_3)\bar{m}_3 = 2c_3 + b_6 + y_{15},$$

which is impossible for $c_3 \in Supp\{\bar{m}_3y_4\}$ and $Supp\{\bar{m}_3k_5\}$. The lemma follows. □

Lemma 2.22 *There exists no NITA such that* $(c_3c_9, c_3c_9) = 2$.

Proof If $(c_3c_9, c_3c_9) = 2$, then $c_3c_9 = \bar{y}_{15} + b_{12}$, $b_{12} \in B$ and $c_3\bar{c}_9 = x_{10} + x_{17}$, $x_{17} \in B$.

Since $\bar{c}_3(c_3c_9) = \bar{c}_3\bar{y}_{15} + \bar{c}_3b_{12} = b_6 + c_9 + 2y_{15} + \bar{c}_3b_{12}$, we have that $b_8c_9 = b_6 + 2y_{15} + \bar{c}_3b_{12}$. On the other hand, since

$$c_9 + b_8c_9 = (c_3\bar{c}_3)c_9 = c_3(\bar{c}_3c_9) = c_3(\bar{x}_{10} + \bar{x}_{17}) = b_6 + y_{15} + c_9 + c_3\bar{x}_{17},$$

we have that $b_8c_9 = b_6 + y_{15} + c_3\bar{x}_{17}$. By the expression b_8y_{15} in Lemma 2.17, $(b_8c_9, y_{15}) = 2$. Hence $y_{15} \in Supp\{c_3\bar{x}_{17}\}$, which implies that $x_{17} \in Supp\{c_3\bar{y}_{15}\}$. By the expressions of c_3b_8 and c_3x_{10} in Lemma 2.17, $(c_3\bar{y}_{15}, b_8) = (c_3\bar{y}_{15}, x_{10}) = 1$. There exists $t_{10} \in N^*B$ such that $c_3\bar{y}_{15} = x_{17} + b_8 + x_{10} + t_{10}$, some $t_{10} \in N^*B$. The expression c_3y_{15} in Lemma 2.17 means that $(c_3y_{15}, c_3y_{15}) = 6$. Hence $(c_3\bar{y}_{15}, c_3\bar{y}_{15}) = 6$. So $t_{10} = x_3 + y_3 + z_4$. This leads to $y_{15} \in Supp\{c_3\bar{y}_3\}$, a contradiction. The lemma follows. $\square$

Lemma 2.23 $c_3c_9 = d_3 + \bar{c}_9 + \bar{y}_{15}$.

Proof First, by the previous lemma in this section, we know that $(c_3c_9, c_3c_9) = 3$ and c_3c_9 is a sum of irreducible base elements of degree 3, 9 and 15. So we may assume that $c_3c_9 = d_3 + y_9 + \bar{y}_{15}$, where $d_3, y_9 \in B$. Then $c_3\bar{d}_3 = \bar{c}_9$. Checking the associative law for $\bar{c}_3(c_3\bar{d}_3) = (c_3\bar{c}_3)\bar{d}_3$, it follows that

$$\bar{d}_3b_8 = \bar{y}_9 + y_{15}.$$

Since

$$\begin{aligned} b_6(b_3\bar{x}_6) &= b_6b_8 + b_6x_{10} \\ &= c_3 + b_6 + c_9 + 2y_{15} + b_6x_{10}, \\ (b_6\bar{x}_6)b_3 &= 2b_3x_6 + b_3b_9 + b_3x_{15} \\ &= 2c_3 + 2y_{15} + b_3b_9 + b_6 + 2y_{15} + c_9, \end{aligned}$$

we have that

$$b_6x_{10} = b_3b_9 + c_3 + 2y_{15}.$$

Since $(c_3^2)\bar{d}_3 = \bar{c}_3\bar{d}_3 + \bar{b}_6\bar{d}_3$, $c_3(c_3\bar{d}_3) = c_3\bar{c}_9$ and $x_{10} \in Supp\{c_3\bar{c}_9\}$ by Lemma 2.17, we obtain $x_{10} \in Supp\{\bar{b}_6\bar{d}_3\}$. So $(b_6x_{10}, \bar{d}_3) = 1$. Furthermore $(b_3b_9, \bar{d}_3) = 1$. Thus

$$b_3d_3 = \bar{b}_9.$$

Therefore

$$\bar{b}_3\bar{b}_9 = \bar{b}_3(b_3d_3) = (\bar{b}_3b_3)d_3 = d_3 + d_3b_8 = d_3 + y_9 + \bar{y}_{15},$$

which implies that

$$\begin{aligned} b_3b_9 &= \bar{d}_3 + \bar{y}_9 + y_{15}, \\ b_6x_{10} &= \bar{d}_3 + c_3 + \bar{y}_9 + 3y_{15}. \end{aligned}$$

Since $(r_3b_3)d_3 = c_9 + \bar{b}_6d_3$ and $(b_3d_3)r_3 = r_3\bar{b}_9$, we have that $c_9 \in Supp\{r_3\bar{b}_9\}$. But $y_{15} \in Supp\{r_3\bar{b}_9\}$ by Lemma 2.17. Thus there exists $m_3 \in B$ such that

$$r_3\bar{b}_9 = m_3 + c_9 + y_{15}.$$

Hence $b_9 \in Supp\{r_3\bar{c}_9\}$. But $(r_3\bar{c}_9, x_{15}) = 1$ by Lemma 2.17. Therefore

$$r_3\bar{c}_9 = n_3 + b_9 + x_{15}, \quad n_3 \in B.$$

By r_3 real, it follows that

$$r_3n_3 = \bar{c}_9.$$

Furthermore,

$$\begin{aligned} r_3(r_3\bar{c}_9) &= r_3x_{15} + r_3b_9 + r_3n_3 \\ &= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15} + \bar{c}_9 + \bar{y}_{15} + \bar{m}_3 + \bar{c}_9, \\ r_3^2\bar{c}_9 &= \bar{c}_9 + b_8\bar{c}_9, \end{aligned}$$

so that

$$b_8\bar{c}_9 = \bar{b}_6 + 2\bar{c}_9 + 3\bar{y}_{15} + \bar{m}_3.$$

Hence $(b_8y_{15}, c_9) = 3$.

By the expression b_8y_{15} in Lemma 2.17, we have $(\bar{b}_3d_9, c_9) = 1$. From the expression b_3t_{15} in Lemma 2.17, one has that $(t_{15}, b_3c_9) = 1$. Hence there exists $u_3 \in B$ such that

$$b_3c_9 = u_3 + d_9 + t_{15}.$$

Thus $\bar{b}_3u_3 = c_9$. Moreover,

$$c_3b_9 = (\bar{b}_3\bar{d}_3)c_3 = \bar{b}_3(c_3\bar{d}_3) = \bar{b}_3\bar{c}_9 = \bar{u}_3 + d_9 + t_{15}.$$

Thus $u_3c_3 = \bar{b}_9$.

Since

$$\begin{aligned} \bar{c}_3(b_3\bar{u}_3) &= \bar{c}_3\bar{c}_9 \\ &= \bar{d}_3 + \bar{y}_9 + y_{15}, \\ (b_3\bar{c}_3)\bar{u}_3 &= \bar{b}_3\bar{u}_3 + \bar{x}_6\bar{u}_3, \end{aligned}$$

we have that

$$b_3u_3 = y_9 \quad \text{and} \quad u_3x_6 = \bar{y}_{15} + \bar{d}_3.$$

Hence

$$c_3y_9 = c_3(b_3u_3) = b_3(u_3c_3) = b_3\bar{b}_9.$$

Since $(x_{10}, b_3\bar{b}_9) = 1$ by Lemma 2.17, one has that $(x_{10}, c_3y_9) = 1$, so $(\bar{y}_9, c_3x_{10}) = 1$. Therefore $\bar{y}_9 = c_9$ by the expression c_3x_{10} (see Lemma 2.17). The lemma follows. □

Lemma 2.24 *A NITA generated by c_3 is isomorphic to the algebra of characters of $3 \cdot A_6$: $(Ch(3 \cdot A_6), Irr(3 \cdot A_6))$. Furthermore, we have all the equations of products of base elements in the table subset D listed in Sect. 2.2.*

Proof By Lemmas 2.17 and 2.23 the sub-algebra generated by c_3 satisfies the following equations:

$$c_3\bar{c}_3 = 1 + b_8, \qquad c_3^2 = \bar{c}_3 + \bar{b}_6, \qquad c_3b_6 = \bar{c}_3 + \bar{y}_{15},$$
$$c_3\bar{b}_6 = b_8 + x_{10},\ x_{10} \text{ real}, \qquad c_3x_{10} = b_6 + c_9 + y_{15}, \qquad c_3y_{15} = \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}.$$

Changing c_3 into b_3, b_6 into $\bar{b}_6$, y_{15} into b_{15}, x_{10} into b_{10}, we can see that the above equations are the same as those in Hypothesis 2.1 and Lemma 2.9. Hence the NITA generated by c_3 is exactly the Table Algebra of characters of $3 \cdot A_6$ by Theorem 2.7. Of course, we have equations of products of base elements in the table subset D listed in Sect. 2.2. □

Now we continue our calculations, and we shall construct a new NITA which is not derived from groups.

Theorem 2.9 *If c_3 is nonreal, $(b_3b_8, b_3b_8) = 3$. Then the NITA generated by b_3 with $b_3^2 = c_3 + b_6$ is isomorphic to the NITA in $(A(3 \cdot A_6 \cdot 2), B_{32})$, the NITA of dimension 32 in Sect. 2.2.*

Proof By Lemma 2.24, we may say that we have found a table subset: $D = \{1, b_8, x_{10}, b_5, c_5, c_8, x_9, c_3, \bar{c}_3, d_3, \bar{d}_3, c_9, \bar{c}_9, b_6, \bar{b}_6, y_5, \bar{y}_{15}\}$. To prove that the NITA is isomorphic to NITA of dimension 32 in Sect. 2.2, it is sufficient to find all the basis elements outside D and the remaining expressions of all products of basis elements. Until now we have found many products of basis elements in Lemmas 2.16, 2.17 and 2.23. It remains to find the rest of them. In order to make the proof read easily, we shall divide it into several steps:

Step 1 $b_3b_9 = y_{15} + c_9 + \bar{d}_3$, $b_3d_3 = \bar{b}_9$, $b_3\bar{b}_9 = x_9 + c_8 + x_{10}$, $b_3d_9 = \bar{c}_9 + d_3 + \bar{y}_{15}$, $b_3\bar{d}_3 = d_9$, $b_3\bar{c}_9 = \bar{b}_9 + \bar{x}_{15} + z_3$, $b_3\bar{z}_3 = c_9$, $b_3\bar{x}_{15} = b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}$, $b_3z_3 = x_9$, $b_3y_3 = \bar{c}_9$.

Since by Lemma 2.17

$$\begin{aligned} b_3^2x_{10} &= c_3x_{10} + b_6x_{10} \\ &= b_6 + c_9 + y_{15} + 3y_{15} + c_3 + \bar{d}_3 + c_9, \\ (b_3x_{10})b_3 &= x_6b_3 + x_{15}b_3 + b_9b_3 \\ &= c_3 + y_{15} + b_6 + 2y_{15} + c_9 + b_3b_9, \end{aligned}$$

we have that $b_3b_9 = y_{15} + c_9 + \bar{d}_3$. Furthermore, $b_3d_3 = \bar{b}_9$. So

$$b_3\bar{b}_9 = (b_3d_3)b_3 = b_3^2d_3 = c_3d_3 + b_6d_3 = x_9 + c_8 + x_{10}.$$

Set $b_3\bar{d}_3 = y_9$. Since $b_3d_3 = \bar{b}_9$, then $y_9 \in B$. And

$$b_3y_9 = (b_3\bar{d}_3)b_3 = b_3^2\bar{d}_3 = c_3\bar{d}_3 + b_6\bar{d}_3 = \bar{c}_9 + d_3 + \bar{y}_{15}.$$

We have that $(b_3y_{15}, \bar{y}_9) = 1$. But $b_3y_{15} = s_6 + 2t_{15} + d_9$, thus $d_9 = \bar{y}_9$ and $\bar{b}_3d_3 = d_9$.

Since $(b_3x_{15}, c_9) = (b_3b_9, c_9) = 1$, there exists $z_3 \in B$ such that

$$b_3\bar{c}_9 = \bar{b}_9 + \bar{x}_{15} + z_3.$$

Hence $b_3\bar{z}_3 = c_9$.

Since

$$\begin{aligned}
b_3^2\bar{b}_6 &= c_3\bar{b}_6 + b_6\bar{b}_6\\
&= b_8 + x_{10} + 1 + b_8 + b_5 + c_5 + c_8 + x_9,\\
(b_3\bar{b}_6)b_3 &= b_3\bar{b}_3 + b_3\bar{x}_{15}\\
&= 1 + b_8 + b_3\bar{x}_{15},
\end{aligned}$$

we have that

$$b_3\bar{x}_{15} = b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}.$$

Now we have

$$\begin{aligned}
b_3^2\bar{c}_9 &= c_3\bar{c}_9 + b_6\bar{c}_9\\
&= c_8 + x_9 + x_{10} + 2x_9 + b_8 + x_{10} + c_8 + b_5 + c_5,\\
(b_3\bar{c}_9)b_3 &= b_3\bar{b}_9 + b_3\bar{x}_{15} + b_3z_3\\
&= c_8 + x_9 + x_{10} + b_5 + c_5 + b_8 + c_8 + x_9 + x_{10} + b_3z_3,
\end{aligned}$$

which imply $b_3z_3 = x_9$.

Step 2 $b_3c_9 = t_{15} + d_9 + y_3$, y_3 real, $d_3\bar{x}_6 = t_{15} + \bar{y}_3$, $c_3b_9 = t_{15} + d_9 + y_3$, $c_3y_3 = \bar{b}_9$, $b_3y_3 = \bar{c}_9$ and $b_8t_{15} = r_3 + \bar{y}_3 + 5t_{15} + 2s_6 + 3d_9$.

We have that

$$b_3c_9 = b_3(d_3\bar{c}_3) = (b_3\bar{c}_3)d_3 = \bar{b}_3d_3 + \bar{x}_3d_3 = d_9 + \bar{x}_6d_3.$$

Hence $(b_3c_9, d_9) = 1$. But by Lemma 2.17 $(b_3t_{15}, \bar{c}_9) = 1$. Then there exists $y_3 \in B$ such that

$$b_3c_9 = t_{15} + d_9 + y_3, \qquad d_3\bar{x}_6 = t_{15} + \bar{y}_3.$$

Consequently $b_3\bar{y}_3 = \bar{c}_9$. Since $b_3c_9 = (d_3\bar{c}_3)b_3 = (b_3d_3)\bar{c}_3 = \bar{c}_3\bar{b}_9$, we have that

$$c_3b_9 = t_{15} + d_9 + y_3.$$

Consequently $c_3\bar{y}_3 = \bar{b}_9$. Since

$$
\begin{aligned}
b_3^2 c_9 &= c_3c_9 + b_6c_9 \\
&= d_3 + \bar{c}_9 + \bar{y}_{15} + \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15}, \\
(b_3c_9)b_3 &= b_3t_{15} + b_3d_9 + b_3y_3 \\
&= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15} + d_3 + \bar{c}_9 + \bar{y}_{15} + b_3y_3,
\end{aligned}
$$

then $b_3y_3 = \bar{c}_9$.

Since we have that

$$
\begin{aligned}
(b_3\bar{b}_3)t_{15} &= t_{15} + b_8t_{15} \\
(b_3t_{15})\bar{b}_3 &= \bar{b}_3\bar{b}_6 + \bar{b}_3\bar{c}_9 + 2\bar{b}_3\bar{y}_{15} \\
&= r_3 + t_{15} + d_9 + \bar{y}_3 + t_{15} + 2s_6 + 4t_{15} + 2d_9,
\end{aligned}
$$

so $b_8t_{15} = r_3 + \bar{y}_3 + 5t_{15} + 2s_6 + 3d_9$, which implies that y_3 is real.

Step 3 $b_3d_9 = \bar{c}_9 + \bar{y}_{15} + d_3$, $r_3t_{15} = b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}$, $r_3d_9 = c_8 + x_9 + x_{10}$.

Since

$$
\begin{aligned}
b_3^2 y_{15} &= c_3y_{15} + b_6y_{15} \\
&= \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15} + \bar{c}_3 + d_3 + \bar{b}_6 + 2\bar{c}_9 + 4\bar{y}_{15}, \\
(b_3y_{15})b_3 &= b_3s_6 + 2b_3t_{15} + b_3d_9 \\
&= \bar{c}_3 + \bar{y}_{15} + b_3d_9 + 2\bar{b}_6 + 2\bar{c}_9 + 4\bar{y}_{15},
\end{aligned}
$$

then $b_3d_9 = \bar{c}_9 + \bar{y}_{15} + d_3$.

Since $r_3^2b_8 = (r_3b_8)r_3$ and $r_3^2x_{10} = (r_3x_{10})r_3$, we have that

$$
\begin{aligned}
r_3t_{15} &= b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}, \\
r_3d_9 &= c_8 + x_9 + x_{10}.
\end{aligned}
$$

Step 4 $r_3d_3 = b_9$ and $r_3b_9 = d_3 + \bar{c}_9 + \bar{y}_{15}$.

There are three possibilities: $r_3d_3 = t_9, m_3 + n_6, m_4 + n_5$. Thus

$$
\begin{aligned}
r_3^2 d_3 &= d_3 + d_3b_8 \\
&= d_3 + \bar{c}_9 + \bar{y}_{15}, \\
(r_3d_3)r_3 &= r_3t_9 \\
&\text{or } m_3r_3 + n_6r_3 \\
&\text{or } m_4r_3 + n_5r_3,
\end{aligned}
$$

If $r_3d_3 = m_3 + n_6$, then $m_3r_3 = \bar{c}_9$ and $r_3n_6 = d_3 + \bar{y}_{15}$. But $r_3\bar{y}_{15} = x_6 + 2x_{15} + b_9$. Hence $n_6 = x_6$, so that $(r_3x_6, d_3) = 1$, a contradiction.

If $r_3d_3 = m_4 + n_5$, then $m_4r_3 = d_3 + \bar{c}_9$ and $r_3n_5 = \bar{y}_{15}$. But $(r_3\bar{y}_{15}, n_5) = 0$, a contradiction. Hence $r_3d_3 = t_9$ and $r_3t_9 = d_3 + \bar{c}_9 + \bar{y}_{15}$, which implies that $(r_3\bar{y}_{15}, t_9) = 1$. Then $t_9 = b_9$.

Step 5 $x_6y_{15} = r_3 + 4t_{15} + s_6 + 2d_9 + y_3$, $b_8x_{15} = b_3 + \bar{z}_3 + 2x_6 + 5x_{15} + 3b_9$, $b_8b_9 = \bar{z}_3 + x_6 + 2b_9 + 3x_{15}$, $b_8d_9 = y_3 + s_6 + 2d_9 + 3t_{15}$, $y_3b_8 = d_9 + t_{15}$ and $b_6b_9 = s_6 + 2d_9 + 2t_{15}$ and $b_6x_{15} = r_3 + s_6 + y_3 + 2d_9 + 4t_{15}$.

The above equations follow from $b_3x_6^2 = (b_3x_6)x_6$, $(b_3\bar{b}_3)x_{15} = (b_3x_{15})\bar{b}_3$, $(b_3\bar{b}_3)b_9 = b_3b_9)\bar{b}_3$, $(b_3\bar{b}_3)d_9 = (b_3d_9)\bar{b}_3$, $(b_3\bar{b}_3)y_3 = (b_3y_3)\bar{b}_3$, $b_3^2b_9 = (b_3b_9)b_3$ and $b_3^2x_{15} = (b_3x_{15})b_3$.

Step 6 $c_3d_9 = \bar{b}_9 + z_3 + \bar{x}_{15}$, $b_6d_9 = \bar{x}_6 + 3\bar{b}_9 + 2\bar{x}_{15}$, $z_3\bar{c}_3 = d_9$.

Since

$$
\begin{aligned}
b_3^2d_9 &= c_3d_9 + b_6d_9, \\
(b_3d_9)b_3 &= b_3d_3 + b_3\bar{c}_9 + b_3\bar{y}_{15} \\
&= 2\bar{b}_9 + 3\bar{x}_{15} + z_3 + \bar{x}_6,
\end{aligned}
$$

and $(c_3x_{15}, d_9) = 1$, from which Step 6 follows.

Step 7 $b_8x_{15} = b_3 + \bar{z}_3 + 2x_6 + 5x_{15} + 3b_9$, $b_3x_9 = \bar{z}_3 + b_9 + x_{15}$, $b_3c_8 = x_{15} + b_9$, $b_3b_5 = x_{15}$ and $b_3c_5 = x_{15}$.

By Lemma 2.17 and Step 1, we have

$$
\begin{aligned}
(b_3x_{15})\bar{b}_3 = b_6\bar{b}_3 + 2y_{15}\bar{b}_3 + c_9\bar{b}_3 &= b_3 + x_{15} + x_6 + 2x_{15} + b_9 + b_9 + x_{15} + \bar{z}_3, \\
(b_3\bar{b}_3)x_{15} &= x_{15} + b_8x_{15},
\end{aligned}
$$

hence

$$b_8x_{15} = b_3 + \bar{z}_3 + 2x_6 + 5x_{15} + 3b_9.$$

Consequently, we have the following equations:

$$
\begin{aligned}
b_8^2b_3 &= b_3 + 2b_3b_8 + 2b_3x_{10} + b_3b_5 + b_3c_5 + b_3c_8 + b_3x_9 \\
&= 3b_3 + 2x_6 + 2x_{15} + 2x_6 + 2x_{15} + 2b_9 + b_3b_5 + b_3c_5 + b_3c_8 + b_3x_9, \\
(b_3b_8)b_8 &= b_3b_8 + x_6b_8 + x_{15}b_8 \\
&= b_3 + x_6 + x_{15} + b_3 + x_6 + b_9 + 2x_{15} + b_3 + \bar{z}_3 + 2x_6 + 5x_{15} + 3b_9,
\end{aligned}
$$

which imply that

$$b_3b_5 + b_3c_5 + b_3c_8 + b_3x_9 = 4x_{15} + 2b_9 + \bar{z}_3.$$

By the expression $b_3\bar{x}_{15}$ in Step 3, it can be easily shown that Step 7 follows.

Step 8 $r_3b_5 = t_{15}$, $r_3c_5 = t_{15}$, $r_3c_8 = t_{15} + d_9$, $r_3x_9 = t_{15} + d_9 + y_3$ and $r_3y_3 = x_9$.

Since

$$
\begin{aligned}
b_8^2 r_3 &= r_3 + 2r_3 b_8 + 2r_3 x_{10} + r_3 b_5 + r_3 c_5 + r_3 c_8 + r_3 x_9 \\
&= r_3 + 2r_3 + 2s_6 + 2t_{15} + 2s_6 + 2t_{15} + 2d_9 + r_3 b_5 + r_3 c_5 + r_3 c_8 + r_3 x_9, \\
(r_3 b_8) b_8 &= r_3 b_8 + b_8 s_6 + b_8 t_{15} \\
&= r_3 + s_6 + t_{15} + r_3 + s_6 + 2t_{15} + d_9 + r_3 + y_3 + 5t_{15} + 2s_6 + 3d_9,
\end{aligned}
$$

it holds that

$$r_3 b_5 + r_3 c_5 = r_3 c_8 + r_3 x_9 = 4t_{15} + 2d_9 + y_3.$$

But $(r_3 t_{15}, x_9) = 1$. The first four equations follow from the above equation. Consequently, $r_3 y_3 = x_9$.

Step 9 $r_3 c_9 = \bar{b}_9 + \bar{x}_{15} + z_3$, $r_3 z_3 = c_9$, $x_6 \bar{y}_{15} = \bar{b}_3 + z_3 + \bar{x}_6 + 4\bar{x}_{15} + 2\bar{b}_9$, $x_6 c_9 = 2t_{15} + 2d_9 + s_6$, $x_6 x_{10} = b_9 + b_3 + \bar{z}_3 + 3x_{15}$, $x_6 \bar{x}_6 = 1 + b_8 + b_5 + c_5 + c_8 + x_9$, $x_6 s_6 = 2\bar{b}_6 + \bar{c}_9 + \bar{y}_{15}$ and $x_6 \bar{c}_9 = 2\bar{x}_{15} + \bar{x}_6 + 2\bar{b}_9$.

Since $r_3^2 b_9 = (r_3 b_9) r_3$, we have the first equation, from which the second equation follows. The rest of the equations follow from associativity of $(c_3 b_8)\bar{x}_6 = b_8(c_3 \bar{x}_6)$, $(d_3 b_8)\bar{x}_6 = b_8(d_3 \bar{x}_6)$, $(b_3 \bar{x}_6) x_6 = (b_3 x_6)\bar{x}_6$, $(b_3 \bar{c}_3) x_6 = (b_3 x_6)\bar{c}_3$, $(b_3 c_3) x_6 = b_3(c_3 x_6)$ and $(r_3 c_3) c_9 = r_3(c_3 c_9)$, respectively.

Step 10 $x_6 y_3 = d_3 + \bar{y}_{15}$.

By known equations, we have that $(x_6 y_3, \bar{y}_{15}) = (x_6 y_{15}, y_3) = 1$ and $(x_6 y_3, d_3) = (x_6 \bar{d}_3, y_3) = 1$, from which Step 10 follows.

Step 11 $x_6 b_5 = x_6 + b_9 + x_{15}$, $x_6 c_5 = x_6 + b_9 + x_{15}$, $x_6 c_8 = x_6 + b_9 + 2x_{15} + z_3$ and $z_3 x_6 = x_{10} + c_8$.

Since

$$
\begin{aligned}
b_8^2 x_6 &= x_6 + 2x_6 b_8 + 2x_6 x_{10} + x_6 b_5 + x_6 c_5 + x_6 c_8 + x_6 x_9 \\
&= x_6 + 2b_3 + 2x_6 + 2b_9 + 4x_{15} + 2b_9 + 2b_3 + 2\bar{z}_3 + 6x_{15} + \\
&\quad + x_6 b_5 + x_6 c_5 + x_6 c_8 + x_6 + 2b_9 + 2x_{15}, \\
(b_8 x_6) b_8 &= b_3 b_8 + x_6 b_8 + b_9 b_8 + 2b_8 x_{15} \\
&= b_3 + x_6 + x_{15} + b_3 + x_6 + b_9 + 2x_{15} + \bar{z}_3 + x_6 + 2b_9 + \\
&\quad + 3x_{15} + 2b_3 + 2\bar{z}_3 + 4x_6 + 10x_{15} + 6b_9,
\end{aligned}
$$

we have that

$$x_6 b_5 + x_6 c_5 + x_6 c_8 = 3x_6 + 3b_9 + \bar{z}_3 + 4x_{15}.$$

Table 2.3 Further calculations

Values of inner products	New equations
$(z_3b_9, b_8) = (b_8b_9, \bar{z}_3) = 1$	
$(z_3x_{15}, b_8) = (b_8x_{15}, \bar{z}_3) = 1$	$z_3b_8 = \bar{b}_9 + \bar{x}_{15}$
$(x_6x_{10}, b_9) = (x_6c_5, b_9) = (x_6b_5, b_9) = 1$	
$(x_6b_8, b_9) = (x_6c_8, b_9) = 1, (x_6x_9, b_9) = 2$	$x_6\bar{b}_9 = b_5 + c_5 + x_{10} + b_8 + c_8 + 2x_9$
$(x_6b_8, x_{15}) = 2, (x_6x_{10}, x_{15}) = 3,$	
$(x_6b_5, x_{15}) = (x_6c_5, x_{15}) = 1$	$x_6\bar{x}_{15} = b_5 + c_5 + 2b_8 + 3x_{10} + 2c_8 + 2x_9$
$(x_6c_8, x_{15}) = (x_6x_9, x_{15}) = 2$	
$(x_6c_3, t_{15}) = (x_6\bar{d}_3, t_{15}) = (x_6b_6, t_{15}) = 1$	
$(x_6c_9, t_{15}) = 2, (x_6y_{15}, t_{15}) = 4$	$x_6t_{15} = \bar{c}_3 + d_3 + 2\bar{c}_9 + 4\bar{y}_{15} + \bar{b}_6$
$(x_6\bar{z}_3, \bar{d}_3) = (x_6\bar{y}_{15}, z_3) = 1$	$x_6\bar{z}_3 = \bar{d}_3 + y_{15}$
$(x_6d_3, \bar{x}_{15}) = (x_6\bar{b}_6, \bar{x}_{15}) = (x_6\bar{c}_3, \bar{x}_{15}) = 1$	
$(x_6\bar{c}_9, \bar{x}_{15}) = 2, (x_6\bar{y}_{15}, \bar{x}_{15}) = 4$	$x_6x_{15} = 4y_{15} + b_6 + 2c_9 + d_3 + c_3$

By the expression $x_6\bar{x}_6$ in Step 9, we have that $(x_6b_5, x_6) = (x_6c_5, x_6) = (x_6c_8, x_6) = 1$. The first three equations of Step 11 follow by comparing the degrees. Hence $(x_6z_3, c_8) = 1$. In addition to $(z_3x_6, x_{10}) = (x_6x_{10}, \bar{z}_3) = 1$, from which the last equation follows.

Step 12 $x_6x_9 = x_6 + 2b_9 + 2x_{15}$, $x_6d_9 = \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15}$, $x_6d_3 = z_3 + \bar{x}_{15}$ and $x_6b_9 = 2\bar{c}_9 + \bar{b}_6 + 2\bar{y}_{15}$.

It follows from $(b_3z_3)x_6 = b_3(z_3x_6)$, $(z_3\bar{c}_3)x_6 = \bar{c}_3(z_3x_6)$, $(r_3c_3)\bar{d}_3 = r_3(c_3\bar{d}_3)$ and $(r_3c_3)\bar{b}_9 = (r_3\bar{b}_9)c_3$, respectively.

Step 13 Based on the equations known by us, we can check the inner products of some products of basis elements and obtain Table 2.3.

Step 14 Repeatedly considering the associativity of basis elements, we can obtain new equations. Since the process is trivial and repeated, the concrete process is omitted and the correspondence between the associativity of basis elements and new equations are listed in the following table. Occasionally, we need to check the inner products based on the old and new equations to obtain a newer equation. Table 2.4 is an explanation. □

Table 2.4 Further calculations

Associativity checked or inner products	New equations
$(b_3c_3)\bar{d}_3 = b_3(c_3\bar{d}_3)$	$d_3s_6 = \bar{z}_3 + x_{15}$
$(b_3c_3)y_3 = b_3(c_3y_3)$	$y_3s_6 = c_8 + x_{10}$
$(b_3c_3)z_3 = (b_3z_3)c_3$	$z_3s_6 = \bar{d}_3 + y_{15}$
$(b_3c_3)\bar{z}_3 = b_3(c_3\bar{z}_3)$	$\bar{z}_3s_6 = d_3 + \bar{y}_{15}$
$(b_3c_3)s_6 = b_3(c_3s_6)$	$s_6^2 = 1 + b_5 + c_5 + b_8 + c_8 + x_9$
$(b_3c_3)b_9 = b_3(c_3b_9)$	$b_9s_6 = \bar{b}_6 + 2\bar{c}_9 + 2\bar{y}_{15}$
$(b_3c_3)x_{15} = b_3(c_3x_{15})$	$s_6x_{15} = \bar{c}_3 + d_3 + \bar{b}_6 + 2\bar{c}_9 + 4\bar{y}_{15}$
$(b_3c_3)t_{15} = b_3(c_3t_{15})$	$s_6t_{15} = b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 3x_{10}$
$(b_3c_3)d_9 = b_3(c_3d_9)$	$s_6d_9 = b_5 + c_5 + 2x_9 + b_8 + c_8 + x_{10}$
$(b_3c_3)x_{10} = b_3(c_3x_{10})$	$s_6x_{10} = r_3 + y_3 + d_9 + 3t_{15}$
$(b_3c_3)b_5 = b_3(c_3b_5)$	$s_6b_5 = s_6 + d_9 + t_{15}$
$(b_3c_3)c_5 = b_3(c_3c_5)$	$s_6c_5 = s_6 + t_{15} + d_9$
$(b_3c_3)c_8 = b_3(c_3c_8)$	$s_6c_8 = y_3 + s_6 + d_9 + 2t_{15}$
$(b_3c_3)x_9 = b_3(c_3x_9)$	$s_6x_9 = s_6 + 2d_9 + 2t_{15}$
$(b_3c_3)c_9 = b_3(c_3c_9)$	$s_6c_9 = \bar{x}_6 + 2\bar{b}_9 + 2\bar{x}_{15}$
$(s_6b_3, \bar{y}_{15}) = 1$	
$(s_6b_9, \bar{y}_{15}) = 2$, $(s_6x_{15}, \bar{y}_{15}) = 4$	
$(s_6\bar{z}_3, \bar{y}_{15}) = (s_6x_6, \bar{y}_{15}) = 1$	$s_6y_{15} = 2\bar{b}_9 + 4\bar{x}_{15} + z_3 + \bar{x}_6 + \bar{b}_3$
$(b_3d_3)b_9 = d_3(b_3b_9)$	$b_9\bar{b}_9 = 1 + b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 2x_{10}$
$(b_3d_3)\bar{b}_9 = d_3(b_3\bar{b}_9)$	$b_9^2 = \bar{d}_3 + c_3 + 2b_6 + 2c_9 + 3y_{15}$
$(b_3d_3)\bar{x}_{15} = d_3(b_3\bar{x}_{15})$	$b_9x_{15} = 6y_{15} + 3c_9 + \bar{d}_3 + 2b_6 + c_3$
$(b_3d_3)x_{15} = d_3(b_3x_{15})$	$b_9\bar{x}_{15} = 2b_5 + 2c_5 + 3c_8 + 3b_8 + 4x_{10} + 3x_9$
$(b_3d_3)t_{15} = d_3(b_3t_{15})$	$b_9t_{15} = d_3 + \bar{c}_3 + 6\bar{y}_{15} + 3\bar{c}_9 + 2\bar{b}_6$
$(b_3d_3)d_9 = d_3(b_3d_9)$	$b_9d_9 = \bar{c}_3 + d_3 + 2\bar{b}_6 + 2\bar{c}_9 + 3\bar{y}_{15}$
$(b_3d_3)y_3 = d_3(b_3y_3)$	$b_9y_3 = \bar{c}_3 + \bar{y}_{15} + \bar{c}_9$
$(b_3d_3)z_3 = d_3(b_3z_3)$	$b_9\bar{z}_3 = c_9 + c_3 + y_{15}$
$(b_3d_3)\bar{z}_3 = d_3(b_3\bar{z}_3)$	$b_9z_3 = b_8 + x_9 + x_{10}$
$(b_3d_3)x_{10} = b_3(d_3x_{10})$	$b_9x_{10} = \bar{z}_3 + b_3 + 2b_9 + x_6 + 4x_{15}$
$(b_3d_3)b_5 = b_3(d_3b_5)$	$\bar{b}_9c_9 = b_3 + \bar{z}_3 + 2x_6 + 3x_{15} + 2b_9$
$(b_3d_3)c_5 = b_3(d_3c_5)$	$b_9c_9 = r_3 + y_3 + 2s_6 + 2d_9 + 3t_{15}$
$(b_3d_3)b_6 = b_3(d_3b_6)$	$\bar{b}_9b_6 = 2b_9 + 2x_{15} + x_6$
$(b_3d_3)c_8 = b_3(d_3c_8)$	$\bar{b}_9y_{15} = 6x_{15} + b_3 + \bar{z}_3 + 2x_6 + 3b_9$
$(b_3d_3)x_9 = b_3(d_3x_9)$	$\bar{b}_9x_9 = 2\bar{b}_9 + 3\bar{x}_{15} + z_3 + \bar{b}_3 + 2\bar{x}_6$
$(b_3d_3)\bar{y}_{15} = b_3(d_3\bar{y}_{15})$	$\bar{b}_9\bar{y}_{15} = r_3 + y_3 + 2s_6 + 6t_{15} + 3d_9$
$(b_3d_3)d_3 = b_3(d_3^2)$	$\bar{b}_9d_3 = r_3 + t_{15} + d_9$
$(b_3d_3)c_3 = b_3(d_3c_3)$	$\bar{b}_9c_3 = \bar{z}_3 + b_9 + x_{15}$

(*continued on the next page*)

Table 2.4 (Continued)

Associativity checked or inner products	New equations
$(b_3d_3)\bar{d}_3 = b_3(d_3\bar{d}_3)$	$b_9d_3 = \bar{b}_3 + \bar{b}_9 + \bar{x}_{15}$
$(b_3d_3)c_3 = b_3(d_3c_3)$	$b_9\bar{c}_3 = z_3 + \bar{b}_9 + \bar{x}_{15}$
$(b_3b_5)x_{15} = (b_3x_{15})b_5$	$x_{15}^2 = 2c_3 + 2\bar{d}_3 + 4b_6 + 6c_9 + 9y_{15}$
$(b_3b_5)\bar{x}_{15} = (b_3\bar{x}_{15})b_5$	$x_{15}\bar{x}_{15} = 1 + 6x_9 + 3b_5 + 3c_5 + 5b_8 + +5c_8 + 6x_{10}$
$(b_3b_5)t_{15} = (b_3t_{15})b_5$	$x_{15}t_{15} = 4\bar{b}_6 + 6\bar{c}_9 + 9\bar{y}_{15} + +2\bar{c}_3 + 2d_3$
$(b_3b_5)d_9 = (b_3d_9)b_5$	$x_{15}d_9 = \bar{c}_3 + d_3 + 2\bar{b}_6 + 3\bar{c}_9 + 6\bar{y}_{15}$
$(b_3b_5)z_3 = (b_3z_3)b_5$	$x_{15}z_3 = b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}$
$(b_3b_5)b_5 = b_3b_5^2$	$x_{15}b_5 = \bar{z}_3 + b_3 + x_6 + 2b_9 + 3x_{15}$
$(b_3b_5)x_{10} = (b_3x_{10})b_5$	$x_{15}x_{10} = b_3 + \bar{z}_3 + 3x_6 + 4b_9 + 6x_{15}$
$(b_3b_5)c_5 = (b_5c_5)b_3$	$x_{15}c_5 = b_3 + \bar{z}_3 + x_6 + 2b_9 + 3x_{15}$
$(b_3b_5)c_8 = (b_5c_8)b_3$	$x_{15}c_8 = b_3 + \bar{z}_3 + 2x_6 + 3b_9 + 5x_{15}$
$(b_3b_5)x_9 = (b_5x_9)b_3$	$x_{15}x_9 = b_3 + \bar{z}_3 + 2x_6 + 3b_9 + 6x_{15}$
$(b_3b_5)d_3 = (b_5d_3)b_3$	$x_{15}d_3 = \bar{x}_6 + 2\bar{x}_{15} + \bar{b}_9$
$(b_3b_5)\bar{d}_3 = (b_5\bar{d}_3)b_3$	$x_{15}d_3 = s_6 + d_9 + 2t_{15}$
$(b_3b_5)c_9 = (b_5c_9)b_3$	$x_{15}c_9 = r_3 + y_3 + 2s_6 + 6t_{15} + 3d_9$
$(b_3b_5)\bar{c}_9 = (b_5\bar{c}_9)b_3$	$x_{15}\bar{b}_9 = \bar{b}_3 + z_3 + 2\bar{x}_6 + 3\bar{b}_9 + 6\bar{x}_{15}$
$(b_3b_5)\bar{b}_6 = (b_5\bar{b}_6)b_3$	$x_{15}\bar{b}_6 = \bar{b}_3 + z_3 + 2\bar{b}_9 + 4\bar{x}_{15} + \bar{x}_6$
$(b_3b_5)y_{15} = (b_5y_{15})b_3$	$x_{15}y_{15} = 2y_3 + 2r_3 + 4s_6 + 6d_9 + 9t_{15}$
$(b_3b_6)t_{15} = (b_3t_{15})b_6$	$t_{15}^2 = 1 + 5b_8 + 5c_8 + 3b_5 + 3c_5 + 6x_{10} + 6x_9$
$(b_3b_6)d_9 = (b_3d_9)b_6$	$d_9t_{15} = 2b_5 + 2c_5 + 3b_8 + 3c_8 + 3x_9 + 4x_{10}$
$(b_3b_6)y_3 = (b_3y_3)b_6$	$y_3t_{15} = b_5 + c_5 + b_8 + c_8 + x_9 + x_{10}$
$(b_3b_6)z_3 = (b_3z_3)b_6$	$z_3t_{15} = b_6 + c_9 + 2y_{15}$
$(b_3b_6)x_{10} = (b_6x_{10})b_3$	$t_{15}x_{10} = r_3 + y_3 + 4d_9 + 3s_6 + 6t_{15}$
$(b_3b_6)b_5 = (b_6b_5)b_3$	$t_{15}b_5 = r_3 + y_3 + s_6 + 2d_9 + 3t_{15}$
$(b_3b_6)c_5 = (b_6c_5)b_3$	$t_{15}c_5 = r_3 + y_3 + s_6 + 2d_9 + 3t_{15}$
$(b_3b_6)c_8 = (b_6c_8)b_3$	$t_{15}c_8 = r_3 + y_3 + 2s_6 + 3d_9 + 5t_{15}$
$(b_3b_6)x_9 = (b_6x_9)b_3$	$t_{15}x_9 = r_3 + y_3 + 3d_9 + 2s_6 + 6t_{15}$
$(b_3b_6)d_3 = (b_6d_3)b_3$	$t_{15}d_3 = x_6 + b_9 + 2x_{15}$
$(b_3b_6)c_9 = (b_6c_9)b_3$	$t_{15}c_9 = \bar{b}_3 + z_3 + 3\bar{b}_9 + 2\bar{x}_6 + 6\bar{x}_{15}$
$(b_3b_6)b_6 = b_3b_6^2$	$t_{15}b_6 = \bar{b}_3 + z_3 + \bar{x}_6 + 2\bar{b}_9 + 4\bar{x}_{15}$
$(b_3b_6)y_{15} = (b_6y_{15})b_3$	$t_{15}y_{15} = 2\bar{b}_3 + 2z_3 + 4\bar{x}_6 + 6\bar{b}_9 + 9\bar{x}_{15}$
$(b_3\bar{d}_3)d_9 = (b_3d_9)\bar{d}_3$	$d_9^2 = 1 + b_5 + c_5 + 2b_8 + 2c_8 + 2x_9 + 2x_{10}$
$(b_3\bar{d}_3)x_{10} = (b_3x_{10})\bar{d}_3$	$d_9x_{10} = y_3 + r_3 + s_6 + 2d_9 + 4t_{15}$
$(b_3\bar{d}_3)b_5 = (b_3b_5)\bar{d}_3$	$d_9b_5 = s_6 + d_9 + 2t_{15}$
$(b_3\bar{d}_3)c_5 = (b_3c_5)\bar{d}_3$	$d_9c_5 = s_6 + d_9 + 2t_{15}$

(*continued on the next page*)

Table 2.4 (Continued)

Associativity checked or inner products	New equations
$(b_3\bar{d}_3)c_8 = (b_3c_8)\bar{d}_3$	$d_9c_8 = r_3 + s_6 + 2d_9 + 3t_{15}$
$(b_3\bar{d}_3)d_3 = (b_3d_3)\bar{d}_3$	$d_9d_3 = b_3 + b_9 + x_{15}$
$(b_3\bar{d}_3)c_9 = (b_3c_9)\bar{d}_3$	$d_9c_9 = b_3 + z_3 + 2\bar{x}_6 + 2\bar{b}_9 + 3\bar{x}_{15}$
$(b_3\bar{d}_3)y_{15} = (b_3y_{15})\bar{d}_3$	$d_9y_{15} = \bar{b}_3 + z_3 + 2\bar{x}_6 + 3\bar{b}_9 + 6\bar{x}_{15}$
$(b_3\bar{d}_3)x_9 = (\bar{d}_3x_9)b_3$	$d_9x_9 = r_3 + y_3 + 2s_6 + 2d_9 + 3t_{15}$
$(x_6d_3)z_3 = (z_3x_6)d_3$	$z_3^2 = d_3 + \bar{b}_6$
$(x_6d_3)\bar{z}_3 = (\bar{z}_3x_6)d_3$	$z_3\bar{z}_3 = 1 + b_8$
$(x_6\bar{d}_3)z_3 = (y_3x_6)\bar{d}_3$	$y_3^2 = 1 + c_8$
$(x_6d_3)y_3 = (y_3x_6)d_3$	$y_3z_3 = \bar{d}_3 + b_6$
$(x_6d_3)\bar{d}_3 = (d_3\bar{d}_3)x_6$	$z_3\bar{d}_3 = \bar{z}_3 + x_6$
$(x_6d_3)\bar{c}_9 = (d_3\bar{c}_9)x_6$	$z_3\bar{c}_9 = r_3 + d_9 + t_{15}$
$(x_6d_3)\bar{b}_6 = (d_3\bar{b}_6)x_6$	$z_3\bar{b}_6 = y_3 + t_{15}$
$(x_6d_3)b_6 = (d_3b_6)x_6$	$z_3b_6 = \bar{z}_3 + x_{15}$
$(r_3x_{10}, s_6) = (s_6b_8, r_3) = 1$	$r_3s_6 = x_{10} + b_8$
$(d_9y_{15}, \bar{x}_6) = 2, (d_9b_6, \bar{x}_6) = 1, (d_9c_9, \bar{x}_6) = 2$	$x_6d_9 = \bar{b}_6 + 2\bar{y}_{15} + 2\bar{c}_9$
$(x_6y_3, d_3) = (x_6t_{15}, d_3) = 1$	$x_6\bar{d}_3 = y_3 + t_{15}$
$(r_3c_3)\bar{d}_3 = r_3(c_3\bar{d}_3)$	$x_6d_3 = z_3 + \bar{x}_{15}$
$(\bar{x}_6b_6, b_9) = 1, (\bar{b}_9y_{15}, x_6) = (\bar{b}_9c_9, x_6) = 2$	$x_6b_9 = b_6 + 2y_{15} + 2c_9$
$(x_6\bar{b}_9, x_9) = (x_6\bar{x}_{15}, x_9) = 2, (x_6\bar{x}_6, x_9) = 1$	$x_6x_9 = 2b_9 + x_6 + 2x_{15}$
$(b_9\bar{b}_9, c_8) = 2, (b_9\bar{x}_{15}, c_8) = 1, (x_6c_8, b_9) = 1$	$b_9c_8 = 2b_9 + 3x_{15} + x_6 + b_3$
$(b_9\bar{b}_9, b_8) = 2, (b_9\bar{x}_{15}, b_8) = 3,$	
$(b_9z_3, b_8) = (x_6b_8, b_9) = 1$	$b_9b_8 = 2b_9 + 3x_{15} + \bar{z}_3 + x_6$
$(b_9\bar{b}_9, c_5) = (x_6c_5, b_9) = 1, (b_9\bar{x}_{15}, c_5) = 2$	$c_5b_9 = b_9 + 2x_{15} + x_6$
$(x_{15}c_9, y_3) = (x_{15}b_6, y_3) = 1, (x_{15}y_{15}, y_3) = 2$	$x_{15}y_3 = \bar{b}_6 + \bar{c}_9 + 2\bar{y}_{15}$
$(b_3b_5)\bar{z}_3 = (b_3\bar{z}_3)b_5$	$x_{15}\bar{z}_3 = b_6 + c_9 + 2y_{15}$
$(b_3b_5)\bar{d}_3 = (b_5\bar{d}_3)b_3$	$x_{15}\bar{d}_3 = s_6 + 2t_{15} + d_9$
$(b_3b_5)\bar{c}_9 = (b_5\bar{c}_9)b_3$	$x_{15}\bar{c}_9 = \bar{b}_3 + z_3 + 3\bar{b}_9 + 2\bar{x}_6 + 6\bar{x}_{15}$
$(b_3b_5)y_{15} = (b_5y_{15})b_3$	$x_{15}\bar{y}_{15} = 9\bar{x}_{15} + 2z_3 + 2\bar{b}_3 + 4\bar{x}_6 + 6\bar{b}_9$
$(x_6t_{15}, c_3) = (b_9t_{15}, \bar{c}_3) = 1, (x_{15}t_{15}, \bar{c}_3) = 2$	$t_{15}c_3 = \bar{x}_6 + \bar{b}_9 + 2\bar{x}_{15}$
$(d_9b_8, y_3) = (d_9x_{10}, y_3) = (d_9x_9, y_3) = 1$	$d_9y_3 = x_9 + b_8 + x_{10}$
$(d_9y_{15}, z_3) = (d_9c_3, z_3) = (d_9c_9, z_3) = 1$	$d_9z_3 = c_3 + c_9 + y_{15}$
$(b_3\bar{b}_3)d_9 = (b_3d_9)\bar{b}_3$	$d_9b_8 = y_3 + s_6 + 2d_9 + 3t_{15}$
$(b_9d_9, \bar{c}_3) = (d_9z_3, c_3) = (x_{15}d_9, \bar{c}_3) = 1$	$d_9c_3 = \bar{b}_9 + z_3 + \bar{x}_{15}$
$(d_9b_8, y_3) = (t_{15}b_8, y_3) = 1$	$y_3b_8 = d_9 + t_{15}$
$(s_6x_{10}, y_3) = (t_{15}x_{10}, y_3) = (d_9x_{10}, y_3) = 1$	$y_3x_{10} = s_6 + t_{15} + d_9$

(*continued on the next page*)

Table 2.4 (Continued)

Associativity checked or inner products	New equations
$(t_{15}y_3, b_5) = 1$	$y_3b_5 = t_{15}$
$(t_{15}y_3, c_5) = 1$	$y_3c_5 = t_{15}$
$(t_{15}y_3, c_8) = (s_6y_3, c_8) = (y_3c_8, y_3) = 1$	$y_3c_8 = s_6 + t_{15} + y_3$
$(t_{15}y_3, x_9) = (d_9y_3, x_9) = (r_3y_3, x_9) = 1$	$y_3x_9 = t_{15} + d_9 + r_3$
$(b_9y_3, \bar{c}_3) = 1$	$y_3c_3 = \bar{b}_9$
$(b_9y_3, \bar{d}_3) = 1$	$y_3d_3 = \bar{b}_9$
$(b_3y_3, c_9) = (b_9c_9, y_3) = (x_{15}c_9, y_3) = 1$	$y_3c_9 = \bar{b}_3 + \bar{b}_9 + \bar{x}_{15}$
$(x_{15}y_3, b_6) = (y_3z_3, b_6) = 1$	$y_3b_6 = \bar{x}_{15} + z_3$
$(x_6y_3, \bar{y}_{15}) = (b_9y_3, \bar{y}_{15}) = 1, (x_{15}y_3, \bar{y}_{15}) = 2$	$y_3y_{15} = \bar{x}_6 + \bar{b}_9 + 2\bar{x}_{15}$
$(x_6x_{10}, z_3) = (b_9x_{10}, z_3) = (x_{15}x_{10}, z_3) = 1$	$z_3x_{10} = \bar{x}_6 + \bar{b}_9 + \bar{x}_{15}$
$(x_{15}b_5, z_3) = 1$	$z_3b_5 = \bar{x}_{15}$
$(x_{15}c_5, z_3) = 1$	$z_3c_5 = \bar{x}_{15}$
$(x_6c_8, z_3) = (x_{15}c_8, z_3) = (z_3\bar{z}_3, c_8) = 1$	$z_3c_8 = \bar{x}_6 + \bar{x}_{15} + z_3$
$(b_3x_9, z_3) = (b_9x_9, z_3) = (x_{15}x_9, z_3) = 1$	$z_3x_9 = \bar{b}_3 + \bar{b}_9 + \bar{x}_{15}$
$(b_9c_3, z_3) = 1$	$z_3c_3 = b_9$
$(s_6\bar{d}_3, z_3) = (d_3y_3, \bar{z}_3) = 1$	$z_3d_3 = s_6 + y_3$
$(\bar{b}_3c_9, \bar{z}_3) = (\bar{b}_9c_9, \bar{z}_3) = (x_{15}\bar{c}_9, z_3) = 1$	$z_3y_{15} = b_3 + b_9 + x_{15}$
$(x_6\bar{y}_{15}, z_3) = (b_9\bar{y}_{15}, z_3) = 1, (x_{15}\bar{y}_{15}, z_3) = 2$	$z_3y_{15} = x_6 + b_9 + 2x_{15}$
$(s_6y_{15}, z_3) = (d_9y_{15}, z_3) = 1, (t_{15}y_{15}, z_3) = 2$	$z_3\bar{y}_{15} = s_6 + 2t_{15} + d_9$

2.8 Structure of NITA Generated by b_3 and Satisfying $b_3^2 = c_3 + b_6$, $c_3 \neq b_3, \bar{b}_3$, $(b_3b_8, b_3b_8) = 3$ and c_3 Real

In this section we discuss NITA satisfying the following hypothesis.

Hypothesis 2.4 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ of degree 3 and without non-identity basis element of degree 1 or 2, such that*:

$$\bar{b}_3b_3 = 1 + b_8$$

and

$$b_3^2 = c_3 + b_6, \quad c_3 = \bar{c}_3$$

where $b_i, c_i, d_i \in B$ are of degree i.

Lemma 2.25 *Let (A, B) satisfy Hypothesis 2.4, then*

1) $c_3^2 = 1 + b_8$,
2) $\bar{b}_3c_3 = b_3 + x_6, x_6 \neq \bar{x}_6 \in B$,
3) $b_8b_3 = x_6 + b_6\bar{b}_3$,
4) $b_8 + b_8^2 = c_3\bar{b}_6 + b_6c_3 + b_6\bar{b}_6$,

5) $\bar{b}_6c_3 = x_6\bar{b}_3$,
6) $c_3b_8 = b_6 + x_6b_3$,
7) $(b_8b_3, b_8b_3) = 3, 4$.

Proof By Hypothesis 2.4, we have $(\bar{b}_3\bar{c}_3, b_3) = (c_3, b_3^2) = 1$. So $(\bar{b}_3\bar{b}_3, c_3^2) = (\bar{b}_3c_3, \bar{b}_3c_3) \geq 2$, hence

$$c_3^2 = 1 + b_8. \tag{1}$$

Furthermore, we have

$$\bar{b}_3c_3 = b_3 + x_6. \tag{2}$$

If $x_6 = \bar{x}_6$, then $1 = (\bar{b}_3c_3, b_3c_3) = (c_3^2, b_3^2)$, a contradiction to Hypothesis 2.4. By the associative law $(\bar{b}_3b_3)b_3 = \bar{b}_3b_3^2$ and Hypothesis 2.4, one concludes that $b_3 + b_8b_3 = c_3\bar{b}_3 + b_6\bar{b}_3$. So $b_8b_3 = x_6 + b_6\bar{b}_3$ holds by (2). By the associative law $(\bar{b}_3b_3)^2 = \bar{b}_3^2b_3^2$ and Hypothesis 2.4 and (1), we see that

$$1 + 2b_8 + b_8^2 = (c_3 + \bar{b}_6)(c_3 + b_6),$$

$$1 + 2b_8 + b_8^2 = c_3^2 + c_3b_6 + \bar{b}_6c_3 + \bar{b}_6b_6,$$

$$b_8^2 + b_8 = c_3b_6 + \bar{b}_6c_3 + \bar{b}_6b_6.$$

By the associative law $(\bar{b}_6b_3)b_3 = \bar{b}_3b_3^2$ and Hypothesis 2.4 and (2), it holds that $b_3 + b_8b_3 = c_3\bar{b}_3 + b_6\bar{b}_3$; hence

$$b_8b_3 = x_6 + b_6\bar{b}_3. \tag{3}$$

By Hypothesis 2.4, $(b_6\bar{b}_3, b_3) = (b_6, b_3^2) = 1$, so by (3)

$$(b_8b_3, b_8b_3) \geq 3,$$

we shall show that $(b_8b_3, b_8b_3) = 3, 4$. By the associative law and Hypothesis 2.4 and (1) and (2), we have that $(\bar{b}_3^2)c_3 = \bar{b}_3(\bar{b}_3c_3)$, $c_3^2 + \bar{b}_6c_3 = b_3\bar{b}_3 + x_6\bar{b}_3$; thus

$$\bar{b}_6c_3 = x_6\bar{b}_3. \tag{4}$$

By (3), $(x_6\bar{b}_3, b_8) = (x_6, b_3b_8) = 1$ holds. So $(x_6\bar{b}_3, x_6\bar{b}_3) \geq 2$. We shall show that $(x_6\bar{b}_3, x_6\bar{b}_3) = 2, 3$. If $(x_6\bar{b}_3, x_6\bar{b}_3) = 4$, then $x_6\bar{b}_3 = b_8 + g_3 + w_3 + v_4$; hence

$$b_3g_3 = x_6 + w_3, \qquad b_3w_3 = x_6 + k_3, \qquad b_3v_4 = x_6 + z_6.$$

Therefore $1 \leq (b_3g_3, b_3w_3) = (b_3\bar{b}_3, \bar{g}_3w_3)$. We obtain $w_3 = g_3$ by Hypothesis 2.4, hence $2 = (x_6\bar{b}_3, w_3) = (x_6, b_3w_3)$, so that $b_3w_3 = 2x_6$, a contradiction by (4). Now we can state that $(\bar{b}_6c_3, \bar{b}_6c_3) = (\bar{b}_6b_6, c_3^2) = 2, 3$. By (1), it follows that $(\bar{b}_6b_6, b_3\bar{b}_3) = (b_6\bar{b}_3, b_6\bar{b}_3) = 2$ or 3. Therefore $(b_8b_3, b_8b_3) = 3, 4$ by (3).

By the associative law $(\bar{b}_3c_3)b_3 = c_3(\bar{b}_3b_3)$ and Hypothesis 2.4, we get $b_3^2 + x_6b_3 = c_3 + c_3b_8$ and $c_3b_8 = b_6 + x_6b_3$. ∎

Lemma 2.26 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8b_3, b_8b_3) = 3$, *then we get*:

1) $b_6\bar{b}_3 = b_3 + x_{15}$,
2) $b_8b_3 = x_6 + b_3 + x_{15}$, $x_6 \neq \bar{x}_6$,
3) $\bar{x}_6b_3 = b_8 + x_{10} = b_6c_3$, $x_6 \neq b_6$, $x_{10} = \bar{x}_{10}$,
4) $x_6c_3 = \bar{b}_3 + \bar{x}_{15}$,
5) $b_3x_6 = c_3 + y_{15}$,
6) $b_8c_3 = b_6 + c_3 + y_{15}$, $b_6 = \bar{b}_6$, $y_{15} = \bar{y}_{15}$.

Proof By our assumption in this lemma and part 3) in Lemma 2.25, and Hypothesis 2.4:

$$b_6\bar{b}_3 = b_3 + x_{15}, \tag{1}$$

then

$$b_8b_3 = x_6 + b_3 + x_{15}$$

by (1),

$$(b_6\bar{b}_6, b_3\bar{b}_3) = (b_6\bar{b}_3, b_6\bar{b}_3) = 2.$$

Then by 1) in Lemma 2.25,

$$2 = (b_6\bar{b}_6, c_3^2) = (\bar{b}_6c_3, \bar{b}_6c_3). \tag{2}$$

By 6) in Lemma 2.25,

$$(\bar{b}_6c_3, b_8) = (c_3b_8, b_6) = 1,$$

so by (2)

$$\bar{b}_6c_3 = b_8 + x_{10}. \tag{3}$$

So by 5) in Lemma 2.25

$$x_6\bar{b}_3 = b_8 + x_{10}; \tag{4}$$

so by (1) we get that $x_6 \neq b_6$. By the associative law $(c_3^2)\bar{b}_3 = (c_3\bar{b}_3)c_3$ and 1), 2) in Lemma 2.25 and (2), it follows that

$$\bar{b}_3 + \bar{b}_3b_8 = b_3c_3 + x_6c_3,$$

$$\bar{b}_3 + \bar{x}_6 + \bar{b}_3 + \bar{x}_{15} = \bar{b}_3 + \bar{x}_6 + x_6c_3,$$

so

$$x_6c_3 = \bar{b}_3 + \bar{x}_{15}. \tag{5}$$

By (4) and 2) in Lemma 2.25:

$$2 = (\bar{x}_6b_3, x_6b_3) = (x_6b_3, x_6b_3), \qquad (b_3x_6, c_3) = (x_6, \bar{b}_3c_3) = 1.$$

Therefore

$$b_3x_6 = c_3 + y_{15}. \tag{6}$$

By the associative law $(b_3\bar{b}_3)c_3 = (\bar{b}_3c_3)b_3$ and 2) in Lemma 2.25 and Hypothesis 2.4, one has

$$c_3 + b_8c_3 = b_3^2 + x_6b_3,$$
$$c_3 + b_8c_3 = c_3 + b_6 + c_3 + y_{15},$$
$$b_8c_3 = b_6 + c_3 + y_{15}.$$

b_8, c_3 are reals, then

$$b_6 = \bar{b}_6, \qquad y_{15} = \bar{y}_{15}.$$

By (3), we have $x_{10} = \bar{x}_{10}$. □

Lemma 2.27 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8b_3, b_8b_3) = 3$, *then Lemma* 2.26 *holds and we get*:

1) $x_{10}b_3 = x_{15} + x_6 + x_9$, $x_{10} = \bar{x}_{10}$,
2) $x_6b_8 = 2x_{15} + b_3 + x_6 + x_9$,
3) $\bar{x}_{15}c_3 = 2x_{15} + x_6 + x_9$,
4) $y_{15}\bar{b}_3 = 2x_{15} + x_6 + x_9$,
5) $x_6b_6 = 2\bar{x}_6 + \bar{x}_{15} + \bar{x}_9$,
6) $c_3x_{10} = b_6 + y_{15} + y_9$,
7) $x_6^2 = 2b_6 + y_{15} + y_9$,
8) $x_{15}b_3 = 2y_{15} + b_6 + y_9$,
9) $b_8b_6 = 2y_{15} + b_6 + c_3 + y_9$, $y_9 = \bar{y}_9$,
10) $b_8x_{15} = 4x_{15} + 2x_6 + 2x_9 + b_3 + y_9\bar{b}_3$.

Proof By the associative law $(b_3^2)b_6 = (b_3b_6)b_3$ and Hypothesis 2.4 and 1), 3) in Lemma 2.26, we get

$$c_3b_6 + b_6^2 = \bar{b}_3b_3 + \bar{x}_{15}b_3,$$
$$b_8 + x_{10} + b_6^2 = 1 + b_8 + \bar{x}_{15}b_3.$$

Hence

$$x_{10} + b_6^2 = 1 + \bar{x}_{15}b_3. \tag{1}$$

By 3) in Lemma 2.26,

$$(b_3x_{10}, x_6) = (x_{10}, \bar{b}_3x_6) = 1,$$

hence by (1),

$$(x_{10}b_3, x_{15}) = (b_3\bar{x}_{15}, x_{10}) = 1.$$

So

$$x_{10}b_3 = x_{15} + x_6 + x_9,$$

where x_9 is a linear combination of B. We shall show that $x_9 \in B$. If $x_{10}b_3$ has an element of degree 3, then

$$1 = (x_{10}b_3, y_3) = (x_{10}; \bar{b}_3 y_3),$$

hence $\bar{b}_3 y_3 = x_{10}$, a contradiction. So

$$x_{10}b_3 = x_{15} + x_6 + x_9, \tag{2}$$

where $x_9 \in B$. By the associative law and Hypothesis 2.4 and 2), 3) in Lemma 2.26, we get

$$\begin{aligned}
(b_3\bar{x}_6)\bar{b}_3 &= \bar{x}_6(b_3\bar{b}_3),\\
b_8\bar{b}_3 + x_{10}\bar{b}_3 &= \bar{x}_6 + \bar{x}_6 b_8,\\
\bar{x}_6 + \bar{x}_6 b_8 &= \bar{x}_6 + \bar{b}_3 + \bar{x}_{15} + \bar{x}_{15} + \bar{x}_6 + \bar{x}_9.
\end{aligned}$$

Hence

$$\bar{x}_6 b_8 = 2\bar{x}_{15} + \bar{b}_3 + \bar{x}_6 + \bar{x}_9. \tag{3}$$

By the associative law $(\bar{b}_3c_3)b_8 = \bar{b}_3(b_8c_3)$ and 1), 2), 6) in Lemma 2.26 and 2) in Lemma 2.25, we obtain

$$\begin{aligned}
b_3b_8 + x_6b_8 &= b_6\bar{b}_3 + c_3\bar{b}_3 + y_{15}\bar{b}_3,\\
x_6 + b_3 + x_{15} + 2x_{15} + b_3 + x_6 + x_9 &= b_3 + x_{15} + b_3 + x_6 + y_{15}\bar{b}_3.
\end{aligned}$$

So

$$y_{15}\bar{b}_3 = 2x_{15} + x_6 + x_9. \tag{4}$$

By the associative law $(\bar{b}_3c_3)b_8 = (\bar{b}_3b_8)c_3$ and 1), 2), 6) in Lemma 2.26 and in Lemma 2.25 and (3), the following equations hold:

$$\begin{aligned}
b_3b_8 + x_6b_8 &= \bar{x}_6c_3 + \bar{b}_3c_3 + \bar{x}_{15}c_3,\\
x_6 + b_3 + x_{15} + 2x_{15} + b_3 + x_6 + x_9 &= b_3 + x_{15} + b_3 + x_6 + \bar{x}_{15}c_3.
\end{aligned}$$

So

$$\bar{x}_{15}c_3 = 2x_{15} + x_6 + x_9. \tag{5}$$

By the associative law $(\bar{b}_3c_3)b_6 = c_3(\bar{b}_3b_6)$ and 2) in Lemma 2.25 and 1) in Lemma 2.26 and (5), one should have

$$\begin{aligned}
b_3b_6 + x_6b_6 &= b_3c_3 + x_{15}c_3,\\
\bar{b}_3 + \bar{x}_{15} + x_6b_6 &= \bar{b}_3 + \bar{x}_6 + 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9,\\
x_6b_6 &= 2\bar{x}_6 + \bar{x}_{15} + \bar{x}_9. \qquad (6)
\end{aligned}$$

By the associative law and 5), 4) in Lemma 2.26 and 1) in Lemma 2.25, we get $(b_3x_6)c_3 = b_3(x_6c_3)$, $c_3^2 + y_{15}c_3 = b_3\bar{b}_3 + b_3\bar{x}_{15}$ and $y_{15}c_3 = b_3\bar{x}_{15}$. Now by (2) it follows that $(\bar{x}_{15}b_5, x_{10}) = (b_3x_{10}, x_{15}) = 1$. So

$$1 = (y_{15}c_3, x_{10}) = (y_{15}, c_3x_{10}). \tag{7}$$

By 3) in Lemma 2.26,

$$(c_3x_{10}, b_6) = (c_3b_6, x_{10}) = 1. \tag{8}$$

By (2) and 1) in Lemma 2.25, we have $(c_3x_{10}, c_3x_{10}) = (c_3^2, x_{10}^2) = (b_3\bar{b}_3, x_{10}^2) = (b_3x_{10}, b_3x_{10}) = 3$. Therefore by (7) and (8), the following equation holds:

$$c_3x_{10} = b_6 + y_{15} + y_9, \tag{9}$$

where $y_9 \in B$. By the associative law and 3), 5), 6) in Lemma 2.26 and 2) in Lemma 2.25 and (9), we have that $(\bar{x}_6b_3)c_3 = \bar{x}_6(b_3c_3)$ and

$$b_8c_3 + x_{10}c_3 = \bar{x}_6\bar{b}_3 + \bar{x}_6^2,$$

$$b_6 + c_3 + y_{15} + b_6 + y_{15} + y_9 = c_3 + y_{15} + \bar{x}_6^2,$$

so

$$x_6^2 = 2b_6 + y_{15} + y_9. \tag{10}$$

By the associative law and 3), 4), 6) in Lemma 2.26 and Hypothesis 2.4 and (9), we have $(\bar{x}_6b_3)c_3 = (\bar{x}_6c_3)b_3$. Hence

$$b_8c_3 + x_{10}c_3 = b_3^2 + x_{15}b_3,$$

$$b_6 + c_3 + y_{15} + b_6 + y_{15} + y_9 = c_3 + b_6 + x_{15}b_3,$$

so

$$x_{15}b_3 = 2y_{15} + b_6 + y_9. \tag{11}$$

By the associative law and 2), 5), 6) in Lemma 2.26 and (11) and the Hypothesis 2.4, $b_3(b_8b_3) = b_8(b_3^2)$ holds, from which we conclude

$$x_6b_3 + b_3^2 + x_{15}b_3 = b_8c_3 + b_8b_6,$$

$$c_3 + y_{15} + c_3 + b_6 + 2y_{15} + b_6 + y_9 = b_6 + c_3 + y_{15} + b_8b_6,$$

so

$$b_8b_6 = 2y_{12} + b_6 + c_3 + y_9. \tag{12}$$

By the associative law and 1), 2) in Lemma 2.26 and (4) and 2) in Lemma 2.25:

$$(b_6\bar{b}_3)b_8 = \bar{b}_3(b_6b_8), \qquad b_3b_8 + x_{15}b_8 = 2y_{12}\bar{b}_3 + b_6\bar{b}_3 + c_3\bar{b}_3 + y_9\bar{b}_3.$$

□

Lemma 2.28 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8 b_3, b_3 b_8) = 3$, *then Lemma* 2.27 *holds and* $x_9 c_3 = \bar{x}_{15} + d_3 + d_9$, $d_3 \neq c_3, b_3, \bar{b}_3$.

Proof By 3) in Lemma 2.27,

$$(c_3 x_9, \bar{x}_{15}) = (x_9, c_3 \bar{x}_{15}) = 1,$$

so

$$c_3 x_9 = \bar{x}_{15} + \alpha, \tag{1}$$

where α is a linear combination at B. We shall show that α has neither degree 4 nor two elements of degree 3. If $x_4 \in \alpha$, then $1 \leq (c_3 x_9, x_4) = (x_9, c_3 x_4)$, hence $1 = (x_9, c_3 x_4)$. So

$$c_3 x_4 = x_9 + m_3. \tag{2}$$

So

$$(c_3 m_3, x_4) = (m_3, c_3 x_4) = 1,$$

hence

$$c_3 m_3 = y_4 + k_5. \tag{3}$$

Hence

$$2 = (c_3 m_3, c_3 m_3) = (c_3^2, \bar{m}_3 m_3),$$

therefore $\bar{m}_3 m_3 = 1 + b_8$. So by the associative law and (3) and 6) in Lemma 2.26, we get $(\bar{m}_3 m_3) c_3 = \bar{m}_3 (m_3 c_3)$, $c_3 + c_3 b_8 = y_4 \bar{m}_3 + k_5 \bar{m}_3$. Hence

$$2c_3 + b_6 + y_{15} = y_4 \bar{m}_3 + k_5 \bar{m}_3. \tag{4}$$

By (3)

$$(y_4 \bar{m}_3, c_3) = (y_4, m_3 c_3) = 1, \qquad (k_5 \bar{m}_3, c_3) = (k_5, m_3 c_3) = 1;$$

hence by (4), we get a contradiction. If $x_3, y_3 \in \alpha$, then by (1),

$$1 = (c_3 x_9, x_3) = (x_9, c_3 x_3),$$

$$1 = (c_3 x_9, y_3) = (x_9, c_3 y_3).$$

So

$$c_3 x_3 = x_9, \qquad c_3 y_3 = x_9,$$

hence

$$1 = (c_3 x_3, c_3 y_3) = (c_3^2, \bar{x}_3 y_3).$$

Therefore by 1) in Lemma 2.25, $\bar{x}_3 y_3 = b_8 + 1$, so $x_3 = y_3$; then by (1) $2 = (c_3 x_9, x_3) = (x_9, c_3 x_3)$, a contradiction. Therefore by (1) one of the following holds: a) $\alpha = d_3 + d_9$ or b) $\alpha = d_5 + d_7$ or c) $\alpha = d_6 + k_6$ or d) $\alpha = 2d_6$ or e) $\alpha = d_{12}$.

We shall show that a) is the only possibility. Assume b) holds, then by (1) we get $c_3x_9 = \bar{x}_{15} + d_5 + d_7$. So $(c_3d_5, x_9) = (d_5, c_3x_9) = 1$. Therefore $c_3d_5 = x_9 + \beta$, where $\beta = 2x_3$ or $\beta = x_3 + y_3$ or $\beta = m_6$. We shall show that these possibilities are impossible if $\beta = 2x_3$, then $c_3d_5 = x_9 + 2x_3$. Therefore $(c_3x_3, d_5) = (x_3, c_3d_5) = 2$, so $c_3x_3 = 2d_5$, a contradiction. If $\beta = x_3 + y_3$, then $c_3d_5 = x_9 + x_3 + y_3$. Hence $(c_3x_3, d_5) = (x_3, c_3d_5) = 1$, $(c_3y_3, d_5) = (y_3, c_3d_5) = 1$, so $1 \leq (c_3x_3, c_3y_3) = (c_3^2, \bar{x}_3y_3)$. Then by 1) in Lemma 2.25 $x_3 = y_3$ a contradiction. If $\beta = m_6$, then $c_3d_5 = x_9 + m_6$, so by (1) and (6), we have $(c_3^2)d_5 = c_3(c_3d_5), d_5 + b_8d_5 = x_9c_3 + m_6c_3$ and $b_8d_5 = \bar{x}_{15} + d_7 + m_6c_3$. So $(b_8x_{15}, \bar{d}_5) = (b_8d_5, \bar{x}_{15}) = 1$, hence by 10) in Lemma 2.27: $(y_9\bar{b}_3, \bar{d}_5) = 1$ and by 8) in Lemma 2.27: $(y_9\bar{b}_3, x_{15}) = (y_9, b_3x_{15}) = 1$, so $y_9\bar{b}_3 = x_{15} + \bar{d}_5 + k_7$, where k_7 is a linear combination. When $k_7 = e_3 + m_4$, then $y_9\bar{b}_3 = x_{15} + \bar{d}_5 + e_3 + t_3$, hence $(b_3m_4, y_9) = (m_4, \bar{b}_3y_9) = 1$, so $b_3m_4 = y_9 + t_3$, hence $(t_3\bar{b}_3, m_4) = (t_3, b_3m_4) = 1$, therefore

$$t_3\bar{b}_3 = m_4 + t_5, \tag{5}$$

so $2 = (t_3\bar{b}_3, t_3\bar{b}_3) = (t_3\bar{t}_3, b_3\bar{b}_3)$, so by Hypothesis 2.4 $\bar{t}_3t_3 = 1 + b_8$. Hence by the associative law and (5) we obtain:

$$(\bar{t}_3t_3)b_3 = (\bar{t}_3b_3)t_3, \qquad b_3 + b_8b_3 = \bar{m}_4t_3 + \bar{t}_5t_3.$$

By 2) in Lemma 2.26, it follows that $2b_3 + x_6 + x_{15} = \bar{m}_4t_3 + \bar{t}_5t_3$, so $\bar{m}_4t_3 = 2b_3 + x_6$. Hence $5 = (\bar{m}_4t_3, \bar{m}_4t_3) = (\bar{m}_4m_4, \bar{t}_5t_3)$, and then $\bar{m}_4m_4 = 1 + 4b_8$, a contradiction. If

$$y_9\bar{b}_3 = x_{15} + \bar{d}_5 + k_7, \tag{6}$$

where $k_7 \in B$, then $(b_3\bar{d}_5, y_9) = (\bar{d}_5, \bar{b}_3y_9) = 1$; hence $b_3\bar{d}_5 = y_9 + \gamma$, where $\gamma = v_6$ or $\gamma = v_3 + w_3$. When $b_3\bar{d}_5 = y_9 + v_3 + w_3$, then $(b_3\bar{v}_3, d_5) = (b_3\bar{d}_5, v_3) = 1$. Also $(b_3\bar{m}_3, d_5) = 1$, so $1 \leq (b_3\bar{v}_3, b_3\bar{m}_3) = (b_3\bar{b}_3, v_3\bar{m}_3)$, by Hypothesis 2.4, $v_3 = w_3$, then

$$2 = (b_3\bar{d}_5, v_3) = (\bar{d}_5, \bar{b}_3v_3),$$

so

$$\bar{b}_3v_3 = 2\bar{d}_5,$$

a contradiction. When $b_3\bar{d}_5 = y_9 + v_6$, then by the associative law and (6) and our assumption:

$$\begin{aligned}(b_3^2)\bar{d}_5 &= b_3(b_3\bar{d}_5),\\ \bar{d}_5c_3 + \bar{d}_5b_6 &= y_9b_3 + v_6b_3,\\ \bar{x}_9 + \bar{m}_6 + \bar{d}_5b_6 &= \bar{x}_{15} + d_5 + \bar{k}_7 + v_6b_3.\end{aligned}$$

Hence

$$v_6b_3 = \bar{x}_9 + \bar{m}_6 + t_3, \tag{7}$$

so $(t_3\bar{b}_3, v_6) = (t_3, b_3 v_6) = 1$, hence $t_3\bar{b}_3 = v_6 + p_3$. Therefore $2 = (\bar{b}_3 t_3, \bar{b}_3 t_3) = (t_3\bar{t}_3, b_3\bar{b}_3)$, Now by the Hypothesis 2.4 we get

$$t_3\bar{t}_3 = 1 + b_8. \tag{8}$$

Again by the associative law:

$$b_3(t_3\bar{t}_3) = t_3(b_3\bar{t}_3), \qquad b_3 + b_3 b_8 = t_3\bar{v}_6 + t_3\bar{p}_3;$$

and by 2) in Lemma 2.26, we have $2b_3 + x_6 + x_{15} = t_3\bar{v}_6 + t_3\bar{p}_3$, but $(t_3\bar{v}_6, b_3) = (t_3\bar{b}_3, v_6) = 1$, $(t_3\bar{p}_3, b_3) = (t_3\bar{b}_3, p_3) = 1$. This means $\bar{v}_6 t_3 = b_3 + x_{15}$, $t_3\bar{p}_3 = b_3 + x_6$. Then by (7) and (8), it follows that $2 = (\bar{v}_6 t_3, \bar{v}_6 t_3) = (\bar{v}_6 v_6, \bar{t}_3 t_3) = (\bar{v}_6 v_6, \bar{b}_3 b_3) = (v_6 b_3, v_6 b_5) = 3$, a contradiction.

Assume c) holds. Then by (1) one has that $c_3 x_9 = \bar{x}_{15} + d_6 + k_6$, so $(c_3 d_6, x_9) = (d_6, c_3 x_9) = 1$; hence $c_3 d_6 = x_9 + \gamma$, where $\gamma = m_9$ or $\gamma = m_3 + v_3 + k_3$ or $\gamma = v_3 + y_6$ or $\gamma = v_4 + v_5$ or $\gamma = 2v_3 + m_3$ or $\gamma = 3v_3$. If $\gamma = m_9$, then $c_3 d_6 = x_9 + m_9$, so by the associative law and 1) in Lemma 2.26:

$$\begin{aligned}
(c_3^2)d_6 &= (c_3 d_6)c_3,\\
d_6 + d_6 b_8 &= x_9 c_3 + m_9 c_3,\\
d_6 + d_6 b_8 &= \bar{x}_{15} + d_6 + k_6 + m_9 c_3,\\
d_6 b_8 &= \bar{x}_{15} + k_6 + m_9 c_3.
\end{aligned} \tag{9}$$

We shall prove that $d_6 \neq x_6, \bar{x}_6, k_6 \neq x_6, \bar{x}_6$. By 4) in Lemma 2.26 and our assumption if $d_6 = x_6$,

$$1 = (c_3 x_9, x_6) = (x_9, c_3 x_6) = 0, \qquad 1 = (c_3 x_9, \bar{x}_6) = (c_3 x_6, \bar{x}_9) = 0,$$

a contradiction, so $d_6 \neq x_6, \bar{x}_6$. In the same way $k_6 \neq x_6, \bar{x}_6$. By (9), $(b_8 x_{15}, \bar{d}_6) = (b_8 d_6, \bar{x}_{15}) = 1$, in the same way $(b_8 x_{15}, \bar{k}_6) = (b_8 k_6, \bar{x}_{15}) = 1$, then by 8), 10) in Lemma 2.27, $(y_9\bar{b}_3, x_{15}) = (y_9, b_3 x_{15}) = 1$, $(y_9\bar{b}_3, \bar{d}_6) = 1$, $(y_9\bar{b}_3, \bar{k}_6) = 1$. So

$$y_9\bar{b}_3 = x_{15} + \bar{d}_6 + \bar{k}_6. \tag{10}$$

So by the associative law and 2) in Lemma 2.25:

$$\begin{aligned}
(\bar{b}_3 c_3)y_9 &= c_3(\bar{b}_3 y_9),\\
b_3 y_9 + y_9 x_6 &= x_{15}c_3 + \bar{d}_6 c_3 + \bar{k}_6 c_3,\\
\bar{x}_{15} + d_6 + k_6 + y_9 x_6 &= 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + \bar{x}_9 + \bar{m}_9 + \bar{k}_6 c_3,\\
d_6 + k_6 + y_9 x_6 &= \bar{x}_{15} + \bar{x}_6 + 2\bar{x}_9 + \bar{m}_9 + \bar{k}_6 c_3.
\end{aligned}$$

And by 3) in Lemma 2.27, (10) and the fact that $c_3 d_6 = x_9 + m_9$, since $k_6, d_6 \neq x_6$, then $1 \leq (\bar{k}_6 c_3, d_6) = (\bar{k}_6, c_3 d_6)$, a contradiction to the assumption that $c_3 d_6 = x_9 + m_9$. If $\gamma = m_3 + v_3 + k_3$, then $c_3 d_3 = x_9 + m_3 + v_3 + k_3$, so $(c_3 m_3, d_3) =$

$(m_3, c_3d_3) = 1$, $(c_3v_3, d_3) = (v_3, c_3d_3) = 1$. Hence $1 \le (c_3m_3, c_3v_3) = (m_3\bar{v}_3, c_3^2)$, then by 1) in Lemma 2.25 $m_3 = v_3$, a contradiction. If $\gamma = v_3 + y_6$, then $c_3d_6 = x_9 + v_3 + y_6$, so $(c_3v_3, d_6) = (v_3, c_3d_6) = 1$. Hence $c_3v_3 = d_6 + t_3$, so by the associative law and 1) in Lemma 2.25, we obtain equations $c_3^2v_3 = c_3(c_3v_3)$ and

$$v_3 + v_3b_8 = d_6c_3 + t_3c_3,$$

$$v_3 + v_3b_8 = x_9 + v_3 + y_9 + t_3c_3.$$

Hence $v_3b_8 = x_9 + y_6 + t_3c_3$. Since $(t_3c_3, v_3) = (t_3, c_3v_3) = 1$, then $(v_3b_8, v_3b_8) \ge 4$. On the other hand, $2 = (c_3v_3, c_3v_3) = (c_3^2, \bar{v}_3v_3)$, so by 1) in Lemma 2.25, $\bar{v}_3v_3 = 1 + b_8 = \bar{b}_3b_3$. Therefore $(v_3b_8, v_3b_8) = (b_8^2, v_3\bar{v}_3) = (b_8^2, \bar{b}_3b_3) = (b_8b_3, b_8b_3)$, by 2) in Lemma 2.26, $(b_8b_3, b_8b_3) = 3$, a contradiction. If $\gamma = v_4 + v_5$, then $c_3d_6 = x_4 + v_4 + v_5$, so by the associative law and 1) in Lemma 2.25:

$$c_3^2d_6 = c_3(c_3d_6),$$

$$d_6 + d_6b_8 = x_9c_3 + v_4c_3 + v_5c_3.$$

Hence by our assumption we have $d_6 + d_6b_8 = \bar{x}_{15} + d_6 + k_6 + v_4c_3 + v_5c_3$, thus

$$d_6b_8 = \bar{x}_{15} + k_6 + v_4c_3 + v_5c_3. \tag{11}$$

We shall show that $d_6 \ne x_6, \bar{x}_6, k_6 \ne x_6, \bar{x}_6$. By 4) in Lemma 2.26, if $d_6 = x_6$, or $d_6 = \bar{x}_6$, then $1 = (c_3x_9, x_6) = (x_9, c_3x_6) = 0$, $1 = (c_3x_9, \bar{x}_6) = (c_3x_6, \bar{x}_9) = 0$, a contradiction. In the same way $k_6 \ne x_6, \bar{x}_6$. By (12) and 10) in Lemma 2.27,

$$(b_8x_{15}, \bar{d}_6) = (b_8d_6, \bar{x}_{15}) = 1.$$

In the same way $(b_8x_{15}, \bar{k}_6) = 1$ and by 10), 8) in Lemma 2.27 $(y_9\bar{b}_3, \bar{k}_6) = 1$, $(y_9\bar{b}_3, \bar{d}_6) = 1$, $(y_9\bar{b}_3, x_{15}) = 1$, so $y_9\bar{b}_3 = x_{15} + \bar{k}_6 + \bar{d}_6$. Hence by the associative law an 2) in Lemma 2.25:

$$(\bar{b}_3c_3)y_9 = (\bar{b}_3y_9)c_3,$$

$$b_3y_9 + x_6y_9 = x_{15}c_3 + \bar{k}_6c_3 + \bar{d}_6c_3,$$

so by 3) in Lemma 2.27 and our assumption:

$$\bar{x}_{15} + k_6 + d_6 + x_6y_9 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + \bar{k}_6c_3 + \bar{d}_6c_3,$$

$$k_6 + d_6 + x_6y_9 = \bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + \bar{v}_4 + \bar{v}_5 + \bar{x}_9 + \bar{k}_6c_3.$$

Hence $1 = (\bar{k}_6c_3, d_6) = (\bar{k}_6, c_3d_6)$, a contradiction to our assumption. If $\gamma = 2v_3 + m_3$, or $\gamma = 3v_3$, then $c_3d_6 = x_9 + 2v_3 + m_3$ or $c_3d_6 = x_9 + 3v_3$. In two cases, it always follows that $(c_3v_3, d_6) = (v_3, c_3d_6) \ge 2$, a contradiction. Assume then by (1) $c_3x_9 = \bar{x}_{15} + 2d_6$. Hence $(c_3d_6, x_9) = (d_6, c_3x_9) = 2$, therefore $c_3d_6 = 2x_9$. So by the associative law and 1) in Lemma 2.25, we have that $(c_3^2)d_6 = c_3(c_3d_6)$

$$\begin{aligned} d_6 + d_6 b_8 &= 2x_9 c_3 \\ &= 2\bar{x}_{15} + 4d_6, \\ d_6 b_8 &= 2\bar{x}_{15} + 3d_6. \end{aligned}$$

So

$$(b_8 x_{15}, \bar{d}_6) = (b_8 d_6, \bar{x}_{15}) = 2. \tag{12}$$

By 4) in Lemma 2.26, if $x_6 = d_6$, then $2 = (c_3 x_9, x_6) = (x_9, c_3 x_6) = 0$, then $d_6 \neq x_6$, also $d_6 \neq \bar{x}_6$. Then by (12) and 8), 10) in Lemma 2.27: $(y_9 \bar{b}_3, x_{15}) = (y_9, b_3 x_{15}) = 1$, $(y_9 \bar{b}_3, \bar{d}_6) = 2$. So $y_9 \bar{b}_3 = x_{15} + 2\bar{d}_6$. Hence by the associative law and 2) in Lemma 2.25:

$$\begin{aligned} y_9(\bar{b}_3 c_3) &= (y_9 \bar{b}_3) c_3, \\ y_9 b_3 + y_9 x_6 &= x_{15} c_3 + 2\bar{d}_6 c_3, \end{aligned}$$

then by 3) in Lemma 2.27, $\bar{x}_{15} + 2d_6 + y_9 x_6 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + 2\bar{x}_9$, a contradiction. Assume e) then by 1), $c_3 x_9 = \bar{x}_{15} + d_{12}$. Then by 1) in Lemma 2.25, $2 = (x_9 c_3, x_9 c_3) = (x_9 \bar{x}_9, c_3^2) = (x_9 \bar{x}_9, b_3 \bar{b}_3) = (x_9 \bar{b}_3, x_9 \bar{b}_3)$. So by 1) in Lemma 2.27, $(x_9 \bar{b}_3, x_{10}) = (x_9, b_3 x_{10}) = 1$, hence $x_9 \bar{b}_3 = x_{10} + x_{17}$(13). By the associative law and 1) in Lemma 2.25: $(c_3^2) x_9 = (b_3 \bar{b}_3) x_9 = b_3(\bar{b}_3 x_9) = (c_3 x_9) c_3$, $x_{10} b_3 + b_3 x_{17} = \bar{x}_{15} c_3 + d_{12} c_3$. By 1), 3) and Lemma 2.27, $x_{15} + x_6 + x_9 + b_3 x_{17} = 2x_{15} + x_6 + x_9 + d_{12} c_3$, $b_3 x_{17} = x_{15} + d_{12} c_3$. Then $(x_{15} \bar{b}_3, x_{17}) = (x_{15}, b_3 x_{17}) = 1$ and by 2) in Lemma 2.26 and 1), 8) in Lemma 2.27:

$$\begin{aligned} (\bar{b}_3 x_{15}, b_8) &= (x_{15}, b_3 b_8) = 1, \\ (\bar{b}_3 x_{15}, x_{10}) &= (x_{15}, b_3 x_{10}) = 1, \\ (\bar{b}_3 x_{15}, \bar{b}_3 x_{15}) &= (x_{15} b_3, x_{15} b_3) = 6. \end{aligned}$$

Therefore the only possibility is $\bar{b}_3 x_{15} = b_8 + x_{10} + x_{17} + m_3 + z_3 + z_4$. So $(b_3 m_3, x_{15}) = (m_3, \bar{b}_3 x_{15}) = 1$, a contradiction. Hence the only possibility is that $\alpha = d_3 + d_9$, hence by 1) $x_9 c_3 = \bar{x}_{15} + d_3 + d_9$. If $d_3 = c_3$, then $(c_3^2, x_9) = (c_3, c_3 x_9) = 1$, a contradiction to 1) in Lemma 2.25. If $d_3 = b_3$, then $(c_3 b_3, x_9) = (b_3, c_3 x_9) = 1$ a contradiction to 2) in Lemma 2.25 in the same way. If $d_3 = \bar{b}_3$, we get a contradiction so $d_3 \neq c_3, b_3, \bar{b}_3$. □

Lemma 2.29 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8 b_3, b_8 b_3) = 3$, *then Lemma* 2.28 *holds and we obtain the following*:

1) $d_3 c_3 = x_9$, $x_9 \neq \bar{x}_9$,
2) $d_3 b_8 = \bar{x}_{15} + \bar{x}_9$,
3) $b_3 \bar{d}_3 = y_9$, $y_9 = \bar{y}_9$, $y_9 \neq x_9$,
4) $x_9 c_3 = \bar{x}_{15} + d_3 + \bar{x}_9$,
5) $y_9 \bar{b}_3 = \bar{d}_3 + x_9 + x_{15}$,
6) $\bar{d}_3 b_6 = d_3 + \bar{x}_{15}$,

7) $b_8x_{15} = 5x_{15} + 3x_9 + 2x_6 + b_3 + \bar{d}_3$,
8) $x_9b_8 = 3x_{15} + 2x_9 + x_6 + \bar{d}_3$,
9) $y_{15}x_6 = 4\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6 + \bar{b}_3 + d_3$,
10) $b_6x_{15} = 4\bar{x}_{15} + \bar{x}_6 + 2\bar{x}_9 + \bar{b}_3 + d_3$,
11) $d_3y_{15} = 2x_{15} + x_6 + x_9$,
12) $x_9x_{15} = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3$,
13) $d_3b_8 = \bar{x}_{15} + \bar{x}_9$,
14) $\bar{d}_3d_3 = 1 + c_8$, $c_8 \neq b_8$,
15) $d_3^2 = b_6 + y_3$, $d_3 \neq c_3, b_3, \bar{b}_3$, $y_3 \neq b_3, c_3, \bar{b}_3$,
16) $b_3d_3 = z_9$, $z_9 \neq y_9, x_9, \bar{x}_9$, $z_9 = \bar{z}_9$,
17) $y_3^2 = 1 + c_8$.

Proof By Lemma 2.28, $(c_3d_3, x_9) = (d_3, c_3x_9) = 1$, hence

$$d_3c_3 = x_9. \tag{1}$$

By the associative law and 1) in Lemma 2.28, we have $d_3(c_3^2) = x_9c_3$ and $d_3 + d_3b_8 = \bar{x}_{15} + d_3 + d_9$, which leads to

$$d_3b_8 = \bar{x}_{15} + d_9. \tag{2}$$

So $(b_8x_{15}, \bar{d}_3) = (b_8d_3, \bar{x}_{15}) = 1$ and by 10) in Lemma 2.27, $(y_9\bar{b}_3, \bar{d}_3) = 1 = (y_9, b_3\bar{d}_3)$. So

$$b_3\bar{d}_3 = y_9. \tag{3}$$

By the associative law $(\bar{b}_3b_3)\bar{d}_3 = \bar{b}_3(b_3\bar{d}_3)$ and Hypothesis 2.4, one has $\bar{d}_3 + \bar{d}_3b_8 = y_9\bar{b}_3$. By (2), we get

$$y_9\bar{b}_3 = \bar{d}_3 + \bar{d}_9 + x_{15}. \tag{4}$$

By the associative law and Hypothesis 2.4 and (3) we have $(b_3^2)\bar{d}_3 = (b_3\bar{d}_3)b_3 = y_9b_3$, $\bar{d}_3c_3 + \bar{d}_3b_6 = y_9b_3 = d_3 + d_9 + \bar{x}_{15}$. So

$$\bar{d}_3c_3 = d_9, \tag{5}$$

$$\bar{d}_3b_6 = d_3 + \bar{x}_{15}. \tag{6}$$

Hence by (1) we obtain $\bar{d}_9 = x_9$. By Lemma 2.28, we may assume

$$x_9c_3 = \bar{x}_{15} + d_3 + \bar{x}_9.$$

By 10) in Lemma 2.27 and (4) we get

$$b_8x_{15} = 5x_{15} + 3x_9 + 2x_6 + b_3 + \bar{d}_3. \tag{7}$$

By the associative law and 1) in Lemma 2.25 and Lemma 2.28:

$$(c_3^2)x_9 = (c_3x_9)c_3,$$

$$x_9 + x_9 b_8 = \bar{x}_{15} c_3 + d_3 c_3 + \bar{x}_9 c_3.$$

By (1) and 3) in Lemma 2.27 and Lemma 2.28, we obtain

$$x_9 + x_9 b_8 = 2x_{15} + x_6 + x_9 + x_{15} + \bar{d}_3 + x_9,$$

so

$$x_9 b_8 = 3x_{15} + 2x_9 + x_6 + \bar{d}_3. \tag{8}$$

By the associative law and 4) in Lemma 2.27 and 2) in Lemma 2.25, we have that $y_{15}(\bar{b}_3 c_3) = (y_{15}\bar{b}_3)c_3$, $y_{15}b_3 + y_{15}x_6 = 2x_{15}c_3 + x_6 c_3 + x_9 c_3$. Moreover by 4) in Lemma 2.26 and 3), 4) in Lemma 2.27 and Lemma 2.28, we get

$$y_{15}b_3 + y_{15}x_6 = 4\bar{x}_{15} + 2\bar{x}_6 + 2\bar{x}_9 + \bar{b}_3 + \bar{x}_{15} + \bar{x}_{15} + d_3 + \bar{x}_9,$$

then

$$y_{15}x_6 = 4\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6 + \bar{b}_3 + d_3. \tag{9}$$

By the associative law and 8) in Lemma 2.27 and Hypothesis 2.4, we obtain $(b_3^2)x_{15} = (b_3 x_{15})b_3, x_{15}c_3 + b_6 x_{15} = 2y_{15}b_3 + b_6 b_3 + y_9 b_3$, By 3), 4) in Lemma 2.27 and 10) in Lemma 2.26 and (4), we have $2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + b_6 x_{15} = 4\bar{x}_{15} + 2\bar{x}_6 + 2\bar{x}_9 + \bar{b}_3 + \bar{x}_{15} + d_3 + \bar{x}_9 + \bar{x}_{15}$, so $b_6 x_{15} = 4\bar{x}_{15} + \bar{x}_6 + 2\bar{x}_9 + \bar{b}_3 + d_3$. By the associative law and 6) in Lemma 2.26 and (1), (6) and (8), we have $(c_3 b_8)d_3 = (c_3 d_3)b_8 = x_9 b_8$. Hence

$$\begin{aligned} b_6 d_3 + c_3 d_3 + y_{15}d_3 &= 3x_{15} + 2x_9 + x_6 + \bar{d}_3, \\ \bar{d}_3 + x_{15} + x_9 + y_{15}d_3 &= 3x_{15} + 2x_9 + x_6 + \bar{d}_3, \\ y_{15}d_3 &= 2x_{15} + x_6 + x_9. \end{aligned}$$

So by the associative law and (1), it follows that $(d_3 c_3)y_{15} = c_3(d_3 y_{15})$ and $x_9 y_{15} = 2x_{15}c_3 + x_9 c_3 + x_6 c_3$. Hence by 3) in Lemma 2.27 and 4) in Lemma 2.26, we obtain that

$$x_9 c_3 = \bar{x}_{15} + d_3 + \bar{x}_9, \qquad x_9 x_{15} = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3.$$

By the associative law 1) in Lemma 2.25 and Lemma 2.28 and (1), one has that $d_3(c_3^2) = (d_3 c_3)c_3$, $d_3 + d_3 b_8 = c_3 x_9 = \bar{x}_{15} + d_3 + \bar{x}_9$, and $d_3 b_8 = \bar{x}_{15} + \bar{x}_9$. Then we have by (6) that $(d_3^2, b_6) = (d_3, \bar{d}_3 b_6) = 1$, so

$$d_3^2 = b_6 + y_3, \tag{10}$$

hence $(d_3\bar{d}_3, d_3\bar{d}_3) = (d_3^2, d_3^2) = 2$. Now by (1) and 1) in Lemma 2.25, we get $(d_3 c_3, d_3 c_3) = (\bar{d}_3 d_3, c_3^2) = 1$. Hence

$$d_3\bar{d}_3 = 1 + c_8, \tag{11}$$

where $c_8 \neq b_8$ by the main theorem in [CA], so $d_3 \neq c_3, b_3, \bar{b}_3$. Also $y_3 \neq c_3, b_3, \bar{b}_3$, since if $y_3 = c_3$, then $1 = (d_3^2, c_3) = (\bar{d}_3 c_3, d_3)$; therefore $(\bar{d}_3 d_3, c_3\bar{d}_3) = (\bar{d}_3 c_3, \bar{d}_3 c_3) \geq 2$, a contradiction. In the same way if $y_3 = b_3, \bar{b}_3$. If $x_9 = \bar{x}_9$, then by (1) we have $1 = (c_3 d_3, c_3\bar{d}_3) = (\bar{d}_3^2, c_3^2)$, a contradiction to 1) in Lemma 2.25 and that $d_3^2 = b_6 + y_3$, so $x_9 \neq \bar{x}_9$. By (11) and Hypothesis 2.4, it holds that $1 = (\bar{d}_3 d_3, \bar{b}_3 b_3) = (d_3 b_3, d_3 b_3)$, hence $d_3 b_3 = z_9$. By (10) and Hypothesis 2.4, we obtain $1 = (d_3^2, \bar{b}_3^2) = (d_3 b_3, \bar{d}_3\bar{b}_3)$, so $z_9 = \bar{z}_9$, hence $z_9 \neq x_9, \bar{x}_9$. If $z_9 = y_9$, then by (3) we have $1 = (d_3 b_3, b_3\bar{d}_3) = (d_3^2, \bar{b}_3 b_3)$, a contradiction to (10) and Hypothesis 2.4; hence $z_9 \neq y_9$. By the associative law and (11), (6) and (10):

$$(d_3 b_6)\bar{d}_3 = b_6(d_3\bar{d}_3),$$

$$\bar{d}_3^2 + x_{15}\bar{d}_3 = b_6 + c_8 b_6,$$

then

$$c_8 b_6 = \bar{y}_3 + x_{15}\bar{d}_3$$

since c_8, b_6 are reals and

$$(x_{15}\bar{d}_3, y_3) = (x_{15}, d_3 y_3) = 0.$$

So $y_3 = \bar{y}_3$, by (10), $(\bar{d}_3 y_3, d_3) = 1$, hence

$$2 \leq (\bar{d}_3 y_3, \bar{d}_3 y_3) = (d_3\bar{d}_3, y_3^2),$$

so $y_3^2 = 1 + c_8$. □

Lemma 2.30 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8 b_3, b_8 b_3) = 3$, then Lemma 2.29 holds and we obtain the following*:

1) $b_8^2 = 1 + 2b_8 + 2x_{10} + z_4 + c_8 + b_5 + c_5$,
2) $\bar{x}_{15} b_3 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5$, *either b_5, c_5 are reals or $b_5 = \bar{c}_5$*,
3) $b_6^2 = 1 + b_8 + z_9 + c_8 + b_5 + c_5$,
4) $\bar{x}_6 x_6 = 1 + b_8 + c_8 + z_9 + b_5 + c_5$,
5) $x_{10}^2 = 1 + 2b_8 + 2x_{10} + 2c_8 + 2b_5 + 2c_5 + 3z_9$,
6) $x_{10}\bar{x}_6 = \bar{b}_3 + 3\bar{x}_{15} + d_3 + \bar{x}_9$,
7) $b_8 x_{10} = 2b_8 + x_{10} + \bar{x}_9 b_3 + c_8 + z_9 + b_5 + c_5$,
8) $c_3 y_{15} = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5$,
9) $c_3 b_5 = y_{15}$,
10) $c_3 c_5 = y_{15}$.

Proof First we note that either b_5 and c_5 are reals or $b_5 = \bar{c}_5$ holds. But in these two cases, it always follows that $\overline{b_5 + c_5} = \bar{b}_5 + \bar{c}_5 = b_5 + c_5$. By 2) in Lemma 2.26 and 1), 8) in Lemma 2.27, we have $(\bar{x}_{15} b_3, \bar{x}_{15} b_3) = 6$, $(\bar{x}_{15} b_3, x_{10}) = (b_3 x_{10}, x_{15}) = 1$, $(\bar{x}_{15} b_3, b_8) = (b_3 b_8, x_{15})$, then

$$\bar{x}_{15} b_3 = x_{10} + b_8 + x + y + z + w, \tag{1}$$

where $|x+y+z+w|=27$. Here the case $\bar{x}_{15}b_3 = x_{10}+b_8+2x$ is impossible since $|2x|=27$ implies that $|x|=13.5$. But x is an integer, a contradiction. By the associative law $(\bar{b}_3b_8)b_3 = b_8(\bar{b}_3b_3)$ and Hypothesis 2.4 and 2), 3) in Lemma 2.26, we get

$$\bar{x}_6b_3+\bar{b}_3b_3+\bar{x}_{15}b_3 = b_8+b_8^2, \qquad b_8+x_{10}+1+b_8+\bar{x}_{15}b_3 = b_8+b_8^2,$$
$$1+x_{10}+b_8+\bar{x}_{15}b_3 = b_8^2.$$

Thus

$$b_8^2 = 1+2b_8+2x_{10}+x+y+z+w. \tag{2}$$

So by 4) in Lemma 2.25 and 3) in Lemma 2.26,

$$b_6^2 = 1+b_8+x+y+z+w. \tag{3}$$

By the associative law $(\bar{b}_3c_3)(b_3c_3) = (\bar{b}_3b_3)c_3^2$ and 1), 2) in Lemma 2.25 and Hypothesis 2.4 and 3) in Lemma 2.26:

$$(b_3+x_6)(\bar{b}_3+\bar{x}_6) = (1+b_8)^2 = 1+2b_8+b_8^2,$$
$$b_3\bar{b}_3+b_3\bar{x}_6+\bar{b}_3x_6+x_6\bar{x}_6 = 1+2b_8+b_8^2.$$

So by (2) and (1) and 3) in Lemma 2.26:

$$x_6\bar{x}_6 = 1+b_8+x+y+z+w. \tag{4}$$

By the associative law $(\bar{x}_6b_8)b_3 = (\bar{x}_6b_3)b_8$ and 3) in Lemma 2.26 and 2) in Lemma 2.27:

$$2\bar{x}_{15}b_3+\bar{b}_3b_3+\bar{x}_6b_3+\bar{x}_9b_3 = b_8^2+x_{10}b_8,$$

then by 3) in Lemma 2.26 and (1) and (2) we have

$$\begin{aligned}&2x_{10}+2b_8+2x+2y+2z+2w+1+b_8+b_8+x_{10}+\bar{x}_9b_3\\ &\quad = 1+2b_8+2x_{10}+x+y+z+w+x_{10}b_8,\end{aligned}$$

so

$$2b_8+x_{10}+\bar{x}_9b_3+x+y+z+w = x_{10}b_8. \tag{5}$$

By the associative law $(\bar{x}_6^2)b_3 = \bar{x}_6(\bar{x}_6b_3)$ and 7) in Lemma 2.27 and 3) in Lemma 2.26, we get $2b_6b_3+y_{15}b_3+y_9b_3 = \bar{x}_6b_8+\bar{x}_6x_{10}$. Then by 1) in Lemma 2.26 and 2), 4) in Lemma 2.27 and 5) in Lemma 2.29, we have the equation:

$$2\bar{b}_3+2\bar{x}_{15}+2\bar{x}_{15}+\bar{x}_6+\bar{x}_9+d_3+\bar{x}_{15}+\bar{x}_9 = 2\bar{x}_{15}+\bar{b}_3+\bar{x}_6+\bar{x}_9+x_{10}\bar{x}_6,$$

so

$$\bar{b}_3+3\bar{x}_{15}+d_3+\bar{x}_9 = x_{10}\bar{x}_6. \tag{6}$$

By the associative $(\bar{x}_6 b_3)x_{10} = b_3(\bar{x}_6 x_{10})$ and 3) in Lemma 2.26, we get $b_8 x_{10} + x_{10}^2 = \bar{b}_3 b_3 + 3\bar{x}_{15} b_3 + d_3 b_3 + \bar{x}_9 b_3$. Then by (3) and Hypothesis 2.4 and (1) and 5) in Lemma 2.29, it follows that

$$\begin{aligned} &2b_8 + x_{10} + \bar{x}_9 b_3 + x + y + z + w + x_{10}^2 \\ &\quad = 1 + b_8 + 3x_{10} + 3b_8 + 3x + 3y + 3z + 3w + z_9 + \bar{x}_9 b_3. \end{aligned}$$

Hence

$$x_{10}^2 = 1 + 2b_8 + 2x_{10} + z_9 + 2x + 2y + 2z + 2w. \tag{7}$$

By 13) and 14) in Lemma 2.29, one has that $(d_3\bar{d}_3, b_8^2) = (d_3 b_8, d_3 b_8) = 2$, hence we obtain by (2) that $(b_8^2, c_8) = 1$. Without loss of generality, say $x = c_8$. By (4) and (7), it holds that $11 \le (x_{10}^2, x_6\bar{x}_6) = (x_{10}\bar{x}_6, x_{10}\bar{x}_6)$. But by (6), $(x_{10}\bar{x}_6, x_{10}\bar{x}_6) = 12$ holds, thus we may assume $y = z_9$. By (1), $(\bar{b}_3\alpha, \bar{x}_{15}) = (\alpha, b_3\bar{x}_{15}) = 1$ holds, where $\alpha \in \{x, y, z, w\}$. So $|\alpha| \ge 5$. Hence $z = b_5$ and $w = c_5$. Hence by (1), (3), (4), (5), (6) and (7), we obtain

$$\begin{aligned} \bar{x}_{15}\bar{b}_3 &= x_{10} + b_8 + c_8 + z_9 + b_5 + c_5, \\ b_8^2 &= 1 + 2b_8 + 2x_{10} + c_8 + z_9 + b_5 + c_5, \\ b_6^2 &= 1 + b_8 + c_8 + z_9 + b_5 + c_5, \\ x_6\bar{x}_6 &= 1 + b_8 + c_8 + z_9 + b_5 + c_5, \\ x_{10}^2 &= 1 + 2b_8 + 2x_{10} + 3z_9 + 2c_8 + 2b_5 + 2c_5, \\ b_8 x_{10} &= 2b_8 + x_{10} + \bar{x}_9 b_3 + c_8 + z_9 + b_5 + c_5. \end{aligned}$$

So by the associative law $(\bar{b}_3\bar{x}_6)c_3 = \bar{x}_6(\bar{b}_3 c_3)$ and 1), 2) in Lemma 2.25 and 3), 5) in Lemma 2.26:

$$\begin{aligned} c_3^2 + c_3 y_{15} &= b_3\bar{x}_6 + x_6\bar{x}_6, \\ 1 + b_8 + c_3 y_{15} &= b_8 + x_{10} + 1 + b_8 + c_8 + z_9 + b_5 + c_5, \\ c_3 y_{15} &= x_{10} + b_8 + c_8 + z_9 + b_5 + c_5. \end{aligned}$$

So $(c_3 b_5, y_{15}) = (b_5, c_3 y_{15}) = 1$, $(c_3 c_5, y_{15}) = (y_{15} c_3, c_5) = 1$, hence $c_3 b_5 = y_{15}$, $c_3 c_5 = y_{15}$. □

Lemma 2.31 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8 b_3, b_8 b_3) = 3$, then Lemma 2.30 holds and we obtain the following*:

1) $b_3 z_9 = x_9 + \bar{d}_3 + x_{15}$,
2) $\bar{x}_9 b_3 = z_9 + x_{10} + c_8$,
3) $b_8 x_{10} = 2b_8 + 2x_{10} + 2z_9 + 2c_8 + b_5 + c_5$,
4) $b_3 c_8 = x_9 + x_{15}$,
5) $d_3 x_6 = x_{10} + c_8$,

6) $d_3b_6 = \bar{d}_3 + x_{15}$,
7) $d_3c_8 = \bar{x}_{15} + \bar{x}_6 + d_3$,
8) $y_3\bar{d}_3 = \bar{x}_6 + d_3$,
9) $d_3\bar{x}_6 = y_{15} + y_3$.

Proof By the associative law $(b_3^2)d_3 = b_3(b_3d_3)$ and 6), 4) in Lemma 2.29 and Hypothesis 2.4, we get $c_3d_3 + b_6d_3 = b_3z_9$. Then by 1), 6) in Lemma 2.29 it follows that

$$x_9 + \bar{d}_3 + x_{15} = b_3z_9. \tag{1}$$

Then $(\bar{b}_3x_9, z_9) = (b_3z_9, x_9) = 1$. And by 1) in Lemma 2.27, $(\bar{x}_9b_3, x_{10}) = (b_3x_{10}, x_9) = 1$ holds. It follows by 4) in Lemma 2.29 and 1) in Lemma 2.25 and Hypothesis 2.4 that $(\bar{x}_9b_3, \bar{x}_9b_3) = (\bar{x}_9x_9, \bar{b}_3b_3) = (\bar{x}_9x_9, c_3^2) = (x_9c_3, x_9c_3) = 3$. Hence

$$\bar{x}_9b_3 = z_9 + x_{10} + d_8. \tag{2}$$

By 1) and 5) in Lemma 2.30, we get

$$(b_8^2, x_{10}^2) = 18 = (b_8x_{10}, b_8x_{10}). \tag{3}$$

By 7) in Lemma 2.30 and (2), we have $b_8x_{10} = 2b_8 + 2x_{10} + 2z_9 + c_8 + b_5 + c_5 + d_8$, then by (3) we obtain that $c_8 = d_8$, so

$$b_8x_{10} = 2b_8 + 2x_{10} + 2z_9 + 2c_8 + b_5 + c_5. \tag{4}$$

Now we have $(b_3c_8, x_9) = (b_3\bar{x}_9, c_8) = 1$ by (2) and $(b_3c_8, x_{15}) = (b_3\bar{x}_{15}, c_8) = 1$ by 2) in Lemma 2.30, so

$$b_3c_8 = x_9 + x_{15}. \tag{5}$$

By the associative law $d_3(c_3\bar{b}_3) = (d_3c_3)\bar{b}_3$ and 1) in Lemma 2.29 and 2) in Lemma 2.25 and (2), we see that $d_3(c_3\bar{b}_3) = (d_3c_3)\bar{b}_3$, $d_3b_3 + d_3x_6 = x_9\bar{b}_3 = z_9 + x_{10} + c_8$. Hence by 14) in Lemma 2.29 we have

$$d_3x_6 = x_{10} + c_8. \tag{6}$$

By the associative law $(b_3^2)d_3 = (b_3d_3)b_3$ and 14) in Lemma 2.29 and (1) and Hypothesis 2.4, it follows $d_3c_3 + d_3b_6 = z_9b_3 = x_9 + \bar{d}_3 + x_{15}$. Hence by 1) in Lemma 2.29 it holds that:

$$d_3b_6 = \bar{d}_3 + x_{15}. \tag{7}$$

By the associative law $(\bar{d}_3d_3)d_3 = \bar{d}_3(d_3^2)$ and 12) and 13) in Lemma 2.29, we have $d_3 + d_3c_8 = b_6\bar{d}_3 + y_3\bar{d}_3$, and get $d_3c_8 = \bar{x}_{15} + y_3\bar{d}_3$ by (7) and $(d_3c_8, \bar{x}_6) = (d_3x_6, c_8) = 1$ by (6). By 12) in Lemma 2.29, $(d_3c_8, d_3) = (c_8, \bar{d}_3d_3) = 1$ holds, hence

$$d_3c_8 = \bar{x}_{15} + \bar{x}_6 + d_3. \tag{8}$$

Therefore

$$y_3\bar{d}_3 = \bar{x}_6 + d_3. \tag{9}$$

Moreover $(d_3\bar{x}_6, y_3) = (\bar{x}_6, \bar{d}_3 y_3) = 1$, and by 11) in Lemma 2.29, it follows that $(d_3\bar{x}_6, y_{15}) = 1$, so $d_3\bar{x}_6 = y_{15} + y_3$. □

Lemma 2.32 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8b_3, b_8b_3) = 3$, then Lemma 2.31 holds and we obtain the following*:

1) $x_9b_3 = y_9 + y_{15} + y_3$,
2) $x_9\bar{d}_3 = y_{15} + c_3 + y_9$,
3) $x_{15}\bar{d}_3 = y_{15} + y_6 + y_9$,
4) $c_8b_6 = 2y_{15} + y_3 + y_9 + b_6$,
5) $z_9b_6 = 2y_{15} + 2y_9 + b_6$,
6) $x_{10}b_6 = 3y_{15} + y_3 + c_3 + y_9$,
7) $y_3b_6 = c_8 + x_{10}$,
8) $d_3x_{15} = b_8 + z_9 + b_5 + c_5 + x_{10} + c_8$,
9) $y_{15}b_6 = 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5$,
10) $y_9b_6 = b_8 + 2z_9 + b_5 + c_5 + x_{10} + c_8$,
11) $\bar{x}_{15}x_{15} = 5b_8 + 6x_{10} + 6z_9 + 5c_8 + 3b_5 + 3c_5 + 1$,
12) $b_6b_5 = y_{15} + y_9 + b_6$,
13) $b_3y_3 = \bar{x}_9$.

Proof By the associative law $d_3(c_3b_3) = (d_3c_3)b_3 = x_9b_3$ and 1) in Lemma 2.29 and 2) in Lemma 2.25, we have that $d_3\bar{b}_3 + d_3\bar{x}_6 = x_9b_3$ and by 3) in Lemma 2.29 that $y_9 + d_3\bar{x}_6 = x_9b_3$. Finally, we obtain

$$x_9b_3 = y_9 + y_{15} + y_3, \tag{1}$$

by 9) in Lemma 2.31.

By the associative law $b_3(c_8\bar{d}_3) = \bar{d}_3(b_3c_8)$ and 4), 7) in Lemma 2.31, we get $x_{15}b_3 + b_3x_6 + b_3\bar{d}_3 = x_9\bar{d}_3 + x_{15}\bar{d}_3$, and by 5) in Lemma 2.26 and 3) in Lemma 2.29 and 8) in Lemma 2.27, we get $2y_{15}+b_6+y_9+c_3+y_{15} = x_9\bar{d}_3+x_{15}\bar{d}_3$, $x_9\bar{d}_3 + x_{15}\bar{d}_3 = 3y_{15} + 2y_9 + b_6 + c_3$. Now from 10), 11), 1) in Lemma 2.29, we see that $(x_9\bar{d}_9, c_3) = (x_9, d_3c_3) = 1$ and $(x_9\bar{d}_3, y_{15}) = (x_9, d_3y_{15}) = 1$ and $(x_9\bar{d}_3, b_6) = (x_9, d_3b_6) = 0$, hence

$$x_9\bar{d}_3 = y_{15} + c_3 + y_9, \tag{2}$$

which gives

$$x_{15}\bar{d}_3 = 2y_{15} + y_9 + b_6. \tag{3}$$

By the associative law $(\bar{d}_3d_3)b_6 = \bar{d}_3(d_3b_6)$ and 6), 12) in Lemma 2.29, one has that $b_6 + c_8b_6 = \bar{d}_3^2 + \bar{d}_3x_{15}$. And by (3) and 13) in Lemma 2.29, we obtain $b_6 + c_8b_6 = b_6 + y_3 + 2y_{15} + y_9 + b_6$, hence

$$c_8b_6 = 2y_{15} + y_3 + y_9 + b_6. \tag{4}$$

By the associative law $(d_3 b_3)b_6 = d_3(b_3 b_6)$ and 1) in Lemma 2.26 and 3) and 16) in Lemma 2.29, we obtain $z_9 b_6 = d_3\bar{b}_3 + d_3\bar{x}_{15}$, and hence by (3) and 3) in Lemma 2.29, we have

$$z_9 b_6 = 2y_{15} + 2y_9 + b_6. \tag{5}$$

By the associative law $(x_6 b_6)d_3 = b_6(x_6 d_3)$ and 5) in Lemma 2.27 and 5) in Lemma 2.31, we have $2\bar{x}_6 d_3 + \bar{x}_{15} d_3 + \bar{x}_9 d_3 = x_{10} b_6 + c_8 b_6$. Then by 9) in Lemma 2.31 and (2), (3) and (4), it follows that

$$2y_{15} + 2y_3 + 2y_{15} + y_9 + b_6 + y_{15} + c_3 + y_9 = x_{10} b_6 + 2y_{15} + y_3 + y_9 + b_6.$$

Hence

$$x_{10} b_6 = 3y_{15} + y_3 + c_3 + y_8. \tag{6}$$

Now we have by (4) $(y_3 b_6, c_8) = (y_3, b_6 c_8) = 1$, $(y_3 b_6, x_{10}) = (y_3, b_6 x_{10}) = 1$ by (4) and (6), from which we get

$$y_3 b_6 = c_8 + x_{10}. \tag{7}$$

By the associative law $(d_3^2)b_6 = d_3(d_3 b_6)$ and 15) in Lemma 2.29 and 6) in Lemma 2.31, we obtain $b_6^2 + y_3 b_6 = d_3\bar{d}_3 + x_{15} d_3$. Moreover by 14) in Lemma 2.29 and 3) in Lemma 2.30 and (7), we have

$$\begin{aligned} 1 + b_8 + z_9 + c_8 + b_5 + c_5 + x_{10} + c_8 &= 1 + c_8 + x_{15} d_3, \\ b_8 + z_9 + b_5 + c_5 + x_{10} + c_8 &= x_{15} d_3. \end{aligned} \tag{8}$$

By the associative law and 5) in Lemma 2.26 and 5) in Lemma 2.27:

$$\begin{aligned} b_3(x_6 b_6) &= b_6(b_3 x_6), \\ 2\bar{x}_6 b_3 + \bar{x}_{15} b_3 + \bar{x}_9 b_3 &= b_6 c_3 + y_{15} b_6. \end{aligned}$$

By 2) in Lemma 2.31 and 2) in Lemma 2.30 and 3) in Lemma 2.26:

$$\begin{aligned} 2b_8 + 2x_{10} + x_{10} + b_8 + z_9 + c_8 + b_5 + c_5 + x_{10} + z_9 + c_8 &= b_8 + x_{10} + y_{15} b_6, \\ y_{15} b_6 &= 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5. \end{aligned} \tag{9}$$

By 3) in Lemma 2.29 and 1) in Lemma 2.26:

$$\begin{aligned} (b_3 b_6)\bar{d}_3 = (b_3\bar{d}_3)b_6 &= y_9 b_6, \\ \bar{b}_3\bar{d}_3 + \bar{x}_{15}\bar{d}_3 &= y_9 b_6, \end{aligned}$$

by 16) in Lemma 2.29 and (8)

$$b_8 + 2z_9 + b_5 + c_5 + x_{10} + c_8 = y_9 b_6. \tag{10}$$

By the associative law and 8) in Lemma 2.27 and 1) in Lemma 2.26:

$$(b_3 b_6)x_{15} = b_6(b_3 x_{15}),$$
$$\bar{b}_3 x_{15} + \bar{x}_{15} x_{15} = 2y_{15} b_6 + b_6^2 + y_9 b_6,$$

so by (10), (9) and 2), 3) in Lemma 2.30

$$\begin{aligned} x_{10}+b_8+z_9+c_8+b_5+c_5+\bar{x}_{15}x_{15} &= 4b_8+6x_{10}+4z_9+4c_8+2c_5+1+b_8+z_9 \\ &\quad +c_8+b_5+c_5+b_8+2z_9+c_8+x_{10}+b_5+c_5, \\ \bar{x}_{15}x_{15} &= 1+5b_8+6x_{10}+6z_9+5c_8+3b_5+4c_5. \end{aligned}$$

By (9) and (10), we get

$$(b_6 b_5, y_9) = (b_6 y_9, b_5) = 1, \qquad (b_6 b_5, y_{15}) = (b_5, y_{15} b_6) = 1$$

and by (3)

$$(b_5 b_6, b_6) = (b_5, b_6^2) = 1,$$

so

$$b_5 b_6 = y_9 + y_{15} + b_6.$$

By (1)

$$(b_3 y_3, \bar{x}_9) = (b_3 x_9, y_3) = 1,$$

then $b_3 y_3 = \bar{x}_9$. □

Lemma 2.33 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8 b_3, b_8 b_3) = 3$, then Lemma 2.32 holds and we obtain the following*:

1) $y_9 d_3 = b_3 + x_{15} + x_9$,
2) $z_9 d_3 = \bar{b}_3 + \bar{x}_{15} + \bar{x}_9$,
3) $z_9 y_9 = 3y_{15} + 2b_6 + 2y_9 + c_3 + y_3$,
4) $c_8 b_8 = 2x_{10} + 2z_9 + b_8 + c_8 + b_5 + c_5$,
5) $y_9 c_3 = z_9 + x_{10} + c_8$,
6) $\bar{x}_{15} x_6 = 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5$,
7) $x_{10} d_3 = \bar{x}_{15} + \bar{x}_6 + \bar{x}_9$,
8) $d_3 x_9 = x_{10} + b_8 + z_9$,
9) $y_3 y_9 = z_9 + b_8 + x_{10}$,
10) $b_6 y_9 = 2z_9 + b_8 + x_{10} + b_5 + c_5 + c_8$,
11) $x_{15}^2 = 9y_{15} + 4b_6 + 6y_9 + 2c_3 + 2y_3$,
12) $x_{15} x_6 = c_3 + 4y_{15} + b_6 + 2y_9 + y_3$.

Proof By the associative law and 13), 3) in Lemma 2.29:

$$(b_3 \bar{d}_3)d_3 = (\bar{b}_3 d_3)d_3 = \bar{b}_3(d_3^2), \qquad y_9 d_3 = b_6 \bar{b}_3 + y_3 \bar{b}_3,$$

then by 1) in Lemma 2.26 and 13) in Lemma 2.32,

$$y_9 d_3 = b_3 + x_{15} + x_9. \tag{1}$$

By the associative law and 16), 15) in Lemma 2.29:

$$(d_3^2)b_3 = (d_3 b_3)d_3,$$
$$b_6 b_3 + y_3 b_3 = z_9 d_3,$$

then by 13) in Lemma 2.32 and 1) in Lemma 2.26,

$$z_9 d_3 = \bar{b}_3 + \bar{x}_{15} + \bar{x}_9. \tag{2}$$

By 16) in Lemma 2.29 and (1):

$$(b_3 d_3) y_9 = z_9 y_9 = (d_3 y_9) b_3 = b_3^2 + x_{15} b_3 + x_9 b_3,$$

by 8) in Lemma 2.27 and Hypothesis 2.4 and 1) in Lemma 2.32:

$$c_3 + b_6 + 2y_{15} + b_6 + y_9 + y_9 + y_{15} + y_3 = z_9 y_9,$$
$$3y_{15} + 2b_6 + 2y_9 + c_3 + y_3 = z_9 y_9. \tag{3}$$

By the associative law and Hypothesis 2.4 and 4) in Lemma 2.31:

$$(b_3 \bar{b}_3)c_8 = \bar{b}_3(b_3 c_8),$$
$$c_8 + c_8 b_8 = x_9 \bar{b}_3 + x_{15} \bar{b}_3,$$

then by 2) in Lemma 2.31 and 2) in Lemma 2.30:

$$c_8 b_8 = 2x_{10} + 2z_9 + b_8 + c_8 + b_5 + c_5. \tag{4}$$

By the associative law and 2) in Lemma 2.25 and 3) in Lemma 2.29:

$$(b_3 c_3)\bar{d}_3 = (b_3 \bar{d}_3)c_3,$$
$$\bar{b}_3 \bar{d}_3 + \bar{x}_6 \bar{d}_3 = y_9 c_3,$$

then by 16) in Lemma 2.29 and 5) in Lemma 2.31:

$$y_9 c_3 = z_9 + x_{10} + c_8. \tag{5}$$

By the associative law and 7) in Lemma 2.27 and 4) in Lemma 2.26:

$$(x_6^2)c_3 = x_6(x_6 c_3),$$
$$2b_6 c_3 + y_{15} c_3 + y_9 c_3 = \bar{b}_3 x_6 + \bar{x}_{15} x_6,$$

then by (5) and 3) in Lemma 2.26 and 8) in Lemma 2.30:

$$2b_8 + 2x_{10} + x_{10} + b_8 + z_9 + c_8 + b_5 + c_5 + z_9 + x_{10} + c_8 = b_8 + x_{10} + \bar{x}_{15} x_6,$$
$$\bar{x}_{15} x_6 = 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5. \tag{6}$$

By the associative law, 9) in Lemma 2.31 and 3) in Lemma 2.26:

$$(\bar{x}_6 b_3)d_3 = (\bar{x}_6 d_3)b_3,$$
$$b_8 d_3 + x_{10} d_3 = y_{15} b_3 + y_3 b_3,$$

then by 4) in Lemma 2.27 and 13) in Lemma 2.32

$$b_8 d_3 + x_{10} d_3 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 + \bar{x}_9,$$

by 13) in Lemma 2.29

$$x_{10} d_3 = \bar{x}_{15} + \bar{x}_6 + \bar{x}_9. \tag{7}$$

So

$$(d_3 x_9, x_{10}) = (d_3 x_{10}, \bar{x}_9) = 1,$$

and by 13) in Lemma 2.29,

$$(d_3 x_9, b_8) = (d_3 b_8, \bar{x}_9) = 1;$$

and by (2)

$$(d_3 x_9, z_9) = (d_3 z_9, \bar{x}_9) = 1,$$

so

$$d_3 x_9 = x_{10} + b_8 + z_9. \tag{8}$$

By the associative law and 15) in Lemma 2.29 and (1):

$$(d_3^2) y_9 = d_3(d_3 y_9),$$
$$b_6 y_9 + y_3 y_9 = b_3 d_3 + x_{15} d_3 + x_9 d_3,$$

then by 16) in Lemma 2.29 and (8) and 8) in Lemma 2.32:

$$b_6 y_9 + y_3 y_9 = z_9 + b_8 + z_9 + b_5 + c_5 + x_{10} + c_8 + x_{10} + b_8 + z_9,$$
$$b_6 y_9 + y_3 y_9 = 3z_9 + 2b_8 + b_5 + c_5 + c_8 + 2x_{10}.$$

By 5), 6) in Lemma 2.32 and 9) in Lemma 2.27:

$$(b_6 y_9, b_8) = (y_9, b_6 b_8) = 1,$$
$$(b_6 y_9, x_{10}) = (y_9, b_6 x_{10}) = 1,$$
$$(b_6 y_9, z_9) = (y_9, z_9 b_6) = 2;$$

hence

$$y_3 y_9 = z_9 + b_8 + x_{10}. \tag{9}$$

Therefore

$$b_6 y_9 = 2z_9 + b_8 + x_{10} + b_5 + c_5 + c_8. \tag{10}$$

By the associative law and 10) in Lemma 2.29 and 1) in Lemma 2.26:

$$(b_6\bar{b}_3)x_{15} = \bar{b}_3(b_6x_{15}), \qquad b_3x_{15} + x_{15}^2 = 4\bar{x}_{15}\bar{b}_3 + \bar{x}_6\bar{b}_3 + 2\bar{x}_9\bar{b}_3 + \bar{b}_3^2 + d_3\bar{b}_3,$$

by 8) in Lemma 2.27 and 5) in Lemma 2.26 and 1) in Lemma 2.32 and Hypothesis 2.4 and 3) in Lemma 2.29:

$$x_{15}^2 = 6y_{15} + 3b_6 + 3y_9 + c_3 + y_{15} + 2y_9 + 2y_{15} + 2y_3 + c_3 + b_6 + y_9,$$
$$x_{15}^2 = 9y_{15} + 4b_6 + 6y_9 + 2c_3 + 2y_3.$$

By the associative law and 5) in Lemma 2.27 and 1) in Lemma 2.26:

$$(\bar{x}_6b_6)b_3 = \bar{x}_6(b_6b_3),$$
$$2x_6b_3 + x_{15}b_3 + x_9b_3 = \bar{b}_3\bar{x}_6 + \bar{x}_{15}\bar{x}_6,$$

then by 5) in Lemma 2.26 and 8) in Lemma 2.27 and 1) in Lemma 2.32:

$$2c_3 + 2y_{15} + 2y_{15} + b_6 + y_9 + y_{15} + y_3 = c_3 + y_{15} + \bar{x}_{15}\bar{x}_6,$$
$$x_{15}x_6 = c_3 + 4y_{15} + b_6 + 2y_9 + y_3.$$

□

Lemma 2.34 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8b_3, b_8b_3) = 3$, *then Lemma* 2.33 *holds, and we obtain the following*:

1) $y_3c_3 = z_9$,
2) $z_9c_3 = y_9 + y_{15} + y_3$,
3) $y_3b_8 = y_9 + y_{15}$,
4) $z_9b_8 = 2z_9 + 2x_{10} + 2c_8 + b_8 + b_5 + c_5$,
5) $z_9^2 = 1 + 2x_{10} + 2z_9 + 2b_8 + 2c_8 + b_5 + c_5$,
6) $c_8^2 = 1 + 2c_8 + 2x_{10} + b_8 + z_9 + b_5 = c_5$,
7) $c_8c_3 = y_{15} + y_9$,
8) $x_9^2 = 2b_6 + 3y_{15} + 2y_9 + y_3 + c_3$,
9) $c_8y_3 = b_6 + y_3 + y_{15}$,
10) $x_{10}y_3 = b_6 + y_3 + y_{15}$,
11) $d_3y_3 = \bar{d}_3 + x_6$,
12) $\bar{x}_6y_3 = \bar{d}_3 + x_{15}$,
13) $x_{15}y_3 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9$.

Proof By the associative law and Hypothesis 2.4 and 13) in Lemma 2.32:

$$y_3(b_3^2) = \bar{x}_9b_3,$$
$$y_3c_3 + y_3b_6 = \bar{x}_9b_3,$$

then by 2) in Lemma 2.31

$$y_3c_3 + y_3b_6 = z_9 + x_{10} + c_8,$$

then by 7) in Lemma 2.32

$$y_3c_3 = z_9. \tag{1}$$

By 1), 16) in Lemma 2.29 and the associative law and 1) in Lemma 2.32:

$$b_3(c_3d_3) = (b_3c_3)d_3 = (b_3d_3)c_3 = z_9c_3,$$
$$b_3x_9 = z_9c_3,$$

then

$$z_9c_3 = y_9 + y_{15} + y_3. \tag{2}$$

So by (1) and 1) in Lemma 2.25 and the associative law:

$$y_3(c_3^2) = z_9c_3,$$
$$y_3 + y_3b_8 = y_9 + y_{15} + y_3.$$

Thus

$$y_3b_8 = y_9 + y_{15}. \tag{3}$$

By (2) and the associative law and 1) in Lemma 2.25:

$$z_9(c_3^2) = y_9c_3 + y_{15}c_3 + y_3c_3,$$
$$z_9 + z_9b_8 = y_9c_3 + y_{15}c_3 + y_3c_3,$$

so by (1) and 8) in Lemma 2.30 and 5) in Lemma 2.33:

$$z_9 + z_9b_8 = z_9 + x_{10} + c_8 + x_{10} + b_8 + c_8 + z_9 + b_5 + c_5 + z_9,$$
$$z_9b_8 = 2z_9 + 2x_{10} + 2c_8 + b_8 + b_5 + c_5. \tag{4}$$

By 17) in Lemma 2.29 and 1) in Lemma 2.25 and (1):

$$y_3^2c_3^2 = z_9^2,$$
$$(1 + c_8)(1 + b_8) = z_9^2,$$
$$1 + b_8 + c_8 + c_8b_8 = z_9^2,$$

by 4) in Lemma 2.33

$$z_9^2 = 1 + 2x_{10} + 2z_9 + 2b_8 + 2c_8 + b_5 + c_5. \tag{5}$$

By the associative law and 14) in Lemma 2.29 and 7) in Lemma 2.31:

$$(\bar{d}_3d_3)c_8 = \bar{d}_3(d_3c_8),$$
$$c_8 + c_8^2 = \bar{d}_3\bar{x}_{15} + \bar{x}_6\bar{d}_3 + d_3\bar{d}_3,$$

then by 5) in Lemma 2.31 and 8) in Lemma 2.32:

$$c_8 + c_8^2 = b_8 + z_9 + b_5 + c_5 + x_{10} + c_8 + x_{10} + c_8 + 1 + c_8,$$
$$c_8^2 = 1 + 2c_8 + 2x_{10} + b_8 + z_9 + b_5 + c_5. \tag{6}$$

By 1), 12) in Lemma 2.29 and the associative law:

$$\bar{d}_3(d_3c_3) = c_3 + c_8c_3,$$
$$\bar{d}_3x_9 = c_3 + c_8c_3,$$

then by 2) in Lemma 2.32

$$c_8c_3 = y_{15} + y_9. \tag{7}$$

By the associative law and 13) in Lemma 2.32 and 17) in Lemma 2.29 and Hypothesis 2.4:

$$b_3^2y_3^2 = \bar{x}_9^2,$$
$$(c_3 + b_6)(1 + c_8) = \bar{x}_9^2,$$
$$c_3 + b_6 + c_3c_8 + b_6c_8 = \bar{x}_9^2,$$

then by (7) and 4) in Lemma 2.32:

$$c_3 + b_6 + y_{15} + y_9 + 2y_{15} + y_3 + y_9 + b_6 = \bar{x}_9^2,$$
$$x_9^2 = 2b_6 + 3y_{15} + 2y_9 + y_3 + c_3.$$

By the associative law and 4), 7) in Lemma 2.32 and 17) in Lemma 2.29:

$$(y_3^2)b_6 = y_3(y_3b_6),$$
$$b_6 + c_8b_6 = c_8y_3 + x_{10}y_3,$$
$$2y_{15} + y_3 + y_9 + 2b_6 = c_8y_3 + x_{10}y_3,$$

by 9) in Lemma 2.33, 4) in Lemma 2.32 and 15) in Lemma 2.29: $(c_8y_3, b_6) = (y_3, c_8b_6) = 1$, $(c_8y_3, y_3) = (c_8, y_3^2) = 1$, $(c_8y_3, y_9) = (c_8, y_3y_9) = 0$. So

$$c_8y_3 = b_6 + y_3 + y_{15}, \tag{8}$$

then

$$x_{10}y_3 = b_6 + y_9 + y_{15}. \tag{9}$$

By the associative law and 8) in Lemma 2.31 and 17) in Lemma 2.29:

$$(y_3^2)\bar{d}_3 = y_3(y_3\bar{d}_3),$$
$$\bar{d}_3 + \bar{d}_3c_8 = \bar{x}_6y_3 + d_3y_3,$$

by 7) in Lemma 2.31, $2\bar{d}_3 + x_{15} + x_6 = \bar{x}_6 y_3 + d_3 y_3$, by 15) in Lemma 2.29, $(d_3 y_3, \bar{d}_3) = (d_3^2, y_3) = 1$. So

$$d_3 y_3 = \bar{d}_3 + x_6, \tag{10}$$

then

$$\bar{x}_6 y_3 = \bar{d}_3 + x_{15}. \tag{11}$$

By the associative law and 6) in Lemma 2.31 and 7) in Lemma 2.32:

$$(d_3 b_6) y_3 = d_3 (y_3 b_6),$$
$$\bar{d}_3 y_3 + x_{15} y_3 = c_8 d_3 + x_{10} d_3,$$

then by 7) in Lemma 2.33 and 7) in Lemma 2.31 and (10):

$$d_3 + \bar{x}_6 + x_{15} y_3 = \bar{x}_{15} + \bar{x}_6 + d_3 + \bar{x}_{15} + \bar{x}_6 + \bar{x}_9,$$
$$y_{15} y_3 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9.$$

□

Lemma 2.35 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8 b_3, b_8 b_3) = 3$, *then Lemma* 2.34 *holds and we obtain the following*:

(1) $z_9 x_{10} = 3x_{10} + 2b_8 + 2z_9 + 2c_8 + b_5 + c_5$,
(2) $z_9 c_8 = 2x_{10} + 2b_8 + 2z_9 + b_5 + c_5 + c_8$,
(3) $c_8 x_{10} = 2x_{10} + 2c_8 + 2b_8 + 2z_9 + b_5 + c_5$,
(4) $y_9^2 = 1 + 2z_9 + 2b_8 + 2c_8 + 2x_{10} + b_5 + c_5$,
(5) $x_9 c_8 = 3x_{15} + x_6 + 2x_9 + b_3$,
(6) $\bar{x}_9 y_3 = b_3 + x_9 + x_{15}$,
(7) $\bar{x}_9 x_9 = 1 + 2b_8 + 2x_{10} + 2c_8 + 2z_9 + b_5 + c_5$,
(8) $x_9 y_9 = d_3 + 3\bar{x}_{15} + 2\bar{x}_9 + \bar{b}_3 + 2\bar{x}_6$,
(9) $z_9 y_3 = y_9 + c_3 + y_{15}$,
(10) $x_9 z_9 = 3x_{15} + 2x_9 + 2x_6 + b_3 + \bar{d}_3$.

Proof By the associative law and 7) in Lemma 2.33 and 16) in Lemma 2.29:

$$(b_3 d_3) x_{10} = b_3 (d_3 x_{10}),$$
$$z_9 x_{10} = \bar{x}_{15} b_3 + \bar{x}_6 b_3 + \bar{x}_9 b_3.$$

Then by 2) in Lemma 2.30 and 3) in Lemma 2.26 and 2) in Lemma 2.31:

$$z_9 x_{10} = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5 + b_8 + x_{10} + z_9 + x_{10} + c_8,$$
$$z_9 x_{10} = 3x_{10} + 2b_8 + 2z_9 + 2c_8 + b_5 + c_5. \tag{1}$$

By the associative law and 16) in Lemma 2.29 and 7) in Lemma 2.31:

$$(b_3 d_3) c_8 = b_3 (d_3 c_8),$$
$$z_9 c_8 = \bar{x}_{15} b_3 + \bar{x}_6 b_3 + d_3 b_3.$$

Then by 2) in Lemma 2.30 and 3) in Lemma 2.26 and 16) in Lemma 2.29, we get $z_9c_8 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5 + b_8 + x_{10} + z_9$ and

$$z_9c_8 = 2x_{10} + 2b_8 + 2z_9 + b_5 + c_5 + c_8. \tag{2}$$

By the associative law and 7) in Lemma 2.33 and 14) in Lemma 2.29:

$$(\bar{d}_3d_3)x_{10} = \bar{d}_3(d_3x_{10}),$$
$$x_{10} + c_8x_{10} = \bar{d}_3\bar{x}_{15} + \bar{d}_3\bar{x}_6 + \bar{d}_3\bar{x}_9,$$

then by 5) in Lemma 2.31 and 8) in Lemma 2.32 and 8) in Lemma 2.33, it follows that $x_{10} + c_8x_{10} = b_8 + z_9 + b_5 + c_5 + x_{10} + c_8 + x_{10} + c_8 + x_{10} + b_8 + z_9$ and

$$c_8x_{10} = 2x_{10} + 2c_8 + 2b_8 + 2z_9 + b_5 + c_5. \tag{3}$$

By the associative law and 3), 15) in Lemma 2.29 and Hypothesis 2.4, one has

$$b_3^2\bar{d}_3^2 = y_9^2,$$
$$(c_3 + b_6)(b_6 + y_3) = y_9^2,$$
$$c_3b_6 + c_3y_3 + b_6^2 + b_6y_3 = y_9^2,$$

then by 3) in Lemma 2.26 and 1) in Lemma 2.34 and 3) in Lemma 2.30 and 7) in Lemma 2.32, we have $y_9^2 = b_8 + x_{10} + z_9 + 1 + b_8 + z_9 + c_8 + b_5 + c_5 + c_8 + x_{10}$. So

$$y_9^2 = 1 + 2z_9 + 2b_8 + 2c_8 + 2x_{10} + b_5 + c_5. \tag{4}$$

By the associative law and 13) in Lemma 2.32 and 9) in Lemma 2.34:

$$(b_3y_3)c_8 = b_3(y_3c_8),$$
$$\bar{x}_9c_8 = b_6b_3 + y_3b_3 + y_{15}b_3,$$

then by 13) in Lemma 2.32 and 1) in Lemma 2.26 and 4) in Lemma 2.27, $\bar{x}_9c_8 = \bar{b}_3 + \bar{x}_{15} + \bar{x}_9 + 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9$ and

$$\bar{x}_9c_8 = \bar{b}_3 + 3\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6. \tag{5}$$

By the associative law and 13) in Lemma 2.32 and 17) in Lemma 2.29:

$$(y_3^2)b_3 = y_3(y_3b_3),$$
$$b_3 + c_8b_3 = \bar{x}_9y_3,$$

then 4) in Lemma 2.31

$$\bar{x}_9y_{13} = b_3 + x_9 + x_{15}. \tag{6}$$

By the associative law and 13) in Lemma 2.32 and (6):

$$(b_3 y_3)x_9 = b_3(y_3 x_9),$$
$$\bar{x}_9 x_9 = b_3\bar{b}_3 + b_3\bar{x}_9 + b_3\bar{x}_{15},$$

then by Hypothesis 2.4 and 2) in Lemma 2.31 and 2) in Lemma 2.30:

$$\bar{x}_9 x_9 = 1 + b_8 + z_9 + x_{10} + c_8 + x_{10} + b_8 + c_8 + z_9 + b_5 + c_5,$$
$$\bar{x}_9 x_9 = 1 + 2b_8 + 2x_{10} + 2x_{10} + 2c_8 + 2z_9 + b_5 + c_5. \quad (7)$$

By the associative law and 13) in Lemma 2.32 and 9) in Lemma 2.33:

$$(b_3 y_3)y_9 = b_3(y_3 y_9),$$
$$\bar{x}_9 y_9 = z_9 b_3 + b_8 b_3 + x_{10} b_3,$$

then by 1) in Lemma 2.31 and 2) in Lemma 2.26 and 1) in Lemma 2.27:

$$\bar{x}_9 y_9 = x_9 + \bar{d}_3 + x_{15} + x_6 + b_3 + x_{15} + x_{15} + x_6 + x_9 = 2x_9 + 3x_{15} + 2x_6 + \bar{d}_3 + b_3. \quad (8)$$

By the associative law and 16) in Lemma 2.29 and 11) in Lemma 2.34:

$$(b_3 d_3)y_3 = (d_3 y_3)b_3,$$
$$z_9 y_3 = \bar{d}_3 b_3 + x_6 b_3,$$

then by 3) in Lemma 2.29 and 5) in Lemma 2.26

$$z_9 y_3 = y_9 + c_3 + y_{15}. \quad (9)$$

By the associative law and 13) in Lemma 2.32 and (9):

$$(z_9 y_3)b_3 = z_9(y_3 b_3),$$
$$y_9 b_3 + c_3 b_3 + y_{15} b_3 = z_9 \bar{x}_9,$$

then by 5) in Lemma 2.29 and by 2) in Lemma 2.35 and 4) in Lemma 2.27:

$$z_9\bar{x}_9 = d_3 + \bar{x}_9 + \bar{x}_{15} + \bar{b}_3 + \bar{x}_6 + 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9 = 3\bar{x}_{15} + 2\bar{x}_9 + 2\bar{x}_6 + \bar{b}_3 + d_3. \quad (10)$$

□

Lemma 2.36 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8 b_3, b_8 b_3) = 3$, then Lemma 2.25 holds and we obtain the following*:

(1) $x_{15} c_8 = 5x_{15} + 2x_6 + 3x_9 + b_3 + \bar{d}_3$,
(2) $x_{15} x_9 = 6y_{15} + 3y_9 + 2b_6 + c_3 + y - 3$,
(3) $\bar{x}_9 x_{15} = 3b_8 + 4x_{10} + 3z_9 + 3c_8 + 2b_5 + 2c_5$,
(4) $x_{15} y_{15} = 4\bar{x}_6 + 9\bar{x}_{15} + 6\bar{x}_9 + 2\bar{b}_3 + 2d_3$,

(5) $x_{15}y_9 = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3$,
(6) $\bar{x}_9 x_6 = 2z_9 + x_{10} + b_8 + c_8 + b_5 + c_5$,
(7) $x_6 x_9 = 2y_9 + 2z_{15} + b_6$,
(8) $x_6 y_9 = 2\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6$,
(9) $x_6 z_9 = 2x_{15} + 2x_9 + x_6$,
(10) $x_6 c_8 = 2x_{15} + x_6 + x_9 + \bar{d}_3$,
(11) $y_{15}y_9 = 4x_{10} + 3b_8 + 3z_9 + 3c_8 + 2c_5 + 2b_5$.

Proof By the associative law and 6), 7) in Lemma 2.31:

$$(d_3 b_6)c_8 = b_6(d_3 c_8),$$
$$\bar{d}_3 c_8 + x_{15}c_8 = \bar{x}_{15}b_6 + \bar{x}_6 b_6 + d_3 b_6,$$

then by 7) in Lemma 2.31 and 6), 10) in Lemma 2.29, 5) in Lemma 2.27:

$$x_{15} + x_6\bar{d}_3 + x_{15}c_8 = 4x_{15} + x_6 + 2x_9 + b_3 + \bar{d}_3 + 2x_6 + x_{15} + x_9 + \bar{d}_3 + x_{15},$$
$$x_{15}c_8 = 5x_{15} + 2x_6 + 3x_9 + b_3 + \bar{d}_3. \tag{1}$$

By the associative law and 8) in Lemma 2.33 and 6) in Lemma 2.31:

$$(d_3 b_6)x_9 = b_6(d_3 x_9),$$
$$\bar{d}_3 x_9 + x_{15}x_9 = x_{10}b_6 + b_8 b_6 + z_9 b_6,$$

then by and 9) in Lemma 2.27 and 2), 5), 6) in Lemma 2.32:

$$y_{15} + c_3 + y_9 + x_{15}x_9 = 3y_{15} + y_3 + c_3 + y_9 + 2y_{15} + b_6 + c_3 + y_9$$
$$+ 2y_{15} + 2y_9 + b_6,$$
$$x_{15}x_9 = 6y_{15} + 3y_9 + 2b_6 + c_3 + y_3. \tag{2}$$

By 13) in Lemma 2.32 and 13) in Lemma 2.34:

$$b_3(y_3 x_{15}) = (b_3 y_3)x_{15} = \bar{x}_9 x_{15},$$
$$2\bar{x}_{15}b_3 + \bar{x}_6 b_3 + \bar{x}_9 b_3 = \bar{x}_9 x_{15}.$$

and 2) in Lemma 2.31 and 3) in Lemma 2.26 and 2) in Lemma 2.30:

$$2x_{10} + 2b_8 + 2c_8 + 2z_9 + 2b_5 + 2c_5 + b_8 + x_{10} + z_9 + x_{10} + c_8 = \bar{x}_9 x_{15},$$
$$\bar{x}_9 x_{15} = 3b_8 + 4x_{10} + 3z_9 + 3c_8 + 2b_5 + 2c_5. \tag{3}$$

By 11), 12) in Lemma 2.33 and by (2) and 8) in Lemma 2.27 and 3) in Lemma 2.32:

$$(x_{15}y_{15}, \bar{x}_6) = (x_{15}x_6, y_{15}) = 4, \qquad (x_{15}y_{15}, \bar{x}_{15}) = (x_{15}^2, y_{15}) = 9,$$
$$(x_{15}y_{15}, \bar{x}_9) = (x_{15}x_9, y_{15}) = 6, \qquad (x_{15}y_{15}, \bar{b}_3) = (x_{15}b_3, y_{15}) = 2,$$
$$(x_{15}y_{15}, d_3) = (x_{15}\bar{d}_3, y_{15}) = 2.$$

Then

$$x_{15}y_{15} = 4\bar{x}_6 + 9\bar{x}_{15} + 6\bar{x}_9 + 2\bar{b}_3 + 2d_3. \tag{4}$$

By (2) and 11), 12) in Lemma 2.33 and 8) in Lemma 2.27 and 3) in Lemma 2.32:

$$(x_{15}y_9, \bar{x}_{15}) = (y_9, x_{15}^2) = 6, \qquad (x_{15}y_9, \bar{x}_6) = (x_{15}x_6, y_9) = 2,$$
$$(x_{15}y_9, \bar{x}_9) = (x_{15}x_9, y_9) = 3, \qquad (x_{15}y_9, \bar{b}_3) = (x_{15}b_3, y_9) = 1,$$
$$(x_{15}y_9, d_3) = (x_{15}\bar{d}_3, y_9) = 1.$$

Then

$$x_{15}y_9 = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3. \tag{5}$$

By the associative law and 13) in Lemma 2.32 and 12) in Lemma 2.34:

$$b_3(y_3x_6) = (b_3y_3)x_6 = \bar{x}_9x_6,$$
$$b_3d_3 + b_3\bar{x}_{15} = \bar{x}_9x_6,$$

then by 16) in Lemma 2.29 and 2) in Lemma 2.30

$$2z_9 + x_{10} + b_8 + c_8 + b_5 + c_5 = \bar{x}_9x_6. \tag{6}$$

By the associative law and 13) in Lemma 2.32 and 12) in Lemma 2.34:

$$b_3(y_3\bar{x}_6) = \bar{x}_9\bar{x}_6,$$
$$b_3\bar{d}_3 + x_{15}b_3 = \bar{x}_9\bar{x}_6,$$

so by 3) in Lemma 2.29 and 8) in Lemma 2.27

$$\bar{x}_6\bar{x}_9 = 2y_9 + 2y_{15} + b_6. \tag{7}$$

By the associative law and 3) in Lemma 2.29 and 9) in Lemma 2.31:

$$b_3(\bar{d}_3x_6) = (b_3\bar{d}_3)x_6 = x_6y_9,$$
$$b_3y_{15} + b_3y_3 = x_6y_9. \tag{8}$$

By 16) in Lemma 2.29 and 5) in Lemma 2.31 and the associative law:

$$x_6(b_3d_3) = b_3(x_6d_3),$$
$$x_6z_9 = x_{10}b_3 + c_8b_3,$$

then by 1) in Lemma 2.27 and 4) in Lemma 2.31:

$$x_6z_9 = x_{15} + x_6 + x_9 + x_9 + x_{15} = 2x_{15} + 2x_9 + x_6. \tag{9}$$

By (1) and 4) in Lemma 2.30 and 5) in Lemma 2.35 and 7) in Lemma 2.31:

$$(x_6c_8, x_{15}) = (x_6, c_8x_{15}) = 2, \qquad (x_6c_8, x_6) = (x_6\bar{x}_6, c_8) = 1,$$
$$(x_6c_8, x_9) = (x_6, c_8x_9) = 1, \qquad (x_6c_8, \bar{d}_3) = (c_8d_3, \bar{x}_6) = 1,$$

then

$$x_6c_8 = 2x_{15} + x_6 + x_9 + \bar{d}_3. \tag{10}$$

By the associative law and 9) in Lemma 2.31 and 1) in Lemma 2.33:

$$(d_3\bar{x}_6)y_9 = (d_3y_9)\bar{x}_6,$$
$$y_{15}y_9 + y_9y_3 = b_3\bar{x}_6 + x_{15}\bar{x}_6 + x_9\bar{x}_6,$$

then by (6) and 6), 9) in Lemma 2.33 and 3) in Lemma 2.26:

$$\begin{aligned} y_{15}y_9 + z_9 + b_8 + x_{10} &= b_8 + x_{10} + 2b_8 + 3x_{10} + 2z_9 + 2c_8 \\ &\quad + b_5 + c_5 + 2z_9 + x_{10} + b_8 + c_8 + b_5 + c_5, \\ y_{15}y_9 &= 4x_{10} + 3b_8 + 3z_9 + 3c_8 + 2c_5 + 2b_5. \end{aligned}$$

□

Lemma 2.37 *Let (A, B) satisfy Hypothesis 2.4 and $(b_8b_3, b_8b_3) = 3$, then Lemma 2.36 holds and we obtain the following:*

1) $y_3y_{15} = x_{10} + b_8 + b_5 + c_5 + c_8 + z_9$,
2) $y_{15}^2 = 1 + 3b_5 + 6x_{10} + 6z_9 + 5b_8 + 3c_5 + 5c_8$,
3) $x_{10}y_{15} = 6y_{15} + 3b_6 + 4y_9 + c_3 + y_3$,
4) $z_9x_{10} = 3x_{10} + b_5 + c_5 + 2b_8 + 2z_9 + 2c_8$,
5) $y_9x_{10} = 4y_{15} + 2y_9 + y_3 + b_6 + c_3$,
6) $y_9c_8 = 3y_{15} + 2y_9 + b_6 + c_3$,
7) $z_9y_{15} = 6y_{15} + 2b_6 + 3y_9 + c_3 + y_3$,
8) $x_{10}x_{15} = 3x_6 + 6x_{15} + b_3 + \bar{d}_3 + 4x_9$,
9) $x_{10}x_9 = 4x_{15} + x_6 + 2x_9 + b_3 + \bar{d}_3$,
10) $b_6x_9 = 2\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6$,
11) $b_8y_9 = 3y_{15} + y_3$,
12) $\bar{x}_6d_3 = y_{15} + y_3$,
13) $b_8y_{15} = 5y_{15} + 2b_6 + 3y_9 + c_3 + y_3$,
14) $x_{15}z_9 = 2x_6 + 6x_{15} + b_3 + \bar{d}_3 + 3x_9$,
15) $x_9y_{15} = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3$,
16) $y_{15}c_8 = 2b_6 + 5y_{15} + c_3 + y_3 + 3y_9$.

Proof By the associative law and 9) in Lemma 2.31 and 15) in Lemma 2.29 and 7) in Lemma 2.27:

$$d_3^2\bar{x}_6^2 = y_{15}^2 + 2y_{15}y_3 + y_3^2,$$
$$(b_6 + y_3)(2b_6 + y_{15} + y_9) = y_{15}^2 + 2y_{15}y_3 + y_3^2,$$
$$2b_6^2 + y_{15}b_6 + b_6y_9 + 2b_6y_3 + y_3y_{15} + y_3y_9 = y_{15}^2 + 2y_{15}y_3 + y_3^2,$$

by 3) in Lemma 2.30 and 7), 9), 10) in Lemma 2.32 and 9) in Lemma 2.33 and 17) in Lemma 2.29:

$$y_{15}^2 + 1 + c_8 + y_{15}y_3 = 2 + 2b_8 + 2z_9 + 2c_8 + 2b_5 + 2c_5 + 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5 + b_8 + 2z_9 + b_5 + c_5 + x_{10} + c_8. \quad (1)$$

By the associative law and 11), 15) in Lemma 2.29:

$$(d_3^2)y_{15} = (d_3y_{15})d_3, \qquad b_6y_{15} + y_3y_{15} = 2x_{15}d_3 + x_6d_3 + x_9d_3,$$

so by 8), 9) in Lemma 2.32 and 5) in Lemma 2.31 and 8) in Lemma 2.33:

$$2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5 + y_3y_{15} = 2b_8 + 2z_9 + 2b_5 + 2c_5 + 2x_{10} + 2c_8 + x_{10} + c_8 + x_{10} + b_8 + z_9,$$
$$y_3y_{15} = x_{10} + b_8 + b_5 + c_5 + c_8 + z_9. \quad (2)$$

Then by (1),

$$y_{15}^2 = 1 + 3b_5 + 6x_{10} + 6z_9 + 5b_8 + 3c_5 + 5c_8. \quad (3)$$

By 9) in Lemma 2.32 and 11) in Lemma 2.36 and 8) in Lemma 2.30 and (2):

$$(x_{10}y_{15}, y_{15}) = (x_{10}, y_{15}^2) = 6, \qquad (x_{10}y_{15}, b_6) = (x_{10}, y_{15}b_6) = 3,$$
$$(x_{10}y_{15}, y_9) = (x_{10}, y_{15}y_9) = 4, \qquad (x_{10}y_{15}, c_3) = (x_{10}, y_{15}c_3) = 1,$$
$$(y_{15}x_{10}, y_3) = (x_{10}, y_{15}y_3) = 1.$$

So

$$x_{10}y_{15} = 6y_{15} + 3b_6 + 4y_9 + c_3 + y_3. \quad (4)$$

By 16) in Lemma 2.29 and 7) in Lemma 2.33:

$$b_3(d_3x_{10}) = (b_3d_3)x_{10} = z_9x_{10},$$
$$\bar{x}_{15}b_3 + \bar{x}_6b_3 + \bar{x}_9b_3 = z_9x_{10},$$

then by 2) in Lemma 2.30 and 3) in Lemma 2.26 and 2) in Lemma 2.31:

$$x_{10} + b_8 + c_8 + z_9 + b_5 + c_5 + b_8 + x_{10} + z_9 + x_{10} + c_8 = z_9x_{10},$$
$$z_9x_{10} = 3x_{10} + b_5 + c_5 + 2b_8 + 2z_9 + 2c_8. \quad (5)$$

□

Lemma 2.38 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8b_3, b_8b_3) = 3$, *then* $b_5 \neq \bar{c}_5$.

Proof Assume $b_5 = \bar{c}_5$. By 2) in Lemma 2.30, $(\bar{b}_3b_5, \bar{x}_{15}) = (b_5, b_3\bar{x}_{15}) = 1$, $(b_3b_5, x_{15}) = (b_3\bar{x}_{15}, \bar{b}_5) = 1$, then

$$\bar{b}_3b_5 = \bar{x}_{15}, \tag{1}$$
$$b_3b_5 = x_{15}. \tag{2}$$

By 9), 10) in Lemma 2.32 and 3) in Lemma 2.30:

$$(b_6b_5, y_{15}) = (b_5, b_6y_{15}) = 1,$$
$$(b_6b_5, y_9) = (b_5, b_6y_9) = 1,$$
$$(b_6b_5, b_6) = (b_5, b_6^2) = 1,$$

then

$$b_6b_5 = y_{15} + y_9 + b_6. \tag{3}$$

By 4) in Lemma 2.30 and 6) in Lemma 2.33 and 6) in Lemma 2.36:

$$(x_6b_5, x_6) = (b_5, \bar{x}_6x_6) = 1,$$
$$(x_6b_5, x_{15}) = (b_5, \bar{x}_6x_{15}) = 1,$$
$$(x_6b_5, x_9) = (b_5, \bar{x}_6x_9) = 1,$$

then

$$x_6b_5 = x_6 + x_{15} + x_9. \tag{4}$$

By 6) in Lemma 2.33 and 6) in Lemma 2.36 and 4) in Lemma 2.30:

$$(\bar{x}_6b_5, \bar{x}_{15}) = (b_5, x_6\bar{x}_{15}) = 1,$$
$$(\bar{x}_6b_5, \bar{x}_9) = (\bar{x}_6x_9, \bar{b}_5) = 1,$$
$$(\bar{x}_6b_5, \bar{x}_6) = (\bar{x}_6x_6, \bar{b}_5) = 1$$

then

$$\bar{x}_6b_5 = \bar{x}_{15} + \bar{x}_9 + \bar{x}_6. \tag{5}$$

By 6) in Lemma 2.33 and 8), 11) in Lemma 2.32 and 2) in Lemma 2.30 and 3) in Lemma 2.36:

$$(x_{15}b_5, x_6) = (b_5, \bar{x}_{15}x_6) = 1,$$
$$(x_{15}b_5, \bar{d}_3) = (x_{15}d_3, \bar{b}_5) = 1,$$
$$(x_{15}b_5, x_{15}) = (b_5, \bar{x}_{15}x_{15}) = 3,$$
$$(x_{15}b_5, x_9) = (b_5, \bar{x}_{15}x_9) = 2,$$
$$(x_{15}b_5, b_3) = (x_{15}\bar{b}_3, \bar{b}_5) = 1,$$

then

$$x_{15}b_5 = x_6 + 3x_{15} + 2x_9 + b_3 + \bar{d}_3. \tag{6}$$

By 6) in Lemma 2.33 and 8), 11) in Lemma 2.32 and 3) in Lemma 2.36 and 2) in Lemma 2.30:

$$\begin{aligned}
(\bar{x}_{15}b_5, \bar{x}_6) &= (b_5, x_{15}\bar{x}_6) = 1,\\
(\bar{x}_{15}b_5, d_3) &= (b_5, x_{15}d_3) = 1,\\
(\bar{x}_{15}b_5, \bar{x}_{15}) &= (\bar{x}_{15}x_{15}, \bar{b}_5) = 3,\\
(\bar{x}_{15}b_5, \bar{x}_9) &= (\bar{x}_{15}x_9, \bar{b}_5) = 1,\\
(\bar{x}_{15}b_5, \bar{b}_3) &= (\bar{x}_{15}b_3, \bar{b}_5) = 1,
\end{aligned}$$

then

$$\bar{x}_{15}b_5 = \bar{x}_6 + 3\bar{x}_{15} + 2\bar{x}_9 + \bar{b}_3 + d_3. \tag{7}$$

By 3), 6) in Lemma 2.36 and 7) in Lemma 2.35:

$$\begin{aligned}
(x_9b_5, x_{15}) &= (x_9\bar{x}_{15}, \bar{b}_5) = 2,\\
(x_9b_5, x_6) &= (b_5, \bar{x}_9x_6) = 1,\\
(x_9b_5, x_9) &= (x_9\bar{x}_9, \bar{b}_5) = 1,
\end{aligned}$$

then

$$x_9b_5 = 2x_{15} + x_6 + x_9. \tag{8}$$

By 3), 6) in Lemma 2.36 and 7) in Lemma 2.35:

$$\begin{aligned}
(\bar{x}_9b_5, \bar{x}_{15}) &= (\bar{x}_9x_{15}, \bar{b}_5) = 2,\\
(\bar{x}_9b_5, \bar{x}_6) &= (b_5, \bar{x}_9x_6) = 1,\\
(x_9b_5, x_9) &= (x_9\bar{x}_9x_9, \bar{b}_5) = 1,
\end{aligned}$$

then

$$\bar{x}_9b_5 = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9. \tag{9}$$

By 9) in Lemma 2.32 and 1), 2) in Lemma 2.37 and 11) in Lemma 2.36 and 8) in Lemma 2.30

$$\begin{aligned}
(y_{15}b_5, b_6) &= (b_5, y_{15}b_6) = 1,\\
(y_{15}b_5, y_{15}) &= (b_5, y_{15}^2) = 3,\\
(y_{15}b_5, y_3) &= (y_{15}y_3, \bar{b}_5) = 1,\\
(y_{15}b_5, c_3) &= (b_5, y_{15}c_3) = 1,\\
(y_{15}b_5, y_9) &= (b_5, y_{15}y_9) = 2,
\end{aligned}$$

then

$$y_{15}b_5 = b_6 + 3y_{15} + 2y_9 + c_3 + y_3. \tag{10}$$

By 11) in Lemma 2.36 and 10) in Lemma 2.32 and 4) in Lemma 2.35:

$$(y_9b_5, y_{15}) = (b_5, y_9y_{15}) = 2,$$
$$(y_9b_5, b_6) = (b_5, y_9b_6) = 1,$$
$$(y_9b_5, y_9) = (b_5, y_9^2) = 1,$$

then

$$y_9b_5 = 2y_{15} + b_6 + y_9. \tag{11}$$

By 8) in Lemma 2.32,

$$(d_3b_5, \bar{x}_{15}) = (d_3x_{15}, \bar{b}_5) = 1,$$
$$(\bar{d}_3b_5, x_{15}) = (b_5, d_3x_{15}) = 1,$$

then

$$d_3b_5 = \bar{x}_{15}, \tag{12}$$
$$\bar{d}_3b_5 = x_{15}. \tag{13}$$

Then by 16) in Lemma 2.29 and associative law:

$$b_3(b_5d_3) = (b_3d_3)b_5, \qquad \bar{x}_{15}b_3 = z_9b_5,$$

then by 2) in Lemma 2.30,

$$z_9b_5 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5. \tag{14}$$

By 1) in Lemma 2.37, $(y_3b_5, y_{15}) = (b_5, y_3y_{15}) = 1$, then

$$y_3b_5 = y_{15}. \tag{15}$$

Then $1 = (y_3b_5, y_3\bar{b}_5) = (b_5^2, y_3^2)$, then by 17) in Lemma 2.29,

$$(b_5^2, c_8) = 1, \tag{16}$$

By (1), (2) we get $1 = (b_3\bar{b}_5, b_3b_5) = (b_3\bar{b}_3, b_5^2)$, then by Hypothesis 2.4,

$$(b_5^2, b_8) = 1. \tag{17}$$

By (2), (12) $1 = (\bar{d}_3\bar{b}_5, b_3b_5) = (\bar{d}_3\bar{b}_3, b_5^2)$, then by 16) in Lemma 2.29 $(b_5^2, z_9) = 1$, so by (16), (17), we obtain

$$b_5^2 = c_8 + b_8 + z_9. \tag{18}$$

By (1) $1=(\bar{b}_3 b_5, \bar{b}_3 b_5)=(\bar{b}_3 b_3, \bar{b}_5 b_5)$, then by Hypothesis 2.4,

$$(b_5\bar{b}_5, b_8)=0. \tag{19}$$

By (2), (4), $1=(x_6 b_5, b_3 b_5)=(x_6\bar{b}_3, \bar{b}_5 b_5)$, then by 3) in Lemma 2.26 and (19):

$$(\bar{b}_5 b_5, x_{10})=1.$$

By (13), $3=(b_6 b_5, b_6 b_5)=(b_6^2, \bar{b}_5 b_5)$, so by 3) in Lemma 2.30, $m_5=b_5$, or $m_5=\bar{b}_5$, in the two cases $(\bar{b}_5 b_5, b_5)\neq 0$ or $(\bar{b}_5 b_5, \bar{b}_5)\neq 0$, a contradiction to (18). □

Lemma 2.39 *Let* (A, B) *satisfy Hypothesis* 2.4 *and* $(b_8 b_3, b_8 b_3)=3$, *then Lemma* 2.37 *holds and we obtain the following*:

1) $b_3 b_5 = x_{15}$;
2) $b_3 b_5 = x_{15}$;
3) $b_6 b_5 = b_6 + y_{15} + y_9$;
4) $b_6 c_5 = b_6 + y_{15} + y_9$;
5) $b_5 x_6 = x_{15} + x_9 + x_6$;
6) $c_5 x_6 = x_{15} + x_9 + x_6$;
7) $x_{15} b_5 = x_6 + \bar{d}_3 + 3x_{15} + b_3 + 2x_9$;
8) $x_{15} c_5 = x_6 + \bar{d}_3 + 3x_{15} + b_3 + 2x_9$;
9) $x_9 b_5 = 2x_{15} + x_6 + x_9$;
10) $x_9 c_5 = 2x_{15} + x_6 + x_9$;
11) $y_{15} b_5 = b_6 + y_3 + 3y_{15} + 2y_9 + c_3$;
12) $y_{15} c_5 = b_6 + y_3 + 3y_{15} + 2y_9 + c_3$;
13) $y_9 b_5 = 2y_{15} + b_6 + y_9$;
14) $y_9 c_5 = 2y_{15} + b_6 + y_9$;
15) $d_3 b_5 = \bar{x}_{15}$;
16) $d_3 c_5 = \bar{x}_{15}$;
17) $z_9 b_5 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5$;
18) $z_9 c_5 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5$;
19) $y_3 b_5 = y_{15}$;
20) $y_3 c_5 = y_{15}$;
21) $c_5 b_5 = c_8 + b_8 + z_9$;
22) $b_8 b_5 = x_{10} + b_8 + c_8 + c_5 + z_9$;
23) $b_8 c_5 = x_{10} + b_8 + c_8 + c_5 + z_9$;
24) $b_5^2 = 1 + b_5 + z_9 + x_{10}$;
25) $c_5^2 = 1 + c_5 + z_9 + x_{10}$;
26) $b_5 x_{10} = 2x_{10} + b_8 + z_9 + b_5 + c_8$;
27) $c_5 x_{10} = 2x_{10} + b_8 + z_9 + c_5 + c_8$;
28) $c_8 b_5 = b_8 + z_9 + c_5 + x_{10} + c_8$;
29) $c_8 c_5 = b_8 + z_9 + x_{10} + c_8 + b_5$.

Proof By 2) in Lemma 2.30, $(b_3 b_5, x_{15})=(b_3\bar{x}_{15}, b_5)=1$, $(b_3 c_5, x_{15})=(b_3\bar{x}_{15}, c_5)=1$, then

$$b_3 b_5 = x_{15}, \tag{1}$$

$$b_3c_5 = x_{15}. \tag{2}$$

By 9), 10) in Lemma 2.32 and 3) in Lemma 2.30:

$$(b_6b_5, b_6) = (b_6^2, b_5) = 1, \qquad (b_6c_5, b_6) = (b_6^2, c_5) = 1,$$
$$(b_6b_5, y_{15}) = (b_5, b_6y_{15}) = 1, \qquad (b_6c_5, y_{15}) = (c_5, b_6y_{15}) = 1,$$
$$(b_6b_5, y_9) = (b_5, b_6y_9) = 1, \qquad (b_6c_5, y_9) = (c_5, b_6y_9) = 1,$$

then

$$b_5b_6 = b_6 + y_{15} + y_9, \tag{3}$$
$$b_6c_5 = y_{15} + y_9 + b_6. \tag{4}$$

By 4) in Lemma 2.30 and 6) in Lemma 2.33 and 6) in Lemma 2.36:

$$(b_5x_6, x_6) = (b_5, \bar{x}_6x_6) = 1, \qquad (c_5x_6, x_6) = (c_5, \bar{x}_6x_6) = 1,$$
$$(b_5x_6, x_{15}) = (x_6\bar{x}_{15}, b_5) = 1, \qquad (c_5x_6, x_{15}) = (x_6\bar{x}_{15}, c_5) = 1,$$
$$(b_5x_6, x_9) = (x_6\bar{x}_9, b_5) = 1, \qquad (c_5x_6, x_9) = (x_6\bar{x}_9, c_5) = 1,$$

then

$$b_5x_6 = x_6 + x_{15} + x_9, \tag{5}$$
$$c_5x_6 = x_{15} + x_9 + x_6. \tag{6}$$

By 6) in Lemma 2.33 and 8), 11) in Lemma 2.32 and 2) in Lemma 2.30 and 3) in Lemma 2.36:

$$(x_{15}b_5, x_6) = (b_5, \bar{x}_{15}x_6) = 1, \qquad (x_{15}c_5, x_6) = (c_5, \bar{x}_{15}x_6) = 1,$$
$$(x_{15}b_5, \bar{d}_3) = (x_{15}d_3, b_5) = 1, \qquad (x_{15}c_5, \bar{d}_3) = (x_{15}d_3, c_5) = 1,$$
$$(x_{15}b_5, x_{15}) = (x_{15}\bar{x}_{15}, b_5) = 3, \qquad (x_{15}c_5, x_{15}) = (x_{15}\bar{x}_{15}, c_5) = 3,$$
$$(x_{15}b_5, b_3) = (b_5, \bar{x}_{15}b_3) = 1, \qquad (x_{15}c_5, b_3) = (c_5, \bar{x}_{15}b_3) = 1,$$
$$(x_{15}b_5, x_9) = (b_5, \bar{x}_{15}x_9) = 2, \qquad (x_{15}c_5, x_9) = (x_{15}\bar{x}_9, c_5) = 2,$$

then

$$x_{15}b_5 = x_6 + \bar{d}_3 + 3x_{15} + b_3 + 2x_9, \tag{7}$$
$$x_{15}c_5 = x_6 + \bar{d}_3 + 3x_{15} + b_3 + 2x_9. \tag{8}$$

By 3), 6) in Lemma 2.36 and 7) in Lemma 2.35:

$$(x_9b_5, x_{15}) = (b_5, \bar{x}_9x_{15}) = 2, \qquad (x_9c_5, x_{15}) = (c_5, \bar{x}_9x_{15}) = 2,$$
$$(x_9b_5, x_6) = (b_5, \bar{x}_9x_6) = 1, \qquad (x_9c_5, x_6) = (c_5, \bar{x}_9x_6) = 1,$$
$$(x_9b_5, x_9) = (b_5, \bar{x}_9x_9) = 1, \qquad (x_9c_5, x_9) = (c_5, \bar{x}_9x_9) = 1,$$

then

$$x_9b_5 = 2x_{15} + x_6 + x_9, \tag{9}$$

$$x_9c_5 = 2x_{15} + x_6 + x_9. \tag{10}$$

By 9) in Lemma 2.32 and 1), 2) in Lemma 2.37 and 11) in Lemma 2.36 and 8) in Lemma 2.30:

$$\begin{aligned}
(y_{15}b_5, b_6) &= (y_{15}b_6, b_5) = 1, & (y_{15}c_5, b_6) &= (y_{15}b_6, c_5) = 1,\\
(y_{15}b_5, y_3) &= (y_{15}y_3, b_5) = 1, & (y_{15}c_5, y_3) &= (y_{15}y_3, c_5) = 1,\\
(y_{15}b_5, y_{15}) &= (b_5, y_{15}^2) = 3, & (y_{15}c_5, y_{15}) &= (y_{15}^2, c_5) = 3,\\
(y_{15}b_6, c_3) &= (y_{15}c_3, b_6) = 1, & (y_{15}c_5, c_3) &= (y_{15}c_3, c_5) = 1,\\
(y_{15}b_5, y_9) &= (y_{15}y_9, b_5) = 2, & (y_{15}c_5, y_9) &= (c_5, y_{15}y_9) = 2,
\end{aligned}$$

then

$$y_{15}b_5 = b_6 + y_3 + 3y_{15} + 2y_9 + c_3, \tag{11}$$

$$y_{15}c_5 = b_6 + y_3 + 3y_{15} + 2y_9 + c_3. \tag{12}$$

By 11) in Lemma 2.36 and 10) in Lemma 2.32 and 4) in Lemma 2.35:

$$\begin{aligned}
(y_9b_5, y_{15}) &= (b_5, y_9y_{15}) = 2, & (y_9c_5, y_{15}) &= (c_5, y_9y_{15}) = 2,\\
(y_9b_5, b_6) &= (y_9b_6, b_5) = 1, & (y_9c_5, b_6) &= (y_9b_6, c_5) = 1,\\
(y_9b_5, y_9) &= (b_5, y_9^2) = 1, & (y_9c_5, y_9) &= (c_5, y_9^2) = 1,
\end{aligned}$$

then

$$y_9b_5 = 2y_{15} + b_6 + y_9, \tag{13}$$

$$y_9c_5 = 2y_{15} + b_6 + y_9. \tag{14}$$

By 8) in Lemma 2.32, $(d_3b_5, \bar{x}_{15}) = (d_3x_{15}, b_5) = 1$, $(d_3c_5, \bar{x}_{15}) = (d_3x_{15}, c_5) = 1$. Then

$$d_3b_5 = \bar{x}_{15}, \tag{15}$$

$$d_3c_5 = \bar{x}_{15}. \tag{16}$$

So by the associative law and 16) in Lemma 2.29, we get $(b_3d_3)b_5 = b_3(d_3b_5)$, $(b_3d_3)c_5 = b_3(d_3c_5)$, $z_9b_5 = b_3\bar{x}_{15}$, $z_9c_5 = b_3\bar{x}_{15}$, then by 2) in Lemma 2.30,

$$z_9b_5 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5, \tag{17}$$

$$z_9c_5 = x_{10} + b_8 + c_8 + z_9 + b_5 + c_5. \tag{18}$$

By 1) in Lemma 2.37, $(y_3b_5, y_{15}) = (b_5, y_3y_{15}) = 1$, $(y_3c_5, y_{15}) = (c_5, y_3y_{15}) = 1$, then

$$y_3b_5 = \bar{y}_{15}, \tag{19}$$

$$y_3c_5 = \bar{y}_{15}. \tag{20}$$

So $(y_3^2, b_5^2) = (y_3b_5, y_3b_5) = 1$, $(y_3^2, c_5b_5) = (y_3b_5, y_3c_5) = 1$, then by 17) in Lemma 2.29, $(c_5b_5, c_8) = 1$. By (1), (2), $(\bar{b}_3b_3, b_5c_5) = (b_3b_5, b_3c_5) = 1$. then by Hypothesis 2.4, $(c_5b_5, b_8) = 1$. By (17),

$$c_5b_5 = c_8 + b_8 + z_9. \tag{21}$$

By the associative and Hypothesis 2.4 and (1), (2), it follows that $(b_3\bar{b}_3)b_5 = (b_3b_5)\bar{b}_3$, $(b_3\bar{b}_3)c_5 = (b_3c_5)\bar{b}_3$, $b_5 + b_8b_5 = x_{15}\bar{b}_3$, $c_5 + b_8c_5 = x_{15}\bar{b}_3$, then by 2) in Lemma 2.30,

$$b_8b_5 = x_{10} + b_8 + c_8 + c_5 + z_9, \tag{22}$$

$$b_8c_5 = x_{10} + b_8 + c_8 + b_5 + z_9. \tag{23}$$

By 5) in Lemma 2.30, $(x_{10}b_5, x_{10}) = (b_5, x_{10}^2) = 2$, $(x_{10}c_5, x_{10}) = (c_5, x_{10}^2) = 2$. By (22) and (23), $(x_{10}b_5, b_8) = (x_{10}, b_5b_8) = 1$, $(x_{10}c_5, b_8) = (x_{10}, c_5b_8) = 1$. By (17), (18), $(x_{10}b_5, z_9) = (x_{10}, z_9b_5) = 1$, $(x_{10}c_5, b_8) = (x_{10}, c_5b_8) = 1$, $(b_5^2, z_9) = (b_5, b_5z_9) = 1$ and $(c_5^2, z_9) = (c_5, c_5z_9) = 1$.

By (19), (20), we obtain $(y_3^2, b_5^2) = (y_3b_5, y_3b_5) = 1$, $(y_3^2, c_5^2) = (y_3c_5, y_3c_5) = 1$, then by 17) in Lemma 2.29 $(b_5^2, c_8) = 0$, $(c_5^2, c_8) = 0$.

By (1) and (2), we see that $(\bar{b}_3b_3, b_5^2) = (b_3b_5, b_3b_5) = 1$, $(\bar{b}_3b_3, c_5^2) = (b_3c_5, b_3c_5) = 1$, then by Hypothesis 2.4, $(b_5^2, b_8) = 0$, $(c_5^2, b_8) = 0$ holds.

By (5), (6), (15) and (16), $(b_5^2, d_3x_6) = (b_5\bar{d}_3, b_5x_6) = 1$, $(c_5^2, d_3x_6) = (c_5\bar{d}_3, c_5x_6) = 1$, so by 5) in Lemma 2.31, $(b_5^2, x_{10}) = 1$, $(c_5^2, x_{10}) = 1$.

By (5), (21) and (6) we have the following inner-products: $(b_5^2, \bar{x}_6x_6) = (b_5x_6, b_5x_6) = 3$, $(c_5^2, \bar{x}_6x_6) = (c_5x_6, c_5x_6) = 3$. $(c_5^2, b_5) = (c_5, c_5b_5) = 0$ and $(b_5^2, c_5) = (b_5, b_5c_5) = 0$, then by 4) in Lemma 2.30 $(b_5^2, b_5) = 1$, $(c_5^2, c_5) = 1$, hence

$$b_5^2 = 1 + b_5 + z_9 + x_{10}, \qquad c_5^2 = 1 + b_5 + z_9 + x_{10}.$$

So $(b_5x_{10}, b_5) = (x_{10}, b_5^2) = 1$, $(c_5x_{10}, c_5) = (x_{10}, c_5^2) = 1$.

By 14) in Lemma 2.29 and (15), (16), one has $b_5(\bar{d}_3d_3) = \bar{d}_3(b_5d_3)$, $c_5(\bar{d}_3d_3) = \bar{d}_3(c_5d_3)$. Hence $b_5 + c_8b_5 = \bar{x}_{15}\bar{d}_3$, $c_5 + c_8c_5 = \bar{d}_3\bar{x}_{15}$.

Therefore by 8) in Lemma 2.32, we have the equations $c_8b_5 = b_8 + z_9 + c_5 + x_{10} + c_8$ and $c_8c_5 = b_8 + z_9 + x_{10} + c_8 + b_5$. Consequently $(b_5x_{10}, c_8) = (x_{10}, b_5c_8) = (x_{10}, c_5c_8) = 1$.

Finally, we get $b_5x_{10} = c_8 + 2x_{10} + b_8 + z_9 + b_8 + z_9 + b_5$ and $c_5x_{10} = 2x_{10} + b_8 + z_9 + c_5 + c_8$. ∎

Theorem 2.10 *Let (A, B) be a NITA generated by a nonreal element $b_3 \in B$ of degree 3 and without non-identity basis element of degree 1 or 2, such that: $\bar{b}_3 b_3 = 1 + b_8$ and $b_3^2 = c_3 + b_6$, $c_3 = \bar{c}_3$, $(b_8 b_3, b_8 b_3) = 3$. Then (A, B) is a Table Algebra of dimension 22:*

$$B = \{b_1, \bar{b}_3, b_3, c_3, b_6, x_6, \bar{x}_6, b_8, x_{15}, \bar{x}_{15}, x_{10}, x_9, \bar{x}_9, y_{15}, y_9, d_3, \bar{d}_3, z_9, c_8, b_5, c_5, y_3\}$$

and B has increasing series of table subsets $\{b_1\} \subseteq \{b_1, b_8, x_{10}, z_9, c_8, b_5, c_5\} \subseteq B$ defined by:

1) $b_3\bar{b}_3 = 1 + b_8$;
2) $b_3^2 = c_3 + b_6$;
3) $b_3 c_3 = \bar{b}_3 + \bar{x}_6$;
4) $b_3 b_6 = \bar{b}_3 + \bar{x}_{15}$;
5) $b_3 x_6 = c_3 + y_{15}$;
6) $b_3 \bar{x}_6 = b_8 + x_{10}$;
7) $b_3 b_8 = x_6 + b_3 + x_{15}$;
8) $b_3 x_{15} = 2y_{15} + b_6 + y_9$;
9) $b_3 x_{10} = x_{15} + x_6 + x_9$;
10) $b_3 \bar{x}_{15} = x_{10} + b_8 + z_9 + c_8 + b_5 + c_5$;
11) $b_3 x_9 = y_9 + y_{15} + y_3$;
12) $b_3 \bar{x}_9 = x_{10} + z_9 + c_8$;
13) $b_3 y_{15} = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9$;
14) $b_3 y_9 = d_3 + \bar{x}_{15} + \bar{x}_9$;
15) $b_3 d_3 = z_9$;
16) $b_3 \bar{d}_3 = y_9$;
17) $b_3 z_9 = x_9 + \bar{d}_3 + x_{15}$;
18) $b_3 c_8 = x_{15} + x_9$;
19) $b_3 b_5 = x_{15}$;
20) $b_3 c_5 = x_{15}$;
21) $b_3 y_3 = \bar{x}_9$;
22) $c_3^2 = 1 + b_8$;
23) $c_3 b_6 = b_8 + x_{10}$;
24) $c_3 x_6 = \bar{b}_3 + \bar{x}_{15}$;
25) $c_3 b_8 = b_6 + c_3 + y_{15}$;
26) $c_3 x_{15} = 2\bar{x}_{15} + \bar{x}_6 + \bar{x}_9$;
27) $c_3 x_{10} = b_6 + y_{15} + y_9$;
28) $c_3 x_9 = d_3 + \bar{x}_{15} + \bar{x}_9$;
29) $c_3 y_{15} = x_{10} + b_8 + z_9 + c_8 + b_5 + c_5$;
30) $c_3 y_9 = z_9 + x_{10} + c_8$;
31) $c_3 d_3 = x_9$;
32) $c_3 z_9 = y_3 + y_9 + y_{15}$;
33) $c_3 c_8 = y_9 + y_{15}$;
34) $c_3 b_5 = y_{15}$;
35) $c_3 c_5 = y_{15}$;

36) $c_3y_3 = z_9$;
37) $b_6^2 = 1 + b_8 + z_9 + c_8 + b_5 + c_5$;
38) $b_6x_6 = 2\bar{x}_6 + \bar{x}_{15} + \bar{x}_9$;
39) $b_6b_8 = 2y_{15} + b_6 + y_9 + c_3$;
40) $b_6x_{15} = 4\bar{x}_{15} + \bar{x}_6 + 2\bar{x}_9 + \bar{b}_3 + d_3$;
41) $b_6x_{10} = c_3 + 3y_{15} + y_9 + y_3$;
42) $b_6x_9 = 2\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6$;
43) $b_6y_{15} = 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5$;
44) $b_6y_9 = b_8 + 2z_9 + c_8 + x_{10} + b_5 + c_5$;
45) $b_6d_3 = \bar{d}_3 + x_{15}$;
46) $b_6z_9 = 2y_{15} + 2y_9 + b_6$;
47) $b_6c_8 = b_6 + y_3 + y_9 + 2y_{15}$;
48) $b_6b_5 = y_{15} + y_9 + b_6$;
49) $b_6c_5 = y_{15} + b_6 + y_9$;
50) $b_6y_3 = x_{10} + c_8$;
51) $x_6^2 = 2b_6 + y_{15} + y_9$;
52) $x_6\bar{x}_6 = 1 + b_8 + z_9 + c_8 + b_5 + c_5$;
53) $x_6b_8 = b_3 + 2x_{15} + x_6 + x_9$;
54) $x_6x_{15} = 4y_{15} + b_6 + 2y_9 + c_3 + y_3$;
55) $x_6x_{10} = b_3 + 3x_{15} + \bar{d}_3 + x_9$;
56) $x_6\bar{x}_{15} = 2b_8 + 3x_{10} + 2z_9 + 2c_8 + b_5 + c_5$;
57) $x_6x_9 = 2y_{15} + 2y_9 + b_6$;
58) $x_6\bar{x}_9 = x_{10} + b_8 + 2z_9 + c_8 + b_5 + c_5$;
59) $x_6y_{15} = 4\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6 + \bar{b}_3 + d_3$;
60) $x_6y_9 = 2\bar{x}_{15} + 2\bar{x}_9 + \bar{x}_6$;
61) $x_6d_3 = x_{10} + c_8$;
62) $x_6\bar{d}_3 = y_{15} + y_3$;
63) $x_6z_9 = 2x_9 + 2x_{15} + x_6$;
64) $x_6c_8 = 2x_{15} + x_6 + x_9 + \bar{d}_3$;
65) $x_6b_5 = x_6 + x_{15} + x_9$;
66) $x_6c_5 = x_{15} + x_9 + x_6$;
67) $x_6y_3 = d_3 + \bar{x}_{15}$;
68) $b_8^2 = 1 + 2b_8 + 2x_{10} + z_9 + c_8 + b_5 + c_5$;
69) $b_8x_{10} = 2x_{10} + 2z_9 + 2c_8 + 2b_8 + b_5 + c_5$;
70) $b_8x_9 = 3x_{15} + 2x_9 + x_6 + \bar{d}_3$;
71) $b_8y_{15} = 5y_{15} + 2b_6 + 3y_9 + c_3 + y_3$;
72) $b_8y_9 = 3y_{15} + 2y_9 + b_6 + y_3$;
73) $b_8d_3 = \bar{x}_{15} + \bar{x}_9$;
74) $b_8z_9 = 2x_{10} + b_8 + 2c_8 + 2z_9 + b_5 + c_5$;
75) $b_8c_8 = 2x_{10} + 2z_9 + b_8 + c_8 + b_5 + c_5$;
76) $b_8b_5 = b_8 + z_9 + c_5 + c_8 + x_{10}$;
77) $b_8c_5 = b_8 + x_{10} + c_8 + z_9 + b_5$;
78) $b_8y_3 = y_9 + y_{15}$;
79) $b_8x_{15} = 5x_{15} + 2x_6 + 3x_9 + b_3 + \bar{d}_3$;
80) $x_{15}^2 = 9y_{15} + 4b_6 + 6y_9 + 2c_3 + 2y_3$;

81) $x_{15}x_{10} = 3x_6 + 6x_{15} + b_3 + \bar{d}_3 + 4x_9$;
82) $x_{15}\bar{x}_{15} = 5b_8 + 6x_{10} + 6z_9 + 5c_8 + 3b_5 + 3c_5 + 1$;
83) $x_{15}x_9 = 6y_{15} + 3y_9 + 2b_6 + c_3 + y_3$;
84) $x_{15}\bar{x}_9 = 3b_8 + 4x_{10} + 3z_9 + 3c_8 + 2b_5 + 2c_5$;
85) $x_{15}y_{15} = 4\bar{x}_6 + 9\bar{x}_{15} + 6\bar{x}_9 + 2\bar{b}_3 + 2d_3$;
86) $x_{15}y_9 = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3$;
87) $x_{15}d_3 = b_8 + z_9 + b_5 + c_5 + x_{10} + c_8$;
88) $x_{15}\bar{d}_3 = 2y_{15} + y_9 + b_6$;
89) $x_{15}z_9 = 2x_6 + 6x_{15} + b_3 + \bar{d}_3 + 3x_9$;
90) $x_{15}c_8 = 5x_{15} + 2x_6 + 3x_9 + b_3 + \bar{d}_3$;
91) $x_{15}b_5 = x_6 + 3x_{15} + 2x_9 + b_3 + \bar{d}_3$;
92) $x_{15}c_5 = x_6 + 3x_{15} + 2x_9 + b_3 + \bar{d}_3$;
93) $x_{15}y_3 = \bar{x}_6 + 2\bar{x}_{15} + \bar{x}_9$;
94) $x_{10}^2 = 1 + 2b_8 + 2x_{10} + 2c_8 + 2b_5 + 2c_5 + 3z_9$;
95) $x_{10}x_9 = b_3 + 4x_{15} + \bar{d}_3 + 2x_9 + x_6$;
96) $x_{10}y_{15} = 6y_{15} + 3b_6 + 4y_9 + c_3 + y_3$;
97) $x_{10}y_9 = 4y_{15} + 2y_9 + y_3 + b_6 + c_3$;
98) $x_{10}d_3 = \bar{x}_{15} + \bar{x}_6 + \bar{x}_9$;
99) $x_{10}z_9 = 3x_{10} + b_5 + c_5 + 2b_8 + 2z_9 + 2c_8$;
100) $x_{10}c_8 = 2x_{10} + 2c_8 + 2b_8 + 2z_9 + b_5 + c_5$;
101) $x_{10}b_5 = b_8 + 2x_{10} + z_9 + c_8 + b_5$;
102) $x_{10}c_5 = b_8 + z_9 + c_5 + 2x_{10} + c_8$;
103) $x_{10}y_3 = y_{15} + y_9 + b_6$;
104) $x_9^2 = y_3 + c_3 + 2b_6 + 3y_{15} + 2y_9$;
105) $x_9\bar{x}_9 = 1 + 2b_8 + 2x_{10} + 2c_8 + 2z_9 + b_5 + c_5$;
106) $x_9y_{15} = 6\bar{x}_{15} + 2\bar{x}_6 + 3\bar{x}_9 + \bar{b}_3 + d_3$;
107) $x_9y_9 = d_3 + 3\bar{x}_{15} + 2\bar{x}_9 + \bar{b}_3 + 2\bar{x}_6$;
108) $x_9d_3 = x_{10} + b_8 + z_9$;
109) $x_9\bar{d}_3 = y_9 + c_3 + y_{15}$;
110) $x_9z_9 = 3x_{15} + 2x_9 + 2x_6 + b_3 + \bar{d}_3$;
111) $x_9b_5 = 2x_{15} + x_6 + x_9$;
112) $x_9c_5 = 2x_{15} + x_6 + x_9$;
113) $x_9y_3 = \bar{b}_3 + \bar{x}_9 + \bar{x}_{15}$;
114) $y_{15}^2 = 1 + 3b_5 + 6x_{10} + 6z_9 + 5b_8 + 3c_5 + 5c_8$;
115) $y_{15}y_9 = 4x_{10} + 3b_8 + 3z_9 + 3c_8 + 2c_5 + 2b_5$;
116) $y_{15}d_3 = 2x_{15} + x_6 + x_9$;
117) $y_{15}z_9 = 6y_{15} + 2b_6 + 3y_9 + c_3 + y_3$;
118) $y_{15}c_8 = 2b_6 + 5y_{15} + c_3 + y_3 + 3y_9$;
119) $y_{15}b_5 = b_6 + 3y_{15} + 2y_9 + c_3 + y_3$;
120) $y_{15}c_5 = b_6 + c_3 + 3y_{15} + y_3 + 2y_9$;
121) $y_{15}y_3 = b_5 + b_8 + z_9 + c_5 + x_{10} + c_8$;
122) $y_9^2 = 1 + 2z_9 + 2b_8 + 2c_8 + 2x_{10} + b_5 + c_5$;
123) $y_9d_3 = b_3 + x_{15} + x_9$;
124) $y_9z_9 = 3y_{15} + 2b_6 + 2y_9 + c_3 + y_3$;
125) $y_9c_8 = 3y_{15} + 2y_9 + b_6 + c_3$;

126) $y_9b_5 = 2y_{15} + b_6 + y_9$;
127) $y_9c_5 = 2y_{15} + y_9 + b_6$;
128) $y_9y_3 = z_9 + b_8 + x_{10}$;
129) $d_3\bar{d}_3 = 1 + c_8$;
130) $d_3^2 = b_6 + y_3$;
131) $d_3z_9 = \bar{b}_3 + \bar{x}_{15} + \bar{x}_9$;
132) $d_3c_8 = \bar{x}_{15} + d_3 + \bar{x}_6$;
133) $d_3b_5 = \bar{x}_{15}$;
134) $d_3c_5 = \bar{x}_{15}$;
135) $d_3y_3 = \bar{d}_3 + x_6$;
136) $z_9^2 = 1 + 2x_{10} + 2z_9 + 2b_8 + 2c_8 + b_5 + c_5$;
137) $z_9c_8 = 2x_{10} + 2b_8 + 2z_9 + b_5 + c_5 + c_8$;
138) $z_9b_5 = x_{10} + b_8 + z_9 + c_8 + b_5 + c_5$;
139) $z_9c_5 = x_{10} + b_8 + z_9 + c_8 + b_5 + c_5$;
140) $z_9y_3 = y_9 + c_3 + y_{15}$;
141) $c_8^2 = 1 + 2c_8 + 2x_{10} + b_8 + z_9 + b_5 + c_5$;
142) $c_8b_5 = b_8 + z_9 + c_5 + x_{10} + c_8$;
143) $c_8c_5 = b_8 + z_9 + x_{10} + c_8 + b_5$;
144) $c_8y_3 = b_6 + y_3 + y_{15}$;
145) $b_5^2 = 1 + b_5 + x_{10} + z_9$;
146) $b_5c_5 = b_8 + z_9 + c_8$;
147) $b_5y_3 = y_{15}$;
148) $c_5^2 = 1 + z_9 + x_{10} + c_5$;
149) $c_5y_3 = y_{15}$;
150) $y_3^2 = 1 + c_8$.

Proof The equations hold by Lemmas 2.25–2.37 and 2.39. □

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