

Chapter 2

Stability Analysis and Stabilization—Polytopic Representation Approach

2.1 Introduction

In this chapter, the problems of analysis and synthesis of linear control systems with input saturations are addressed by using the polytopic models for representing the saturated closed-loop system introduced in Sect. 1.7.1.

The basic idea behind using polytopic differential inclusions consists in guaranteeing some properties for a polytopic system that can be inherited by the actual saturated closed-loop system. The main advantage in this case is that stability and stabilization conditions can be obtained in a more tractable way, since the hard saturation nonlinearity disappears in the polytopic representation. However, it should be highlighted a crucial difference with respect to the case where polytopic differential inclusions are used to model uncertainties in a robust control framework. In that case, the polytopic model is in general valid in a global sense (i.e. it is valid in all state space) [45]. On the other hand, the key point when dealing with saturated systems consists in ensuring that the region of stability or the reachable region for the trajectories obtained using the polytopic model are included in the region of validity of the model, as discussed in Sect. 1.7.1. In other words, we should keep in mind that the polytopic differential inclusion represents the saturated system only locally.

First, we focus on the asymptotic stability analysis, i.e. the internal stability of the closed-loop system. Several conditions to ensure the local asymptotic stability of the closed-loop system are derived in the form of BMIs or LMIs. From these conditions, numerical algorithms (based on convex optimization problems) are proposed to determine regions of stability as large as possible, i.e. to obtain estimates of the region of attraction of the closed-loop system. We are mainly concerned with a quadratic approach and the consequent determination of ellipsoidal regions of stability in an LMI framework. In the end of the chapter, considering discrete-time systems, the determination of polyhedral regions of stability is also addressed. The results in this case are based on linear programming techniques.

From the proposed analysis conditions, results allowing the synthesis of control laws taking explicitly into account the actuator saturation possibility are derived.

The synthesis of state feedback control laws aiming at enlarging an estimate of the region of attraction or at ensuring some time-domain performance with a guaranteed region of asymptotic stability (RAS) is addressed. Considering the action of exogenous input signals on the system, solutions to the problem of designing a control law aiming at maximizing the tolerance of the closed-loop system to disturbances or at maximizing the disturbance rejection are also presented. Although the results are mainly focused on the state feedback control laws, conditions regarding the design of observer-based and dynamic output feedback control laws are briefly presented.

The extensions of the approach to address the analysis and synthesis in a robust context are briefly discussed considering norm-bounded and polytopic uncertainties. Finally, the discrete-time counterpart of the results is presented. In this case, the determination of polyhedral regions of stability is also considered.

2.2 Asymptotic Stability Analysis

In this section, we focus on the internal asymptotic stability analysis problem formulated in Sect. 1.6.1. We consider that the saturated closed-loop system is given by

$$\dot{x}(t) = Ax(t) + B \text{sat}(Kx(t)) \quad (2.1)$$

where matrices A and B are supposed to be real constant matrices of appropriate dimensions. The gain K is also supposed to be a given real constant matrix of appropriate dimensions computed from an adequate design technique, such that all the eigenvalues of matrix $(A + BK)$ are placed in the left half complex plane. As seen in Sect. 1.3, (2.1) generically models closed-loop systems obtained from linear state feedback or output feedback control laws.

For simplicity, the saturation nonlinearity is assumed to be symmetric, that is $u_{\max} = u_{\min} = u_0$ with u_0 being a positive vector, i.e. $u_0 \succ 0$.

2.2.1 Ellipsoidal Sets of Stability

As seen in Chap. 1, due to actuator saturation, depending on the initial state the trajectory of the closed-loop system (2.1) may diverge, even if $(A + BK)$ is Hurwitz. Actually, the problem of finding the set of all initial conditions whose corresponding trajectories converge asymptotically to the origin, i.e. the exact determination of the region (or basin) of attraction of the closed-loop system is a very challenging issue which is still unsolved in the general case. The idea is therefore to estimate this region computing what we call “regions of stability” or “regions of guaranteed safe behavior”. These regions correspond to sets of admissible initial conditions. Hence, if the initial condition belongs to a set of stability, the convergence of the associated trajectory to the origin is ensured.

A practical way of determining regions of stability comes from Lyapunov Theory [215]. It is well known that if $V(x)$ is a Lyapunov function, such that $\dot{V}(x) < 0$, $\forall x \in \mathcal{D} \subset \mathbb{R}^n$, along the trajectories of the system, and \mathcal{D} is a set containing the origin in its interior, then the level sets of $V(x)$, defined as

$$\mathcal{S}_V = \{x \in \mathbb{R}^n; V(x) \leq c\}$$

with $c \in \mathbb{R}$, $c > 0$, are regions of asymptotic stability for the closed-loop system, provided that $\mathcal{S}_V \subset \mathcal{D}$ (see also Appendix A).

In particular, considering quadratic Lypaunov functions, i.e.

$$V(x) = x'Px \quad \text{with } P = P' > 0$$

the associated level sets are given by ellipsoidal domains defined as follows:

$$\mathcal{E}(P, \eta) = \{x \in \mathbb{R}^n; x'Px \leq \eta^{-1}\} \quad (2.2)$$

with $\eta > 0$.

Hence, the following lemma is instrumental for the derivation of results to determine estimates of the basin of attraction, using ellipsoidal domains.

Lemma 2.1 *If there exist a positive definite symmetric matrix P and a scalar η such that*

$$(Ax(t) + B \text{sat}(Kx(t)))'Px(t) + x(t)'P(Ax(t) + B \text{sat}(Kx(t))) < 0, \quad \forall x(t) \in \mathcal{D} \quad (2.3)$$

and

$$\mathcal{E}(P, \eta) \subset \mathcal{D} \quad (2.4)$$

then the set $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for system (2.1).

The condition in Lemma 2.1 ensures in fact that the time derivative of the quadratic function $V(x(t)) = x(t)'Px(t)$ is strictly negative $\forall x(t) \in \mathcal{E}(P, \eta)$. Hence, from the Lyapunov theory, it follows that $\forall x(0) \in \mathcal{E}(P, \eta)$, $x(t) \in \mathcal{E}(P, \eta)$ and $\lim_{t \rightarrow \infty} x(t) = 0$. In other words, the ellipsoid $\mathcal{E}(P, \eta)$ is a positively invariant set and any trajectory initialized in this ellipsoid converges to the origin. We can say in this case that system (2.1) is locally (or regionally) asymptotically stable in $\mathcal{E}(P, \eta)$.

Moreover, since the inequality (2.3) is strictly verified, there exists a scalar $\sigma > 0$ such that

$$(Ax(t) + B \text{sat}(Kx(t)))'Px(t) + x(t)'P(Ax(t) + B \text{sat}(Kx(t))) < \sigma x(t)'Px(t) \quad (2.5)$$

which ensures that

$$V(x(t)) \leq e^{-\sigma t} V(x(0))$$

i.e. the trajectories converge exponentially to the origin with a contraction rate σ . In this case, we say that the ellipsoid $\mathcal{E}(P, \eta)$ is also “contractive” or “ σ -contractive” with respect to the trajectories of system (2.1).

Based on these considerations, the problem we want to solve can be stated as follows.

Problem 2.1 Find a matrix P and a positive scalar η such that the set $\mathcal{E}(P, \eta)$ is a region of asymptotic stability for the closed-loop system (2.1).

Suppose that $\mathcal{E}(P, \eta)$ is a solution to the above analysis problem. Obviously, it follows that the sets $\mathcal{E}(P, \lambda\eta)$, with $\lambda > 1$ are also RAS. It may also exist sets $\mathcal{E}(P, \lambda\eta)$, with $0 < \lambda < 1$ that are RAS. Hence, given a matrix P , a relevant problem is to find the minimal value of λ for which $\mathcal{E}(P, \lambda\eta)$ is effectively a RAS.

On the other hand, when we are concerned with the determination of an estimate of the region (basin) of attraction, the idea is to find P and η such that $\mathcal{E}(P, \eta)$ best fits in R_A or, equivalently, that leads to a maximized region of stability considering some size criterion. In other words, we want to find the Lyapunov level set $\mathcal{E}(P, \eta)$ as large as possible. Our second problem originates from this remark.

Problem 2.2 Find a matrix P and a positive scalar η such that $\mathcal{E}(P, \eta)$ is maximal considering some size criterion.

The way to measure the size of the set $\mathcal{E}(P, \eta)$ and the associated criteria for maximizing it will be discussed in the sequel.

2.2.2 Polytopic Approach I

According to Sect. 1.7.1.1, system (2.1) can be written as follows:

$$\dot{x}(t) = (A + B\Gamma(\alpha(x(t)))K)x(t) \quad (2.6)$$

where $\Gamma(\alpha(x(t)))$ is a diagonal matrix whose diagonal elements are defined for $i = 1, \dots, m$ as

$$0 < \alpha_{(i)}(x(t)) = \min\left(1, \frac{u_{0(i)}}{|K_{(i)}x(t)|}\right) \leq 1 \quad (2.7)$$

Note that when $\alpha_{(i)}(x)$ approaches 0 there is almost no feedback injected in input $u_{(i)}$, whereas $\alpha_{(i)} = 1$ means that $u_{(i)}$ does not saturate.

It follows that for all $x(t)$ belonging to the region

$$S(|K|, u_0^\alpha) = \{x \in \mathbb{R}^n; |Kx| \leq u_0^\alpha\} \quad (2.8)$$

with $u_{0(i)}^\alpha = \frac{u_{0(i)}}{\alpha_{(i)}}$, $i = 1, \dots, m$, one verifies

$$0 < \alpha_{l(i)} \leq \alpha_{(i)}(x(t)) \leq 1$$

Thus, $\dot{x}(t)$ can be determined from an appropriate convex linear combination of matrices $\mathbb{A}_j = A + B\Gamma_j(\alpha_l)K$ at time t , that is,

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t)) \mathbb{A}_j x(t) \quad (2.9)$$

with $\sum_{j=1}^{2^m} \lambda_j(x(t)) = 1$, $\lambda_j(x(t)) \geq 0$.

It follows that all the trajectories of (2.1) that are confined in the region $S(|K|, u_0^\alpha)$, can be generated as trajectories of the system (2.9). As a consequence of this fact, the idea is to use the polytopic differential inclusion (2.9) to determine regions of stability for the saturated system (2.1).

Consider that such a region is an ellipsoid, i.e. it is defined by $\mathcal{E}(P, \eta)$. Hence, if

1. $V(x) = x'Px$ is strictly decreasing along the trajectories of the polytopic system (2.9) and
2. $\mathcal{E}(P, \eta) \subset S(|K|, u_0^\alpha)$,

then it follows that $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for system (2.1). The following formal result can therefore be stated [141, 172].

Proposition 2.1 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$, a vector $\alpha_l \in \mathbb{R}^m$ and a positive scalar η satisfying:*

$$(A + B\Gamma_j(\alpha_l)K)'P + P(A + B\Gamma_j(\alpha_l)K) < 0, \quad j = 1, \dots, 2^m \quad (2.10)$$

$$\begin{bmatrix} P & \alpha_{l(i)}K'_{(i)} \\ \alpha_{l(i)}K_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.11)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.12)$$

then the ellipsoid $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for the saturated system (2.1).

Proof Suppose that condition (2.10) is verified $\forall j = 1, \dots, 2^m$. Then, for any scalars $0 \leq \lambda_j \leq 1$, such that $\sum_{j=1}^{2^m} \lambda_j = 1$, it follows that

$$\sum_{j=1}^{2^m} \lambda_j ((A + B\Gamma_j(\alpha_l)K)'P + P(A + B\Gamma_j(\alpha_l)K)) < 0$$

Hence, $\forall x \neq 0$, it follows that

$$x' \left[\left(\sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K) \right)' P + P \left(\sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K) \right) \right] x < 0$$

Since for any $x(t) \in S(|K|, u_0^\alpha)$,

$$\dot{x}(t) = Ax(t) + B \text{sat}(Kx(t)) = \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K)$$

with appropriate λ_j , such that $0 \leq \lambda_j \leq 1$ and $\sum_{j=1}^{2^m} \lambda_j = 1$, if (2.10) is verified, we can therefore conclude that

$$\begin{aligned} & (Ax(t) + B \text{sat}(Kx(t)))' Px(t) + x(t)' P (Ax(t) + B \text{sat}(Kx(t))) < 0 \\ & \forall x(t) \in S(|K|, u_0^\alpha) \end{aligned}$$

On the other hand, the satisfaction of conditions (2.11) and (2.12) implies that $\mathcal{E}(P, \eta)$ is included in the set $S(|K|, u_0^\alpha)$. Thus, from Lemma 2.1 it follows that $\mathcal{E}(P, \eta)$ is a RAS. \square

In some cases, as we will see in the sequel of this book, we can be interested in conditions involving P^{-1} instead of P . Hence setting $W = P^{-1}$ the following corollary to Proposition 2.1 can be stated.

Corollary 2.1 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a vector $\alpha_l \in \mathbb{R}^m$ and a positive scalar η satisfying:*

$$W(A + B\Gamma_j(\alpha_l)K)' + (A + B\Gamma_j(\alpha_l)K)W < 0, \quad j = 1, \dots, 2^m \quad (2.13)$$

$$\begin{bmatrix} W & \alpha_{l(i)} W K'_{(i)} \\ \alpha_{l(i)} K_{(i)} W & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.14)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.15)$$

then the ellipsoid $\mathcal{E}(W^{-1}, \eta)$ is a region of asymptotic stability (RAS) for the saturated system (2.1).

Proof The proof directly follows from the proof of Proposition 2.1. Relation (2.13) is obtained by right- and left-multiplying inequality (2.10) by W . Relation (2.14) is deduced by multiplying both sides of (2.11) by $\text{Diag}(W, 1)$. \square

Remark 2.1 Proposition 2.1 and Corollary 2.1 give only sufficient conditions for the solution of Problem 2.1. In fact, the matrix inequality (2.10) ensures that the polytopic system is quadratically asymptotically stable in a global sense, which is not a necessary condition to ensure the asymptotic stability of system (2.1) in $\mathcal{E}(P, \eta)$. Note that, as mentioned in Remark 1.6, all the trajectories of system (2.1) that lie in $S(|K|, u_0^\alpha)$ are trajectories of (2.9), but the converse is not true.

Remark 2.2 The problem of finding a positively invariant and contractive ellipsoidal set $\mathcal{E}(P, \eta)$ included in the region of linear behavior of the closed-loop system, R_L , can also be addressed with conditions of Proposition 2.1. For this it suffices to consider $\alpha_{l(i)} = 1, i = 1, \dots, m$. In this case, all the trajectories starting in $\mathcal{E}(P, \eta)$ will not generate control saturation, i.e., $\mathcal{E}(P, \eta)$ is a set of admissible initial conditions for which the behavior of the system is guaranteed to be linear. However, when we are interested in computing estimates of the region of attraction of the saturated system (2.1), it is important to have a set $\mathcal{E}(P, \eta)$ that spreads over the region of nonlinear behavior of the system, i.e., where control saturations are effectively active.

2.2.3 Polytopic Approach II

Define the set

$$S(|H|, u_0) = \{x \in \mathbb{R}^n; |Hx| \leq u_0\}$$

Recall the definition of matrices Γ_j^+ and Γ_j^- given in Sect. 1.7.1.2:

- Γ_j^+ are diagonal matrices whose diagonal elements take the value 1 or 0, $j = 1, \dots, 2^m$.
- $\Gamma_j^- = I_m - \Gamma_j^+$, $j = 1, \dots, 2^m$.

As pointed in Sect. 1.7.1.2, if $x(t) \in S(|H|, u_0)$, there exists $\sum_{j=1}^{2^m} \lambda_j(x(t)) = 1$, $0 \leq \lambda_j(x(t)) \leq 1$, such that

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t)) (A + B\Gamma_j^+ K + B\Gamma_j^- H)x(t) \quad (2.16)$$

Hence, following the same reasoning as performed in Sect. 2.2.2, if

1. $V(x) = x'Px$ is strictly decreasing along the trajectories of the polytopic system (2.16) and
2. $\mathcal{E}(P, \eta) \subset S(|H|, u_0)$,

then it follows that $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for system (2.6). The following formal result can therefore be stated [188, 193].

Proposition 2.2 *If there exist a symmetric positive definite matrix $W \in \Re^{n \times n}$, a matrix $Q \in \Re^{m \times n}$ and a positive scalar η satisfying:*

$$\begin{aligned} W(A + B\Gamma_j^+ K)' + Q'\Gamma_j^- B' + (A + B\Gamma_j^+ K)W \\ + B\Gamma_j^- Q < 0, \quad j = 1, \dots, 2^m \end{aligned} \quad (2.17)$$

$$\begin{bmatrix} W & Q'_{(i)} \\ Q_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.18)$$

then the ellipsoid $\mathcal{E}(P, \eta) = \{x \in \Re^n; x'Px \leq \eta^{-1}\}$, with $P = W^{-1}$, is a region of asymptotic stability (RAS) for the saturated system (2.1).

Proof By considering the change of variable $Q = HW$, the proof follows the same steps as the ones of Proposition 2.1 and Corollary 2.1. \square

Remark 2.1 also holds for the result of Proposition 2.2. In this case, it follows that all the trajectories of system (2.1) in $S(|H|, u_0)$ are trajectories of system (2.16), but the converse is not true. However, it is important to observe that the conditions of Proposition 2.2 encompass the ones of Proposition 2.1. Actually, if we choose $H = \text{diag}(\alpha_l)K$, the result of Corollary 2.1 is recovered. This fact means that Proposition 2.2 leads to less conservative results, although it is still only a sufficient condition to ensure that $\mathcal{E}(P, \eta)$ is a RAS.

2.2.4 Polytopic Approach III

Recall the definition of the sets \mathcal{S}_j , $j = 1, \dots, 2^m$, of indices $i \in \mathcal{M} = \{1, \dots, m\}$ regarding the situations combining saturated and non-saturated entries, as given in Sect. 1.7.1.3.

- By definition, the combination associated to the case in which all the control inputs are not saturated corresponds to $\mathcal{S}_1 = \emptyset$.
- For the j th combination, if the i th control input is saturated then $i \in \mathcal{S}_j$.

Associated to each set \mathcal{S}_j , $j = 2, \dots, 2^m$, define the set

$$S(|H_j|, u_0) = \{x \in \mathbb{R}^n; |H_{j(i)}x| \leq u_{0(i)}, \forall i \in \mathcal{S}_j\}$$

As pointed out in Sect. 1.7.1.3, if $x(t) \in \bigcap_{j=2}^{2^m} S(|H_j|, u_0)$, there exists $\sum_{j=1}^{2^m} \lambda_j(x(t)) = 1$, $0 \leq \lambda_j(x(t)) \leq 1$ such that

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t)) \left(A + \sum_{i \in \mathcal{S}_j^c} B_i K_{(i)} + \sum_{i \in \mathcal{S}_j} B_i H_{j(i)} \right) x(t) \quad (2.19)$$

with B_i standing for the i th column of B .

Hence, following the same reasoning as in Sect. 2.2.2, if

1. $V(x) = x'Px$ is strictly decreasing along the trajectories of the polytopic system (2.19) and
2. $\mathcal{E}(P, \eta) \subset \bigcap_{j=2}^{2^m} S(|H_j|, u_0)$,

then it follows that $\mathcal{E}(P, \eta)$ is a RAS for system (2.1).

The following formal result can therefore be stated [1].

Proposition 2.3 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, row vectors $Q_{j(i)} \in \mathbb{R}^{1 \times n}$, $j = 2, \dots, 2^m$, $i \in \mathcal{S}_j$, and a positive scalar η satisfying*

$$\begin{aligned} & \left(AW + \sum_{i \in \mathcal{S}_j^c} B_i K_{(i)} W + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)} \right) \\ & + \left(AW + \sum_{i \in \mathcal{S}_j^c} B_i K_{(i)} W + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)} \right)' < 0, \quad \forall j = 1, \dots, 2^m \end{aligned} \quad (2.20)$$

$$\begin{bmatrix} W & Q'_{j(i)} \\ Q_{j(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad \forall j = 2, \dots, 2^m, \forall i \in \mathcal{S}_j \quad (2.21)$$

then the ellipsoid $\mathcal{E}(P, \eta)$ with $P = W^{-1}$, is a region of asymptotic stability (RAS) for the saturated system (2.1).

Proof By considering the change of variables $Q_{j(i)} = H_{j(i)}W$, the proof follows the same steps as the ones of Proposition 2.1 and Corollary 2.1. \square

Remark 2.1 also holds for the result of Proposition 2.3. In this case, it follows that all the trajectories of system (2.1) in $\bigcap_{j=2}^{2^m} S(|H_j|, u_0)$ are trajectories of the polytopic system (2.19), but the converse is not true. Moreover, the conditions of Proposition 2.3 encompass the ones of Propositions 2.1 and 2.2. Note that if we select a fixed $Q_{j(i)} = Q_{(i)}$, $\forall j$, conditions of Proposition 2.2 are recovered. This fact means that Proposition 2.3 leads in general to less conservative results, although it is still only a sufficient condition to ensure that $\mathcal{E}(P, \eta)$ is a RAS.

Remark 2.3 In [1] and [417] different proofs for Proposition 2.3, not directly based on differential polytopic inclusion, are provided.

2.2.5 Optimization Problems

The results stated in Propositions 2.1, 2.2 and 2.3, allow to address the following optimization problems:

P1: Considering P and η given, test if $\mathcal{E}(P, \eta)$ is a RAS.

P2: Considering a given P , minimize η for which $\mathcal{E}(P, \eta)$ is a RAS.

P3: Find P and η such that the RAS $\mathcal{E}(P, \eta)$ is maximized considering some size criterion.

The problem *P1* is a feasibility problem, i.e., we have to test if the conditions in Propositions 2.1, 2.2 or 2.3 are actually satisfied.

Problem *P2* corresponds to the following optimization problem:

$$\begin{aligned} \min \quad & \eta \\ \text{subject to} \quad & \text{inequalities (2.10)–(2.12) or (2.17)–(2.18) or (2.20)–(2.21)} \end{aligned} \quad (2.22)$$

Finally, problem *P3* can be seen as a search of the best ellipsoidal estimate for the region of attraction, considering a given criterion. In order to address this problem, an optimization criterion, given by a function $f(\mathcal{E}(P, \eta))$, associated to the ellipsoidal region of stability that we want to determine must be conveniently defined. The minimization of this function implies the maximization of a geometric characteristic of the region, such as the volume, the length of the minor axis or even the maximization of the ellipsoid in certain directions. The maximization of the ellipsoid $\mathcal{E}(P, \eta)$ can therefore be tackled through the following generic optimization problem:

$$\begin{aligned} \min \quad & f(\mathcal{E}(P, \eta)) \\ \text{subject to} \quad & \text{inequalities (2.10)–(2.12) or (2.17)–(2.18) or (2.20)–(2.21)} \end{aligned} \quad (2.23)$$

At this point it should be noticed that inequalities (2.10)–(2.12) are BMIs if P and α_l are decision variables and LMIs if P or α_l are a priori fixed. On the other hand, the

inequalities (2.17)–(2.18) and (2.20)–(2.21) are LMIs in variables W , Q (or $Q_{j(i)}$) and η . Hence, if $f(\mathcal{E}(P, \eta))$ is a convex function, it turns that the optimization problem can be solved in a convex framework. These issues are discussed in the following sections.

2.2.5.1 Size Criteria

The objective function $f(\mathcal{E}(P, \eta))$, as above mentioned, should be associated to a geometric characteristic of the ellipsoidal domain. The most used size criteria are detailed in the sequel. Some of these criteria have to be associated to some auxiliary LMIs. In this case, the auxiliary LMIs must be added to the constraints of the optimization problem (2.23).

Volume Maximization The volume of the ellipsoid is proportional to

$$\sqrt{\det(P^{-1}\eta^{-1})} = \sqrt{\det(W\eta^{-1})}$$

Then it is possible to maximize the size of the ellipsoid by minimizing the function $\log(\det(\eta P))$ [45]. In this case, we can consider

$$f(\mathcal{E}(P, \eta)) = \log(\det(\eta P)) = \log(\eta^n \det(P)) = n \log(\eta) + \log(\det(P))$$

Note that such a function is convex in the decision variables P and η . When W is a decision variable it suffices to consider the minimization of the following function:

$$f(\mathcal{E}(W^{-1}, \eta)) = \log(\det(\eta W^{-1})) = n \log(\eta) - \log(\det(W))$$

which is linear and convex in the decision variables W and η .

It should be pointed out that the volume maximization can lead to ellipsoids that are “flat” in some directions. In this case, although the volume is maximized, the ellipsoidal region may be a bad estimate of the basin of attraction in those directions.

Minor Axis Maximization Noting that

$$\mathcal{E}(P, \eta) = \{x \in \mathbb{R}^n; x'(\eta P)x \leq 1\}$$

the minimization of the greatest eigenvalue of ηP corresponds to maximize the minor axis of the ellipsoid $\mathcal{E}(P, \eta)$ [45].

Considering that $(\eta P)^{-1} = \eta^{-1}W$, it follows that $\lambda_{\max}(\eta P) = 1/\lambda_{\min}(\eta^{-1}W)$. Thus, depending on whether W or P are decision variables, the optimization criterion becomes

$$f(\mathcal{E}(W^{-1}, \eta)) = -\lambda_{\min}(\eta^{-1}W) \quad \text{or} \quad f(\mathcal{E}(P, \eta)) = \lambda_{\max}(\eta P) \quad (2.24)$$

Note that $\lambda_{\max}(\eta P) = \eta \lambda_{\max}(P)$. However, in general, P and η are simultaneously decision variables and it is desirable to have a linear criterion, in order to deal with convex optimization problems (this will made be clear in the next section).

In this case, we can consider the following linear criteria and additional constraints to be added in the optimization problem (2.23):

$$\begin{aligned} & \min \beta_0 \eta + \beta_1 \lambda \\ & \text{subject to} \quad \text{inequalities (2.10)–(2.12)} \\ & \quad \quad \quad P \leq \lambda I_n \end{aligned}$$

or

$$\begin{aligned} & \max -\beta_0 \eta + \beta_1 \lambda \\ & \text{subject to} \quad \text{inequalities (2.13)–(2.15) or (2.17)–(2.18) or (2.20)–(2.21)} \\ & \quad \quad \quad W \geq \lambda I_n \end{aligned}$$

where β_0 and β_1 are tuning parameters allowing to weight the effects of η and λ in the effective size of the minor axis.

Trace Minimization Each eigenvalue of P is associated to the length of one ellipsoid axis. Since the trace of matrix P is the sum of its eigenvalues, its minimization leads to ellipsoids that tend to be homogeneous in all directions. In fact, in this case, all the axis length have the same weight in the criterion.

Taking into account that $\text{trace}(\eta P) = \eta \text{trace}(P)$, the following optimization function can be chosen:

$$f(\mathcal{E}(P, \eta)) = \beta_0 \eta + \beta_1 \text{trace}(P)$$

where β_0 and β_1 are weighting parameters.

When W is a decision variable one can consider the function

$$f(\mathcal{E}(W^{-1}, \eta)) = \beta_0 \eta + \beta_1 \text{trace}(M_W)$$

along with the following constraints in the optimization problems:

$$\begin{bmatrix} M_W & I_n \\ I_n & W \end{bmatrix} > 0; \quad M_W = M'_W > 0 \quad (2.25)$$

Note that (2.25) ensures that $P < M_W$ and, in consequence, that $\text{trace}(P) < \text{trace}(M_W)$. Thus, the minimization of the $\text{trace}(M_W)$ implies the minimization of $\text{trace}(P)$.

Maximization Along Certain Directions It is possible that some preferential directions in which the system will be initialized or driven by the actions of disturbances are known. Thus, it is of interest to find a stability region as large as possible along these directions [130, 133, 137, 193].

With this aim, let us consider the set of vectors that define the directions in which the ellipsoid should be maximized:

$$\mathcal{D} = \{d_1, \dots, d_s\}, \quad d_i \in \mathbb{R}^n, \quad i = 1, \dots, s$$

For simplicity, consider $\eta = 1$. The idea in this case is to maximize scaling factors θ_i , $i = 1, \dots, s$, such that

$$(\theta_i d'_i) P (\theta_i d_i) \leq 1$$

or still

$$\begin{bmatrix} 1 & \theta_i d_i' \\ \theta_i d_i & W \end{bmatrix} \geq 0$$

Thus, the following optimization criterion and additional constraints can be considered:

$$\begin{aligned} & \min \sum_{i=1}^s \beta_i \bar{\theta}_i \\ & \text{subject to} \quad \text{inequalities (2.10)–(2.12)} \\ & \quad d_i' P d_i \leq \bar{\theta}_i, \quad i = 1, \dots, s \end{aligned}$$

with $\bar{\theta}_i = \theta_i^{-2}$, or

$$\begin{aligned} & \max \sum_{i=1}^s \beta_i \theta_i \\ & \text{subject to} \quad \text{inequalities (2.13)–(2.15) or (2.17)–(2.18) or (2.20)–(2.21)} \\ & \quad \begin{bmatrix} 1 & \theta_i d_i' \\ \theta_i d_i & W \end{bmatrix} \geq 0, \quad i = 1, \dots, s \end{aligned}$$

where β_i are tuning parameters corresponding to a weight factor associated to the maximization in direction d_i . As a particular case it is possible to consider $\theta_i = \theta$, $i = 1, \dots, s$, i.e. the same scaling factor or “amplification” in all directions.

2.2.5.2 BMI × LMI Problems

The inequalities (2.11) and (2.12) are LMIs in P , α_l and η . Note, however, that matrix inequality (2.10) is bilinear (i.e. it is a BMI) in decision variables P and α_l , due to the products involving these variables. This fact renders difficult the resolution of the optimization problem (2.23). The same type of remarks can be done with respect to inequalities (2.13) and (2.14), considering variables W and α_l .

In [129] it has been shown that many of the problems considered in the robust control literature can be formulated as BMIs. This is the case of the conditions in Proposition 2.1 and Corollary 2.1. A way to overcome this problem consists in relaxing the BMI constraints in an LMI form by fixing variables. In this case, a (probably) suboptimal solution can be searched by an iterative algorithm, where in each step some variables are fixed and an LMI problem is solved. An example of this procedure is given by the algorithm below.

Algorithm 2.1

1. Initialize α_l .
2. Solve the following optimization problem for P and η :

$$\begin{aligned} & \min f(\mathcal{E}(P, \eta)) \\ & \text{subject to} \quad \text{inequalities (2.10)–(2.12)} \end{aligned} \tag{2.26}$$

3. Keep the previous value of P , solve the following problem for η and α_l :

$$\begin{aligned} \min \quad & \eta \\ \text{subject to} \quad & \text{inequalities (2.10)–(2.12)} \end{aligned} \quad (2.27)$$

4. Go to step 2 until no significant change on the size of the ellipsoid $\mathcal{E}(P, \eta)$ is obtained. When no significant change arises then stop.

In particular, for the single-input case, it is also possible to seek the optimal solution of the considered optimization problem by performing a bisection (line) search on α_l . For the general multi-input case, P or α_l are fixed and a convex optimization problem with LMI constraints are solved as described in Algorithm 2.1. Two issues arise in this case: how to choose the initial vector α_l and how exactly to decrease the components of α_l if needed? One simple way of handling these issues is to apply trial and error procedures. On the other hand, since the matrix $A + BK$ is Hurwitz by assumption, there will always exist a solution to (2.26) for $\alpha_l = 1_m$. Hence, if we start the algorithm with $\alpha_l = 1_m$, the convergence of the algorithm is ensured. This follows from the fact that an optimal solution for one step is also a feasible solution for the next step. Although conservative (in the sense that, in general, the optimal solution is not achieved) this kind of approach solves, in part, the problem of the choice of vector α_l by using robust and available packages to solve LMIs [45, 231]. Of course, taking different initial vectors α_l , the proposed algorithm can converge toward different values.

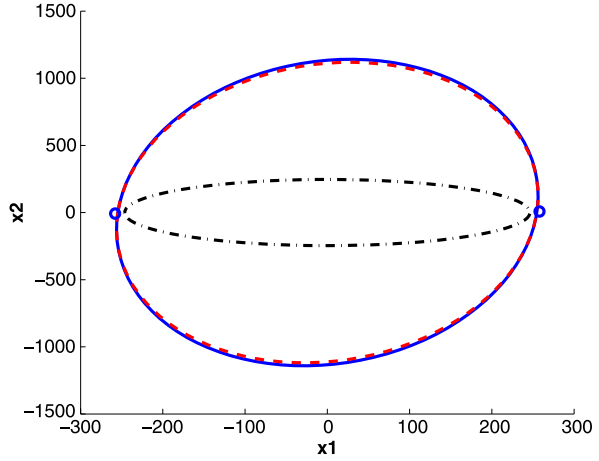
On the other hand, the inequalities of Propositions 2.2 and 2.3 are true LMIs in variables W , Q (or $Q_{j(i)}$) and η . In this case, considering the criteria and the additional constraints presented in Sect. 2.2.5.1, it follows that problem (2.23) is a convex one and can be directly and efficiently solved by using available LMI solvers. In addition, without loss of generality, we can consider $\eta = 1$. Note that if W , Q and η satisfy LMIs (2.17) and (2.18), by multiplying both LMIs by η^{-1} it follows that $\bar{W} = \eta^{-1}W$, $\bar{Q} = \eta^{-1}Q$ satisfy:

$$\begin{aligned} \bar{W}(A + B\Gamma_j^+K)' + \bar{Q}'\Gamma_j^-B' + (A + B\Gamma_j^+K)\bar{W} + B\Gamma_j^-\bar{Q} &< 0, \quad j = 1, \dots, 2^m \\ \begin{bmatrix} \bar{W} & \bar{Q}'_{(i)} \\ \bar{Q}_{(i)} & u_{0(i)}^2 \end{bmatrix} &\geq 0, \quad i = 1, \dots, m \end{aligned}$$

The same reasoning can be done by considering LMIs (2.20) and (2.21).

Remark 2.4 The conditions stated in Propositions 2.1, 2.2 and 2.3 regard only the guarantee of local (or regional) asymptotic stability of the closed-loop system. Indeed, the polytopic models are well defined only in $S(|K|, u_0^\alpha)$, $S(|H|, u_0)$ or $\bigcap_{j=2}^{2^m} S(|H_j|, u_0)$, respectively. In the case open-loop system is stable, the origin of the closed-loop system can be in fact globally asymptotically stable. In this case, the optimization problems can lead to numerical solutions with $\alpha_{l(i)}$ and the elements of H with values tending to zero. This fact means that the obtained ellipsoid tends to cover all the state space and its size is in fact limited by the numerical precision of the solver.

Fig. 2.1 Example 2.1—regions of stability obtained with polytopic approaches I (dot-dashed line), II (solid line) and III (dashed)



Example 2.1 Consider the system studied in [218], which is described by the following data:

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix}; \quad B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.7283 & -0.0338 \\ -0.0135 & -1.3583 \end{bmatrix}; \quad u_0 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

Consider first the polytopic approach I. In this case, we apply Algorithm 2.1 considering the maximization of the minor axis of the ellipsoidal region of stability. To apply the algorithm, we initialize $\alpha_l = 1_m$. Therefore, one obtains

$$\alpha_l = \begin{bmatrix} 0.0275 \\ 0.0034 \end{bmatrix}; \quad P = 10^{-4} \times \begin{bmatrix} 0.1608 & 0.0001 \\ 0.0001 & 0.1592 \end{bmatrix}; \quad \eta = 1$$

Considering now the same problem, but with the polytopic approach II we obtain

$$P = 10^{-4} \times \begin{bmatrix} 0.1542 & -0.0038 \\ -0.0038 & 0.0078 \end{bmatrix}; \quad Q = 10^3 \times \begin{bmatrix} -1.2732 & -0.0135 \\ 0.2362 & -0.0155 \end{bmatrix}$$

Finally, with approach III the following is obtained:

$$P = 10^{-4} \times \begin{bmatrix} 0.1541 & -0.0038 \\ -0.0038 & 0.0081 \end{bmatrix}; \quad Q_2 = 10^3 \times \begin{bmatrix} -1.2741 & -0.0400 \\ 0 & 0 \end{bmatrix}$$

$$Q_3 = 10^3 \times \begin{bmatrix} 0 & 0 \\ 0.0000 & -0.0007 \end{bmatrix}; \quad Q_4 = 10^3 \times \begin{bmatrix} -1.2698 & 0.0759 \\ 0.2787 & -0.0167 \end{bmatrix}$$

Figure 2.1 depicts the obtained ellipsoidal regions of asymptotic stability (RAS). It is important to note that the saturated system has two additional unstable equilibrium points at

$$x_{\text{eq}} = \pm \begin{bmatrix} 257.931 \\ 7.931 \end{bmatrix}$$

This fact allows to see that the provided estimates of the region of attraction are relatively good since the boundary of the three ellipsoids are very close to the equilibrium points above mentioned. The estimates obtained with the polytopic approaches II and III are clearly better. This enforces the fact that these approaches encompass the first one and provide, in general, less conservative estimates of the region of attraction. It should, however, be noticed that the directions in which the ellipsoid obtained with approach II and III are larger are not directly considered in the optimization criteria, i.e. considering the maximization of the minor axis of the ellipsoids, all the approaches practically lead to the same optimal value of the criterion (the size of the minor axis).

On the other hand, there is no practical difference between the obtained ellipsoid with approaches II and III. Nonetheless, the value of the optimal criterion for approach III (-6.4800×10^4) is slightly smaller than that one for approach II (-6.4763×10^4). This shows, as expected, a slight reduction in conservatism. Actually, this small difference can be justified by the fact that the considered example is of low dimension (second order, two inputs). Hence, the contribution of the extra degrees of freedom introduced by the different matrices Q_j in polytopic approach III is less effective.

Example 2.2 Recall the balancing pointer system studied in Example 1.1:

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}; \quad K = \begin{bmatrix} 13 & 7 \end{bmatrix}$$

Suppose that the control inputs are limited by $u_0 = 5$. The conditions of Proposition 2.2 are applied to find estimates of the region of attraction of the closed-loop saturated system. The results obtained with the following size criteria:

- maximization of $\lambda_{\min}(W)$
- minimization of $\text{trace}(W^{-1})$
- maximization in the direction $[-4.5 \ 6]'$

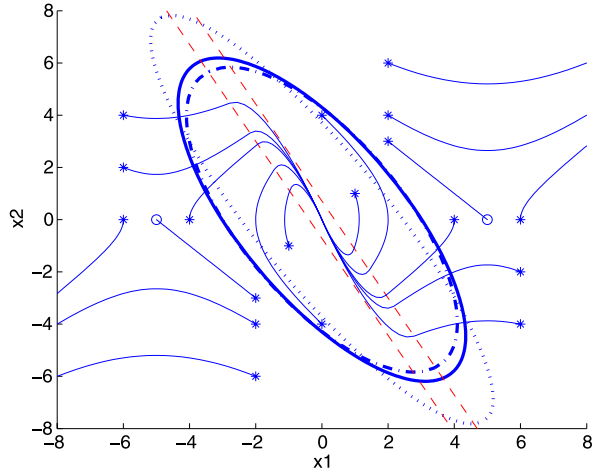
are shown in Fig. 2.2. The region of linearity R_L is also plotted in the figure (it is delimited by the dashed lines). Note that in all cases, the estimate of the region of attraction spreads over the region where the behavior of the system is actually nonlinear.

2.3 External Stability

In this section, we focus on the analysis of the external stability of the saturated closed-loop system when the exogenous signal $w(t)$ is present. This signal can be considered as a disturbance, a reference to track or a combination of both. Hence, we consider the closed-loop system is generically represented as

$$\dot{x}(t) = Ax(t) + B \text{sat}(Kx(t)) + B_w w(t) \quad (2.28)$$

Fig. 2.2 Example 2.2—region of asymptotic stability (RAS) considering: maximization of $\lambda_{\min}(W)$ (---), minimization of $\text{trace}(W^{-1})$ (—), maximization in the direction $[-4.5 \ 6]'$ (\cdots); R_L (---); equilibrium points (\circ); initial conditions ($*$)



As illustrated in Example 1.2, due to the control signal saturation, in general it is not possible to ensure that the trajectories of system (2.28) will be bounded for any bounded signal $w(t)$. Hence, it is important to characterize admissible sets of exogenous signals as well as of initial conditions, for which the trajectories are guaranteed bounded. Furthermore, if $w(t)$ is vanishing, the convergence to the origin (equilibrium point of interest) must be ensured. In other words, for the admissible exogenous signals and initial conditions, the corresponding trajectories will never leave the region of attraction of system (2.1).

In the sequel we derive some results to address Problem 1.5. In particular, we consider two classes of disturbances: amplitude bounded and energy bounded ones. The results are based on a quadratic approach. The conditions are derived considering the polytopic model II. Formulations considering the polytopic approach I and III follow exactly the same reasoning and can be straightforwardly obtained.

2.3.1 Amplitude Bounded Exogenous Signals

Consider that the exogenous signal $w(t)$ belongs to the following set:

$$\mathcal{W} = \{w \in \mathbb{R}^q; w' R w \leq \delta^{-1}\} \quad (2.29)$$

with $R = R' > 0$ and $\delta > 0$.

In this case, $w(t)$ is bounded by a quadratic norm which reflects amplitude bounds on $w(t)$ [193, 353]. Note that \mathcal{L}_∞ -norm constraints can be straightforwardly considered in this way. For instance, if R is a diagonal matrix, it follows that $|w_i(t)| \leq \sqrt{1/(\delta r_i)}$, with r_i denoting the i th diagonal element of R , $i = 1, \dots, q$.

Considering a quadratic Lyapunov function $V(x) = x' P x$ and the application of the S-procedure (see Appendix C), a sufficient condition to obtain a solution

to Problem 1.5 is achieved if the following relation is satisfied $\forall x \in \mathcal{E}(P, \eta)$ and $\forall w \in \mathcal{W}$ [45]:

$$\dot{V}(x) + \tau_1(x'Px - \eta^{-1}) + \tau_2(\delta^{-1} - w'Rw) < 0, \quad \tau_1 > 0, \quad \tau_2 > 0 \quad (2.30)$$

In fact, relation (2.30) ensures that $\dot{V}(x) < 0$, $\forall x(t) \notin \text{int } \mathcal{E}(P, \eta)$ and $\forall w(t) \in \mathcal{W}$. In this case, suppose that at time $t = t_1$, $x(t_1) \in \partial \mathcal{E}(P, \eta)$ and $w(t) \in \mathcal{W}$. It follows that $\dot{V}(x(t_1)) < 0$, which implies that $x(t_1 + \Delta t) \in \text{int } \mathcal{E}(P, \eta)$. Thus, we can conclude that (2.30) ensures that the trajectories initialized in $\mathcal{E}(P, \eta)$ do not escape from this domain $\forall w(t) \in \mathcal{W}$. In other words, the ellipsoid $\mathcal{E}(P, \eta)$ is said a \mathcal{W} -positively invariant set [33] or a *robustly positively invariant set* [36] with respect to the closed-loop system (2.28).

In particular, if

$$\dot{V}(x) + \tau_1 x'Px - \tau_2 w'Rw < 0 \quad (2.31)$$

and

$$\delta^{-1}\tau_2 - \eta^{-1}\tau_1 < 0 \quad (2.32)$$

are satisfied then condition (2.30) holds. On the other hand, when $w = 0$, it follows that (2.31) implies that

$$\dot{V}(x) < -\tau_1 x'Px < 0$$

which ensures that $\dot{V}(x) < 0$, $\forall x \in \mathcal{E}(P, \eta)$. Hence, we can conclude that $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for the saturated system. This means that if $w(t)$ vanishes, the trajectory will converge asymptotically to the origin. Moreover, this convergence is exponential with a rate given by τ_1 .

Considering a polytopic modeling for the saturated system (2.28), the idea therefore consists in ensuring (2.30) along the trajectories of a polytopic system. In addition, the positive invariant set must be included in the region of validity of the polytopic differential inclusion. From this reasoning, the following result can be stated considering the polytopic model II, and, without loss of generality, $\eta = 1$.

Proposition 2.4 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Q \in \mathbb{R}^{m \times n}$ and positive scalars τ_1 and τ_2 satisfying:*

$$\begin{bmatrix} W(A + B\Gamma_j^+K)' + Q'\Gamma_j^-B' + (A + B\Gamma_j^+K)W + B\Gamma_j^-Q + \tau_1W & B_w \\ B_w' & -\tau_2R \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.33)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.34)$$

$$\tau_2 - \delta\tau_1 < 0 \quad (2.35)$$

then

1. $\forall w \in \mathcal{W}$, the trajectories of the system (2.28) do not leave the set $\mathcal{E}(P, 1)$;
2. for $w = 0$, $\mathcal{E}(P, 1)$ is a region of asymptotic stability (RAS) for the system (2.28).

Proof Left- and right-multiplying inequality (2.33) by the block diagonal matrix $\text{Diag}(P, I)$, with $P = W^{-1}$ it follows that (2.33) is equivalent to

$$\begin{bmatrix} (A + B\Gamma_j^+ K)'P + H'\Gamma_j^- B'P + P(A + B\Gamma_j^+ K) + PB\Gamma_j^- H + \tau_1 P & PB_w \\ B_w'P & -\tau_2 R \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.36)$$

with $H = QP$. Now, left- and right-multiplying (2.36), $j = 1, \dots, 2^m$, respectively, by $[x(t)' \ w(t)']$ and $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$, by convexity, it follows that

$$\dot{V}(x) + \tau_1 x' P x - \tau_2 w' R w < 0$$

is verified along the trajectories of the polytopic system:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t)) (\mathbb{A}_j x(t) + B_w w(t)) \quad (2.37)$$

This fact along with the satisfaction of (2.35) implies that (2.30) is verified. Hence, if the set $\mathcal{E}(P, 1)$ is included in the region $S(|H|, u_0)$, it follows that (2.33) implies that (2.30) is verified along the trajectories of the saturated system (2.28).

Finally, left- and right-multiplying (2.34) by $\text{Diag}(P, I)$, it follows that this LMI ensures that $\mathcal{E}(P, 1) \subset S(|H|, u_0)$, which concludes the proof. \square

Given a set \mathcal{W} , the condition stated in Proposition 2.4 can be used to check if the trajectories of the system are bounded considering that $w(t) \in \mathcal{W}$. In this case R and the scalar δ are given, and an optimization problem can be formulated to determine a region of admissible initial states, $\mathcal{E}(P, 1)$ as large as possible, considering some size criterion.

Considering now that a set of admissible initial conditions is given. In particular, let \mathcal{X}_0 be this set and assume that it is a polyhedral or an ellipsoid set (or a union of them) described, respectively, as follows.

(a) Polyhedral set:

$$\mathcal{X}_0 = \text{Co}\{v_1, \dots, v_{n_v}\}, \quad v_s \in \mathbb{R}^n, \quad \forall s = 1, \dots, n_v \quad (2.38)$$

where each vector v_s denotes a vertex of the polyhedral set. In this case the constraint $\mathcal{X}_0 \subset \mathcal{E}(P, 1)$ can be expressed in LMI form as follows [45]:

$$\begin{bmatrix} 1 & v_s' \\ v_s & W \end{bmatrix} \geq 0, \quad \forall s = 1, \dots, n_v \quad (2.39)$$

(b) Ellipsoidal set:

$$\mathcal{X}_0 = \{x \in \mathbb{R}^n; x' P_0 x \leq 1\} \quad (2.40)$$

where $P_0 = P_0' \in \mathbb{R}^{n \times n}$. In this case the constraint $\mathcal{X}_0 \subset \mathcal{E}(P, 1)$ can be expressed in LMI form as follows [45]:

$$\begin{bmatrix} P_0 & I_n \\ I_n & W \end{bmatrix} \geq 0 \quad (2.41)$$

Table 2.1 Example 2.3—
minimization of $\text{trace}(R)$

θ	$\text{trace}(R)$	$(\tau_1; \tau_2)$
50	0.0736	(0.4485; 0.3788)
100	0.1828	(0.1697; 0.1)
130	0.6307	(0.1697; 0.1)

It should be pointed out that, even if R is given, inequality (2.33) is not a “true” LMI in variables W and τ_1 . However, a feasible solution to the problem can be directly searched by fixing the scalar τ_1 and solving an LMI feasibility problem.

On the other hand, given \mathcal{X}_0 , an interesting problem consists in maximizing the set \mathcal{W} , for which the trajectories are guaranteed bounded. In this case, without loss of generality, we can assume $\delta = 1$ and consider one of the size criteria for ellipsoidal sets, expressed in terms of matrix R , proposed in Sect. 2.2.5. For instance, the following optimization problem can be formulated:

$$\begin{aligned} \min \quad & \text{trace}(R) \\ \text{subject to} \quad & \text{inequalities (2.33)–(2.35), (2.39) and/or (2.41)} \end{aligned} \quad (2.42)$$

However, note that (2.42) is not a “true” LMI problem. Actually, there is a product between the variables R and τ_2 and also between W and τ_1 . A way to overcome these bilinearities is to perform a search on a grid defined by parameters τ_1 and τ_2 . In this case, for each fixed (τ_1, τ_2) , an LMI optimization problem can be solved.

Example 2.3 Consider the system treated in Example 2.1, with $B_w = B$, and the following set of admissible initial conditions:

$$\mathcal{X}_0 = \text{Co} \left\{ \theta \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \theta \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

We aim at maximizing a set of admissible amplitude bounded disturbances, given by

$$\mathcal{W} = \{w \in \mathbb{R}^q; w' R w \leq 1\}$$

for which the trajectories of the system are guaranteed to be bounded in an ellipsoidal set $\mathcal{E}(P, 1) \supset \mathcal{X}_0$. For this, we consider the optimization problem (2.42). The optimal value of this problem is obtained by solving LMI problems in a grid on τ_1 and τ_2 . Table 2.1 shows the obtained optimal values for $\text{trace}(R)$, considering different values for θ . The values of τ_1 and τ_2 for which the optimal solution is achieved in each case are also shown.

The domains \mathcal{W} and $\mathcal{E}(P, 1)$, obtained for the cases in Table 2.1 are depicted in Figs. 2.3 and 2.4. The trade-off between the admissible set of disturbances and the admissible set of initial conditions is clearly visible. The larger is the former, the smaller tends to be the latter. This means that, in general, the larger are the initial conditions to be considered, the smaller will be the admissible bounds on the tolerated disturbances, for which we can ensure that the closed-loop trajectories are bounded.

Fig. 2.3 Example 2.3—
obtained domains of
admissible disturbances
considering: $\theta = 50$ (—),
 $\theta = 100$ (---), $\theta = 130$ (- - -)

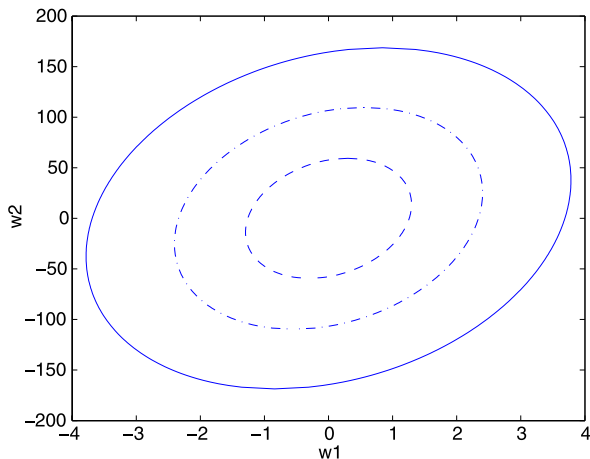
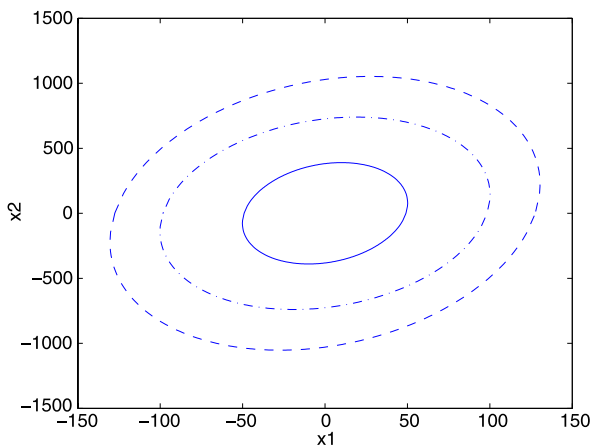


Fig. 2.4 Example 2.3—
obtained domains of
admissible initial states
considering: $\theta = 50$ (—),
 $\theta = 100$ (---), $\theta = 130$ (- - -)



Considering $\theta = 50$, the following matrices are obtained:

$$R = \begin{bmatrix} 0.0735 & -0.0004 \\ -0.0004 & 0.0000 \end{bmatrix}; \quad W = 10^5 \times \begin{bmatrix} 0.0251 & 0.0389 \\ 0.0389 & 1.5193 \end{bmatrix}$$

For this case, consider now an initial condition $x(0) = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$ and the following disturbance constant amplitude signal:

$$w(t) = \begin{bmatrix} 0 \\ -164.3495 \end{bmatrix}, \quad \forall t \geq 0$$

In this case note that $x(0)'Px(0) = 1$ and $w(t)'Rw(t) = 1$. The response of the system is shown in Fig. 2.5. It appears that the state converges to an equilibrium point outside the linearity region. Note that control signal $u_{(2)}(t)$ remains saturated in steady state. Such a behavior is also depicted in Fig. 2.6. The dashed lines cor-

Fig. 2.5 Example 2.3—time-domain simulation considering $x(0) = [50 \ 50]'$ and $w(t) = [0 \ -164.3495]'$

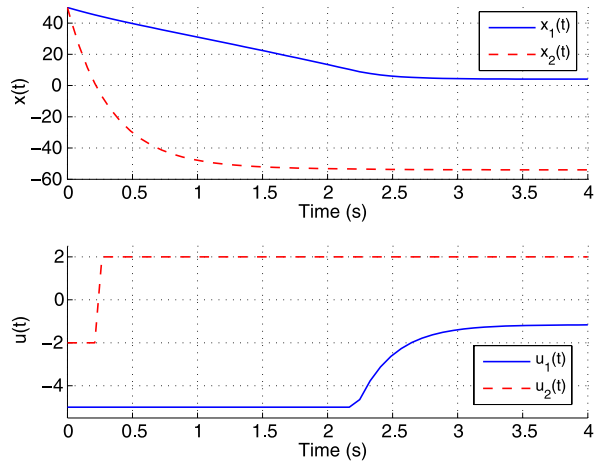
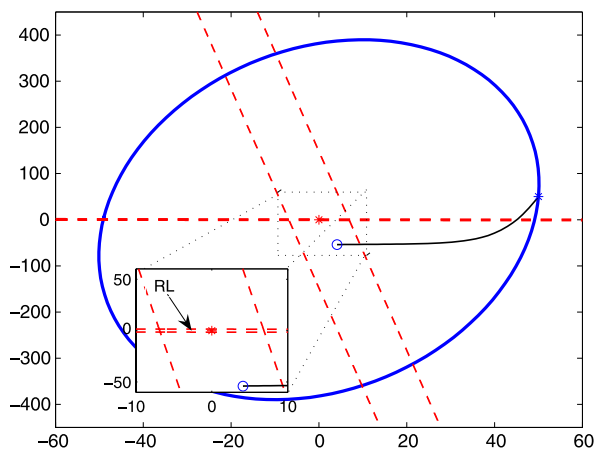


Fig. 2.6 Example 2.3—state trajectory considering $x(0) = [50 \ 50]'$ and $w(t) = [0 \ -164.3495]'$, the set $\mathcal{E}(P, 1)$ (—) and the equilibrium point (o)



responds to the lines $K_{(1)}x = \pm u_0$ and $K_{(2)}x = \pm u_0$. As expected, note that the trajectory does not leave the set $\mathcal{E}(P, 1)$.

2.3.2 Energy Bounded Exogenous Signals

A generic energy measure of the signal $w(t)$ is given by

$$E_w = \int_0^\infty w(\tau)' R w(\tau) d\tau \quad (2.43)$$

with $R = R' > 0$.

Consider that the exogenous signal $w(t)$ is energy bounded, i.e. it belongs to the following set of functions:

$$\mathcal{W} = \left\{ w : [0, \infty) \rightarrow \mathbb{R}^q; \int_0^\infty w(\tau)' R w(\tau) d\tau \leq \delta^{-1} \right\} \quad (2.44)$$

for some $\delta > 0$. In this case, the energy of $w(t)$ is bounded by δ^{-1} . If $R = I_q$, the energy measure (2.43) corresponds to the square of the \mathcal{L}_2 -norm of the signal $w(t)$, i.e

$$\|w\|_2 = \sqrt{\int_0^\infty w(\tau)' w(\tau) d\tau}$$

In this case, \mathcal{W} represents a set of \mathcal{L}_2 -bounded disturbances.

Consider a quadratic Lyapunov function $V(x) = x' P x$. If [45]

$$\dot{V}(x) - w' R w < 0 \quad (2.45)$$

is verified along the trajectories of system (2.28), then

$$V(x(T)) - V(x(0)) - \int_0^T w(t)' R w(t) dt < 0, \quad \forall T$$

Hence, $\forall x(0) \in \mathcal{E}(P, \beta)$ and $w(t) \in \mathcal{W}$, it follows that [89, 280]

- $V(x(T)) < \|w\|_2^2 + V(x(0)) \leq \delta^{-1} + \beta^{-1}$, $\forall T > 0$, i.e. the trajectories of the system do not leave the set $\mathcal{E}(P, (\delta^{-1} + \beta^{-1})^{-1})$;
- if $w(t) = 0$, $\forall t > t_1 \geq 0$, then $\dot{V}(x(t)) < 0$, which ensures that $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

Based on the discussion above, the following formal result considering the polytopic model II can be stated.

Proposition 2.5 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Q \in \mathbb{R}^{m \times n}$ and a positive scalar μ , satisfying*

$$\begin{bmatrix} W(A + B\Gamma_j^+ K)' + Q'\Gamma_j^- B' + (A + B\Gamma_j^+ K)W + B\Gamma_j^- Q & B_w \\ B_w' & -R \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.46)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & \mu u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.47)$$

$$\delta - \mu \geq 0 \quad (2.48)$$

then:

1. $\forall w(t) \in \mathcal{W}$ and $\forall x(0) \in \mathcal{E}(P, \beta)$, with $0 \leq \beta^{-1} \leq \mu^{-1} - \delta^{-1}$, the trajectories of the system (2.28) do not leave the set $\mathcal{E}(P, \mu)$, with $P = W^{-1}$;
2. $\mathcal{E}(P, \mu)$ is a region of asymptotic stability (RAS) for the system (2.28).

Proof Left- and right-multiplying the LMI (2.46) by the block diagonal matrix $\text{Diag}(P, I)$, with $P = W^{-1}$ and considering $H = QP$, it follows that (2.46) is equivalent to:

$$\begin{bmatrix} (A + B\Gamma_j^+ K)'P + H'\Gamma_j^- B'P + P(A + B\Gamma_j^+ K) + PB\Gamma_j^- H & PB_w \\ B_w'P & -R \end{bmatrix} < 0 \quad (2.49)$$

Moreover by left- and right-multiplying (2.49) by $[x(t)' w(t)']$ and $\begin{bmatrix} x(t) \\ w(t) \end{bmatrix}$, respectively, by convexity, it follows that (2.45) is verified along the trajectories of the polytopic system (2.37). Hence, if the set $\mathcal{E}(P, \mu)$ is included in the region $S(|H|, u_0)$, it follows that (2.46) implies that (2.45) is verified along the trajectories of the saturated system. Relation (2.48) ensures that $\mu^{-1} - \delta^{-1} \geq 0$, i.e. there exists $0 \leq \beta^{-1} \leq \mu^{-1} - \delta^{-1}$. Finally, the LMI (2.47) ensures that $\mathcal{E}(P, \mu) \subset S(|H|, u_0)$, which concludes the proof. \square

As commented in Sect. 1.6.3, for a given set of admissible initial states, we may be interested in estimating the maximal set of admissible exogenous signals \mathcal{W} , for which the trajectories are bounded and the internal asymptotic stability is preserved (disturbance tolerance analysis problem). This problem can be addressed by casting the conditions stated in Proposition 2.5 in some optimization problems as follows.

In particular, if $x(0) = 0$ (i.e. the system is in equilibrium), it follows that $\delta^{-1} = \mu^{-1}$ and the problem of maximization of the set \mathcal{W} can be addressed by solving the following convex problem:

$$\begin{aligned} \min \quad & \mu \\ \text{subject to} \quad & \text{inequalities (2.46)–(2.47)} \end{aligned} \quad (2.50)$$

If $x(0) \neq 0$, a trade-off between the size of the admissible initial conditions and the maximal level of admissible disturbances, given by δ^{-1} , appears [62, 280]. The larger is the admissible δ (i.e. the lower is the disturbance admissible energy), the larger is the “allowable” set of admissible initial conditions, given by $\mathcal{E}(P, \beta)$.

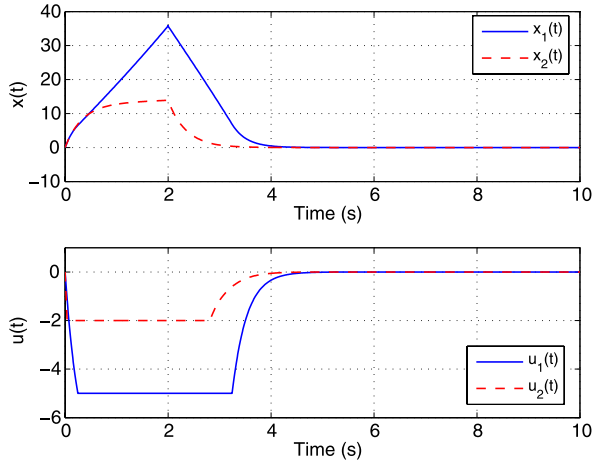
Differently from the amplitude bounded case discussed in Sect. 2.3.1, the set of admissible states does not coincide with the reachable set $\mathcal{E}(P, \beta)$, although both are defined from matrix P . Moreover, the set $\mathcal{E}(P, \mu)$ is not an invariant set. In fact, if $x(0) \in \mathcal{E}(P, \mu)$ but $x(0) \notin \mathcal{E}(P, \beta)$, there is no guarantee that the trajectories will lie in $\mathcal{E}(P, \mu)$, $\forall w \in \mathcal{W}$.

Hence, considering a generic given set of admissible initial states \mathcal{X}_0 , described for instance as in (2.40) or (2.38), the solution of the problem of finding an estimate of the maximal bound on the admissible disturbances is not direct. In this case, two steps have to be performed:

1. solve (2.50);
2. from the obtained μ computed in the first step, compute $\beta^{-1} = \mu^{-1} - \delta^{-1}$, and test if $\mathcal{X}_0 \subset \mathcal{E}(P, \beta)$.

Of course, the scalar β^{-1} in step 2 must be positive. Otherwise, we can conclude that conditions of Proposition 2.5 fail in providing a guarantee of external stability

Fig. 2.7 Example 2.4—response to $x(0) = 0$ and $w(t)$ as defined in (2.51), with $\alpha = 40.8228$



for the saturated system (2.28). Note that if \mathcal{X}_0 is described as (2.40) or as (2.38), it can be easily checked if $\mathcal{X}_0 \subset \mathcal{E}(P, \beta)$.

Example 2.4 Consider the same system as in Example 2.3. We analyse now the case in which the disturbance is energy bounded, i.e.

$$\mathcal{W} = \left\{ w : [0, \infty) \rightarrow \Re; \int_0^\infty w(\tau)' w(\tau) d\tau \leq \delta^{-1} \right\}$$

In order to evaluate the disturbance tolerance of the closed-loop system, we consider the optimization problem (2.50), which gives, as optimal value

$$\mu = 3.0003 \times 10^{-4}$$

Hence, if the initial condition is zero, i.e. $x(0) = 0$, from Proposition 2.5, we can ensure that the closed-loop trajectories are bounded for $\|w\|_2 \leq 57.7322$, i.e. $\delta^{-1} = 3.3330 \times 10^3$. It should, however, be highlighted that this value is only an estimate of the actual allowable maximal admissible energy bounded disturbance.

Assuming $x(0) = 0$, Figs. 2.7 and 2.8, show the time response and the trajectory of the state for a disturbance given by

$$w(t) = \begin{cases} \alpha & \text{if } 0 \leq t \leq 2 \\ 0 & \text{if } t > 2 \end{cases} \quad (2.51)$$

with $\alpha = 40.8228$. In this case, it follows effectively that $\|w\|_2 = 57.7322$. It can be noticed that both control signals saturate, then once the disturbance action stops (from $t = 2$) the state converges to the origin. Moreover note that, as expected, the trajectory is bounded, but it is relatively “far” from the boundary of the computed reachable set $\mathcal{E}(P, \mu)$. Actually, optimization problem (2.50) may provide a very conservative estimate of the actual maximal disturbance for which the trajectories are bounded. This comes from the fact that Proposition 2.5 provides only a sufficient condition and that the polytopic model is also a conservative representation of the true saturated system. On the other hand, it is not possible to ensure

Fig. 2.8 Example 2.4—state trajectory for $x(0) = 0$, $w(t)$ as defined in (2.51), with $\alpha = 40.8228$ and the set $\mathcal{E}(P, \mu)$

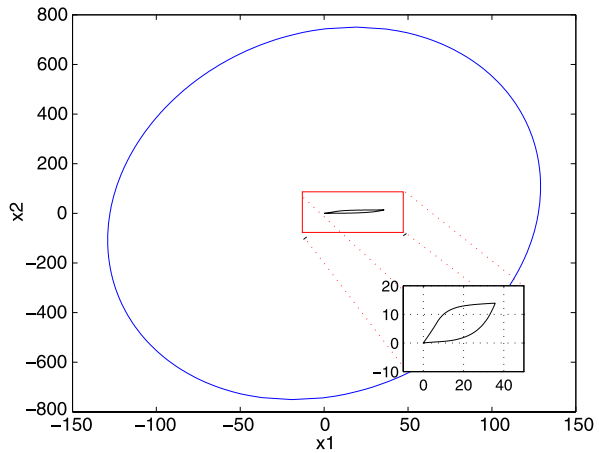
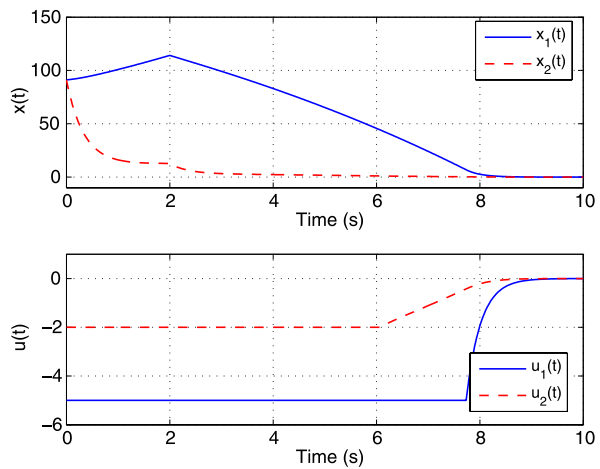


Fig. 2.9 Example 2.4—response to $x(0) = [91.1859 \ 91.1859]'$ and $w(t)$ as defined in (2.51), with $\alpha = 28.8661$



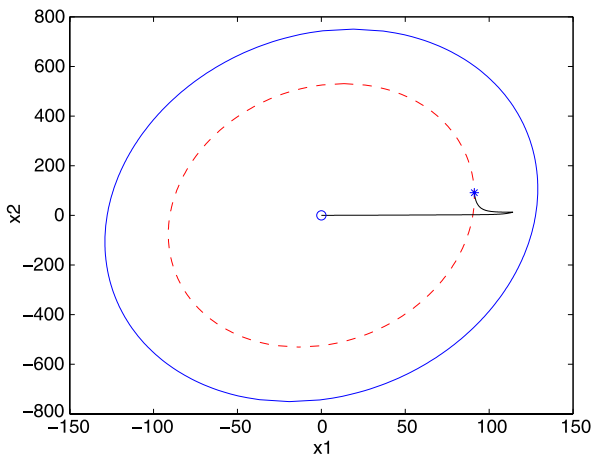
that there is no other particular signal with the same energy that takes the trajectories outside $\mathcal{E}(P, \mu)$ and, eventually, outside the region of attraction of the system.

Let us consider now a trade-off between the initial conditions and disturbances by setting:

$$\delta^{-1} = (2\mu)^{-1}; \quad \beta^{-1} = (2\mu)^{-1}$$

The simulation results considering $x(0) = [91.1859 \ 91.1859]'$ and $w(t)$ defined as in (2.51), with $\alpha = 28.8661$, are depicted in Figs. 2.9 and 2.10. Note that the initial condition is on the boundary of the set $\mathcal{E}(P, \beta)$ and $\|w\|_2^2 = \delta^{-1}$. As expected, the trajectory leaves the set $\mathcal{E}(P, \beta)$, but it does not leave the set $\mathcal{E}(P, \mu)$. Furthermore, when the disturbance goes to zero, the trajectory converges asymptotically to the origin.

Fig. 2.10 Example 2.4—
trajectory for $x(0) =$
[91.1859 91.1859] $'$ and $w(t)$
as defined in (2.51), with
 $\alpha = 28.8661$, and sets
 $\mathcal{E}(P, \beta)$ and $\mathcal{E}(P, \mu)$



2.4 Stabilization

In this section the problems discussed in Sects. 1.6.2 and 1.6.4, regarding the synthesis of control laws taking into account the saturation effects, are addressed, by using the polytopic models. As previously discussed, this class of model is only locally valid. Hence, we are mainly concerned by stating local (regional) stabilization conditions.

In particular, we consider the synthesis of three types of control laws: state feedback, observed-based state feedback and dynamic output feedback. The stabilization conditions are derived in a quadratic framework, which allows to compute the controllers from BMI/LMI optimization problems.

2.4.1 State Feedback Stabilization

Regarding Problems 1.4 and 1.6, in this section we focus on the synthesis of a saturating state feedback control law:

$$u(t) = \text{sat}(Kx(t)) \quad (2.52)$$

In other words, the objective concerns the computation of the matrix K in order to ensure the regional asymptotic stability and the external stability of the closed-loop systems (2.1) and (2.28), respectively.

2.4.1.1 Regional Asymptotic Stabilization

Basically, considering a quadratic approach, given a set of admissible initial states \mathcal{X}_0 , to solve Problem 1.4, the idea is to compute K such that:

- $\dot{V}(x(t)) < 0$ along the trajectories of the polytopic model considered;

- $\mathcal{E}(P, \eta)$ is contained in the region of validity of the polytopic differential inclusion;
- $\mathcal{X}_0 \subseteq \mathcal{E}(P, \eta)$.

Hence, since $\mathcal{E}(P, \eta)$ is a region of asymptotic stability and $\mathcal{X}_0 \subseteq \mathcal{E}(P, \eta)$, it is ensured that $\forall x(0) \in \mathcal{X}_0, \lim_{t \rightarrow \infty} x(t) = 0$. Note, however, that \mathcal{X}_0 is not necessarily a positive invariant set.

On the other hand, we can be interested in computing K in order to maximize the region of attraction of the origin. This problem can be indirectly addressed by computing the gain K in order to maximize an ellipsoidal estimate $\mathcal{E}(P, \eta)$ of the region of attraction.

Regarding the stability conditions stated in Propositions 2.1, 2.2 and 2.3, as well as in Corollary 2.1, we can observe that there exists a product between the variable associated to the quadratic Lyapunov function (P or W) and the gain matrix K . This bilinearity is unsuitable to solve the problem in a convex framework, considering LMI solvers. This apparent difficulty is easily overcome, by performing a classical linearizing change of variables [26, 45], which corresponds to the introduction of an auxiliary variable Y defined as follows:

$$Y = KW$$

Hence, once we solve the matrix inequalities for W and Y , the stabilizing gain is given by $K = YW^{-1}$.

A stabilization result, considering the polytopic approach I, can therefore be straightforwardly obtained from Corollary 2.1 as follows [141].

Proposition 2.6 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{m \times n}$, a vector $\alpha_l \in \mathbb{R}^m$ and a positive scalar η satisfying the following matrix inequalities:*

$$WA' + AW + B\Gamma_j(\alpha_l)Y + Y'\Gamma_j(\alpha_l)'B' < 0, \quad \forall j = 1, \dots, 2^m \quad (2.53)$$

$$\begin{bmatrix} W & \alpha_{l(i)}Y'_{(i)} \\ \alpha_{l(i)}Y_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, m \quad (2.54)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.55)$$

then, for $K = YW^{-1}$, the set $\mathcal{E}(P, \eta)$, with $P = W^{-1}$, is a region of asymptotic stability (RAS) for the closed-loop system (2.1).

Considering polytopic approach II, the result of Proposition 2.2 can be extended as follows to cope with the synthesis of the stabilizing gain K [193].

Proposition 2.7 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, matrices $Y \in \mathbb{R}^{m \times n}$ and $Q \in \mathbb{R}^{m \times n}$ and a positive scalar η satisfying:*

$$WA' + (Y'\Gamma_j^+ + Q'\Gamma_j^-)B' + AW + B(\Gamma_j^+Y + \Gamma_j^-Q) < 0 \\ j = 1, \dots, 2^m \quad (2.56)$$

$$\begin{bmatrix} W & Q'_{(i)} \\ Q_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.57)$$

then, for $K = YW^{-1}$, the set $\mathcal{E}(P, \eta)$, with $P = W^{-1}$, is a region of asymptotic stability (RAS) for the closed-loop system (2.1).

Analogously, considering polytopic approach III, the following synthesis result can be stated [1].

Proposition 2.8 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Y \in \mathbb{R}^{m \times n}$, row vectors $Q_{j(i)} \in \mathbb{R}^{1 \times n}$, $j = 2, \dots, 2^m$, $i \in \mathcal{S}_j$, and a positive scalar η satisfying:*

$$\begin{aligned} & \left(AW + \sum_{i \in \mathcal{S}_j^c} B_i Y_{(i)} + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)} \right) \\ & + \left(AW + \sum_{i \in \mathcal{S}_j^c} B_i Y_{(i)} + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)} \right)' < 0, \quad \forall j = 1, \dots, 2^m \end{aligned} \quad (2.58)$$

$$\begin{bmatrix} W & Q'_{j(i)} \\ Q_{j(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad \forall j = 2, \dots, 2^m, \forall i \in \mathcal{S}_j \quad (2.59)$$

then, for $K = YW^{-1}$, the ellipsoid $\mathcal{E}(P, \eta)$, with $P = W^{-1}$, is a region of asymptotic stability (RAS) for the saturated system (2.1).

The results of Propositions 2.6, 2.7 and 2.8 can be straightforwardly cast in convex optimization problems aiming at the synthesis of K , as follows. When using approaches II and III, as pointed out in Sect. 2.2.5, the set of inequalities can be normalized with respect to η . We can therefore fix $\eta = 1$ without any additional conservatism. This is done in the sequel, i.e. we consider a normalized ellipsoid $\mathcal{E}(P, 1)$.

A. Regional Guaranteed Stability Suppose that the set \mathcal{X}_0 is given. In particular, this set can be described by the union of polyhedral and ellipsoidal sets described as in (2.38) and (2.40), respectively.

Hence, the problem of finding K that ensures the asymptotic stability in \mathcal{X}_0 can be addressed by solving a feasibility problem of the set of inequalities of Propositions 2.6, 2.7 or 2.8 added to LMI constraints like (2.39) and (2.41).

Note that due to the product between Y and α_l , the set of matrix inequalities in Proposition 2.6 are bilinear. Thus, the feasibility problem can be addressed by iteratively solving LMI feasibility problems for α_l or Y a priori fixed. On the other hand, the set of inequalities in Proposition 2.7 and 2.8 are linear on the variables W , Y and Q (or $Q_{j(i)}$) and a direct LMI feasibility problem can be solved.

B. Maximization of an Estimate of the Region of Attraction In this case the idea is to compute K that leads to a region $\mathcal{E}(P, 1)$ as large as possible considering a size criterion, as discussed in Sect. 2.2.5.1.

For instance, the following optimization problems can be considered.

- Maximization of the minor axis of $\mathcal{E}(P, 1)$:

$$\begin{aligned}
 & \max \lambda \\
 & \text{subject to } W > \lambda I_n \\
 & \lambda > 0
 \end{aligned} \tag{2.60}$$

inequalities (2.53)–(2.55) or (2.56)–(2.57) or (2.58)–(2.59)

- Maximization of $\mathcal{E}(P, 1)$ in certain directions

Sometimes, we are interested in maximizing the ellipsoidal set in some specific directions or following a given shape set \mathcal{X}_0 . In particular, this shape set can be described as the union (or the convex hull) of polyhedral and ellipsoidal sets as defined in (2.40) or (2.38). Considering a scaling factor β , the idea is to maximize β such that $\beta\mathcal{X}_0 \subset \mathcal{E}(P, 1)$ and subject to conditions of Propositions 2.6, 2.7 and 2.8. This can be accomplished by the following optimization problem [133, 137, 193]:

$$\begin{aligned}
 & \max \beta \\
 & \text{subject to } \begin{bmatrix} 1 & \beta v'_i \\ \beta v_i & W \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, n_v \quad \text{and/or} \\
 & \begin{bmatrix} P_0 & \beta I_n \\ \beta I_n & W \end{bmatrix} \geq 0
 \end{aligned} \tag{2.61}$$

inequalities (2.53)–(2.55) or (2.56)–(2.57) or (2.58)–(2.59)

C. Optimization of the Actuator Size The control bounds are in general related to the size and the cost of the actuators. A simple (and frequently used in industry) strategy to tackle the problem of saturation is to oversize the actuators, i.e. specify actuators with large output limits. However, in general this leads to actuators that are large, which has serious implication in embedded systems, more expensive and less efficient (in terms of energy consumption), which increases project and production costs.

Hence the idea is to design the control law in order to achieve a guaranteed stability for a given set of admissible initial conditions \mathcal{X}_0 , with less costly actuators. In other words, the control law and the control bounds are simultaneously computed to ensure the stability of the closed-loop system in a pre-specified set.

From the conditions stated in Propositions 2.6, 2.7 or 2.8, and supposing \mathcal{X}_0 given as the union (or the convex hull) of polyhedral and ellipsoidal sets as defined in (2.40) or (2.38), this problem can be addressed from the solution of the following optimization problem.

$$\begin{aligned}
& \min \sum_{i=1}^m c_i \bar{u}_{0(i)} \\
& \text{subject to} \quad \begin{bmatrix} 1 & v'_i \\ v_i & W \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, n_v \quad \text{and/or} \\
& \quad \quad \quad \begin{bmatrix} P_0 & I_n \\ I_n & W \end{bmatrix} \geq 0 \\
& \text{inequalities} \quad \left\{ \begin{array}{l} (2.53), (2.55) \text{ and } \begin{bmatrix} W & \alpha_{l(i)} Y'_{(i)} \\ \alpha_{l(i)} Y_{(i)} & \bar{u}_{0(i)} \end{bmatrix} \geq 0, \\ \quad \quad \quad \forall i = 1, \dots, m \quad \text{or} \\ (2.56) \text{ and } \begin{bmatrix} W & Q'_{(i)} \\ Q_{(i)} & \bar{u}_{0(i)} \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, m \quad \text{or} \\ (2.58) \text{ and } \begin{bmatrix} W & Q'_{j(i)} \\ Q_{j(i)} & \bar{u}_{0(i)} \end{bmatrix} \geq 0, \\ \quad \quad \quad \forall j = 2, \dots, 2^m, \forall i \in \mathcal{S}_j \end{array} \right.
\end{aligned} \tag{2.62}$$

with the coefficients c_i being the weights of the size/cost of each actuator in the composition of the cost function. In this case, it follows that $u_{0(i)} = \sqrt{\bar{u}_{0(i)}}$, $i = 1, \dots, m$.

2.4.1.2 Performance Issues

Suppose there exists a feasible solution for the inequalities of Propositions 2.6, 2.7 or 2.8 such that the inclusion $\mathcal{X}_0 \subset \mathcal{E}(P, 1)$ is satisfied.

A quite natural objective is the search, among all possible stabilizing gains (i.e. all the feasible solutions to the set of inequalities), of an optimal solution taking into account some performance requirement.

In the case where we are interested in the time-domain performance improvement, the idea is, in general, to use the available control to drive the initial state to the origin as fast as possible and with a good damping. Considering a Lyapunov function $V(x(t))$, the estimate of the convergence rate of a nonlinear system can be obtained by an upper bound of $\dot{V}(x(t))$. However, the exigence of the same performance level when the system presents control saturation and when it operates linearly is not realistic. Considering that a performance level is achieved with the unsaturated control law ($u(t) = Kx(t)$), in general it is not possible to keep the same performance level when saturation effectively occurs since the control availability is reduced. Hence, to impose the same performance level when the system operates inside and outside of the linearity region can lead to very conservative solutions. The idea is then to consider different performance criteria depending on the region of operation [141, 144].

The time-domain performance specification in the linearity region may be achieved by placing the poles of $(A + BK)$ in a suitable region of the left half

complex plane. This kind of region can be generically described as an LMI region as follows [68]:

$$\mathcal{D}_p = \{s \in \mathbb{C}; (L + sZ + \bar{s}Z') < 0\} \quad (2.63)$$

where $L = L' \in \mathbb{R}^{l \times l}$, $Z \in \mathbb{R}^{l \times l}$ and s is a complex number with its conjugate \bar{s} .

Henceforth, we assume that the time-domain requirements in the region of linear behavior are satisfied if the poles $(A + BK)$ are located in a region \mathcal{D}_p .

The following result can be used to incorporate this kind of performance constraint in the synthesis of the saturating control law.

Theorem 2.1 [68] *Consider a region \mathcal{D}_p described by (2.63). If there exists a positive definite matrix $W \in \mathbb{R}^{n \times n}$ and a matrix $Y \in \mathbb{R}^{m \times n}$ such that*

$$L \otimes W + Z \otimes (AW + BY) + Z' \otimes (AW + BY)' < 0 \quad (2.64)$$

then $K = YW^{-1}$ places the poles of $(A + BK)$ in \mathcal{D}_p .

The LMI constraint (2.64) can be straightforwardly incorporated in a feasibility problem or even in the optimization problems (2.60) and (2.61) [141]. On the other hand, if there is a feasible solution satisfying the conditions of Propositions 2.6, 2.7 or 2.8 and the inclusion constraint $\mathcal{X}_0 \subseteq \mathcal{E}(P, 1)$, it is possible to search for a gain K in the sense of improving the performance in the region of linearity. Let us consider, for example, the region \mathcal{D}_p be defined as follows:

$$\mathcal{D}_p = \{s \in \mathbb{C}; \Re\{s\} < -\sigma, \sigma > 0\} \quad (2.65)$$

Notice that the larger is σ , the farther from the origin are the poles of $(A + BK)$ and the larger is the speed of convergence of the trajectories to the origin inside the region of linearity. Thus, the following optimization problem can be formulated in order to improve the convergence rate inside the linearity region [130, 137]:

max σ

subject to $WA' + AW + BY + Y'B' + 2\sigma W < 0$

$$\begin{bmatrix} 1 & v_i' \\ v_i & W \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, n_v \quad \text{and/or} \quad (2.66)$$

$$\begin{bmatrix} P_0 & I_n \\ I_n & W \end{bmatrix} \geq 0$$

inequalities (2.53)–(2.55) or (2.56)–(2.57) or (2.58)–(2.59)

Considering (2.53)–(2.55) with a fixed α_l , or directly (2.56)–(2.57) and (2.58)–(2.59), it follows that problem (2.66) is a *generalized eigenvalue problem* (GEVP) [45] and can be easily solved by any standard LMI solver package.

2.4.1.3 Trade-off Between Saturation, Size of the Stability Region and Time-Domain Performance

In some cases one can be interested in finding a gain K such that for all $x(0) \in \mathcal{X}_0$, the trajectories never leave the linearity region. This solution, known as *saturation avoidance problem* (see Sect. 1.6.2.3), can be solved if the following constraints are satisfied:

- (a) $WA' + AW + BY + Y'B' < 0$;
- (b) $\mathcal{E}(P, 1) \subset S(|K|, u_0)$.

A fundamental issue is whether the use of effective saturating control laws can be advantageous or not. In [190, 219] it was shown that, at least in some cases, the use of the saturating control laws does not help in obtaining larger regions of stability. It is, however, very important to highlight that no constraints concerning neither the performance, nor the robustness, are taken into account in this analysis. In this case, although the optimal region of stability is obtained with a linear control law, the closed-loop poles associated to this solution can be very close to the imaginary axis, which implies a very slow behavior [130, 137, 144].

Hence, the idea is to allow saturation in order to ensure the stability for larger sets of admissible states, but in the presence of performance constraints. For instance, if the criterion is the maximization of $\mathcal{E}(P, 1)$ along a shape set \mathcal{X}_0 and a constraint $\sigma(A + BK) \subset \mathcal{D}_p$ is considered (with \mathcal{D}_p defined in (2.65)), the following optimization problem can be solved to compute K :

$$\begin{aligned}
 & \max \beta \\
 & \text{subject to } WA' + AW + BY + Y'B' + 2\sigma W < 0 \\
 & \quad \begin{bmatrix} 1 & \beta v'_i \\ \beta v_i & W \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, n_v \quad \text{and/or} \\
 & \quad \begin{bmatrix} P_0 & \beta I_n \\ \beta I_n & W \end{bmatrix} \geq 0 \\
 & \quad \text{inequalities (2.53)–(2.55) or (2.56)–(2.57) or (2.58)–(2.59)}
 \end{aligned} \tag{2.67}$$

Example 2.5 Let the open-loop system be described by the following matrices:

$$A = \begin{bmatrix} 0 & 1 \\ 10 & -0.1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Consider the control bounds given by $u_0 = 1$. Let the set of admissible initial conditions be given by an hypercube in \mathbb{R}^2 :

$$\mathcal{X}_0 = \{x \in \mathbb{R}^2; -1 \leq x_{(i)} \leq 1, \forall i = 1, \dots, 2\}$$

We consider for the performance specification in the linearity region the placement of the poles of $(A + BK)$ in the region defined in (2.65).

Table 2.2 shows the optimal obtained values for the optimization Problem (2.67) considering the polytopic approach I. For different values of σ , the values of β_{lin}

Table 2.2 Example 2.5—trade-off between the size of the region of stability and the performance constraint in terms of a pole placement

σ	β_{lin}	α_l^*	β^*	K^*	eig_{max}
0	0.0653	1	0.0653	$[-11.56 \ -3.60 \ -0.15]$	-0.0001
1	0.0494	0.76	0.0653	$[-15.29 \ -4.76 \ -0.206]$	-1.334
5	0.0133	0.49	0.0266	$[-50.42 \ -12.28 \ -0.50]$	$-5.08 \pm j3.09$

(obtained considering $\alpha_l = 1$, i.e. $\mathcal{E}(P, 1) \subset S(|K|, u_0) = R_L$) and the optimal value of β with the associated α_l are depicted. It is also shown the obtained gain K and the maximal eigenvalue of $(A + BK)$. The maximum β is obtained for the linear case ($\alpha_l = 1$) with $\sigma = 0$. This is in accordance with the results in [219] and [190]. However, note that in this case the eigenvalues of the linear system are very close to the imaginary axis. Considering $\sigma \neq 0$, the best β is achieved considering saturation, i.e. $\alpha_l \neq 1$.

Example 2.6 Consider the control of two inverted pendulums in cascade where the system matrices are

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 9.8 & 0 & -9.8 & 0 \\ 0 & 0 & 0 & 1 \\ -9.8 & 0 & 2.94 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 0 \\ 1 & -2 \\ 0 & 0 \\ -2 & 5 \end{bmatrix}$$

The control bounds are given by $u_0 = [10 \ 10]'$. Consider the following shape set:

$$\mathcal{X}_0 = \text{Co} \left\{ \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 0 \\ -0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 0 \\ -0.5 \\ 0 \end{bmatrix} \right\}$$

Notice that matrix A is unstable (the eigenvalues of A are 4.0930 , -4.0930 , $\pm 2.0032i$). As a performance requirement we consider the placement of the eigenvalues of $(A + BK)$ in the region \mathcal{D}_p defined in (2.65). Hence the larger is σ , the larger tends to be the rate of convergence of the linear trajectories to the origin.

Considering the above data, and the polytopic approach I, Table 2.3 shows the final values of α_l and β obtained from a stabilization version of the iterative algorithm proposed in Sect. 2.2.5.2 from different initial vectors α_l and scalars σ . β_{initial} and β_{final} denote the optimal value of β obtained by the iterative algorithm from, respectively, α_{initial} and α_{final} .

Regarding Table 2.3 we may notice the following: the smaller are the components of α_l , the larger is the β obtained from (2.67). This illustrates the fact that by allowing saturation we can stabilize the system for a larger set of initial conditions. Besides, the more stringent is the performance requirement (larger σ , in this

Table 2.3 Example 2.6—algorithm performance and trade-off between performance requirement and size of the region of stability

σ	α_{initial}	β_{initial}	α_{final}	β_{final}
1	[1.00 1.00]'	1.9883	[0.7912 0.8950]'	2.3112
1	[0.50 0.50]'	2.3652	[0.4999 0.5000]'	2.3653
1	[0.25 0.25]'	2.5607	[0.2495 0.2500]'	2.5613
2	[1.00 1.00]'	0.9731	[0.7232 0.8430]'	1.2154
2	[0.50 0.50]'	1.2930	[0.4997 0.5000]'	1.2931
2	[0.25 0.25]'	1.5378	[0.2493 0.2500]'	1.5388
3	[1.00 1.00]'	0.5707	[0.6850 0.8039]'	0.7519
3	[0.50 0.50]'	0.8302	[0.5000 0.5000]'	0.8316
3	[0.25 0.25]'	1.0737	[0.2485 0.2500]'	1.0747

case), the smaller is the region of admissible initial states for which we can find a solution.

Comparing the solution obtained by avoiding saturation (i.e. $\alpha_l = [1 \ 1]'$) with the solution that takes into account the nonlinear behavior of the system, we can observe that:

- For $\sigma = 1$, it is possible to obtain a solution for a set of admissible initial states 28% larger.
- For $\sigma = 2$, it is possible to obtain a solution for a set of admissible initial states 58% larger.
- For $\sigma = 3$, it is possible to obtain a solution for a set of admissible initial states 88% larger.

In the examples above, the trade-off between the size of the stability region and the performance of the closed-loop system was illustrated. In general, the more demanding is the required time-domain performance, the smaller will be the region for which the stability can be ensured. Hence, an interesting idea is the application of a “low” gain when the state is far from the equilibrium point and a “high” gain when it approaches the equilibrium [309]. The switching between these gains should be done on a switching surface defined in the state space. This idea is developed in the next sub-section in the context of the present approach.

2.4.1.4 Piecewise Linear State Feedback

The philosophy of the piecewise linear state feedback control (or switched state feedback control) consists in applying higher feedback gains to the system as the state approaches the origin. This is an interesting way to deal with the problem of ensuring the stability for larger regions in the presence of saturation and, at the same time, to improve the rate of convergence of the closed-loop trajectories to the origin. This idea has been applied for instance in [187, 395], considering a saturation avoidance approach, and in [130, 133, 137], considering effective saturation.

The main problem of this kind of control law is to determine appropriate switching sets and the associated gains in order to avoid limit cycles or unstable behavior. In particular, it must be ensured that at a switching instant $t = t_s$, the state $x(t_s)$ belongs to the region of attraction associated to the state feedback gain to be applied. Following this principle, we show now how to compute a stabilizing piecewise *saturating* control law based on the conditions given in Propositions 2.6, 2.7 or 2.8.

Let N be the number of desired switching sets and \mathcal{X}_0 the set of admissible states for which the stability should be ensured. A piecewise *saturating* control law can be computed as follows [137]:

Step 1. Define N homothetical sets to \mathcal{X}_0 as follows:

$$\begin{aligned}\mathcal{X}_q &= \beta_q \mathcal{X}_0, \quad 0 < \beta_q < 1, \quad q = 1, \dots, N \\ \mathcal{X}_N &\subset \mathcal{X}_{N-1} \subset \dots \subset \mathcal{X}_1 \subset \mathcal{X}_0\end{aligned}$$

Step 2. For each $q = 0, \dots, N$, solve a GEVP of type (2.66), by considering \mathcal{X}_q as given and W_q, Y_q, σ as the associated optimal solution.

Step 3. For each $q = 0, \dots, N$ define:

- the feedback matrix: $K_q = Y_q W_q^{-1}$;
- the switching set: $S_q = \{x \in \mathbb{R}^n; x^T W_q^{-1} x \leq 1\}$.

Hence, from Propositions 2.6, 2.7 or 2.8, it follows that the application of the control law defined as

$$u(t) = \begin{cases} \text{sat}(K_0 x(t)) & \text{if } x(t) \in S_0, x(t) \notin \{S_1, S_2, \dots, S_N\} \\ \text{sat}(K_1 x(t)) & \text{if } x(t) \in S_1, x(t) \notin \{S_2, S_3, \dots, S_N\} \\ \vdots & \vdots \\ \text{sat}(K_N x(t)) & \text{if } x(t) \in S_N \end{cases} \quad (2.68)$$

guarantees the asymptotic convergence to the origin of all the trajectories emanating from \mathcal{X}_0 . By construction, if at instant $t = t_s$ the state feedback is switched from K_{q-1} to K_q , the state $x(t_s)$ belongs to the region of attraction associated to $\dot{x}(t) = Ax(t) + B \text{sat}(K_q x(t))$. Note that we consider a different maximized σ for each gain K_q . The idea is to accelerate even more the convergence by allowing the saturation each time the trajectory enters a new region S_q .

Remark 2.5 Differently from [395], the ellipsoids do not need to be nested in the presented approach. Anyway, if for some reason one wants to ensure the nesting property for the switching surfaces, it suffices to consider inclusion constraints of type (2.41).

Remark 2.6 Considering a parameterized Riccati equation approach and the avoidance saturation case, a similar philosophy of switching to “higher” gains as the state approaches the origin is developed in Chap. 5. In that case, each switching set is considered to be included in the region of linearity of the corresponding closed-loop system, i.e. $S_q \subset \mathcal{S}(|K_q|, u_0)$.

2.4.1.5 Regional External Stabilization

Similarly to Sect. 2.4.1.1, considering the linearizing variable change $Y = KW$, stabilization conditions for the case where the system is subject to the action of additive exogenous signals can be straightforwardly derived. In this section we briefly present the resulting stabilization conditions stated considering the polytopic model II. Analogous results can be obtained considering the polytopic models I and III. In addition, some optimization problems are proposed in order to compute the state feedback gain from the proposed conditions.

(A) Amplitude Bounded Exogenous Signals Considering that $w(t)$ is bounded in amplitude, i.e. $w(t) \in \mathcal{W}$ with \mathcal{W} defined in (2.29), the following result follows from Proposition 2.4.

Proposition 2.9 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, matrices $Q, Y \in \mathbb{R}^{m \times n}$ and positive scalars τ_1 and τ_2 satisfying*

$$\begin{bmatrix} WA' + AW + (Y' \Gamma_j^+ + Q' \Gamma_j^-) B' + B(\Gamma_j^+ Y + \Gamma_j^- Q) + \tau_1 W & B_w \\ B_w' & -\tau_2 R \end{bmatrix} < 0 \quad j = 1, \dots, 2^m \quad (2.69)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.70)$$

$$\tau_2 - \delta \tau_1 < 0 \quad (2.71)$$

then the gain $K = YW^{-1}$ is such that:

1. $\forall w \in \mathcal{W}$, the trajectories of system (2.28) do not leave the set $\mathcal{E}(P, 1)$;
2. for $w = 0$, $\mathcal{E}(P, 1)$ is a region of asymptotic stability (RAS) for system (2.28).

From the conditions stated in Proposition 2.9, the following optimization problems can be used to compute a suitable stabilizing state feedback gain.

1. *Maximization of the Disturbance Tolerance with Guaranteed RAS* Suppose that a set of admissible initial states \mathcal{X}_0 is given as a polyhedral or an ellipsoidal set, as defined, respectively, in (2.40) and (2.38), or a combination of both. The idea is to compute K in order to maximize the disturbance tolerance [353], considering that $x(0) \in \mathcal{X}_0$. For the sake of simplicity consider $R = I$. In this case, the objective consists in maximizing the bound δ^{-1} on the exogenous signal, for which it is possible to find a stabilizing gain K . Hence, the following optimization problem can be considered:

$$\begin{aligned} & \min_{\delta, W, Y, \tau_1, \tau_2} \delta \\ & \text{subject to} \quad \text{inequalities (2.39) and/or (2.41), (2.69)–(2.71)} \end{aligned} \quad (2.72)$$

Note that due to the products $\delta \tau_1$ and $\tau_1 W$, the inequalities (2.69) and (2.71) are not LMIs. In order to overcome this drawback, problem (2.72) can be solved by considering the solutions of LMI-based problems formulated on a grid in τ_1 , i.e. a line search considering subproblems with τ_1 fixed.

2. *Maximization of the RAS for a Given Set of Admissible Disturbances* Suppose that matrix R and the scalar δ defining the set of admissible exogenous signals are given. The idea is to find K that maximizes the set $\mathcal{E}(P, 1)$, while ensuring that the trajectories are bounded in this set for every $w \in \mathcal{W}$ and $x(0) \in \mathcal{E}(P, 1)$. Thus, the following optimization problem can be formulated:

$$\begin{aligned} & \max_{W, Y, \tau_1, \tau_2} f(\mathcal{E}(P, 1)) \\ & \text{subject to} \quad \text{inequalities (2.69)–(2.71)} \end{aligned} \quad (2.73)$$

with $f(\mathcal{E}(P, 1))$ being a size criterion as discussed in Sect. 2.2.5.1.

(B) Energy Bounded Exogenous Signals Considering now that $w(t)$ is bounded in energy, i.e. $w(t) \in \mathcal{W}$ with \mathcal{W} defined in (2.44), the following result can be derived from Proposition 2.5.

Proposition 2.10 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrices $Q, Y \in \mathbb{R}^{m \times n}$ and a scalar μ , satisfying*

$$\begin{bmatrix} WA' + AW + (Y' \Gamma_j^+ + Q' \Gamma_j^-) B' + B(\Gamma_j^+ Y + \Gamma_j^- Q) & B_w \\ B_w' & -R \end{bmatrix} < 0 \quad j = 1, \dots, 2^m \quad (2.74)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & \mu u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.75)$$

$$\delta - \mu \geq 0 \quad (2.76)$$

then the gain $K = YW^{-1}$ is such that:

1. $\forall w(t) \in \mathcal{W}$ and $\forall x(0) \in \mathcal{E}(P, \beta)$, with $0 \leq \beta^{-1} \leq \mu^{-1} - \delta^{-1}$ and $P = W^{-1}$, the trajectories of the system (2.28) do not leave the set $\mathcal{E}(P, \mu)$;
2. for $w = 0$, $\mathcal{E}(P, \mu)$ is a region of asymptotic stability (RAS) for the system (2.28).

From Proposition 2.10, it is possible to compute K in order to ensure that the trajectories of the system are bounded for any energy bounded admissible signal $w(t)$. Consider now that the regulated output of the system is given by

$$z(t) = C_z x(t) \quad (2.77)$$

An additional objective can be the determination of an upper bound for the \mathcal{L}_2 -gain from the admissible disturbance w to the regulated output z . In other words, when w is viewed as a disturbance, we want to ensure some level (in the \mathcal{L}_2 sense) of disturbance rejection. In this case, we consider $R = I$. This objective can be achieved by defining a function [45, 215]:

$$\mathcal{J}(t) = \dot{V}(x(t)) - w(t)' w(t) + \frac{1}{\gamma} z(t)' z(t)$$

with $V(x(t)) = x(t)' P x(t)$, $P = P' > 0$. Hence, if $\mathcal{J}(t) < 0$, one obtains

$$\int_0^T \mathcal{J}(t) dt = V(x(T)) - V(x(0)) - \int_0^T w(t)' w(t) dt + \frac{1}{\gamma} \int_0^T z(t)' z(t) dt < 0$$

$$\forall T > 0$$

and it is possible to conclude that:

- $V(x(T)) < V(x(0)) + \|w\|_2^2$,
- for $T \rightarrow \infty$, $\|z\|_2^2 < \gamma(\|w\|_2^2 + V(x(0)))$,
- if $w(t) = 0$, $\forall t \geq t_1 \geq 0$, then $\dot{V}(x(t)) < -\frac{1}{\gamma} z(t)' z(t) < 0$.

The following result can be formulated in this case.

Proposition 2.11 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrices $Q, Y \in \mathbb{R}^{m \times n}$ and scalars μ and γ satisfying*

$$\begin{bmatrix} W A' + A W + (Y' \Gamma_j^+ + Q' \Gamma_j^-) B' + B(\Gamma_j^+ Y + \Gamma_j^- Q) & B_w & W C_z' \\ B_w' & -I & 0 \\ C_z W & 0 & -\gamma I \end{bmatrix} < 0$$

$$j = 1, \dots, 2^m \quad (2.78)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & \mu u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.79)$$

$$\delta - \mu \geq 0 \quad (2.80)$$

then the gain $K = YW^{-1}$ is such that:

1. $\forall w(t)$ such that $\|w(t)\|_2^2 \leq \delta^{-1}$ and $\forall x(0) \in \mathcal{E}(P, \beta)$, with $0 \leq \beta^{-1} \leq \mu^{-1} - \delta^{-1}$:
 - the trajectories of the system (2.28) do not leave the set $\mathcal{E}(P, \mu)$, with $P = W^{-1}$;
 - $\|z\|_2^2 < \gamma(\|w\|_2^2 + V(x(0)))$.
2. If $w(t) = 0$, $\forall t \geq t_1 \geq 0$, then $x(t) \rightarrow 0$, i.e. $\mathcal{E}(P, \mu)$ is a region of asymptotic stability (RAS) for the system (2.28).

Proof Left- and right-multiplying (2.78) by $\text{Diag}(P, I, I)$ and applying Schur's complement in the sequel, it follows that (2.78) is equivalent to

$$\begin{bmatrix} A'P + PA + (K' \Gamma_j^+ + H' \Gamma_j^-) B' P + P B(\Gamma_j^+ K + \Gamma_j^- H) + \gamma^{-1} C' C & P B_w \\ B_w' P & -I \end{bmatrix} < 0, \quad j = 1, \dots, 2^m$$

with $H = QP$, which implies that $\mathcal{J}(t) < 0$ along the trajectories of system (2.28), provided that $\mathcal{E}(P, \mu) \subset S(|H|, u_0)$. This inclusion is ensured by (2.79). The inequality (2.80) ensures that $\beta^{-1} \geq 0$. Hence, we can conclude that:

- For w such that $\|w\|_2^2 \leq \delta^{-1}$ and $x(0) \in \mathcal{E}(P, \beta)$ it follows that $V(x(T)) < V(x(0)) + \|w\|_2^2 \leq \delta^{-1} + \beta^{-1} \leq \mu^{-1}$, $\forall T > 0$, i.e. the trajectories of the system are confined in the set $\mathcal{E}(P, \mu)$.

- For $T \rightarrow \infty$, $\|z\|_2^2 < \gamma(\|w\|_2^2 + V(x(0)))$.
- If $w(t) = 0$, $\forall t \geq t_1 \geq 0$, then $\dot{V}(x(t)) < -\frac{1}{\gamma}z(t)'z(t) < 0$, $\forall x(t) \in \mathcal{E}(P, \mu)$, which ensures that this set is a region of asymptotic stability. \square

Considering that $x(0) = 0$, the result of Proposition 2.11 ensures the \mathcal{L}_2 input-to-output stability of the closed-loop system. In this case $\mu^{-1} = \delta^{-1}$ and the \mathcal{L}_2 -gain of the system is given by $1/\sqrt{\gamma}$, i.e. $\|z\|_2 \leq 1/\sqrt{\gamma}\|w\|_2$. Thus, the following convex optimization problems are of special interest.

1. Maximization of the Disturbance Tolerance

The idea consists in maximizing the bound on the disturbance energy, for which we can ensure that the system trajectories remain bounded. This can be accomplished by the following optimization problem.

$$\begin{aligned} \min \quad & \mu \\ \text{subject to} \quad & \text{inequalities (2.74)–(2.75)} \end{aligned} \quad (2.81)$$

2. Maximization of the Disturbance Rejection

For an *a priori* given bound on the \mathcal{L}_2 -norm of the admissible disturbances (given by $\frac{1}{\delta}$), the idea consists in minimizing the upper bound for the \mathcal{L}_2 -gain from $w(t)$ to $z(t)$. This can be obtained from the solution of the following optimization problem, considering $\mu = \delta$:

$$\begin{aligned} \min \quad & \gamma \\ \text{subject to} \quad & \text{inequalities (2.78)–(2.79)} \end{aligned} \quad (2.82)$$

Remark 2.7 In the case of a non-null initial condition $x(0)$, since $\mu^{-1} = \delta^{-1} + \beta^{-1}$, there is a trade-off between the size of the set of admissible conditions (given basically by β^{-1}), and the size of the admissible norm of the exogenous signal (given by δ^{-1}). In this case, the finite \mathcal{L}_2 -gain from w to z presents a bias term [62]:

$$\|z\|_2^2 \leq \gamma\|w\|_2^2 + \gamma x(0)'Px(0) \leq \gamma(\|w\|_2^2 + \beta)$$

Example 2.7 Recall the system of Example 2.2, with

$$B_w = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

By solving the optimization problem (2.81) we obtain

$$\delta = \mu = 0.3200$$

which means that, provided $x(0) = 0$, the largest bound on the \mathcal{L}_2 disturbance, for which it is possible to determine a stabilizing gain using the result of Proposition 2.10, is given by $1/\sqrt{\delta} = 1.7678$.

In this case, the obtained state feedback matrix is

$$K = 10^4 \times [1.8799 \ 1.8801]$$

Clearly, this matrix is not suitable for implementation neither for simulation because its entries are too large, which can make the closed-loop system too much sensible

Table 2.4 Example 2.7—
minimization of γ

$\delta = \mu$	γ	K
0.32100	318.02	[67.874 67.879]
0.35000	9.0317	[67.772 67.863]
0.40000	2.7262	[68.043 68.046]
0.50000	0.8809	[67.782 67.783]
0.60000	0.4468	[68.327 68.356]
0.70000	0.2723	[68.498 68.498]
0.80000	0.1839	[68.669 68.669]
1.0000	0.1004	[69.017 69.017]
1.2000	0.063276	[69.376 69.376]
1.4000	0.043544	[69.713 69.714]
10.000	0.00063389	[97.131 97.131]
20.000	0.00040000	[101.00 101.00]

to measurement noise, for instance. Note also that the eigenvalues of $(A + BK)$ in this case are given by -0.9999 and $-10^4 \times 1.8800$, i.e. there is a clear dominant mode. In order to overcome this problem, some constraints on the placement of the eigenvalues of $(A + BK)$ can be used. For instance, consider the pole placement in a strip in the complex plane between $-\alpha_1$ and $-\alpha_2$, $\alpha_1 < \alpha_2$, which can be achieved if the following LMI constraints are added to the problem:

$$\begin{aligned}
 WA' + AW + (Y' \Gamma_j^+ + Q' \Gamma_j^-) B' + B(\Gamma_j^+ Y + \Gamma_j^- Q) + 2\alpha_1 W &< 0 \\
 WA' + AW + (Y' \Gamma_j^+ + Q' \Gamma_j^-) B' + B(\Gamma_j^+ Y + \Gamma_j^- Q) + 2\alpha_2 W &> 0
 \end{aligned} \tag{2.83}$$

In this case, considering $\alpha_1 = 1$ and $\alpha_2 = 100$, the following result is obtained:

$$\delta = \mu = 0.3200; \quad K = [67.8306 \quad 67.8349]$$

Note that the value of μ is practically the same while the gain coefficients are much smaller. The eigenvalues of $(A + BK)$ are now -0.9999 and -66.8349 , i.e., the dominant dynamics is unchanged.

Let us now consider a regulated output given by (2.77) with

$$C_z = [1 \quad 1]$$

Considering now the optimization problem (2.82) with the pole placement constraints given in (2.83), Table 2.4 shows the optimal results for different values of $\delta = \mu > 0.3200$.

As pointed before, for $x(0) = 0$, the value of γ corresponds to a bound on the \mathcal{L}_2 -gain between the input w and the output z . Considering in this case that the bound of the admissible disturbances is given by $\delta^{-1} = \mu^{-1}$, it appears a trade-off between the bound of the admissible disturbance and the achievable bound on the

\mathcal{L}_2 -gain. Note that the greater is δ (the smaller is the admissible given disturbance bound), the smaller is γ (i.e., the higher is the disturbance rejection).

It is also important to notice that the best achievable value of γ is 0.0004, i.e. the value of γ “saturates” as δ increases. This value corresponds in fact to the \mathcal{L}_2 -gain bound obtained in the absence of saturation, and corresponds to the solution of the H_∞ stabilization problem for the linear system [45].

2.4.2 Observer-Based Feedback Stabilization

In many practical cases, the state of the system is not fully available for measurement. In this case, an estimated state feedback can be considered. For linear systems, it is well known that the separation principle [64] holds: the state feedback and the observer gain can be computed separately in order to ensure the stability of the augmented closed-loop system.

This philosophy can also be applied to the case of systems presenting saturating actuators. From a state feedback solution to the stabilization problem, one can be interested in computing a state observer in order to ensure the stability of the closed-loop system under an estimated state feedback, while ensuring the same region of stability achieved with the actual state feedback. The main issue in this case is that the region of stability is now associated to the augmented state (composed by the states of the plant plus the states of the observer) and, in general, the dynamics of the plant state and the one of the observer are coupled.

In the sequel we discuss this issue and present solutions to the problem of computing a state-observer-based control law, considering explicitly the control saturation effects.

With this aim, we consider a full-order Luenberger state observer given by the following equation [64]:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - LC(x(t) - \hat{x}(t)) \quad (2.84)$$

where $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of the state and $L \in \mathbb{R}^{n \times p}$ defines the estimation dynamics. The applied control is now given by

$$u(t) = \text{sat}(K\hat{x}(t)) \quad (2.85)$$

In order to derive conditions to compute the stabilizing observer, we give now the representation of the augmented closed-loop system (system + observer) in a different basis of the state space. For the sake of simplicity, we do not consider the time dependence explicitly. Let $e = x - \hat{x}$ be the estimate error and consider the following similarity transformation:

$$\begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ I_n & -I_n \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (2.86)$$

In the new basis the augmented closed-loop system is given by

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \quad (2.87)$$

$$u = \text{sat} \left(\begin{bmatrix} K & -K \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \right) \quad (2.88)$$

The following result is based on the polytopic approach I.

Proposition 2.12 *Suppose there exist matrices $P_1 = P_1' > 0$, $P_1 \in \Re^{n \times n}$, $P_2 = P_2' > 0$, $P_2 \in \Re^{n \times n}$, $U \in \Re^{n \times p}$, $K \in \Re^{m \times n}$ and a vector $\alpha_l \in \Re^m$, satisfying the following matrix inequalities:*

$$\begin{bmatrix} \mathbb{A}_j' P_1 + P_1 \mathbb{A}_j & -P_1 B_j K \\ -(P_1 B_j K)' & A' P_2 + C' U' + P_2 A + UC \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.89)$$

$$\begin{bmatrix} P_1 & 0 & \alpha_{l(i)} K_{(i)}' \\ 0 & P_2 & -\alpha_{l(i)} K_{(i)}' \\ \alpha_{l(i)} K_{(i)} & -\alpha_{l(i)} K_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.90)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.91)$$

where $\mathbb{A}_j = A + B \Gamma_j(\alpha_l) K$ and $B_j = B \Gamma_j(\alpha_l)$. Then the observer-based output feedback control law defined by (2.84)–(2.85), with $L = P_2^{-1} U$ and the gain K , ensures that the set

$$\mathcal{E}(\mathcal{P}, 1) = \{ \tilde{x} \in \Re^{2n}; \tilde{x}' \mathcal{P} \tilde{x} \leq 1 \}$$

with

$$\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix} \quad \text{and} \quad \mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

is a region of asymptotic stability (RAS) for the closed-loop system (2.87)–(2.88).

Proof Define $\mathcal{K} = [K \ -K]$. The set of validity of the polytopic representation I for system (2.87)–(2.88) is then given in terms of the augmented state \tilde{x} :

$$S(|\mathcal{K}|, u_0^\alpha) = \{ \tilde{x} \in \Re^{2n}; |\mathcal{K} \tilde{x}| \leq u_0^\alpha \}$$

Hence, if $\tilde{x} \in S(|\mathcal{K}|, u_0^\alpha)$, for an appropriated convex combination given by λ_j , $j = 1, \dots, 2^m$, it follows that

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \sum_{j=1}^{2^m} \lambda_j \begin{bmatrix} A + B \Gamma_j(\alpha_l) K & -B \Gamma_j(\alpha_l) K \\ 0 & A + LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} \quad (2.92)$$

Consider now the following definitions:

$$\mathcal{A}_j = \begin{bmatrix} A + B \Gamma_j(\alpha_l) K & -B \Gamma_j(\alpha_l) K \\ 0 & A + LC \end{bmatrix}$$

$$\mathcal{V}(\tilde{x}) = \tilde{x}' \mathcal{P} \tilde{x}$$

Then, provided $L = P_2^{-1}U$, if (2.89) and (2.91) are verified, we conclude that

$$\mathcal{P} \sum_{j=1}^{2^m} \lambda_j \mathcal{A}_j + \sum_{j=1}^{2^m} \lambda_j \mathcal{A}_j' \mathcal{P} < 0$$

The matrix inequality (2.90) ensures that $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{K}|, u_0^\alpha)$. Suppose now that $\tilde{x} \in \mathcal{E}(\mathcal{P}, 1)$. Since $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{K}|, u_0^\alpha)$ it follows that $\dot{\tilde{x}}$ can be computed by (2.92) and it follows that

$$\tilde{x}' \mathcal{P} \dot{\tilde{x}} + \dot{\tilde{x}}' \mathcal{P} \tilde{x} < 0$$

Since this reasoning is valid $\forall \tilde{x}(t) \in \mathcal{E}(\mathcal{P}, 1)$, $\tilde{x}(t) \neq 0$, we can conclude that $\tilde{V}(\tilde{x}(t))$ is a locally strictly decreasing Lyapunov function for the system (2.87)–(2.88) in $\mathcal{E}(\mathcal{P}, 1)$ and thus this set is a RAS for the closed-loop system (2.87)–(2.88). \square

The conditions in Proposition 2.12 can therefore be used to compute the observer gain L . Note that, if K is considered given, the inequalities (2.89)–(2.91) are LMIs for a fixed vector α_l .

Consider that K has been previously computed and that the control law $u(t) = \text{sat}(Kx(t))$ ensures that the system (2.1) is asymptotically stable for all $x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n$, with \mathcal{X}_0 given by (2.38) for instance. In particular, if the initial state of the observer is set to zero, i.e. $\hat{x}(0) = 0$, and the initial plant state $x(0) = x_0$, it follows that $e(0) = x_0$, i.e. $\tilde{x}(0) = [x_0' \ x_0']'$. The idea is therefore to compute L in order to ensure that $\forall x(0) \in \mathcal{X}_0$ the trajectories of (2.87)–(2.88) converge asymptotically to the origin. However, since there is a coupling between the plant state and the observer state, this may be impossible. In such a case, we can try to find the best observer gain L , that ensures the closed-loop stability for a set as close as possible of the set \mathcal{X}_0 . This solution can be accomplished using the following optimization problem:

$$\begin{aligned} & \max_{\theta, \alpha_l, L, P_1, P_2} \theta \\ & \text{subject to} \quad \begin{bmatrix} v_i' & v_i' \end{bmatrix} \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix} \begin{bmatrix} v_i \\ v_i \end{bmatrix} < \theta, \quad \forall i = 1, \dots, n_v \\ & \text{inequalities (2.89)–(2.91)} \end{aligned} \quad (2.93)$$

Considering that $\theta = \beta^{-2}$, note that the first inequality in (2.93) ensures that $\beta[v_i' \ v_i']' \in \mathcal{E}(\mathcal{P}, 1)$. Hence, if $\mathcal{X}_0 = \text{Co}\{v_1, \dots, v_{n_v}\}$, $v_i \in \mathbb{R}^n$, $\forall i = 1, \dots, n_v$, the solution of the optimization problem (2.93) ensures that the stability is guaranteed $\forall x(0) \in \beta \mathcal{X}_0$ and $\hat{x}(0) = 0$. In other words the region of asymptotic stability for the augmented system includes the set $\beta \tilde{\mathcal{X}}_0$, with $\tilde{\mathcal{X}}_0 = \text{Co}\left\{\begin{bmatrix} v_i \\ v_i \end{bmatrix}, i = 1, \dots, n_v\right\}$, i.e. $\beta \tilde{\mathcal{X}}_0 \subset \mathcal{E}(\mathcal{P}, 1)$.

Following the same steps, by using the polytopic approach II, the following result can be stated.

Proposition 2.13 *Suppose there exist matrices $W_1 = W_1' > 0$, $W_1 \in \Re^{n \times n}$, $P_2 = P_2' > 0$, $P_2 \in \Re^{n \times n}$, $U \in \Re^{n \times p}$, $K \in \Re^{m \times n}$ and matrices Q_1 and $Q_2 \in \Re^{m \times n}$, satisfying the following matrix inequalities:*

$$\begin{bmatrix} W_1 \mathbb{A}'_j + \mathbb{A}_j W_1 + Q_1' \Gamma_j^- B' + B \Gamma_j^- Q_1 & -B \Gamma_j^+ K + B \Gamma_j^- Q_2 \\ \star & A' P_2 + C' U' + P_2 A + U C \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.94)$$

$$\begin{bmatrix} W_1 & 0 & Q_{1(i)}' \\ 0 & P_2 & Q_{2(i)}' \\ \star & \star & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.95)$$

where $\mathbb{A}_j = A + B \Gamma_j^+ K$. Then the observer-based output feedback control law defined by (2.84)–(2.85), with $L = P_2^{-1} U$ and the gain K , ensures that the set

$$\mathcal{E}(\mathcal{P}, 1) = \{\tilde{x} \in \Re^{2n}; \tilde{x}' \mathcal{P} \tilde{x} \leq 1\}$$

with

$$\tilde{x} = \begin{bmatrix} x \\ e \end{bmatrix} \quad \text{and} \quad \mathcal{P} = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}$$

where $P_1 = W_1^{-1}$, is a region of asymptotic stability (RAS) for the closed-loop system (2.87)–(2.88).

Proof By left- and right-multiplying (2.94) by the matrix $\begin{bmatrix} P_1 & 0 \\ 0 & I \end{bmatrix}$ and considering $H_1 = P_1 Q_1$, $H_2 = Q_2$ and $P_2 L = U$ it follows that (2.94) is equivalent to:

$$\begin{bmatrix} \mathbb{A}'_j P_1 + P_1 \mathbb{A}_j + H_1' \Gamma_j^- B' + B \Gamma_j^- H_1 & -P_1 B \Gamma_j^+ K + P_1 B \Gamma_j^- H_2 \\ \star & (A + LC)' P_2 + P_2 (A + LC) \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.96)$$

From (2.96), we can conclude that

$$\mathcal{P} \sum_{j=1}^{2^m} \lambda_j \mathcal{A}_j + \sum_{j=1}^{2^m} \lambda_j \mathcal{A}_j' \mathcal{P} < 0$$

with

$$\mathcal{A}_j = \begin{bmatrix} A + B \Gamma_j^+ K + B \Gamma_j^- H_1 & -B \Gamma_j^+ K + B \Gamma_j^- H_2 \\ 0 & A + LC \end{bmatrix}$$

On the other hand, by left- and right-multiplying (2.95) by the matrix

$$\begin{bmatrix} P_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

it follows that (2.95) is equivalent to

$$\begin{bmatrix} P_1 & 0 & H'_{1(i)} \\ 0 & P_2 & H'_{2(i)} \\ \star & \star & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m$$

which ensures that $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{H}|, u_0)$, with $\mathcal{H} = [H_1 \ H_2]$.

Hence, from the same arguments as used in the proof of Proposition 2.12, considering the polytopic modeling II for the augmented closed-loop system, it follows that $\mathcal{E}(\mathcal{P}, 1)$ is a RAS for the closed-loop system (2.87)–(2.88). \square

Using the same reasoning as done in the proof of Proposition 2.13, similar results can be straightforwardly derived considering the polytopic model III.

2.4.3 Dynamic Output Feedback Stabilization

In this section we show how to compute a parameter varying dynamic output feedback. Considering the polytopic approach I, the basic idea is to use the variable $\alpha(t)$ as a scheduling parameter for the controller [146, 273]. The approach is therefore similar to LPV control approaches [399].

Consider the strictly proper open-loop system

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.97)$$

$$y(t) = Cx(t) \quad (2.98)$$

and a controller that furnishes an output denoted by $y_c(t)$.

Due to the saturation $u(t) = \Gamma(\alpha(t))y_c(t)$, with $\alpha(t)$ defined as

$$\alpha_{(i)}(t) = \min\left(1, \frac{u_{0(i)}}{|y_{c(i)}(t)|}\right), \quad i = 1, \dots, m \quad (2.99)$$

Based now on the varying parameter $\alpha(t)$, define the following n_c -order nonlinear (or LPV) dynamic output controller:

$$\dot{x}_c(t) = A_c(\alpha(t))x_c(t) + B_c(\alpha(t))y(t) \quad (2.100)$$

$$y_c(t) = C_c x_c(t) + D_c y(t) \quad (2.101)$$

where $A_c(\alpha(t))$ and $B_c(\alpha(t))$ denote real matrices of appropriate dimension whose elements depend on the value of $\alpha(t)$, i.e. they are “scheduled” by $\alpha(t)$.

The control input is therefore given by

$$\begin{aligned} u(t) &= \text{sat}(y_c(t)) = \text{sat}(C_c x_c(t) + D_c y(t)) \\ &= \Gamma(\alpha(t))(C_c x_c(t) + D_c y(t)) \end{aligned}$$

and the closed-loop system can be written as

$$\dot{\tilde{x}}(t) = \mathcal{A}(\alpha(t))\tilde{x}(t) \quad (2.102)$$

with

$$\tilde{x} = \begin{bmatrix} x \\ x_c \end{bmatrix} \quad \text{and} \quad \mathcal{A}(\alpha(t)) = \begin{bmatrix} A + B\Gamma(\alpha(t))D_c C & B\Gamma(\alpha(t))C_c \\ B_c(\alpha(t))C & A_c(\alpha(t)) \end{bmatrix}$$

Define now the matrix $\mathcal{K} = [D_c C \ C_c]$ and the following set:

$$S(|\mathcal{K}|, u_0^\alpha) = \{\tilde{x} \in \mathbb{R}^{n+n_c}; |\mathcal{K}\tilde{x}| \leq u_0^\alpha\}$$

From the polytopic approach I, if $\tilde{x} \in S(|\mathcal{K}|, u_0^\alpha)$ then it follows that

$$u(t) = \sum_{j=1}^{2^m} \lambda_j(\alpha(t)) \Gamma_j(\alpha_l) \mathcal{K} \tilde{x}(t) = \Gamma(\alpha(t)) \mathcal{K} \tilde{x}(t)$$

Based on the congruence transformations proposed in [314], the following proposition can be stated considering $n_c = n$.

Proposition 2.14 *If there exist symmetric positive definite matrices $Y \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times n}$, matrices $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^{n \times p}$, $\hat{C} \in \mathbb{R}^{m \times n}$ and $\hat{D} \in \mathbb{R}^{m \times p}$ and a vector $\alpha_l \in \mathbb{R}^m$, satisfying the following matrix inequalities:*

$$\begin{bmatrix} AX + XA' + B\Gamma_j(\alpha_l)\hat{C} + \hat{C}'\Gamma_j(\alpha_l)B' & \hat{A}' + (A + B\Gamma_j(\alpha_l)\hat{D}C) \\ \hat{A} + (A + B\Gamma_j(\alpha_l)\hat{D}C)' & Y'A + AY + \hat{B}C + C'\hat{B}' \end{bmatrix} < 0$$

$$j = 1, \dots, 2^m \quad (2.103)$$

$$\begin{bmatrix} X & I_n & \alpha_{l(i)}\hat{C}'_{(i)} \\ I_n & Y & \alpha_{l(i)}C'\hat{D}'_{(i)} \\ \alpha_{l(i)}\hat{C}_{(i)} & \alpha_{l(i)}\hat{D}_{(i)}C & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.104)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.105)$$

then the controller (2.100)–(2.101) with

$$D_c = \hat{D}$$

$$C_c = (\hat{C} - D_c C X)(M')^{-1}$$

$$B_c(\alpha(t)) = N^{-1}(\hat{B} - YB\Gamma(\alpha(t))D_c) \quad (2.106)$$

$$A_c(\alpha(t)) = N^{-1}(\hat{A} - NB_c(\alpha(t))CX - YB\Gamma(\alpha(t))C_cM' - Y(A + B\Gamma(\alpha(t))D_c C)X)(M')^{-1}$$

where matrices N and M verify the relation

$$MN' = I - XY \quad (2.107)$$

ensures that the set

$$\mathcal{E}(\mathcal{P}, 1) = \{\tilde{x} \in \mathbb{R}^{2n}; \tilde{x}'\mathcal{P}\tilde{x} \leq 1\} \quad (2.108)$$

with

$$\mathcal{P} = \begin{bmatrix} Y & N \\ N' & F \end{bmatrix}$$

and $F = -N'X(M')^{-1}$, is a region of asymptotic stability (RAS) for the closed-loop system (2.102).

Proof Define the matrices [314]

$$\Pi_1 = \begin{bmatrix} X & I \\ M' & 0 \end{bmatrix} \quad \text{and} \quad \Pi_2 = \begin{bmatrix} I & Y \\ 0 & N' \end{bmatrix}$$

It follows that $\mathcal{P}\Pi_1 = \Pi_2$ and $\Pi_1'\mathcal{P}\Pi_1 = \Pi_1'\Pi_2 = \begin{bmatrix} X & I \\ I & Y \end{bmatrix}$.

Note that if (2.104) is verified, it follows that $X - Y^{-1} > 0$, which implies that $I - XY$ is nonsingular. Then it is always possible to compute square and nonsingular matrices N and M verifying the equation $NM' = I - XY$. This fact also ensures that Π_1 is nonsingular.

Since the inequality (2.103) is verified for $j = 1, \dots, 2^m$, and considering that at the instant t there exists $\lambda_j(\alpha(t))$ such that

$$\sum_{j=1}^{2^m} \lambda_j(\alpha(t)) \Gamma_j(\alpha(t)) = \Gamma(\alpha(t)) \quad (2.109)$$

with $\sum_{j=1}^{2^m} \lambda_j(\alpha(t)) = 1$, $0 \leq \lambda_j(\alpha(t)) \leq 1$, $\forall j = 1, \dots, m$, it follows that

$$\begin{bmatrix} AX + XA' + B\Gamma(\alpha(t))\hat{C} + \hat{C}'\Gamma(\alpha(t))B' & \hat{A}' + (A + B\Gamma(\alpha(t))\hat{D}C) \\ \hat{A} + (A + B\Gamma(\alpha(t))\hat{D}C)' & Y'A + AY + \hat{B}C + C'\hat{B}' \end{bmatrix} < 0 \quad (2.110)$$

Considering the definitions in (2.106) it follows that

$$\begin{aligned} \hat{D} &= D_c \\ \hat{C} &= C_c M' + D_c C X \\ \hat{B} &= N B_c(\alpha(t)) + Y B \Gamma(\alpha(t)) D_c \\ \hat{A} &= N A_c(\alpha(t)) M' + N B_c(\alpha(t)) C X + Y B \Gamma(\alpha(t)) C_c M' \\ &\quad + Y (A + B \Gamma(\alpha(t)) D_c C) X \end{aligned} \quad (2.111)$$

Substituting now (2.111) in (2.110) and after some algebraic manipulations it follows that (2.110) is equivalent to

$$\Pi_1' \mathcal{P} \mathcal{A}(\alpha(t)) \Pi_1 + \Pi_1' \mathcal{A}(\alpha(t))' \mathcal{P} \Pi_1 < 0 \quad (2.112)$$

Thus, since Π_1 is nonsingular, (2.112) is equivalent to

$$\mathcal{P} \mathcal{A}(\alpha(t)) + \mathcal{A}(\alpha(t))' \mathcal{P} < 0 \quad (2.113)$$

By left- and right-multiplying (2.104) by $\begin{bmatrix} (\Pi_1^{-1})' & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} \Pi_1^{-1} & 0 \\ 0 & I \end{bmatrix}$, respectively, we can conclude that (2.104) is equivalent to

$$\begin{bmatrix} \mathcal{P} & \alpha_{l(i)} \mathcal{K}'_{(i)} \\ \alpha_{l(i)} \mathcal{K}_{(i)} & u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.114)$$

Hence, inequality (2.104) ensures the inclusion $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{K}|, u_0^g)$.

Suppose now that $\tilde{x} \in \mathcal{E}(\mathcal{P}, 1)$. Since $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{K}|, u_0^\alpha)$, it follows that $\Gamma(\alpha_l)$ can be computed by (2.109) and, we can conclude that (2.103) implies that

$$\tilde{x}' \mathcal{P} \dot{\tilde{x}} + \dot{\tilde{x}}' \mathcal{P} \tilde{x} < 0$$

Since this reasoning is valid $\forall \tilde{x}(t) \in \mathcal{E}(\mathcal{P}, 1)$, $\tilde{x}(t) \neq 0$, we can conclude that $\tilde{V}(\tilde{x}(t)) = \tilde{x}(t)' \mathcal{P} \tilde{x}(t)$ is a locally strictly decreasing Lyapunov function for the system (2.102) in $\mathcal{E}(\mathcal{P}, 1) \subset S(|\mathcal{K}|, u_0^\alpha)$ and thus this set is a RAS for the closed-loop system. \square

Remark 2.8 As pointed out in the proof of Proposition 2.14, since the inequality (2.104) ensures that $X - Y^{-1} > 0$, it is always possible to compute nonsingular matrices M and N , satisfying $NM' = I - YX$. However, given X and Y the choice of N and M is not unique. For example, for any given scalar κ , we can set $M = \kappa I$ and it follows that $N = (I - YX)\kappa^{-1}$ or we can even compute M and N from a LU or QR decomposition. This means, in fact, that the compensator realization is not unique [314].

Remark 2.9 Note that for computing the controller matrices at time t it is necessary to obtain $\Gamma(\alpha(t))$. Since the matrices C_c and D_c are time invariant and both the output $y(t)$ and the controller state $x_c(t)$ are available, it follows that $y_c(t)$ is available. The matrices $A_c(\alpha(t))$ and $B_c(\alpha(t))$ are in fact dependent on $y(t)$ and $x_c(t)$, since $\alpha(t)$ is computed from these variables. The controller can then be seen as a time-varying system in which the time-variation is “scheduled” by the values of $y(t)$ and $x_c(t)$.

Note that the conditions (2.103)–(2.105) are LMIs for a fixed vector α_l . The result of Proposition 2.14 can then be cast in convex optimization problems to obtain a controller leading to a large region of asymptotic stability for the closed-loop system (2.102) or to ensure the regional stability with respect to a given set of admissible states while guaranteeing some time-domain performance in linear operation.

For instance, consider a shape set $\tilde{\mathcal{X}}_0 = \text{Co}\{v_1, v_2, \dots, v_{n_r}\}$, $v_l \in \mathbb{R}^{2n}$, $l = 1, \dots, n_r$. The idea is to compute the controller in order to satisfy $\beta \tilde{\mathcal{X}}_0 \subset \mathcal{E}(\mathcal{P}, 1)$ with β as large as possible. As discussed in Sect. 2.2.5.1, the vectors v_l can be viewed as directions in which we want to maximize the region of attraction. In particular, it is interesting to maximize the region of stability in directions associated to the states of the plant. In this case the vectors v_l take the form $[v_{l1}' \ 0]'$.

Noticing that $\beta[v_{l1}' \ 0]' \in \mathcal{E}(\mathcal{P}, 1)$ is equivalent to $\beta[v_{l1}' \ 0] \mathcal{P} [v_{l1}' \ 0]' \beta \leq 1$ and considering $\theta = 1/\beta^2$, it follows that $\beta \tilde{\mathcal{X}}_0 \subset \mathcal{E}(\mathcal{P}, 1)$ is equivalent to:

$$v_{l1}' Y v_{l1} \leq \theta, \quad l = 1, \dots, r \quad (2.115)$$

Hence, the maximization of the ellipsoid $\mathcal{E}(\mathcal{P}, 1)$ along the directions $[v_{l1}' \ 0]'$ is equivalent to the minimization of θ .

Note that matrix N , which appears in matrix \mathcal{P} , is not optimized in the above problem. On the other hand, once one obtains matrices X and Y , by maximizing $\mathcal{E}(\mathcal{P}, 1)$ in the space of the system state, it is possible to explore the degrees of

freedom in the choice of matrix N . For instance, $\mathcal{E}(\mathcal{P}, 1)$ can be maximized along the directions given by generic vectors $v_l = [v'_{l1} \ v'_{l2}]'$ with $v_{l1} \in \Re^n$ and $v_{l2} \in \Re^n$, by solving the following problem:

$$\begin{aligned} & \min_{N, \theta} \theta \\ & \text{subject to} \quad \begin{bmatrix} \theta & v'_{l1} & v'_{l1}Y + v'_{l2}N' \\ v_{l1} & X & I_n \\ Yv_{l1} + Nv_{l2} & I_n & Y \end{bmatrix} > 0, \quad l = 1, \dots, r \end{aligned} \quad (2.116)$$

where X and Y are given matrices verifying the conditions of Proposition 2.14.

Actually, the matrix inequality in (2.116) is equivalent to

$$\begin{bmatrix} \theta & [v'_{l1} \ v'_{l2}] \\ \begin{bmatrix} v_{l1} \\ v_{l2} \end{bmatrix} & \mathcal{P}^{-1} \end{bmatrix} > 0, \quad l = 1, \dots, r \quad (2.117)$$

In order to show that, note that

$$\mathcal{P}^{-1} = \begin{bmatrix} X & M \\ M' & U \end{bmatrix}$$

Then, it suffices to left- and right-multiply (2.117) by

$$\Pi_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & I & 0 \\ 0 & Y & N \end{bmatrix}$$

and Π'_3 , respectively, to obtain the matrix inequality in (2.116).

As seen in Sect. 2.4.1.2, in addition to the guarantee of stability for a region as large as possible, the controller can be designed in order to ensure some degree of time-domain performance in a neighborhood of the origin [137]. We consider this neighborhood as the region of linear behavior of the closed-loop system, i.e., the region where the control inputs do not saturate:

$$S(|\mathcal{K}|, u_0) = \{\tilde{x} \in \Re^{2n}; |\mathcal{K}\tilde{x}| \leq u_0\}$$

When the system operates inside $S(|\mathcal{K}|, u_0)$ it follows that $\Gamma(\alpha(t)) = I$ and $A_c(\alpha(t))$ and $B_c(\alpha(t))$ are constant matrices which we denote as A_c and B_c . In this case, the time-domain performance can be achieved if we consider the pole placement of the matrix $\mathcal{A} = \begin{bmatrix} A+B\hat{D}C & BC_c \\ B_cC & A_c \end{bmatrix}$ in a suitable region in the left half complex plane. Considering an LMI framework, the results stated in [314] can be used to place the poles in a called LMI region in the complex plane. For example, if we verify the following LMI:

$$\begin{bmatrix} AX + XA' + B\hat{C} + \hat{C}'B' + 2\sigma X & \hat{A}' + (A + B\hat{D}C) + 2\sigma I \\ \hat{A} + (A + B\hat{D}C)' + 2\sigma I & Y'A + AY + \hat{B}C + C'\hat{B}' + 2\sigma Y \end{bmatrix} < 0 \quad (2.118)$$

then it is easy to show that this inequality ensures that the real part of the poles of \mathcal{A} are less than $-\sigma$, with $\sigma > 0$. Thus, the larger is σ , the faster will be the decay rate of the time response inside the linearity region.

Remark 2.10 The controller (2.100)–(2.101) given by Proposition 2.14 is of order n . In fact, the choice $n_c = n$ leads to a simple and more direct procedure to obtain LMI conditions, since in this case the so-called *rank condition* is directly satisfied (see [314]). However, considering that r states are available for direct measurement a reduced order controller can be obtained. The procedure follows the same lines as the ones proposed in [400].

Remark 2.11 In Proposition 2.14, the same matrices \hat{A} and \hat{B} are supposed to verify all the matrix inequalities stated for each one of the 2^m vertices of the polytope defined by the matrices $\Gamma_j(\alpha_l)$. This can be seen as a source of conservatism. In order to reduce this conservatism, the result can be adapted in order to consider different matrices in each vertex, i.e. \hat{A}_j and \hat{B}_j , $j = 1, \dots, 2^m$. In this case, the matrices $A_c(\alpha(t))$ and $B_c(\alpha(t))$ can be obtained, at each instant, from an appropriate convex combination of matrices \hat{A}_j and \hat{B}_j . The matrices of the dynamic controller can be obtained in this case as follows:

$$\begin{aligned} D_c &= \hat{D}, \\ C_c &= (\hat{C} - D_c C X)(M')^{-1}, \\ B_c(\alpha(t)) &= N^{-1} \left(\sum_{j=1}^{2^m} \lambda_j(t) \hat{B}_j - Y B \Gamma(\alpha(t)) D_c \right), \\ A_c(\alpha(t)) &= N^{-1} \left(\sum_{j=1}^{2^m} \lambda_j(t) \hat{A}_j - N B_c C X - Y B \Gamma(\alpha(t)) C_c M' \right. \\ &\quad \left. - Y (A + B \Gamma(\alpha(t)) D_c C) X \right) (M')^{-1} \end{aligned} \quad (2.119)$$

In this case it is necessary to determine explicitly the coefficients $\lambda_j(t)$ such that $\Gamma(\alpha(t)) = \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l)$. This can be easily accomplished by obtaining a feasible solution for the following linear program:

$$\begin{aligned} \min \quad & \sum_{j=1}^{2^m} \lambda_j(t) \\ \text{subject to} \quad & \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l) = \Gamma(\alpha(t)) \\ & \sum_{j=1}^{2^m} \lambda_j(t) = 1 \\ & 0 \leq \lambda_j(t) \leq 1, \quad j = 1, \dots, 2^m \end{aligned} \quad (2.120)$$

Remark 2.12 Similar results can be obtained using the polytopic approach II, as shown in [400]. For the discrete-time counterpart of the results the reader can refer to [146] and [410], considering the polytopic approaches I and II, respectively. In those papers, it is also established results concerning the external stabilization by

means of a dynamic compensator scheduled by $\alpha(t)$, considering energy bounded disturbances.

2.4.4 Global Stabilization

As pointed in Chap. 1, Sect. 1.6.2.1, the global stabilization of a linear system subject to bounded controls is possible if and only if the system is null controllable [338] (see in particular Theorem 1.2). In other words, the system must be (A, B) -controllable and no eigenvalue of A can have strictly positive real part. Moreover, even if the null controllability assumption is verified, in general, the global stability cannot be obtained with a linear state feedback [99, 254, 337]. On the other hand, if the system is null controllable and if all the eigenvalues of A that lie on the imaginary axis are associated to linear independent eigenvectors, or, equivalently if A is asymptotically or critically stable, then it is always possible to find a global stabilizing control law in the form of a saturating static state feedback [52, 220]. This result and the way of computing this control law are presented in this section, using polytopic model I.

The following theorem shows that a quadratic Lyapunov function is a globally strictly decreasing Lyapunov function for system (2.1) only if this function is also a Lyapunov function for the open-loop system.

Theorem 2.2 *If $V(x) = x'Px$, with $P = P' > 0$, is a Lyapunov function for the closed-loop system (2.1) such that*

- $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^n$, along the trajectories of system (2.1),
- *the only invariant set belonging to the set $\mathcal{E} = \{x \in \mathbb{R}^n; \dot{V}(x) = 0\}$ reduces to the origin $x = 0$,*

then $V(x)$ is necessarily a Lyapunov function for the open-loop system $\dot{x} = Ax$.

Proof Suppose that $\dot{V}(x) \leq 0, \forall x \in \mathbb{R}^n$, along the trajectories of system (2.1). Consider a vector x such that $x'x = \eta$, with $\eta > 0$. By hypothesis $\dot{V}(x)$, along the trajectories of the closed-loop system (2.1) for this vector x , satisfies

$$\dot{V}(x) = x'(A'P + PA)x + 2x'PB \text{sat}(Kx) \leq 0 \quad (2.121)$$

Since (2.121) is valid $\forall x \in \mathbb{R}^n$, it follows that it holds for $\tilde{x} = \rho x$, with $\rho > 0$. Hence, for any positive scalar ρ it follows

$$\dot{V}(x) = \rho^2 x'(A'P + PA)x + 2\rho x'PB \text{sat}(\rho Kx) \leq 0$$

or equivalently

$$x'(A'P + PA)x + \frac{2}{\rho} x'PB \text{sat}(\rho Kx) \leq 0$$

By definition of the saturation term, one can write:

$$x'(A'P + PA)x - \frac{2}{\rho}|x'PB|u_0 \leq x'(A'P + PA)x + \frac{2}{\rho}x'PB \operatorname{sat}(\rho Kx) \leq 0$$

Then one gets:

$$x'(A'P + PA)x \leq \frac{2}{\rho}|x'PB|u_0$$

and therefore, as $x'x = \eta$, when we consider that $\rho \rightarrow \infty$, one obtains $x'(A'P + PA)x \leq 0$. As one can apply this reasoning for any x such that $x'x = \eta$, for any $\eta > 0$, one can conclude that necessarily one verifies $x'(A'P + PA)x \leq 0$, $\forall x \in \mathbb{R}^n$. \square

Before stating the results that allow to determine a global stabilizing saturating state feedback gain, let us introduce the concept of $(A, B)_{P_{ol}}$ -stabilizability.

Definition 2.1 Consider a matrix $P = P' > 0$ such that

$$A'P + PA = -Q_0 \leq 0 \quad (2.122)$$

The open-loop system

$$\dot{x} = Ax + Bu \quad (2.123)$$

is said to be (A, B) -stabilizable with respect to the quadratic Lyapunov function $V(x) = x'Px$, with P solution to (2.122), (shortly denoted as $(A, B)_{P_{ol}}$), if there exists a state feedback gain K such that one of the cases below is satisfied:

- $(A + BK)'P + P(A + BK) = -Q < 0$;
- $(A + BK)'P + P(A + BK) = -Q \leq 0$ and $\{x = 0\}$ is the only invariant set contained in the set $\mathcal{E} = \{x \in \mathbb{R}^n; x'(A + BK)'Px + x'P(A + BK)x = 0\}$.

Remark 2.13 From Definition 2.1, by noting that $x'(A + BK)'Px + x'P(A + BK)x = -x'Q_0x + 2x'PBKx$, it appears that the property of $(A, B)_{P_{ol}}$ -stabilizability implies that $\operatorname{Ker}(Q_0) \cap (\operatorname{Ker}(K) \cup \operatorname{Ker}(B'P)) = \{0\}$.

We use this notion of $(A, B)_{P_{ol}}$ -stabilizability to build a class of global stabilizing controllers.

Theorem 2.3 *If system (2.123) is $(A, B)_{P_{ol}}$ -stabilizable then system (2.1) with $K = -\Gamma(\gamma)B'P$, where $\Gamma(\gamma)$ is a positive diagonal matrix, is globally asymptotically stable.*

Proof Compute the time derivative of $V(x) = x'Px$, along the trajectories of the closed-loop system (2.1), that is along the trajectories of system (2.123) with $K = -\Gamma(\gamma)B'P$:

$$\dot{V}(x) = x'(A'P + PA)x + 2x'PB \operatorname{sat}(-\Gamma(\gamma)B'Px)$$

Considering polytopic model I, from Sect. 1.7.1.1, and more precisely from relations (1.36) and (1.37), the closed-loop system can be written as (1.38) and therefore on gets

$$\begin{aligned}\dot{V}(x) &= x'(A'P + PA)x - 2x'PB\Gamma(\alpha(x))\Gamma(\gamma)B'Px \\ &= -x'Q_0x - 2x'PB\Gamma(\alpha(x))\Gamma(\gamma)B'Px\end{aligned}$$

Hence, since $\Gamma(\alpha(x))$ and $\Gamma(\gamma)$ are positive diagonal matrices, it follows that $-2x'PB\Gamma(\alpha(x))\Gamma(\gamma)B'Px < 0$, for any $x \in \mathfrak{N}^n$, except for $x \in \text{Ker}(B'P)$. From the property of $(A, B)_{P_{\text{ol}}}$ -stabilizability and Remark 2.13, it follows that either $\dot{V}(x) < 0$, for any $x \in \mathfrak{N}^n$, or $\dot{V}(x) \leq 0$ and the only invariant set belonging to the set $\mathcal{E} = \{x \in \mathfrak{N}^n; \dot{V}(x) = 0\}$ reduces to the origin $x = 0$, which proves the global asymptotic stability of the origin of the closed-loop system (2.1). \square

The diagonal elements of the matrix $\Gamma(\gamma)$ can be seen as design parameters. Indeed, they can be chosen to impose some performance or robustness requirement for the linear system $\dot{x} = (A - B\Gamma(\gamma)B'P)x(t)$. For instance, they can be selected to improve (with respect to the open-loop system) the speed of convergence of the trajectories to the origin [52, 220].

2.5 Uncertain Systems

2.5.1 Stability Analysis

Throughout this section, we discuss how to incorporate robustness requirements in the stability analysis and stabilization of systems with saturating inputs presented in the previous sections. This problem has been considered for instance in [112, 172, 175].

We consider that the matrices of system (2.1) are subject to uncertainties represented by a parameter $F \in \mathcal{F}$. Depending on the description of the set \mathcal{F} different stability conditions can be derived. In particular, we focus here on two different types of uncertainties descriptions, namely (see also Appendix B):

- Norm-bounded uncertainty, where

$$\begin{aligned}A(F) &= A_0 + DF(t)E_1, & B(F) &= B_0 + DF(t)E_2 \\ F(t)'F(t) &\leq I_r\end{aligned}\tag{2.124}$$

with constant real matrices D , E_1 and E_2 of appropriate dimensions.

- Polytopic uncertainty, where matrices $A(F)$ and $B(F)$ belong to polytopes of matrices, i.e.

$$\begin{aligned} A(F) \in \mathcal{D}_A &= \left\{ A(F) \in \mathbb{R}^{n \times n}; A(F) = \sum_{i=1}^{N_A} \mu_i A_i, \sum_{i=1}^{N_A} \mu_i = 1, \mu_i \geq 0 \right\} \\ B(F) \in \mathcal{D}_B &= \left\{ B(F) \in \mathbb{R}^{n \times m}; B(F) = \sum_{k=1}^{N_B} v_k B_k, \sum_{k=1}^{N_B} v_k = 1, v_k \geq 0 \right\} \end{aligned} \quad (2.125)$$

In presence of uncertainties the closed-loop system can therefore be written generically as

$$\dot{x}(t) = A(F)x(t) + B(F) \text{sat}(Kx(t)) \quad (2.126)$$

Considering the uncertainty representations above, analogous results to the ones of Proposition 2.1 can be stated for the case of the polytopic model I. In particular, in order to provide straightforward extensions to the synthesis case in the conditions, the results are formulated in the dual framework, i.e. considering the matrix $W = P^{-1}$ as done in Corollary 2.1. The results considering explicitly the matrix P can be found in [172].

Proposition 2.15 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a vector $\alpha_l \in \mathbb{R}^m$ and positive scalars ε and η satisfying*

$$\begin{bmatrix} W & \alpha_{l(i)} W K'_{(i)} \\ \alpha_{l(i)} K_{(i)} W & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.127)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.128)$$

and

- for the norm-bounded uncertainty case

$$\begin{bmatrix} W(A_0 + B_0 \Gamma_j(\alpha_l) K)' + (A_0 + B_0 \Gamma_j(\alpha_l) K) W + \varepsilon D D' & \star \\ E_1 W + E_2 \Gamma_j(\alpha_l) K W & -\varepsilon I \end{bmatrix} < 0 \quad (2.129)$$

$j = 1, \dots, 2^m$

- for the polytopic uncertainty case

$$\begin{aligned} W(A_i + B_k \Gamma_j(\alpha_l) K)' + (A_i + B_k \Gamma_j(\alpha_l) K) W &< 0, \\ i = 1, \dots, N_A, k = 1, \dots, N_B, j = 1, \dots, 2^m \end{aligned} \quad (2.130)$$

then the ellipsoid $\mathcal{E}(W^{-1}, \eta)$ is a region of asymptotic stability (RAS) for the uncertain closed-loop saturated system (2.126).

Proof The proof follows the same lines as the ones of Proposition 2.1 and Corollary 2.1.

- Norm-bounded uncertainty case.

By considering $V(x) = x'Px$, we have to prove that

$$\dot{V}(x) = 2 \sum_{j=1}^{2^m} \lambda_j x' [P(A_0 + B_0 \Gamma_j(\alpha_l)K) + P D F(t)(E_1 + E_2 \Gamma_j(\alpha_l)K)]x$$

is strictly negative. Using the fact that $2u'v \leq \varepsilon u'u + \varepsilon^{-1}v'v$, for all vectors u and v and a positive scalar ε [288], and that $F(t)F(t)' \leq I_r$, it follows that

$$\begin{aligned} & 2x' P D F(t)(E_1 + E_2 \Gamma_j(\alpha_l)K)x \\ & \leq x' \varepsilon P D F(t)F(t)' D' x + x' \varepsilon^{-1} (E_1 + E_2 \Gamma_j(\alpha_l)K)' (E_1 + E_2 \Gamma_j(\alpha_l)K)x \\ & \leq x' \varepsilon P D D' x + x' \varepsilon^{-1} (E_1 + E_2 \Gamma_j(\alpha_l)K)' (E_1 + E_2 \Gamma_j(\alpha_l)K)x \end{aligned}$$

Hence, we can write that

$$\begin{aligned} \dot{V}(x) & \leq \sum_{j=1}^{2^m} \lambda_j x' [P(A_0 + B_0 \Gamma_j(\alpha_l)K) + (A_0 + B_0 \Gamma_j(\alpha_l)K)' P + \varepsilon P D D' P \\ & \quad + (E_1 + E_2 \Gamma_j(\alpha_l)K)' \varepsilon^{-1} (E_1 + E_2 \Gamma_j(\alpha_l)K)]x \end{aligned}$$

Considering now $W = P^{-1}$, left- and right-multiplying (2.129) by $\text{Diag}(P, I)$, it follows that

$$\left[\begin{array}{cc} (A_0 + B_0 \Gamma_j(\alpha_l)K)' P + P(A_0 + B_0 \Gamma_j(\alpha_l)K) + \varepsilon P D D' P & \star \\ E_1 + E_2 \Gamma_j(\alpha_l)K & -\varepsilon I \end{array} \right] < 0$$

(2.131)

By using now the Schur's complement, (2.131) is equivalent to

$$\begin{aligned} & P(A_0 + B_0 \Gamma_j(\alpha_l)K) + (A_0 + B_0 \Gamma_j(\alpha_l)K)' P + \varepsilon P D D' P \\ & + (E_1 + E_2 \Gamma_j(\alpha_l)K)' \varepsilon^{-1} (E_1 + E_2 \Gamma_j(\alpha_l)K) < 0 \end{aligned}$$

which allows one to conclude that the satisfaction of relation (2.129) implies that $\dot{V}(x) < 0$, along the trajectories of the uncertain system (2.1), provided $x(t) \in S(|K|, u_0^\alpha)$.

- Polytopic uncertainty case.

Considering $W = P^{-1}$, left and right-multiplying (2.130) by P , it follows by convexity that

$$\sum_{i=1}^{N_A} \mu_i \left(\sum_{k=1}^{N_B} v_k \left(\sum_{j=1}^{2^m} \lambda_j ((A_i + B_k \Gamma_j(\alpha_l)K)' P + P(A_i + B_k \Gamma_j(\alpha_l)K)) \right) \right) < 0$$

(2.132)

for $\mu_i \geq 0$, $v_k \geq 0$ and $\lambda_j \geq 0$ such that $\sum_{i=1}^{N_A} \mu_i = 1$, $\sum_{k=1}^{N_B} v_k = 1$ and $\sum_{j=1}^{2^m} \lambda_j = 1$.

Hence, from the definition of $A(F)$ and $B(F)$, (2.132) ensures that

$$2 \sum_{j=1}^{2^m} \lambda_j x' P (A(F) + B(F) \Gamma_j (\alpha_l) K) x < 0$$

i.e. $\dot{V}(x) < 0$ along the trajectories of the uncertain system, provided that $x(t) \in S(|K|, u_0^\alpha)$.

Since the relations (2.127), (2.128) are verified, it follows that the set $\mathcal{E}(P, \eta) \subset S(|K|, u_0^\alpha)$ and we can conclude, in both cases, that $\mathcal{E}(P, \eta)$ is a RAS for the uncertain closed-loop saturated system (2.126). \square

Similarly, considering polytopic model II, the uncertain case, counterpart of Proposition 2.2 can be stated as follows:

Proposition 2.16 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Q \in \mathbb{R}^{m \times n}$ and positive scalars ε and η satisfying*

$$\begin{bmatrix} W & Q'_{(i)} \\ Q_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.133)$$

and

- *for the norm-bounded uncertainty case*

$$\begin{bmatrix} W(A_0 + B_0 \Gamma_j^+ K)' + (A_0 + B_0 \Gamma_j^+ K)W + Q' \Gamma_j^- B'_0 + B_0 \Gamma_j^- Q + \varepsilon D D' & \star \\ E_1 W + E_2 \Gamma_j^+ K W + E_2 \Gamma_j^- Q & -\varepsilon I \end{bmatrix} < 0, \quad j = 1, \dots, 2^m \quad (2.134)$$

- *for the polytopic uncertainty case*

$$W(A_i + B_k \Gamma_j^+ K)' + (A_i + B_k \Gamma_j^+ K)W + Q' \Gamma_j^- B'_k + B_k \Gamma_j^- Q < 0, \\ i = 1, \dots, N_A, \quad k = 1, \dots, N_B, \quad j = 1, \dots, 2^m \quad (2.135)$$

then the ellipsoid $\mathcal{E}(W^{-1}, \eta)$ is a region of asymptotic stability (RAS) for the uncertain saturated system (2.1).

Quite similar conditions to the ones of Proposition 2.16 can be derived considering the polytopic approach III and the result given by Proposition 2.3.

Remark 2.14 Considering the conditions given in Propositions 2.15 and 2.16, the same type of optimization problems and associated computational issues discussed in Sect. 2.2.5 apply.

Remark 2.15 The results for the norm-bounded uncertainty case can also be derived considering a LFT representation of the closed-loop system:

$$\begin{cases} \dot{x}(t) = A_0 x(t) + B_0 \text{sat}(Kx(t)) + Dp(t) \\ z(t) = E_1 x(t) + E_2 \text{sat}(Kx(t)) \\ p(t) = F(t)z(t), \quad F(t)'F(t) \leq I_r \end{cases} \quad (2.136)$$

Hence, it follows that $p(t) = F(t)[E_1 \ E_2 \text{sat}(Kx(t))]$.

Consider a Lyapunov candidate function $V(x(t)) = x(t)'Px(t)$, it follows that

$$\begin{aligned} \dot{V}(x(t)) &= x(t)'P \left(A_0 + B_0 \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l) K \right) x(t) \\ &\quad + x(t)' \left(A_0 + B_0 \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l) K \right)' Px(t) + x(t)'P D p(t) \\ &\quad + p(t)'D' P x(t) \end{aligned} \quad (2.137)$$

From (2.136), one has $p(t)'p(t) = z(t)'F(t)'F(t)z(t)$. Since $F(t)'F(t) \leq I_r$, it follows that

$$\begin{aligned} \varepsilon^{-1} &\left(p(t)'p(t) - x(t)' \left(E_1 + E_2 \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l) K \right)' \right. \\ &\quad \times \left. \left(E_1 + E_2 \sum_{j=1}^{2^m} \lambda_j(t) \Gamma_j(\alpha_l) K \right) x(t) \right) \leq 0 \end{aligned} \quad (2.138)$$

for all $\varepsilon > 0$. Thus, from (2.137) and (2.138), the inequalities (2.129) can be obtained by applying the S-Procedure and Schur's complement [45]. Analogous procedure can be made considering polytopic models II and III.

Example 2.8 The linearized equations of a satellite motion, given in [84], yield the following matrices for system (2.1):

$$A(F) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3p(t)^2 & 0 & 0 & 2p(t) \\ 0 & 0 & 0 & 1 \\ 0 & -2p(t) & 0 & 0 \end{bmatrix}; \quad B(F) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Time-varying uncertain parameter $p(t)$ represents the period of rotation and lies within the interval $[0.5, 1.5]$. The saturation levels are $u_{0(1)} = u_{0(2)} = 15$. In order to represent the uncertainty on $p(t)$, we assume that matrix $A(F)$ belongs to the polytope¹

¹ $p(t)$ and $p(t)^2$ are considered independently to satisfy the linearity condition of uncertain parameters in the polytopic modeling, i.e. two uncertain parameters are considered. This leads to a polytope of matrices with four vertices.

$$\mathcal{D}_A = \text{Co} \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.75 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \right. \\ \left. \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.75 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{bmatrix} \right\}.$$

In [84], the authors show that the linear state feedback

$$u = \begin{bmatrix} -19 & -13 & -2 & 0 \\ 0 & 2 & -7 & -8 \end{bmatrix} x \quad (2.139)$$

stabilizes the saturated linear system for any initial condition in the unit sphere, i.e.

$$x(0) \in \{x \in \mathbb{R}^4; x'x \leq 1\},$$

when $p(t)$ is set to 1. However, state feedback (2.139) was not guaranteed to stabilize the uncertain closed-loop system for any initial condition in the unit sphere. Then, we apply the results of Proposition 2.15, using the same steps of Algorithm 2.1 with optimization criterion: $\min \text{Trace}(P) + \eta$. Initializing the Algorithm 2.1 with $\alpha_l = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$ in step 1, Algorithm 2.1 converges to the solution,

$$P = \begin{bmatrix} 0.4192 & 0.1872 & -0.0435 & -0.0037 \\ 0.1872 & 0.2036 & 0.0681 & -0.0219 \\ -0.0435 & 0.0681 & 0.2233 & 0.0293 \\ -0.0037 & -0.0219 & 0.0293 & 0.0761 \end{bmatrix}$$

$$\eta = 0.9149; \quad \alpha_l = \begin{bmatrix} 0.4442 \\ 0.4610 \end{bmatrix}$$

The cuts of ellipsoid $\mathcal{E}(P, \eta)$ of safe initial conditions is plotted in Fig. 2.11. Note that this set encloses the unit sphere. This shows that the state feedback proposed by [84] for the nominal certain case is a robust state feedback for the uncertain system for any initial condition belonging to a domain larger than the unit sphere.

2.5.2 Extensions

To address the external stability analysis problem, such as done in Sect. 2.3, inequalities (2.30), for the bounded amplitude case, and (2.45), for the energy bounded case, should be considered with respect to the trajectories of the uncertain system (2.126). In this case, considering the reasoning done in the proof of Propositions 2.15 and 2.16 to bound $\dot{V}(x(t))$ considering both norm-bound and polytopic uncertainties, extensions of the results stated in Propositions 2.4 and 2.5 can be straightforwardly derived to cope with the external stability analysis problem.

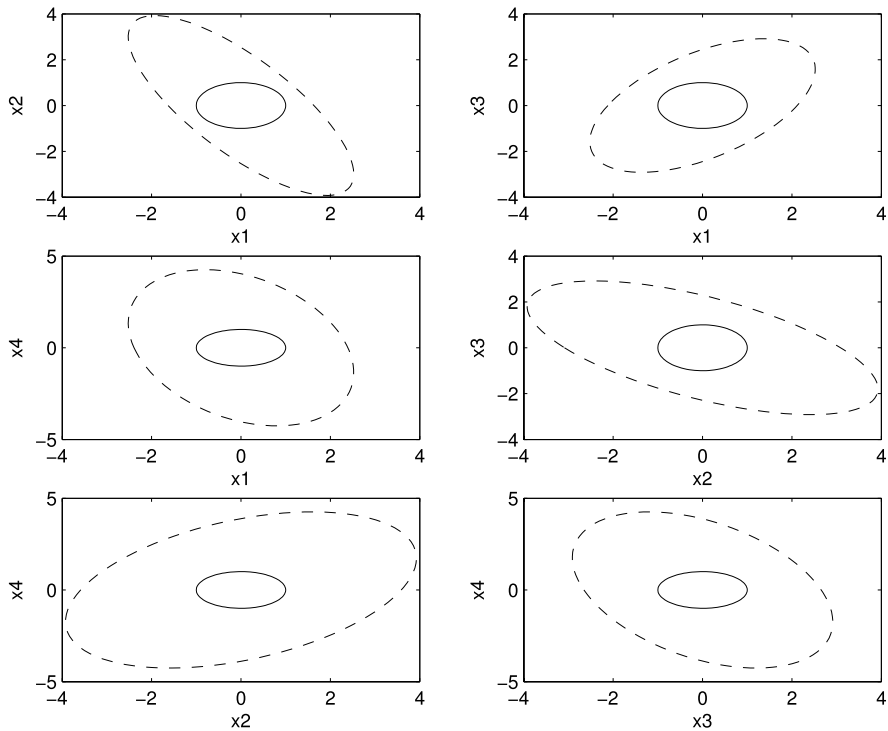


Fig. 2.11 Example 2.8—the region of stability of the uncertain saturated system (---) and the unit ball considered in [84] (—)

The extension of the results to the synthesis problem is also immediate. It suffices to apply the variable change $Y = KW$ in the conditions of Propositions 2.15 and 2.16. In this case, the idea is to compute a stabilizing gain that will ensure the stability for any matrix belonging to the uncertainty set. The same optimization criteria presented in Sect. 2.4.1 can thus be used to compute a robust stabilizing gain to ensure internal or external stability of the uncertain closed-loop system.

2.6 Discrete-Time Case

Consider a linear discrete-time system subject to control saturation generically described by the following state equation:

$$x(k+1) = Ax(k) + B \text{sat}(Kx(k)) \quad (2.140)$$

As in the continuous-time case, a polytopic differential inclusion can be used to model the system dynamics under saturation effects. In fact, considering the polytopic representation I, II or III we have the following, respectively:

- if $x(k) \in S(|K|, u_0^\alpha)$ then

$$x(k+1) = \sum_{j=1}^{2^m} \lambda_j (A + B \Gamma_j(\alpha_l) K) x(k) \quad (2.141)$$

- if $x(k) \in S(|H|, u_0)$ then

$$x(k+1) = \sum_{j=1}^{2^m} \lambda_j (A + B \Gamma_j^+ K + \Gamma_j^- H) x(k) \quad (2.142)$$

- $x(k) \in \bigcap_{j=2}^{2^m} S(|H_j|, u_0)$ then

$$x(k+1) = \sum_{j=1}^{2^m} \lambda_j \left(A + \sum_{i \in \mathcal{S}_j^c} B_i K_{(i)} + \sum_{i \in \mathcal{S}_j} B_i H_{j(i)} \right) x(k) \quad (2.143)$$

with appropriate scalars $0 \leq \lambda_j \leq 1$, $\sum_{j=1}^{2^m} \lambda_j = 1$.

2.6.1 Ellipsoidal Regions of Asymptotic Stability

Ellipsoidal sets can be associated to a discrete-time Lyapunov quadratic function defined as

$$V(x(k)) = x(k)' P x(k)$$

Regarding this function, we define

$$\Delta V(x(k)) = V(x(k+1)) - V(x(k))$$

Following the same reasoning as developed in Sect. 2.2.2, an ellipsoidal set $\mathcal{E}(P, \eta)$ will be a region of asymptotic stability for system (2.140) if $\Delta V(x(k)) < 0$, $\forall x(k) \in \mathcal{E}(P, \eta)$. Hence, considering the polytopic representations (2.141), (2.142) and (2.143) for the system (2.140), we can conclude that $\mathcal{E}(P, \eta)$ is a region of asymptotic stability if:

1. $\Delta V(x(k)) < 0$ along the trajectories of the polytopic system (2.141), (2.142) or (2.143);
2. $\mathcal{E}(P, \eta) \subset S(|K|, u_0^\alpha)$, if polytopic approach I is considered, $\mathcal{E}(P, \eta) \subset S(|H|, u_0)$, if polytopic approach II is used, or $\mathcal{E}(P, \eta) \subset \bigcap_{j=2}^{2^m} S(|H_j|, u_0)$, if polytopic approach III is used.

Hence, the following results can be stated considering the polytopic model I [130, 134].

Proposition 2.17 *If there exist a symmetric positive definite matrix $P \in \Re^{n \times n}$, a vector $\alpha_l \in \Re^m$ and a positive scalar η satisfying*

$$(A + B\Gamma_j(\alpha_l)K)'P(A + B\Gamma_j(\alpha_l)K) - P < 0, \quad j = 1, \dots, 2^m \quad (2.144)$$

$$\begin{bmatrix} P & \alpha_{l(i)}K_{(i)}' \\ \alpha_{l(i)}K_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.145)$$

$$0 < \alpha_{l(i)} \leq 1, \quad i = 1, \dots, m \quad (2.146)$$

then the ellipsoid $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS) for the discrete-time saturated system (2.140).

Proof From the Schur's complement and convexity arguments, if (2.144) is verified for $j = 1, \dots, 2^m$, it follows that

$$\begin{bmatrix} P & \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K)' \\ \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K) & P^{-1} \end{bmatrix} > 0$$

with $\sum_{j=1}^{2^m} \lambda_j = 1, 0 \leq \lambda_j \leq 1$.

Applying Schur's complement again, it follows that

$$\sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K)'P \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K) - P < 0$$

This implies, $\forall x(k) \neq 0$, that

$$x(k)' \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K)'P \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j(\alpha_l)K)x(k) - x(k)'Px(k) < 0$$

Then, provided that $x(k) \in S(|K|, u_0^\alpha)$ it follows that

$$x(k+1)'Px(k+1) - x(k)'Px(k) \leq 0, \quad \forall x(k) \neq 0$$

with respect to the dynamics given by the saturated system (2.140).

On the other hand (2.145) and (2.146) guarantee that $\mathcal{E}(P, \eta) \subset S(|K|, u_0^\alpha)$. Hence, if (2.144), (2.145) and (2.146) are verified we have indeed that $\Delta V(x(k)) < 0, \forall x(k) \in \mathcal{E}(P, \eta)$, which ensures that $\mathcal{E}(P, \eta)$ is a region of asymptotic stability (RAS). \square

Considering the polytopic model (2.142), the following result can be stated [192].

Proposition 2.18 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, a matrix $Q \in \mathbb{R}^{m \times n}$ and a positive scalar η satisfying*

$$\begin{bmatrix} W & W(A + B\Gamma_j^+K)' + Q'\Gamma_j^-B' \\ (A + B\Gamma_j^+K)W + B\Gamma_j^-Q & W \end{bmatrix} > 0 \quad j = 1, \dots, 2^m \quad (2.147)$$

$$\begin{bmatrix} W & Q_{(i)}' \\ Q_{(i)} & \eta u_{0(i)}^2 \end{bmatrix} \geq 0, \quad i = 1, \dots, m \quad (2.148)$$

then the ellipsoid $\mathcal{E}(W^{-1}, \eta)$ is a region of stability (RAS) for the discrete-time saturated system (2.140).

Proof Consider now the change of variables $W = P^{-1}$ and $Q = HW$. Pre- and post-multiplying (2.147) by matrix $\begin{bmatrix} P & 0 \\ 0 & I_n \end{bmatrix}$, it follows that

$$\left[\begin{array}{cc} P & \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j^+ K + B\Gamma_j^- H)' \\ \sum_{j=1}^{2^m} \lambda_j (A + B\Gamma_j^+ K + B\Gamma_j^- H) & W \end{array} \right] > 0$$

From here the proof follows the same steps as in the proof of Proposition 2.17. \square

An analogous result considering the polytopic model (2.143) can be stated as follows [2].

Proposition 2.19 *If there exist a symmetric positive definite matrix $W \in \mathbb{R}^{n \times n}$, row vectors $Q_{j(i)} \in \mathbb{R}^{1 \times n}$, $j = 2, \dots, 2^m$, $i \in \mathcal{S}_j$ and η satisfying*

$$\left[\begin{array}{cc} W & (AW + \sum_{i \in \mathcal{S}_j} B_i K_{(i)} W + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)})' \\ (AW + \sum_{i \in \mathcal{S}_j} B_i K_{(i)} W + \sum_{i \in \mathcal{S}_j} B_i Q_{j(i)}) & W \end{array} \right] > 0, \quad j = 1, \dots, 2^m \quad (2.149)$$

$$\left[\begin{array}{cc} W & Q'_{j(i)} \\ Q_{j(i)} & \eta u_{0(i)}^2 \end{array} \right] \geq 0, \quad \forall j = 2, \dots, 2^m, \forall i \in \mathcal{S}_j \quad (2.150)$$

then the ellipsoid $\mathcal{E}(W^{-1}, \eta)$ is a region of stability (RAS) for the discrete-time saturated system (2.140).

2.6.2 Polyhedral Regions of Asymptotic Stability

In this section, we present a complementary technique to the one presented in the previous section. We will show that if a contractive ellipsoidal region of asymptotic stability has been obtained for system (2.140) using a polytopic differential inclusion, it is possible to obtain a polyhedral region of asymptotic stability that includes the ellipsoidal one. The methodology described below uses some tools based on the positive invariance and contractivity of polyhedra (see also Chap. 4) and the computation of maximal invariant sets (see for instance [34, 126, 214]). In particular, the results can be seen as an extension of the ones presented in [37] and [348].

For the sake of simplicity, we derive the results considering the polytopic approach I. In this case, the idea is to compute the maximal invariant set, with respect to the polytopic model I, included in the region $S(|K|, u_0^\alpha)$. Analogous results can be obtained by using the polytopic approaches II and III.

2.6.2.1 Preliminaries

Let us consider the following definitions:

$$\mathbb{A}_j = A + B\Gamma_j(\alpha_l)K, \quad j = 1, \dots, 2^m$$

and

$$\begin{aligned} S(G_0, \psi_0) &= S(|K|, u_0^\alpha) \\ G_0 &= \begin{bmatrix} K \\ -K \end{bmatrix}; \quad \psi_0 = \begin{bmatrix} u_0^\alpha \\ u_0^\alpha \end{bmatrix} \\ \mathcal{J} &= \{1, \dots, 2^m\}; \quad \mathcal{I}_0 = \{1, \dots, m_0\} \end{aligned}$$

with $m_0 = 2m$.

A series of polytopes $S(G_k, \psi_k)$ can be defined. The first element is $S(G_0, \psi_0)$ and the other elements are generically denoted by $S(G_k, \psi_k)$ where

$$\begin{aligned} G_k &= \begin{bmatrix} G_{k-1} \\ T_k \end{bmatrix}; \quad \psi_k = \begin{bmatrix} \psi_{k-1} \\ r_k \end{bmatrix} \\ T_k &\in \mathbb{R}^{m_k \times n}; \quad G_k \in \mathbb{R}^{l_k \times n}; \quad r_k \in \mathbb{R}^{m_k}; \quad \psi_k \in \mathbb{R}^{l_k} \\ \mathcal{I}_k &= \{1, \dots, m_k\} \end{aligned}$$

By definition $G_0 = T_0$ and $\psi_0 = r_0$.

The determination of T_{k+1} and r_{k+1} , is based on the solution of the following linear programs:

$$LP_k(i, j) = \begin{cases} y_{k(i, j)} = \max_x T_{k(i)} \mathbb{A}_j x, \\ \text{subject to } x \in S(G_k, \psi_k), \end{cases} \quad i \in \mathcal{I}_k, \quad j \in \mathcal{J} \quad (2.151)$$

From the solution of (2.151), consider the definition of the following index sets:

$$\mathcal{J}_{k,i} = \{j \in \mathcal{J}; \quad y_{k(i, j)} > r_{k(i)}\}; \quad \tilde{\mathcal{I}}_k = \{i \in \mathcal{I}_k; \quad \mathcal{J}_{k,i} \neq \emptyset\}$$

Define now the set

$$S(T_{k+1}, r_{k+1})_{i, j} = \{x \in \mathbb{R}^n; \quad T_{k(i)} \mathbb{A}_j x \leq r_{k(i)}\}$$

If $\tilde{\mathcal{I}}_k \neq \emptyset$, consider the construction of the sets

$$\begin{aligned} S(T_{k+1}, r_{k+1})_i &= \bigcap_{j \in \mathcal{J}_{k,i}} S(T_{k+1}, r_{k+1})(i, j) \\ S(T_{k+1}, r_{k+1}) &= \bigcap_{i \in \tilde{\mathcal{I}}_k} S(T_{k+1}, r_{k+1})_i \end{aligned}$$

The set $S(G_{k+1}, \psi_{k+1})$ is therefore defined as follows:

- if $\tilde{\mathcal{I}}_k \neq \emptyset$: $S(G_{k+1}, \psi_{k+1}) = S(G_k, \psi_k) \cap S(T_{k+1}, r_{k+1})$
- if $\tilde{\mathcal{I}}_k = \emptyset$: $S(G_{k+1}, \psi_{k+1}) = S(G_k, \psi_k)$

Denote $\prod^k \mathbb{A}_j$ as any product of any k matrices \mathbb{A}_j .

Recalling that $S(G_0, \psi_0) = S(|K|, u_0^\alpha)$, by construction, the following properties with respect to sets $S(G_k, \psi_k)$ and $S(G_{k+1}, \psi_{k+1})$ hold.

Property 2.1 Considering the definition of the set $S(G_k, \psi_k)$ the following properties hold:

1. $S(G_{k+1}, \psi_{k+1}) \subseteq S(G_k, \psi_k)$,
2. if $x \in S(G_k, \psi_k)$ then $\prod^k \mathbb{A}_j x \in S(|K|, u_0^\alpha)$, or equivalently, $|K \prod^k \mathbb{A}_j x| \leq u_0^\alpha$,
3. $S(G_k, \psi_k) = \{x \in S(|K|, u_0^\alpha); \prod^k \mathbb{A}_j x \in S(|K|, u_0^\alpha)\}$,
4. $S(G_{k+1}, \psi_{k+1}) = \{x \in S(G_k, \psi_k); T_{k(i)} \mathbb{A}_j x \leq r_{k(i)}, \forall i \in \mathcal{I}_k, \forall j \in \mathcal{J}\}$,
5. $S(G_{k+1}, \psi_{k+1}) = \{x \in S(G_k, \psi_k); \mathbb{A}_j x \in S(G_k, \psi_k), \forall j \in \mathcal{J}\}$.

Also by construction, the following lemma can be stated.

Lemma 2.2 *If there exists an index k^* such that \mathcal{J}_{k^*i} is empty for all $i \in \mathcal{I}_{k^*}$ then one gets $S(G_k, \psi_k) = S(G_{k^*}, \psi_{k^*}), \forall k \geq k^*$.*

Remark 2.16 It should be noticed at this point that:

- $S(G_{\tilde{k}}, \psi_{\tilde{k}})$, corresponds to the set of initial states for which the corresponding trajectories of the polytopic system $\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j \mathbb{A}_j x(t)$ remain in $S(|K|, u_0^\alpha)$, for $k = 0$ to $k = \tilde{k}$, i.e. for \tilde{k} steps.
- $S(G_{k^*}, \psi_{k^*})$ corresponds to the set of initial conditions for which the trajectories of the polytopic system $\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j \mathbb{A}_j x(t)$ remain in $S(|K|, u_0^\alpha)$, for all $k \geq 0$.

Before presenting conditions for the existence of such an index k^* , let us give the following useful lemma.

Lemma 2.3 *If pair (K, A) is observable then pairs (K, \mathbb{A}_j) , $j = 1, \dots, 2^m$, are observable.*

Proof Suppose that pair (K, A) is observable, but that (K, \mathbb{A}_j) is not observable. Then there exists a vector x , $x \neq 0$, such that

$$\begin{bmatrix} K \\ K \mathbb{A}_j \\ \vdots \\ K \mathbb{A}_j^{n-1} \end{bmatrix} x = 0$$

Recall that $\mathbb{A}_j = A + B \Gamma_j(\alpha_l) K$, therefore it follows that this vector x satisfies

$$\begin{bmatrix} K \\ K A \\ \vdots \\ K A^{n-1} \end{bmatrix} x = 0$$

That is a contradiction with the hypothesis that the pair (K, A) is observable. \square

2.6.2.2 Maximal Positively Invariant Set in $S(|K|, u_0^\alpha)$

By using results from previous sections, we are now ready to state the following result.

Proposition 2.20 *If pair (K, A) is observable and if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ verifying*

$$\mathbb{A}_j' P \mathbb{A}_j - P < 0, \quad j = 1, \dots, 2^m \quad (2.152)$$

then:

1. *There exists an index k^* such that $S(G_k, \psi_k) = S(G_{k^*}, \psi_{k^*}), \forall k \geq k^*$.*
2. *The set $S(G_{k^*}, \psi_{k^*})$ is positively invariant for the polytopic system*

$$x(k+1) = \sum_{j=1}^{2^m} \lambda_j(x) \mathbb{A}_j x(k) \quad (2.153)$$

3. *The set $S(G_{k^*}, \psi_{k^*})$ is the maximal positively invariant set for the polytopic system (2.153) contained in $S(G_0, \psi_0) = S(|K|, u_0^\alpha)$.*

Proof 1. Define the polyhedral set $S(Q_j, q_j) = \{x \in \mathbb{R}^n; Q_j x \leq q_j\}$ where

$$Q_j = \begin{bmatrix} K \\ K \mathbb{A}_j \\ \vdots \\ K \mathbb{A}_j^{n-1} \\ -K \\ -K \mathbb{A}_j \\ \vdots \\ -K \mathbb{A}_j^{n-1} \end{bmatrix}; \quad q_j = \begin{bmatrix} u_0^\alpha \\ u_0^\alpha \\ \vdots \\ u_0^\alpha \\ u_0^\alpha \\ u_0^\alpha \\ \vdots \\ u_0^\alpha \end{bmatrix}$$

From Lemma 2.3, it follows that the set $S(Q_j, q_j)$ is bounded, $\forall j = 1, \dots, 2^m$. Moreover, by construction one gets $S(G_{n-1}, \psi_{n-1}) \subseteq \bigcap_{j=1}^{2^m} S(Q_j, q_j)$, which implies that there exists an index $\bar{k} \leq n-1$ such that $S(G_{\bar{k}}, \psi_{\bar{k}})$ is bounded.

Suppose that the positive scalar η is chosen in order to satisfy $S(G_{\bar{k}}, \psi_{\bar{k}}) \subset \mathcal{E}(P, \eta)$. Furthermore, since matrix P satisfies (2.152) then, by considering the quadratic function $V(x(k)) = x(k)' P x(k)$, it follows that $V(x(k+1)) - V(x(k)) < 0$ along the trajectories of the polytopic system (2.153). Hence the ellipsoid $\mathcal{E}(P, \eta)$ is a contractive set for the polytopic system (2.153). Thus, there exist an integer $M \geq 0$ and a positive scalar $\sigma_0, 0 < \sigma_0 < 1$ such that

$$\left(\prod_{j=1}^M \mathbb{A}_j \right) S(G_{\bar{k}}, \psi_{\bar{k}}) \subset \left(\prod_{j=1}^M \mathbb{A}_j \right) \mathcal{E}(P, \eta) \subset \mathcal{E}(P, \sigma_0^{-M} \eta) \subset S(|K|, u_0^\alpha) \quad (2.154)$$

where $(\prod_{j=1}^M \mathbb{A}_j) S(G_{\bar{k}}, \psi_{\bar{k}})$ denotes the set of points obtained from the application of $(\prod_{j=1}^M \mathbb{A}_j)$ to any $x \in S(G_{\bar{k}}, \psi_{\bar{k}})$.

Consider $k^* = \bar{k} + M$ and suppose now that $x \in S(G_{k^*}, \psi_{k^*})$. Since $S(G_{k^*}, \psi_{k^*}) \subset S(G_{\bar{k}}, \psi_{\bar{k}})$ and $k^* + 1 > M$, from (2.154) we conclude that

$$\prod_{j=1}^{k^*+1} \mathbb{A}_j x \in S(|K|, u_0^\alpha) \Leftrightarrow \left| K \prod_{j=1}^{k^*+1} \mathbb{A}_j x \right| \leq u_0^\alpha \quad (2.155)$$

On the other hand, note that by construction, each line of T_{k^*} and each component of r_{k^*} are given as $\pm K_{(i)} \prod_{j=1}^{k^*} \mathbb{A}_j$ and $\pm u_{0(i)}^\alpha$, respectively, considering some i , $i = 1, \dots, m$ and some specific product of k^* matrices \mathbb{A}_j . Hence, if (2.155) holds, it follows that

$$T_{k^*} \mathbb{A}_j x \leq r_{k^*}, \quad \forall j \in \mathcal{J}$$

which means that $\tilde{\mathcal{I}}_{k^*} = \emptyset$, and thus

$$S(G_k, \psi_k) = S(G_{k^*}, \psi_{k^*}), \quad \forall k > k^*$$

2. Suppose that $x(k) \in S(G_{k^*}, \psi_{k^*})$. From item 1 and by construction it follows that $G_{k^*} \mathbb{A}_j x(k) \leq \psi_{k^*}$, $\forall j \in \mathcal{J}$. Hence, by convexity, one obtains $G_{k^*} \sum_{j=1}^{2^m} \lambda_j(x) \times \mathbb{A}_j x(k) \leq \psi_{k^*}$, with $0 \leq \lambda_j(x) \leq 1$ and $\sum_{j=1}^{2^m} \lambda_j(x) = 1$. Therefore, it follows that $G_{k^*} x(k+1) \leq \psi_{k^*}$, i.e. $x(k+1) \in S(G_{k^*}, \psi_{k^*})$, which proves the positive invariance of the set $S(G_{k^*}, \psi_{k^*})$.

3. Suppose that $S(G_{k^*}, \psi_{k^*})$ is not the maximal positively invariant set for the polytopic system (2.153) contained in $S(|K|, u_0^\alpha)$. Denote by \mathcal{M} this maximal set. Suppose now that $x \in \mathcal{M}$, but $x \notin S(G_{k^*}, \psi_{k^*})$. Since, \mathcal{M} is positively invariant with respect to the polytopic system (2.153), it follows that

$$\prod_{j=1}^k \mathbb{A}_j x \in S(|K|, u_0^\alpha) \Leftrightarrow \left| K \prod_{j=1}^k \mathbb{A}_j x \right| \leq u_0^\alpha, \quad \forall k \quad (2.156)$$

On the other hand since $\mathcal{M} \subseteq S(|K|, u_0^\alpha)$, it follows that $x \in S(|K|, u_0^\alpha)$. Hence, from (2.156) the Property 2.1 (item 3) and Remark 2.16, we conclude that $x \in S(G_{k^*}, \psi_{k^*})$, which is a contradiction. \square

One can also express the following corollary to Proposition 2.20.

Corollary 2.2 *If the pair (K, A) is observable and if there exists a symmetric positive definite matrix $P \in \mathbb{R}^{n \times n}$ verifying relation (2.152) then:*

1. *The set $S(G_{k^*}, \psi_{k^*})$ is positively invariant for the saturated system (2.140).*
2. *The set $S(G_{k^*}, \psi_{k^*})$ is a region of asymptotic stability for system (2.140). Moreover, the quadratic function $V(x(k)) = x(k)' P x(k)$ is such that $V(x(k+1)) - V(x(k)) < 0$, $\forall x(k) \in S(G_{k^*}, \psi_{k^*})$.*

Proof 1. Recall that the polytopic model (2.153) represents the saturated system (2.140) for $x(k) \in S(|K|, u_0^\alpha)$. Thus since by construction $S(G_{k^*}, \psi_{k^*}) \subset S(|K|, u_0^\alpha)$ and since $S(G_{k^*}, \psi_{k^*})$ is positively invariant for the polytopic model (2.153), it is also a positively invariant set for the saturated system (2.140).

2. By hypothesis, matrix P satisfies relation (2.152), which implies that $V(x(k+1)) - V(x(k)) < 0, \forall x(k) \in S(G_{k^*}, \psi_{k^*})$, with respect to the trajectories of the polytopic system (2.153). Since $S(G_{k^*}, \psi_{k^*})$ is a positively invariant set for the polytopic system and it is included in the region $S(|K|, u_0^\alpha)$, all the trajectories of (2.153) initialized in $S(G_{k^*}, \psi_{k^*})$ converge asymptotically to the origin and never leave $S(|K|, u_0^\alpha)$. Since all the trajectories of the saturated system in $S(|K|, u_0^\alpha)$ are trajectories of the polytopic system (2.153), it follows that all the trajectories of (2.140), initialized in $S(G_{k^*}, \psi_{k^*})$ converge asymptotically to the origin. \square

Remark 2.17 If we define $S(T_{k+1}, r_{k+1})_{i,j} = \{x \in \mathbb{R}^n; T_{k(i)} \mathbb{A}_j x \leq \sigma r_{k(i)}\}$, with $0 < \sigma < 1$, the obtained polyhedron $S(G_{k^*}, \psi_{k^*})$ is σ -contractive with respect to the saturated system [130, 348]. This means that $\forall x(k) \in S(G_{k^*}, \psi_{k^*}), G_{k^*} x(k+1) \leq \sigma \psi_{k^*}$. In this case the function

$$\mathcal{V}(x(t)) = \max_i \left\{ \frac{G_{k^*(i)} x(t)}{\psi_{k^*(i)}} \right\}$$

is a strictly decreasing Lyapunov function for the saturated system in $S(G_{k^*}, \psi_{k^*})$ (see Chap. 4 for details).

From the results above, the implementation of an algorithm to compute $S(G_{k^*}, \psi_{k^*})$ is straightforward. Basically, it suffices to solve and evaluate the linear programs (2.151) and repeat the procedure iteratively until the set $\tilde{\mathcal{I}}_k$ is empty [130, 348].

Algorithm 2.2

1. Set $G_0 = \begin{bmatrix} K \\ -K \end{bmatrix}$, $\psi_0 = \begin{bmatrix} u_0^\alpha \\ u_0^\alpha \end{bmatrix}$, $T_0 = G_0$, $r_0 = \psi_0$, $m_0 = 2m$, $k = 0$.
2. Solve $LP_k(i, j)$ for all $i \in \mathcal{I}_k$, $j \in \mathcal{J}$.
3. Determine $\tilde{\mathcal{I}}_k$. Set $l = 1$.
4. If $\tilde{\mathcal{I}}_k = \emptyset$, then $S(G_k, \psi_k)$ is the maximal admissible set and Stop. Otherwise, go to Step 5.
5. For $i = 1$ to m_k : if $j \in \mathcal{J}_{ki} \neq \emptyset$ then $T_{k+1}(l) = T_{k(i)} \mathbb{A}_j$; $r_{k+1}(l) = \sigma r_{k(i)}$; $l = l + 1$.
6. Set $G_{k+1} = \begin{bmatrix} G_k \\ T_{k+1} \end{bmatrix}$, $\psi_{k+1} = \begin{bmatrix} \psi_k \\ r_{k+1} \end{bmatrix}$. Denote m_{k+1} the number of rows of T_{k+1} . $k = k + 1$. Go to Step 2.

Remark 2.18 In Proposition 2.20 and Corollary 2.2, the vector α_l is supposed to be given. Nevertheless, in order to obtain a larger region $S(|K|, u_0^\alpha)$ and therefore a larger region $S(G_{k^*}, \psi_{k^*})$, it is interesting to find α_l with components as small as possible, for which relation (2.152) is satisfied.

In practice, we can compute first an ellipsoidal RAS using polytopic model I, considering one of the criteria discussed in Sect. 2.2.5. Then, considering the set $S(|K|, u_0^\alpha)$ associated to the ellipsoidal RAS, we compute $S(G_{k^*}, \psi_{k^*})$. Note that the set $S(G_{k^*}, \psi_{k^*})$ will always include the ellipsoidal RAS, because it is the maximal invariant set contained in $S(|K|, u_0^\alpha)$.

Remark 2.19 The (K, A) -observability assumption in Proposition 2.20 and Corollary 2.2 is necessary for guaranteeing that for a certain index \bar{k} , the resulting set $S(G_{\bar{k}}, \psi_{\bar{k}})$ is bounded. Such an assumption can be dropped when the set $S(|K|, u_0^\alpha) = S(G_0, \psi_0)$ is naturally bounded.

Remark 2.20 A more abstracted presentation of these results, based on the concept of one-step set [34, 214], can be found in [238].

Example 2.9 Consider system (2.140) described by the following data [348]:

$$A = \begin{bmatrix} 0.8 & 0.5 \\ -0.4 & 1.2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ K = [0.2888 \quad -1.8350]; \quad u_0 = 7$$

Note that matrix A is unstable because its spectrum is $\sigma(A) = \{1 \pm j0.4\}$.

Considering $\sigma_0 = 0.999$, the minimal α_l for which it is possible to satisfy $\mathbb{A}'_j P \mathbb{A}_j - \sigma_0 P < 0$ is equal to 0.1661. In this case, one obtains

$$P = \begin{bmatrix} 0.00133 & 0.0000 \\ 0.0000 & 0.00198 \end{bmatrix}$$

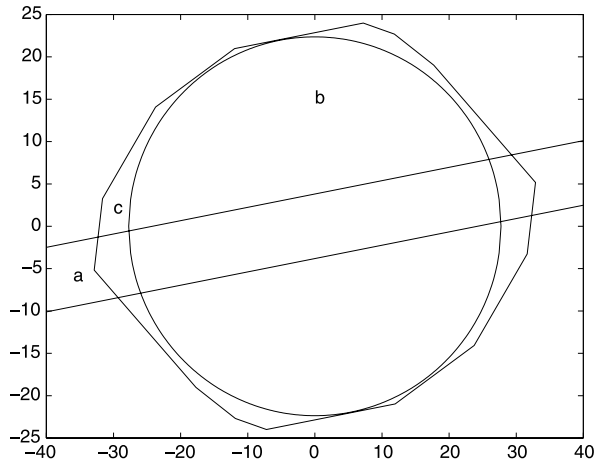
Hence it follows that $\mathcal{E}(P, 1) \subset S(|K|, u_0^\alpha)$ and so we conclude that $\mathcal{E}(P, 1)$ is a RAS for the saturated closed-loop system.

Consider now $\sigma = 0.998$ (see Remark 2.17), we compute the maximal σ -invariant set $S(G_{k^*}, \psi_{k^*})$ contained in $S(|K|, u_0^\alpha)$ by applying Algorithm 2.2. One obtains

$$G_{k^*} = \begin{bmatrix} 0.2888 & -1.835 \\ -0.2888 & 1.835 \\ 0.877 & -1.498 \\ -0.877 & 1.498 \\ 0.4351 & 1.31 \\ -0.4351 & -1.31 \\ 1.229 & -0.9028 \\ -1.229 & 0.9028 \\ 0.8682 & 1.39 \\ -0.8682 & -1.39 \\ 1.301 & -0.1936 \\ -1.301 & 0.1936 \\ 0.2053 & 1.678 \\ -0.2053 & -1.678 \\ 1.084 & 1.188 \\ -1.084 & -1.188 \\ 0.4488 & 1.605 \\ -0.4488 & -1.605 \end{bmatrix}; \quad \psi_{k^*} = \begin{bmatrix} 42.14 \\ 42.14 \\ 42.06 \\ 42.06 \\ 42.06 \\ 42.06 \\ 41.97 \\ 41.97 \\ 41.97 \\ 41.97 \\ 41.89 \\ 41.89 \\ 41.89 \\ 41.89 \\ 41.89 \\ 41.89 \\ 41.81 \\ 41.81 \end{bmatrix}$$

Figure 2.12 shows the ellipsoidal region $\mathcal{E}(P, 1)$, the final polyhedral region $S(G_{k^*}, \psi_{k^*})$ and the region of linearity of the system.

Fig. 2.12 Example 2.9—
ellipsoidal and polyhedral
RAS and the region of
linearity: (a) $S(|K|, u_0)$;
(b) $\mathcal{E}(P, 1)$; (c) $S(G_{k^*}, \psi_{k^*})$



2.6.3 Extensions

2.6.3.1 External Stability

Basically, the conditions for the external stability and stabilization can be derived considering similar approaches to the ones adopted in Sects. 2.3 and 2.4.1.5 as follows.

Amplitude Bounded External Signals In this case we should ensure that

$$\begin{cases} \forall x(k); & x(k)'Px(k) \leq \eta^{-1} \\ \forall w(k); & w(k)'Rw(k) \leq \delta^{-1} \end{cases} \Rightarrow x(k+1)'Px(k+1) \leq \eta^{-1} \quad (2.157)$$

and, in addition, ensure that $\mathcal{E}(P, \eta)$ is contained in $S(|K|, u_0^\alpha)$, $S(|H|, u_0)$ or $\bigcap_{j=2}^{2^m} S(|H_j|, u_0)$, depending on whether polytopic model I, II or III is used.

Applying the S-procedure, it follows that (2.157) is verified if

$$\begin{aligned} & x(k+1)'Px(k+1) - \eta^{-1} + \tau_1(\eta^{-1} - x(k)'Px(k)) \\ & + \tau_2(\delta^{-1} - w(k)'Rw(k)) < 0, \quad \tau_1 > 0, \quad \tau_2 > 0 \end{aligned} \quad (2.158)$$

Hence, to derive analysis or synthesis conditions it suffices to replace the polytopic model dynamics (2.141), (2.142) or (2.143) in (2.158). In [192], a slightly different procedure to show the W -invariance of $\mathcal{E}(P, \eta^{-1})$ is presented.

Energy Bounded External Signals In this case, it is supposed that

$$\sum_{k=0}^{\infty} w(k)'Rw(k) \leq \delta^{-1}$$

As discussed in Sect. 2.3.2, provided that $x(0) \in \mathcal{E}(P, \beta)$, the trajectory boundedness is obtained if the following inequality hold along the trajectories of the polytopic model

$$\Delta V(x(k)) - w(k)'Rw(k) < 0 \quad (2.159)$$

and, in addition, $\mathcal{E}(P, \mu)$, with $\mu^{-1} = \delta^{-1} + \beta^{-1}$, is contained in $S(|K|, u_0^\alpha)$, $S(|H|, u_0)$ or $\bigcap_{j=2}^{2^m} S(|H_j|, u_0)$, depending on whether polytopic model I, II or III is considered.

Note that the verification of (2.159) along the trajectories of system the polytopic system (2.141), (2.142) or (2.143) implies that

$$V(x(T)) \leq \delta^{-1} + V(x(0)) = \delta^{-1} + \beta^{-1} = \mu^{-1}$$

If an \mathcal{L}_2 -gain constraint is considered, then (2.159) should be replaced by

$$\Delta V(x) + \frac{1}{\gamma} z(k)' z(k) - w(k)' R w(k) < 0$$

2.6.3.2 Stabilization

As in the continuous-time case, in order to obtain state feedback synthesis conditions, it suffices to perform the variable transformation $Y = KW$ in the conditions of Propositions 2.17, 2.18 or 2.19. For more details the reader can refer to [137], for the polytopic approach I or [192] for the polytopic approach II. The same optimization problems to compute the control law considered in Sect. 2.4.1 can therefore be applied.

For the discrete-time counterpart of the results considering the time-varying dynamic output feedback controller presented in Sect. 2.4.3, the reader can refer to [146] and [410], considering the polytopic approaches I and II, respectively. In those papers, it is also shown results concerning the external stabilization considering energy bounded disturbances.

2.7 Conclusion

In this chapter we presented several conditions that allow to analyse the stability as well as to compute stabilizing control laws for linear systems with saturating inputs. These conditions have been derived by using some polytopic differential inclusions for the saturated closed-loop system. It should be highlighted that these polytopic differential inclusions model the behavior of the actual closed-loop system only locally. Hence, the stability and stabilization conditions are basically obtained in a local (or regional) context. In fact, to each polytopic differential inclusion we can associate a region of validity of the model. In this case, considering the Lyapunov theory, the basic points to ensure the asymptotic stability are the following:

- determine a Lyapunov function $V(x)$ ensuring the asymptotic stability of the polytopic model;
- determine a level set of $V(x)$ inside the region of validity of the polytopic model.

Then, we can conclude that this level set is a region of asymptotic stability for the saturated closed-loop system. Regarding the external stability, the basic idea consists in ensuring that:

- the trajectories of the polytopic model do not leave a level set associated to a Lyapunov function;
- the level set is inside the region of validity of the polytopic model;
- the level set is also a region of asymptotic stability when the disturbance is vanishing.

The presented results mainly focused on the use of quadratic Lyapunov functions. In this context, the stability and stabilization conditions can be expressed in the form of linear or bilinear matrix inequalities. This is particularly interesting since convex optimization problems can be formulated to address analysis and synthesis problems. For instance, in the analysis context, it is possible to maximize ellipsoidal regions of asymptotic stability (i.e. to find good estimates of the region of attraction) or to maximize estimates of the maximal exogenous signals for which the input-to-state stability is ensured. On the other hand, it is possible to compute stabilizing control laws to maximize the associated region of stability (i.e. implicitly leading to a maximized region of attraction) or the tolerance to disturbances, or even to improve time-domain performance and disturbance rejection with a guaranteed region of stability. In particular, considering the discrete-time case, once an ellipsoidal region of asymptotic stability is determined, it is always possible to compute polyhedral region of asymptotic stability containing the ellipsoidal one. This polyhedral region is also included in the region of validity of the polytopic model and can be determined by linear programming techniques.

It should be pointed out that the conditions derived throughout the chapter are only sufficient. This comes from the fact that all the trajectories of the saturated system are trajectories of the polytopic models, but the converse is not true. Hence an inherent degree of conservatism is introduced in the conditions by the modeling itself. Furthermore, some other sources of conservatism are introduced when dealing with the external stability/stabilization problems, which can lead to conservative bounds on the maximal allowed disturbances and on the \mathcal{L}_2 -gain between the exogenous inputs and the regulated outputs. Although conservative, the conditions are very useful to determine “analytical descriptions” of sets of admissible initial conditions as well as admissible exogenous signals associated to a “guaranteed safe behavior” for the closed-loop system. This is particularly useful in systems with many states, where it is complicated or impossible to characterize the region of attraction of the closed-loop system by intensive simulations.

The main purpose of this chapter has been to provide basic tools to apply polytopic differential inclusions to the analysis and synthesis of linear systems subject to input saturation. From the presented basic results many other problems and extensions have been addressed in the literature considering the same polytopic framework. For instance, among others, we can cite:

- time-delay systems [55, 98, 349, 354];
- rate saturation [16, 144, 417];
- switched and LPV systems [24, 399];
- model predictive control [54, 199, 238].

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