2

Arithmetic functions

The pleasure we obtain from music comes from counting,
but counting unconsciously.
Music is nothing but unconscious arithmetic.
Gottfried Wilhelm Leibniz (1646–1716)

In this chapter we shall define the arithmetic functions Möbius $\mu(n)$, Euler $\phi(n)$, the functions $\tau(n)$ and $\sigma_a(n)$ and, in addition, we shall prove some of their most basic properties and several formulas which are related to them. However, we shall first define some introductory notions.

2.1 Basic definitions

Definition 2.1.1. *An **arithmetic function** is a function $f : \mathbb{N} \to \mathbb{C}$ with domain of definition the set of natural numbers $\mathbb{N}$ and range a subset of the set of complex numbers $\mathbb{C}$.*

Definition 2.1.2. *A function f is called an **additive function** if and only if*

$$f(mn) = f(m) + f(n), \tag{1}$$

*for every pair of coprime integers m, n. In case (1) is satisfied for every pair of integers m, n, which are not necessarily coprime, then the function f is called **completely additive**.*

Definition 2.1.3. *A function f is called a **multiplicative function** if and only if*

$$f(1) = 1 \quad \text{and} \quad f(mn) = f(m)f(n), \tag{2}$$

*for every pair of coprime integers m, n. In case (2) is satisfied for every pair of integers m, n, which are not necessarily coprime, then the function f is called **completely multiplicative**.*

M.Th. Rassias, *Problem-Solving and Selected Topics in Number Theory: In the Spirit of the Mathematical Olympiads*, DOI 10.1007/978-1-4419-0495-9_2,

2.2 The Möbius function

Definition 2.2.1. *The Möbius function $\mu(n)$ is defined as follows:*

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ (-1)^k, & \text{if } n = p_1 p_2 \dots p_k \text{ where } p_1, p_2, \dots, p_k \text{ are } k \text{ distinct primes} \\ 0, & \text{in every other case.} \end{cases}$$

For example, we have

$$\mu(2) = -1,\ \mu(3) = -1,\ \mu(4) = 0,\ \mu(5) = -1,\ \mu(6) = 1$$

Remark 2.2.2. The Möbius function is a *multiplicative function*, since

$$\mu(1) = 1 \ \ \textit{and} \ \ \mu(mn) = \mu(m)\mu(n),$$

for every pair of coprime integers m, n.

However, it is not a *completely multiplicative* function because, for example, $\mu(4) = 0$ and $\mu(2)\mu(2) = (-1)(-1) = 1$.

Theorem 2.2.3.

$$\sum_{d|n} \mu(d) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n > 1, \end{cases}$$

where the sum extends over all positive divisors of the positive integer n.

Proof.

- If $n = 1$, then the theorem obviously holds true, since by the definition of the Möbius function we know that $\mu(1) = 1$.
- If $n > 1$, we can write
$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$
where $p_1, p_2, \dots, p_k$ are distinct prime numbers.
Therefore,
$$\sum_{d|n} \mu(d) = \mu(1) + \sum_{1 \le i \le k} \mu(p_i) + \sum_{\substack{i \ne j \\ 1 \le i, j \le k}} \mu(p_i p_j) + \cdots + \mu(p_1 p_2 \cdots p_k), \quad (1)$$
where generally the sum
$$\sum_{i_1 \ne i_2 \ne \cdots \ne i_\lambda} \mu(p_{i_1} p_{i_2} \cdots p_{i_\lambda})$$
extends over all possible products of λ distinct prime numbers. (By the definition of $\mu(m)$, we know that if in the canonical form of m some prime

number appears multiple times, then $\mu(m) = 0$.) Hence, by (1) and the binomial identity, we obtain

$$\begin{aligned}\sum_{d|n} \mu(d) &= 1 + \binom{k}{1}(-1) + \binom{k}{2}(-1)^2 + \cdots + \binom{k}{k}(-1)^k \\ &= (1-1)^k = 0.\end{aligned}$$

Therefore,

$$\sum_{d|n} \mu(d) = 0, \quad if \quad n > 1.$$

□

Theorem 2.2.4 (The Möbius Inversion Formula). *Let $n \in \mathbb{N}$. If*

$$g(n) = \sum_{d|n} f(d),$$

then

$$f(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) g(d).$$

The inverse also holds.

Proof.

- Generally, for every arithmetic function $m(n)$, it holds

$$\sum_{d|n} m(d) = \sum_{d|n} m\left(\frac{n}{d}\right),$$

since $n/d = d'$ and d' is also a divisor of n. Therefore, it is evident that

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) g(d) = \sum_{d|n} \mu(d) g\left(\frac{n}{d}\right). \tag{1}$$

But

$$\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{d|n} \left(\mu(d) \cdot \sum_{\lambda|\frac{n}{d}} f(\lambda)\right). \tag{2}$$

At this point, we are going to express (2) in an equivalent form, where there will be just one sum at the left-hand side. In order to do so, we must find a common condition for the sums $\sum_{d|n}$ and $\sum_{\lambda|\frac{n}{d}}$. The desired condition is $\lambda d|n$.

Hence, we get

$$\sum_{d|n} \mu(d) g\left(\frac{n}{d}\right) = \sum_{\lambda d|n} \mu(d) f(\lambda).$$

Similarly,

$$\sum_{\lambda|n}\left(f(\lambda)\cdot\sum_{d|\frac{n}{\lambda}}\mu(d)\right)=\sum_{\lambda d|n}\mu(d)f(\lambda).$$

Thus,

$$\sum_{d|n}\mu(d)g\left(\frac{n}{d}\right)=\sum_{\lambda|n}\left(f(\lambda)\cdot\sum_{d|\frac{n}{\lambda}}\mu(d)\right). \tag{3}$$

However, by the previous theorem

$$\sum_{d|\frac{n}{\lambda}}\mu(d)=1 \text{ if and only if } \frac{n}{\lambda}=1,$$

and in every other case the sum is equal to zero. Thus, for $n=\lambda$ we obtain

$$\sum_{\lambda|n}\left(f(\lambda)\cdot\sum_{d|\frac{n}{\lambda}}\mu(d)\right)=f(n). \tag{4}$$

Therefore, by (1), (3) and (4) it follows that if

$$g(n)=\sum_{d|n}f(d),$$

then

$$f(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)g(d).$$

- Conversely, we shall prove that if

$$f(n)=\sum_{d|n}\mu\left(\frac{n}{d}\right)g(d),$$

then

$$g(n)=\sum_{d|n}f(d).$$

We have

$$\begin{aligned}\sum_{d|n}f(d)&=\sum_{d|n}f\left(\frac{n}{d}\right)\\&=\sum_{d|n}\sum_{\lambda|\frac{n}{d}}\mu\left(\frac{n}{\lambda d}\right)g(\lambda)\\&=\sum_{d\lambda|n}\mu\left(\frac{n}{\lambda d}\right)g(\lambda)\\&=\sum_{\lambda|n}g(\lambda)\sum_{d|\frac{n}{\lambda}}\mu\left(\frac{n}{\lambda d}\right).\end{aligned}$$

The sum

$$\sum_{d|\frac{n}{\lambda}} \mu\left(\frac{n}{\lambda d}\right) = 1$$

if and only if $n = \lambda$ and in every other case it is equal to zero. Hence, for $n = \lambda$ we obtain

$$\sum_{d|n} f(d) = g(n). \qquad \square$$

Historical Remark. August Ferdinand Möbius, born on the 17th of November 1790 in Schulpforta, was a German mathematician and theoretical astronomer. He was first introduced to mathematical notions by his father and later on by his uncle. During his school years (1803–1809), August showed a special skill in mathematics. In 1809, however, he started law studies at the University of Leipzig. Not long after that, he decided to quit these studies and concentrate in mathematics, physics and astronomy. August studied astronomy and mathematics under the guidance of Gauss and Pfaff, respectively, while at the University of Göttingen. In 1814, he obtained his doctorate from the University of Leipzig, where he also became a professor.

Möbius's main work in astronomy was his book entitled *Die Elemente den Mechanik des Himmels* (1843) which focused on celestial mechanics. Furthermore, in mathematics, he focused on projective geometry, statics and number theory. More specifically, in number theory, the *Möbius function* $\mu(n)$ and the *Möbius inversion formula* are named after him.

The most famous of Möbius's discoveries was the *Möbius strip* which is a nonorientable two-dimensional surface.

Möbius is also famous for the *five-color problem* which he presented in 1840. The problem's description was to find the least number of colors required to draw the regions of a map in such a way so that no two adjacent regions have the same color (this problem is known today as the *four-color theorem*, as it has been proved that the least number of colors required is four). A. F. Möbius died in Leipzig on the 26th of September, 1868.

Problem 2.2.5. Let f be a multiplicative function and

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}, \quad \textit{where} \quad k \in \mathbb{N},$$

be the canonical form of the positive integer n.

Prove that

$$\sum_{d|n} \mu(d) f(d) = \prod_{i=1}^{k} (1 - f(p_i)).$$

Proof. The nonzero terms of the sum

$$\sum_{d|n} \mu(d) f(d)$$

correspond to divisors d, for which

$$d = p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}, \quad \text{where} \quad q_i = 0 \quad \text{or} \quad 1 \quad \text{and} \quad 1 \le i \le k. \tag{1}$$

Therefore,

$$\sum_{d|n} \mu(d) f(d) = \sum_{q_i = 0 \ or \ 1} (-1)^k f(p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}), \tag{2}$$

where the sum at the right-hand side of (2) extends over all divisors d obeying the property (1). However, if we carry over the operations in the product

$$(1 - f(p_1))(1 - f(p_2)) \cdots (1 - f(p_k)),$$

we get a sum of the form

$$\sum_{q_i = 0 \ or \ 1} (-1)^k f(p_1^{q_1}) f(p_2^{q_2}) \cdots f(p_k^{q_k}) = \sum_{q_i = 0 \ or \ 1} (-1)^k f(p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}).$$

Hence, by (2) it is evident that

$$\begin{aligned} \prod_{i=1}^{k} (1 - f(p_i)) &= \sum_{q_i = 0 \ or \ 1} (-1)^k f(p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}) \\ &= \sum_{d|n} \mu(d) f(d). \end{aligned}$$

□

Remark 2.2.6. In the special case when $f(d) = 1$ for every divisor d of n, it follows that

$$\sum_{d|n} \mu(d) f(d) = \sum_{d|n} \mu(d) = \prod_{i=1}^{k} (1 - 1) = \left\lfloor \frac{1}{n} \right\rfloor,$$ [1]

which is exactly Theorem 2.2.3.

2.3 The Euler function

Definition 2.3.1. *The Euler function $\phi(n)$ is defined as the number of positive integers which are less than or equal to n and at the same time relatively prime to n. Equivalently, the Euler function $\phi(n)$ can be defined by the formula*

$$\phi(n) = \sum_{m=1}^{n} \left\lfloor \frac{1}{\gcd(n, m)} \right\rfloor.$$

[1] $\lfloor r \rfloor$ denotes the *integer part* (also called *integral part*) of a real number r.

For example, we have

$$\phi(1) = 1,\ \phi(2) = 1,\ \phi(3) = 2,\ \phi(6) = 2,\ \phi(9) = 6.$$

Before we proceed on proving theorems concerning the Euler function $\phi(n)$, we shall present two of its most basic properties.

Proposition 2.3.2. *For every prime number p, it holds*

$$\phi(p^k) = p^k - p^{k-1}.$$

Proof. The only positive integers which are less than or equal to p^k and at the same time not relatively prime to p^k are the integers

$$p,\ 2p,\ 3p, \dots,\ p^{k-1}p.$$

Thus, the number of these integers is p^{k-1} and therefore the number of positive integers which are less than or equal to p^k and at the same time relatively prime to p^k are

$$p^k - p^{k-1}. \qquad \square$$

The Euler function $\phi(n)$ is a multiplicative function, since

$$\phi(1) = 1 \ \text{ and } \ \phi(mn) = \phi(m)\phi(n),$$

for every pair of coprime integers m, n.

We shall present the proof of the above fact at the end of this section.

Theorem 2.3.3. *For every positive integer n, it holds*

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}.$$

Proof. In the previous section, we proved that the sum $\sum_{d|n} \mu(d)$ is equal to 1 if $n = 1$ and equal to 0 in any other case. Hence, equivalently we have

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor.$$

Thus, we can write

$$\phi(n) = \sum_{m=1}^{n} \left\lfloor \frac{1}{\gcd(n,m)} \right\rfloor = \sum_{m=1}^{n} \sum_{d|\gcd(n,m)} \mu(d). \tag{1}$$

In the above sums it is evident that

$$1 \le m \le n, \quad d|n,$$

and

$$d|m.$$

Therefore,

$$\sum_{m=1}^{n} \sum_{d|\gcd(n,m)} \mu(d) = \sum_{d|n} \sum_{\lambda=1}^{n/d} \mu(d) = \sum_{d|n} \frac{n}{d}\mu(d). \tag{2}$$

Thus, by (1) and (2) we finally get

$$\phi(n) = \sum_{d|n} \mu(d)\frac{n}{d}. \qquad \square$$

Theorem 2.3.4. *For every positive integer n it holds*

$$\sum_{d|n} \phi(d) = n.$$

Proof. It is clear that every positive integer k which is less than or equal to n has some divisibility relation with n. More specifically, either k and n are coprime or $\gcd(n,k) = d > 1$. Generally, if $\gcd(n,k) = d$, then

$$\left(\frac{n}{d}, \frac{k}{d}\right) = 1.$$

Hence, the number of positive integers for which $\gcd(n,k) = d$ is equal to $\phi(n/d)$. However, since the number of positive integers k with $k \leq n$ is clearly equal to n we obtain

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) = n.$$

But, it is evident that

$$\sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} \phi(d),$$

thus,

$$\sum_{d|n} \phi(d) = n. \qquad \square$$

Remark 2.3.5. Another proof of the above theorem can be given by the use of the Möbius Inversion Formula.

Theorem 2.3.6. *Let n be a positive integer and p_1, p_2,..., p_k be its prime divisors. Then*

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_k}\right)$$

and therefore, for any pair of positive integers n_1, n_2 *it holds*

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2)\frac{d}{\phi(d)},$$

where $d = \gcd(n_1, n_2)$.

Proof. We can write

$$\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right) = 1 + \sum \frac{(-1)^\lambda}{p_{m_1}p_{m_2}\cdots p_{m_\lambda}},$$

where m_i are λ distinct integers in the set $\{1, 2, \ldots, k\}$ and hence the sum extends over all possible products of the prime divisors of n. However, by the definition of the Möbius function we know that

$$\mu(p_{m_1}p_{m_2}\ldots p_{m_\lambda}) = (-1)^\lambda,$$

where $\mu(1) = 1$ and $\mu(r) = 0$ if the positive integer r is divisible by the square of any of the prime numbers $p_1, p_2, \ldots, p_k$. Therefore, we get

$$\begin{aligned}\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right) &= \sum_{d|n}\frac{\mu(d)}{d}\\ &= \frac{\phi(n)}{n}.\end{aligned}$$

Hence,

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_k}\right).$$

We shall now prove that

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2)\frac{d}{\phi(d)}.$$

From the first part of the theorem, it follows

$$\phi(n_1 n_2) = (n_1 n_2)\prod_{p|n_1 n_2}\left(1-\frac{1}{p}\right).$$

But, if $n_1 n_2 = p_1^{q_1} p_2^{q_2} \ldots p_m^{q_m}$, then each of the prime numbers $p_1, p_2, \ldots, p_m$ appears exactly once in the product

$$\prod_{p|n_1 n_2}\left(1-\frac{1}{p}\right).$$

More specifically, distinct primes p appear in distinct factors

$$1 - \frac{1}{p}.$$

On the other hand, in the product

$$\prod_{p|n_1}\left(1 - \frac{1}{p}\right) \cdot \prod_{p|n_2}\left(1 - \frac{1}{p}\right),$$

the prime numbers from the set $\{p_1, p_2, \ldots, p_m\}$ which divide both n_1 and n_2, appear twice.

Hence, according to the above arguments it is evident that

$$\prod_{p|n_1 n_2}\left(1 - \frac{1}{p}\right) = \frac{\prod_{p|n_1}\left(1 - \frac{1}{p}\right) \cdot \prod_{p|n_2}\left(1 - \frac{1}{p}\right)}{\prod_{\substack{p|n_1 \\ p|n_2}}\left(1 - \frac{1}{p}\right)}.$$

Thus, we have

$$\begin{aligned}
\phi(n_1 n_2) &= \frac{n_1 \prod_{p|n_1}\left(1 - \frac{1}{p}\right) \cdot n_2 \prod_{p|n_2}\left(1 - \frac{1}{p}\right)}{\prod_{\substack{p|n_1 \\ p|n_2}}\left(1 - \frac{1}{p}\right)} \\
&= \frac{\phi(n_1)\phi(n_2)}{\prod_{p|d}\left(1 - \frac{1}{p}\right)} \\
&= \frac{\phi(n_1)\phi(n_2)}{\frac{\phi(d)}{d}} \\
&= \phi(n_1)\phi(n_2)\frac{d}{\phi(d)}.
\end{aligned}$$

Therefore,

$$\phi(n_1 n_2) = \phi(n_1)\phi(n_2)\frac{d}{\phi(d)}. \qquad \square$$

Note. For further reading concerning the Möbius and Euler functions the reader is referred to [38].

2.4 The τ-function

Definition 2.4.1. *The function $\tau(n)$ is defined as the number of positive divisors of a positive integer n, including 1 and n. Equivalently, the function $\tau(n)$ can be defined by the formula*

$$\tau(n) = \sum_{\substack{d|n \\ d\geq 1}} 1.$$

Remark 2.4.2. The function $\tau(n)$ is a *multiplicative function*, since

$$\tau(1) = 1 \ \ \textit{and} \ \ \tau(mn) = \tau(m)\tau(n),$$

for every pair of coprime integers m, n.

This property is very useful for the computation of the number of divisors of large integers.

Theorem 2.4.3. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the canonical form of the positive integer n. Then it holds*

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1).$$

Proof. We shall follow the Mathematical Induction Principle.

For $k = 1$ we have

$$\tau(n) = \tau(p_1^{a_1}).$$

Since the divisors of n, where $n = p_1^{a_1}$, are the positive integers $1, p_1, p_1^2, \ldots, p_1^{a_1}$, it is evident that

$$\tau(n) = a_1 + 1.$$

Let $m = p_1^{a_1} p_2^{a_2} \cdots p_{k-1}^{a_{k-1}}$ and assume that

$$\tau(m) = (a_1 + 1)(a_2 + 1) \cdots (a_{k-1} + 1). \tag{1}$$

In order to determine the divisors of the positive integer n, where

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

it suffices to multiply each divisor of m by the powers of the prime p_k (i.e., $p_k^0, p_k^1, p_k^2, \ldots, p_k^{a_k}$).

Therefore, if d_n and d_m denote the positive divisors of n and m, respectively, then

$$\tau(n) = \sum_{d_n|n} 1 = \sum_{d_m|m} 1 + \sum_{d_m p_k|n} 1 + \sum_{d_m p_k^2|n} 1 + \cdots + \sum_{d_m p_k^{a_k}|n} 1$$

and since the number of divisors d_m is $\tau(m)$, we obtain

$$\tau(n) = \tau(m) + \tau(m) + \tau(m) + \cdots + \tau(m) = \tau(m)(a_k + 1). \tag{2}$$

Hence, by (1) and (2) we obtain

$$\tau(n) = (a_1 + 1)(a_2 + 1) \cdots (a_k + 1),$$

which is the desired result. □

Remark 2.4.4. Generally, for every positive integer n with

$$n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k},$$

it holds

$$\tau(n) = \tau(p_1^{a_1})\tau(p_2^{a_2}) \cdots \tau(p_k^{a_k}).$$

EXAMPLES 2.4.5.

$$\begin{aligned}
\tau(126) &= \tau(2 \cdot 3^2 \cdot 7) = (1+1)(2+1)(1+1) = 12\\
\tau(168) &= \tau(2^3 \cdot 3 \cdot 7) = (3+1)(1+1)(1+1) = 16\\
\tau(560) &= \tau(2^4 \cdot 5 \cdot 7) = (4+1)(1+1)(1+1) = 20\\
\tau(1,376,375) &= \tau(5^3 \cdot 7 \cdot 11^2 \cdot 13) = (3+1)(1+1)(2+1)(1+1) = 48.
\end{aligned}$$

2.5 The generalized σ-function

Definition 2.5.1. *The function $\sigma_a(n)$ is defined as the sum of the a-th powers of the positive divisors of a positive integer n, including 1 and n, where a can be any complex number. Equivalently, the function $\sigma(n)$ can be defined by the formula*

$$\sigma_a(n) = \sum_{\substack{d|n \\ d \geq 1}} d^{\,a},$$

where the sum extends over all positive divisors of n.

Remark 2.5.2. For $k = 0$ we obtain

$$\sigma_0(n) = \tau(n).$$

Remark 2.5.3. The function $\sigma_a(n)$ is a *multiplicative function*, since

$$\sigma_a(1) = 1 \;\; \textit{and} \;\; \sigma_a(mn) = \sigma_a(m)\sigma_a(n),$$

for every pair of coprime integers m, n.

Theorem 2.5.4. *Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the canonical form of the positive integer n. Then*

$$\sigma_1(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}.$$

Proof. We shall follow the Mathematical Induction Principle.

For $k = 1$ we obtain

$$\sigma_1(n) = \sigma_1(p_1^{a_1}).$$

But, since the divisors of n, where $n = p_1^{a_1}$, are the integers $1, p_1, p_1^2, \ldots, p_1^{a_1}$, it is evident that

$$\sigma_1(n) = 1 + p_1 + p_1^2 + \cdots + p_1^{a_1} = \frac{p_1^{a_1+1} - 1}{p_1 - 1}.$$

Now let $m = p_1^{a_1} p_2^{a_2} \cdots p_{k-1}^{a_{k-1}}$ and assume that

$$\sigma_1(m) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_{k-1}^{a_{k-1}+1} - 1}{p_{k-1} - 1}. \tag{1}$$

Similarly to the proof of Theorem 2.2.3 let d_n and d_m denote the positive divisors of n and m, respectively, where $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Then we have

$$\begin{aligned}
\sigma_1(n) &= \sum_{d_n | n} d_n \\
&= \sum_{d_m | m} d_m + \sum_{d_m | m} d_m p_k + \sum_{d_m | m} d_m p_k^2 + \cdots + \sum_{d_m | m} d_m p_k^{a_k} \\
&= 1 \cdot \sum_{d_m | m} d_m + p_k \sum_{d_m | m} d_m + p_k^2 \sum_{d_m | m} d_m + \cdots + p_k^{a_k} \sum_{d_m | m} d_m \\
&= (1 + p_k + p_k^2 + \cdots + p_k^{a_k}) \sum_{d_m | m} d_m.
\end{aligned}$$

Therefore, by the above result and relation (1), we obtain

$$\sigma_1(n) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \cdot \frac{p_2^{a_2+1} - 1}{p_2 - 1} \cdots \frac{p_k^{a_k+1} - 1}{p_k - 1}. \qquad \square$$

Remark 2.5.5. For the function $\sigma_a(n)$, it holds

$$\sigma_a(p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}) = \sigma_a(p_1^{a_1}) \sigma_a(p_2^{a_2}) \cdots \sigma_a(p_k^{a_k}).$$

EXAMPLES 2.5.6.

$$\sigma(126) = \sigma(2 \cdot 3^2 \cdot 7) = \frac{2^2 - 1}{2 - 1} \frac{3^3 - 1}{3 - 1} \frac{7^2 - 1}{7 - 1} = 312$$

$$\sigma(168) = \sigma(2^3 \cdot 3 \cdot 7) = \frac{2^4 - 1}{2 - 1} \frac{3^2 - 1}{3 - 1} \frac{7^2 - 1}{7 - 1} = 480$$

$$\sigma(560) = \sigma(2^4 \cdot 5 \cdot 7) = \frac{2^5 - 1}{2 - 1} \frac{5^2 - 1}{5 - 1} \frac{7^2 - 1}{7 - 1} = 1488.$$

Application. We shall use Theorem 2.5.4 in order to prove that *the number of primes is infinite.*

Proof (George Miliakos). Let us suppose that the number of primes is finite. If $m = p_1^{q_1} p_2^{q_2} \cdots p_k^{q_k}$ and $n = m\,! = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ are the canonical forms of m and n, respectively, then it is evident that $a_1 \geq q_1$, $a_2 \geq q_2$,..., $a_k \geq q_k$. Therefore, by Theorem 2.5.4, we obtain

$$\frac{\sigma(n)}{n} = \frac{p_1 - 1/p_1^{a_1}}{p_1 - 1} \cdot \frac{p_2 - 1/p_2^{a_2}}{p_2 - 1} \cdots \frac{p_k - 1/p_k^{a_k}}{p_k - 1}. \tag{1}$$

But, for $q_1, q_2, \ldots, q_k \to \infty$ it follows that $a_1, a_2, \ldots, a_k \to \infty$. Thus, it follows that $n \to \infty$.

Therefore, by (1) we get

$$\lim_{n\to\infty} \frac{\sigma(n)}{n} = \frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdots \frac{p_k}{p_k - 1}. \tag{2}$$

However, it is clear that

$$\frac{n}{1} + \frac{n}{2} + \frac{n}{3} + \cdots + \frac{n}{m} \leq \sigma(n)$$

and hence

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m} \leq \frac{\sigma(n)}{n}. \tag{3}$$

But, it is a standard fact in mathematical analysis that $\sum_{n=1}^{\infty} 1/n = \infty$. Consequently, for $m \to \infty$, by (2) and (3) we obtain

$$\frac{p_1}{p_1 - 1} \cdot \frac{p_2}{p_2 - 1} \cdots \frac{p_k}{p_k - 1} = \infty,$$

which is obviously a contradiction, since we have assumed that the number of primes is finite. Hence, the number of primes must be infinite. □

Springer

http://www.springer.com/978-1-4419-0495-9

Problem-Solving and Selected Topics in Number Theory
In the Spirit of the Mathematical Olympiads
Rassias, M.T.
2011, XIV, 324 p.,
ISBN: 978-1-4419-0495-9