

Chapter 2

Quantum Algebra*

The ways of gods are mysterious, inscrutable and beyond
the comprehension of ordinary mortals.
Julian Schwinger

Quantum algebra was created by Dirac. Its evolution also bears the imprint of the genius of many great mathematicians and physicists such as Weyl, von Neumann, Schwinger, Moyal, Flato, and others. It has inspired developments in deformation theory, representation theory, quantum groups, and many other mathematical themes.

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2.1 The quantum algebra of Dirac

The quotation above is Julian Schwinger’s tribute to Weyl on the occasion of Weyl’s birth centenary*, but it applies with even greater force to the mysterious way in which Dirac and Heisenberg slew all the dragons of classical physics and let quantum theory emerge. We can understand almost all of their thought processes but there will always be a residue of mystery to the moment of creative genius when things suddenly go to a new level of perception and imagination, and everything falls into its place as if by magic.

The term *quantum algebra* appeared for the first time in a paper of Dirac [1a, 1b] which has now become famous. Just a few months earlier Heisenberg [2]

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had come up with the startling and revolutionary idea that in quantum theory physical observables must be represented by hermitian matrices which are in general of infinite order; and that if the physical observable x is represented by the matrix X , then the observable x^n is represented by the matrix X^n ($n = 1, 2, \dots$), the n^{th} power of X . However Heisenberg had realized that the matrices do not obey the commutative rule of multiplication that the classical observables did, and felt that this was a serious flaw in his scheme. Heisenberg had sent the proof sheets of his article to Fowler at Cambridge; and Fowler, who was at that time the thesis advisor to Dirac, passed them on to Dirac. After a study of Heisenberg's paper Dirac realized that the noncommutativity of the quantum observables, which had appeared to Heisenberg as an unwelcome aspect of the new mechanics, was in fact one of its central features, and led to a structure for the new mechanics which was a beautiful and far-reaching generalization of classical mechanics, and which had the classical mechanics as its limiting case in the correspondence limit when $\hbar \rightarrow 0$. Dirac's great conceptual insight was that the quantum observables belong to an algebra which is noncommutative but in which the *commutator*

$$(1) \quad xy - yx =: [x, y]$$

of two elements x, y , which measures the departure from their commutativity, *corresponds to*

$$i\hbar\{x, y\}$$

where

$$\{x, y\} = \sum_{1 \leq r \leq k} \left(\frac{\partial x}{\partial q_r} \frac{\partial y}{\partial p_r} - \frac{\partial x}{\partial p_r} \frac{\partial y}{\partial q_r} \right)$$

is the classical *Poisson Bracket*:

$$(1a) \quad [x, y] = i\hbar\{x, y\}$$

Implicit here is the assumption that the observables in both the classical and quantum theories are denoted by the same symbols but have different multiplicative structures. It is not clear from Dirac's discussion whether the commutator is *exactly equal to* the Poisson Bracket, although for the position and momentum observables

$$q_1, \dots, q_k, p_1, \dots, p_k$$

Dirac postulated the *exact* commutation rules

$$(2) \quad [p_r, q_s] = -i\hbar\delta_{rs} \quad (1 \leq r, s \leq k)$$

Dirac observed that when $\hbar \rightarrow 0$, the right side goes to 0, so that the quantum algebra becomes commutative in the correspondence limit, and coincides with the classical algebra of physical observables. It will turn out eventually that the general commutator *is not equal to the Poisson Bracket* but *coincides with it up to terms of order \hbar^2* .

It is to be noted that the hermitian nature of the elements of the quantum algebra that represent the quantum observables is only implicit in Dirac's formulation. But the requirement (1a) which shows that the commutator of two quantum observables is not an observable but $i\hbar$ times one, is clearly compatible only with the hermitian condition.

The commutation rules (2), were also independently arrived at by Born, Jordan, and Heisenberg, and Weyl [3, 4, 5]. They give only the *kinematical* structure of the quantum algebra. To determine its *dynamical* structure Dirac proceeded as follows. The time derivatives (at a fixed time for instance) of the observables are maps

$$x \longmapsto x' = \frac{dx}{dt}$$

of the quantum algebra into itself with the following properties: item (i) $(ax + by)' = ax' + by' \quad (a, b \in \mathbb{C})$

$$(ii) \quad (xy)' = xy' + x'y$$

(iii) x' is hermitian whenever x is hermitian i.e., they are *derivations* preserving the hermitian property. Dirac verified that in a *matrix algebra* every derivation preserving the hermitian property is of the form

$$(3) \quad x \longmapsto i[a, x]$$

for some hermitian element a (this requires finite dimensionality of the algebra and in that context it is actually a classical result going back to Skolem and Noether). So Dirac was led to the postulate that the dynamics of the quantum system is governed by an observable H such that

$$(4) \quad x' = \frac{dx}{dt} = i[x, H]$$

for all observables x . He naturally identified H with the observable representing the classical Hamiltonian and postulated that it is the same function of the quantum positions and momenta as the classical Hamiltonian is of the classical position and momentum coordinates. This prescription is of course ambiguous as Dirac himself knew (see §5 of [1a]); the lack of commutativity of q_r and p_r makes it unclear what should be the elements of the quantum algebra that correspond to various polynomial expressions in these variables. For the usual Hamiltonians of the form

$$\sum_{1 \leq r \leq k} c_r p_r^2 + V(q_1, \dots, q_k) \quad (c_r > 0 \text{ constants})$$

there is no ambiguity. Incidentally, this point goes back already to Heisenberg's first paper (see [1b] p. 266).

The above discussion makes it very clear that the transition from classical mechanics to quantum mechanics consists in replacing the classical *real* commutative algebra of real physical quantities by a *complex algebra with involution* $*$ (hereafter called the *quantum observable algebra* or the *quantum algebra*) whose real(=self adjoint) elements($x = x^*$) are identified with the physical quantities in the quantum theory. In his preface to the first German edition of his book [6] Hermann Weyl wrote as follows:

... it can justly be maintained that the essence of the new Heisenberg–Schrödinger–Dirac quantum mechanics is to be found in the fact that there is associated with each physical system a set of quantities, constituting a (complex) noncommutative algebra in the technical mathematical sense, the (real) elements of which are the physical quantities themselves (parentheses added).

It is to be noticed that in this initial formulation of Dirac there is no mention of states, Hilbert space, and so on. These would come later and would become part of the definitive framework for quantum mechanics that is due to von Neumann, in which states are identified with the unit vectors (up to a phase) of a complex Hilbert space, observables are the self adjoint operators on this Hilbert space, and the expectation value of an observable A in a state ψ is $(A\psi, \psi)$. If we restrict ourselves to *bounded* observables, then they are the self adjoint elements of the *complex algebra* of all bounded operators of the Hilbert space, thus furnishing what we might call the conventional model for the quantum algebra of Dirac. In illustrating the theory it is better to assume that the Hilbert space is even finite dimensional, as I shall do very often.

The Dirac–Heisenberg insight raised the following absolutely fundamental questions.

1. How can one make more understandable the transition from a commutative real algebra to a complex algebra with involution? Note that only the commutator in the quantum algebra has a physical meaning (of sorts); the multiplication is left hanging, so to speak.
2. How to construct quantum algebras?
3. The quantum algebra of Dirac's theory contains a quantity \hbar . For calculations it is identified with Planck's constant but in order to compare quantum results with classical results it is treated as a *parameter*, and the correspondence principle of Bohr is formulated as the requirement that when $\hbar \rightarrow 0$ the quantum algebra goes over to the classical algebra in the limit. Is there a way to formulate this idea mathematically?
4. To what extent is the quantum algebra uniquely determined?

In what follows we shall explore these questions and try to come up with answers that are currently accepted almost universally.

2.2 The von Neumann perspective

When quantum mechanics was discovered, the break with classical mechanics was so complete, and involved such strange and unexpected relationships between what were thought of as well understood notions like states and observables, that it was extremely difficult to accept it on any other ground except that it worked. If we accept the complex algebra with involution and identify its real elements with the quantum observables, then by general results from functional analysis the algebra can be realized as an operator algebra in a Hilbert space on which the quantum observables act as bounded self adjoint operators, so that one is very close to the usual model. But how to bridge the gap from the classical to the quantum algebra?

It was von Neumann who made the truly profound investigations that finally removed much of the mystery behind this extraordinary leap from classical to quantum mechanics. This is not the place to go into von Neumann's ideas in any detail; they are discussed with his inimitable clarity and succinctness in his wonderful book [7] which is, in my opinion, one of the great landmarks of twentieth century science. They have been discussed in many places [8, 9, 10]. Let me therefore confine myself to a brief outline.

The starting point of von Neumann is that associated to any physical system there is a structure, the set \mathcal{L} of all *experimentally verifiable propositions*.

This set is partially ordered, namely, that $a \leq b$ if a implies b . There is also an involution $a \mapsto a^\perp$ which associates to a its negation. There are also some additional relations between these two concepts that flow directly out of the physical meaning of these concepts; I shall not consider these here because they are technical. The partially ordered set \mathcal{L} was called by von Neumann the *logic of the system*. He observed that if the system considered is *classical* then \mathcal{L} is a *boolean algebra* obeying the rules of *classical logic*. In von Neumann's approach the fundamental fact that distinguishes quantum systems from the classical ones is that the logic \mathcal{L} of a quantum system is *not a boolean algebra*. For technical reasons we shall work with Boolean σ -algebras, which are Boolean algebras where sums of countably many elements exist in the algebra. As long as observations are being made on a single observable or a set of simultaneously measurable observables, one operates within a single Boolean σ -algebra $\mathcal{B} \subset \mathcal{L}$. However a quantum logic \mathcal{L} contains many different Boolean σ -algebras which cannot all be put inside a single Boolean σ -algebra. Examples of this are the observables that refer to the wave and particle aspects of light or electrons, more generally observables which are complementary like the position and the linear momenta. One may say that quantum logics are composed of *interlocking* Boolean σ -algebras, and the manner in which the Boolean σ -algebras combine to generate \mathcal{L} contains the very essence of the *complementary* properties of the quantum system.

Clearly, in trying to understand the transition to quantum logics it will be important, even essential, to try to get a view of all possible logics and select those that are promising enough to serve as models for quantum phenomena. To do this satisfactorily one has to make additional assumptions that may or may not have good physical interpretation. Of course the additional assumptions have to be such that they will encompass many models that are useful in quantum theory. One of the simplest classes of logics is obtained if one is prepared to admit that the logic is isomorphic to a *classical projective geometry*, namely, the partially ordered set of linear subspaces of a finite dimensional vector space over some division ring. The fundamental theorem of classical projective geometry is the statement that under very general circumstances a finite dimensional logic is of this form. The vector space necessarily comes equipped with a definite sesquilinear scalar product such that \perp in the logic corresponds to orthocomplementation in the geometry (this is a theorem of Birkhoff and von Neumann, see [8, 12]. If the division ring is the field \mathbf{C} of complex numbers, we obtain (after some technical arguments) the conven-

tional model. To get infinite dimensional models one has to drop the finiteness condition and use infinite dimensional analogues of the classical theorem of projective geometry. There are many results and this is not the place to go into them; see however the discussions in [8, 9]. I would like to mention the following: my result on geometries of infinite dimension but with atoms which can be identified with the geometry of *finite dimensional* subspaces of an infinite dimensional vector space over a division ring; the theorem of von Neumann that a modular complemented lattice of rank ≥ 4 is isomorphic to the lattice of principal left ideals of a matrix ring over a regular ring (whose definite involutions correspond to the orthocomplementations of the lattice); as well as the theorem of Piron–Amemiya–Araki on the characterization of the standard Hilbertian logic [8, 9].

These results clearly show that the usual models of quantum theory where this vector space is a complex Hilbert space with its scalar product are not unreasonable at all. The road to treat the conventional model in all its details is now wide open; the full story is well known but technically nontrivial. In the context of Hilbert space it requires the Gleason theorem to identify the (pure) states of the system with the points of the *projective space* of the Hilbert space, so that the superposition principle of states that is emphasized by Dirac to be the corner stone of quantum theory is nothing other than the fact that the states correspond bijectively to the points of the projective space of the Hilbert space, and that a state σ is in superposition with two others σ_i ($i = 1, 2$) if and only if the corresponding point p_σ is on the line in the projective space that joins p_{σ_1} and p_{σ_2} . The linear structure of the symmetries in quantum theory is a direct consequence of the classical theorem in projective geometry that all automorphisms of a projective geometry are induced by linear or semilinear transformations of the underlying vector space. This contains the theorem of Wigner that quantum symmetries are induced by unitary or antiunitary maps of the Hilbert space (see [13] where this is proved by elementary arguments without appealing to the fundamental theorems of classical projective geometry). An observable x is identified with the map $E \mapsto x(E)$ where $x(E)$ is the experimental proposition that the value of x is in the set E , thus, a σ -homomorphism from the σ -algebra of borel subsets of the real line into the logic. In the Hilbert space context, one can use the spectral theorems of Hilbert and von Neumann to identify these in turn with self adjoint operators on the Hilbert space. In particular, the bounded observables are the self adjoint elements of the complex algebra of all bounded operators of the Hilbert

space whose involution is the adjoint operation, thus finally coming to the conventional model of the quantum algebra of Dirac–Heisenberg.

In very general mathematical terms, there are three objects: the vector space over \mathbb{C} with a Hilbertian structure, the orthogonal projections in this vector space, and the algebra of operators on this vector space. Classical geometry shows that these structures are all equivalent: from the algebra one can derive the projections as generators of left ideals of the algebra. I personally believe that this is one of the the most direct routes to the mystery of the quantum algebra.

The simplest model of a quantum algebra is the algebra \mathcal{A} of linear operators of a N -dimensional complex Hilbert space \mathcal{H} with the adjoint † as the involution. The quantum observables are then the self adjoint operators of \mathcal{H} , and the values of the observables are the eigenvalues of these self adjoint operators. Every observable has at most N values, and generically, has exactly N distinct values. We can thus think of this algebra as the *quantization* of the classical algebra of functions on a set having N elements. If A is an observable, then by the spectral theorem, if a_i are its distinct eigenvalues and P_i are the orthogonal projections on the eigenspace corresponding to the a_i , the observable has the representation

$$A = \sum_i a_i P_i$$

The case $N = 2$ is especially important as it is the simplest. It arises when one is interested in studying *spin* or *polarization* experiments. The spin refers to the magnetic moment and historically goes back to the first experiments performed by Stern and Gerlach; we shall come to this aspect a little later when we discuss Schwinger’s extremely original approach to the mathematical structure of the observable algebra of quantum kinematics.

In summary, the von Neumann analysis shows that under very general mathematical conditions there are essentially only two models for the set of observables of *finite* physical systems: the classical one of real functions on a set of N elements, or the quantum one of self adjoint elements of the complex algebra with involution of all operators on an N -dimensional Hilbert space, with the adjoint as the involution. It must be noted however that neither the complex (i.e., not self adjoint) elements nor their product in this complex algebra have any physical interpretation.

In my opinion it was only when Schwinger re-examined the foundations of the theory of measurement in the 1960s that a convincing physical interpretation of the full *complex* quantum algebra and its involution was discovered. It turns out that the complex algebra is actually the *algebra of quantum measurements*. Indeed, one starts by introducing *measurement symbols* for the acts of measurements. The product of the symbols corresponds to successive measurements, and the involution to the performance of the measurements in the reverse order. The symbols have certain natural relations among themselves, and so one can introduce the *measurement algebra* as the algebra generated by the measurement symbols with these relations. Under suitable restrictions this algebra is in fact the quantum algebra.

Finally, let me remark that there is the question that has always aroused great interest in the foundations of quantum theory, namely, to understand the role of the *complex* numbers. Since all physical quantities are real why should one work over the complex numbers? There is no definitive answer to this question, except that in almost all parts of the theory it is impossible not to work over \mathbf{C} . Things like matter–antimatter duality, charge conjugation and so on, cannot be discussed without the availability of complex conjugation. Also, even if one starts over the real numbers one can always complexify, and so real systems can always be thought of as complex systems that commute with a conjugation operator. There has been nevertheless some interest in pursuing this matter. See for instance [18], pp. 194–198, and the references therein.

2.3 The measurement algebra of Schwinger

The point of view of Schwinger is most beautifully expressed in his own words [14] (pp. 1–2):

The classical theory of measurement is implicitly based upon the concept of an interaction between the system of interest and the measuring apparatus that can be made arbitrarily small, or at least precisely compensated, so that one can speak meaningfully of an idealized experiment that disturbs no property of the system. The classical representation of physical quantities by numbers is the identification of all properties with the results of such nondisturbing measurements. It is characteristic of atomic phenomena, however, that the interaction between system and instrument cannot be indefinitely weakened. Nor can the disturbance produced by the interaction be compensated precisely since it is only statistically predictable. Accordingly, a measurement on one property can produce unavoidable changes in the value previously assigned to another property, and it is without meaning to ascribe numerical values to all the attributes of a microscopic system. The mathe-

mathematical language that is appropriate to the atomic domain is found in the symbolic transcription of the laws of microscopic measurement.

The starting point of Schwinger's analysis was the famous Stern–Gerlach (SG) experiment on the magnetic moment of atoms. Let us recall that the experiment consisted in heating small pellets of silver to a very high temperature in a closed oven so that the pellets vaporize. Then the vapor is drawn out through an aperture in the oven and made into a thin beam by passing through a number of narrow slits. The high temperature of the vapor guarantees that the magnetic moments of the silver atoms in the vapor are completely randomly distributed. The atoms are then forced to pass through a magnetic field and then stopped by a screen. One would expect the atoms to be deflected differentially and in a continuous manner depending on the values of their magnetic moments, and so it would be natural to expect to get a uniform spread of deposits. Instead, what one finds are two dark spots, sharply localized, and nothing more. One is forced to the interpretation that the apparatus measures whether the magnetic moment is either “up” or “down” (these being defined as the direction of the magnetic field or its opposite), and that the disturbance caused by the apparatus has forced the atoms to split into two ensembles whose magnetic moments are either up or down.

Schwinger's idea was to build a theory of quantum kinematics from a generalization of this example. For the SG experiments the observed results are either $+1$ or -1 , referring to the magnetic moment being either in the direction of the magnetic field or opposite to it. The directions of the magnetic fields are of course completely arbitrary, and one can create a measurement scheme where the atomic ensemble has all its atoms with their magnetic moments aligned one way and then pass through other measurement apparatus whose magnetic fields are at arbitrary angles to this direction of magnetic moment; furthermore, one can repeat such measurements many times in succession. The statistical behavior of the atoms then lead to observable effects which completely describe the kinematics of this system.

To build the structure of the kinematics of general (finite) quantum systems Schwinger considers^{14,15,16} a quantum system analogous to the spin system of SG experiments where a generic observable has N distinct values (for the actual SG case $N = 2$). Following Schwinger we consider a class of *generalized Stern–Gerlach experiments* in which one makes measurements on the system. Typically, a measurement scheme receives ensembles and measures the values of an observable A and then sorts them out according to the values of A . Various types of measurements can now be considered. If A is

an observable with distinct values a_1, a_2, \dots, a_N , then one can conceive of an apparatus that allows only ensembles with the A -value a_j to enter and permits only those ensembles to emerge which have the A -value a_i ; let us tentatively assign the symbol $M(a_i, a_j)$ for this measurement scheme, a special case of which is the symbol $M(a_i, a_i)$ for the measurement that permits only those with A -value a_i to enter and allows these to emerge. More generally, let A, B be two observables, in general incompatible, with values $(a_i), (b_j)$ respectively; then $M(a_i, b_j)$ is the tentative symbol for a measurement scheme that permits only the ensembles with the B -value b_j to enter and allows only those with the A -value a_i to emerge. When such measurements are carried out in succession, say M_1, M_2, \dots, M_r , the composite measurement is denoted by $M_r M_{r-1} \dots M_2 M_1$, the order of measurement being from right to left. The measurement symbols therefore form a monoid with a multiplication, and admits a natural involution † defined by

$$M(a, b)^\dagger = M(b, a), \quad (M_r M_{r-1} \dots M_2 M_1)^\dagger = M_1^\dagger M_2^\dagger \dots M_r^\dagger$$

To construct the full quantum algebra one has to proceed to specify the rule for multiplying these measurement symbols. Schwinger does this and discovers that the measurement algebra, namely the algebra generated by the measurement symbols, is canonically isomorphic to the algebra of all operators in \mathcal{H} in such a manner that the observables can be identified with the self adjoint elements and the involution with the adjoint.

I have said that the symbols $M(a, b)$ are only tentative. Before defining them formally let me follow Schwinger and discuss them a little more. Let us first consider the case when there is only one observable and consequently only the symbols

$$M(a_i, a_j) \quad (1 \leq i, j \leq N)$$

From the physical interpretation it is clear that we should have

$$M(a_i, a_j)M(a_k, a_m) = \delta(j, k)M(a_i, a_m), \quad M(a_i, a_j)^\dagger = M(a_j, a_i)$$

The relations

$$M(a_i, a_i)^2 = M(a_i, a_i), \quad M(a_i, a_i) = M(a_i, a_i)^\dagger$$

which are special cases of the above relations are essentially a formulation of the obvious fact that if we pass an ensemble through the measurement apparatus symbolized by $M(a_i, a_i)$ twice we get the same result as by doing it

just once. Analogous interpretations lead to the more general rules involving all the $M(a_i, a_j)$. The symbols 1 and 0 represent measurement schemes that permit all ensembles to go through without any selection and schemes that permit no ensembles to enter respectively. A measurement scheme that permits ensembles only if their A -values lie in a subset E of the set $\{a_1, \dots, a_N\}$ is denoted by

$$\sum_{i \in E} M(a_i, a_i)$$

If we take E to be the entire set of values of A we should get

$$\sum_i M(a_i, a_i) = 1.$$

Let us now consider measurements involving two observables. The sequence

$$M(a_i, a_i)M(b_j, b_j)M(a_i, a_i)$$

is particularly interesting. It is just the a_i -measurement except for the interposition of the b_j -measurement. From considerations of actual SG experiments one expects only a *positive fraction* of the ensembles to emerge from this scheme; if $p(a_i, b_j)$ is this fraction, Schwinger postulates that

$$M(a_i, a_i)M(b_j, b_j)M(a_i, a_i) = p(a_i, b_j)M(a_i, a_i)$$

and interprets $p(a_i, b_j)$ as the probability of finding an individual member of the ensemble in the state a_i when we know before the measurement that it was in state b_j . The matrix $(p(a_i, b_j))$ is *stochastic*, namely,

$$p(a_i, b_j) \geq 0, \quad \sum_i p(a_i, b_j) = 1.$$

It is at this stage that Schwinger takes a bold step. He does not explain this in [14] or [15] but refers to it as a “leap of imagination” in [16]. He thinks of $M(a_i, a_j)$ as a 2-step process: first the entering a_j -ensemble is annihilated and then the emerging a_i -ensemble is created in its place. So he writes $M(a_i, a_j)$ as a *product* of two new symbols $\langle a_j|$ and $|a_i\rangle$:

$$M(a_i, a_j) = |a_i\rangle\langle a_j|$$

where $\langle a_j|$ refers to the annihilation of the state where A is a_j and $|a_i\rangle$ refers to the creation of the state where A is a_i . The act of measuring A then leads

to the plausible interpretations of the new symbols as follows:

$$\langle a_i | a_j \rangle = \delta(i, j), \quad A | a_j \rangle = a_j | a_j \rangle \quad \langle a_i | A = a_i \langle a_i |$$

It is clear now that $|a_i\rangle$ and $\langle a_j|$ have to be interpreted as *vectors*, and indeed as *dual sets of vectors*. Actually, to prepare the reader for this leap, he writes $|a, b\rangle$ in place of $M(a, b)$ and the leap consists in splitting the symbol as the product $||\rangle\langle|$. The observable A then gets identified as

$$A = \sum_i a_i |a_i\rangle \langle a_i|$$

and we have, for any polynomial function $f(A)$, the same formula

$$f(A) = \sum_i f(a_i) |a_i\rangle \langle a_i|$$

Proceeding formally, the definition of multiplication of the symbols when two incompatible observables A and B are involved takes the form

$$M(a_i, a_j) M(b_k, b_m) = |a_i\rangle \langle a_j| |b_k\rangle \langle b_m| = \langle a_j | b_k \rangle |a_i\rangle \langle b_m|$$

An actual calculation with the multiplication rule defined above shows that

$$p(a_i, b_j) = \langle a_i | b_j \rangle \langle b_j | a_i \rangle.$$

Since probabilities are to be nonnegative, it is natural to ensure this by requiring that

$$\langle a_i | b_j \rangle = \langle b_j | a_i \rangle^{\text{conj}}.$$

Notice now that $p(a, b)$ becomes *symmetric* in a and b , and hence that the matrix $(p(a_i, b_j))$ becomes *doubly stochastic*, i.e.,

$$p(a_i, b_j) = p(b_j, a_i), \quad \sum_i p(a_i, b_j) = \sum_j p(a_i, b_j) = 1.$$

From this point on the treatment is essentially formal and Schwinger reaches his results mentioned above.

One should of course study algebras with symbols such as the $M(a, b)$. We shall do this presently. But before going in this direction it may be of interest to exhibit the measurement symbols in the conventional model. *The measurements symbols will be parametrized by the unit vectors rather than*

the states themselves. The most general measurement symbol is

$$M(\psi, \theta)$$

where ψ, θ are unit vectors in \mathcal{H} . It represents the measurement where only ensembles in the state defined by θ are admitted and only those in the state defined by ψ emerge. For brevity we write $[\theta]$ for the state defined by the unit vector θ . The adjoint is given by

$$(1) \quad M(\psi, \theta)^\dagger = M(\theta, \psi).$$

To take care of the redundancy of phase factors in defining the states we require

$$(2) \quad M(\lambda\psi, \mu\theta) = \lambda\bar{\mu}M(\psi, \theta) \quad (|\lambda| = |\mu| = 1).$$

The multiplication is given by

$$(3) \quad M(\psi, \theta)M(\rho, \tau) = (\theta, \rho)M(\psi, \tau).$$

Finally if (ψ_i) is an orthonormal frame,

$$(4) \quad \sum_i M(\psi_i, \psi_i) = 1.$$

The Schwinger measurement algebra is the complex algebra \mathcal{M} generated by the symbols $M(\psi, \theta)$ with the relations (2),(3) (4). It is clear that this algebra has an involutive conjugate-linear antiautomorphism satisfying (1). We write this involution as \dagger .

What is the structure of this algebra? We first construct a homomorphism of \mathcal{M} into the algebra $L(\mathcal{H})$ of all linear operators of \mathcal{H} . Denote by

$$L_{\psi \otimes \theta} \quad (\psi, \theta \in \mathcal{H}, \|\psi\| = \|\theta\| = 1)$$

the linear operator

$$L_{\psi \otimes \theta} : v \longmapsto (\theta, v)\psi.$$

We must remember here our physicists' convention regarding the scalar product (\cdot) . Then the homomorphism is given by

$$M(\psi, \theta) \longmapsto L_{\psi \otimes \theta}.$$

To verify that this defines a homomorphism of \mathcal{M} into $L(\mathcal{H})$ one must check that the $L_{\psi \otimes \theta}$ satisfy the relations (1)–(4) where we write $L_{\psi \otimes \theta}$ in place of $M(\psi, \theta)$; this however is trivial. Thus we have a well defined homomorphism of \mathcal{M} into $L(\mathcal{H})$ that takes \dagger into the adjoint. It is obvious that this is surjective. The theorem of Schwinger is that it is injective also so that it is a canonical isomorphism.

To prove the injectivity it is enough to prove that $\dim(\mathcal{M}) \leq N^2$ since by the surjectivity we know that the dimension has to be $\geq N^2$. For this purpose we fix an orthonormal frame (ψ_i) in the Hilbert space \mathcal{H} . To the frame (ψ_i) we associate the N^2 symbols

$$M(\psi_i, \psi_j) \quad (1 \leq i, j \leq N).$$

We now have, for any unit vectors θ, τ in \mathcal{H} ,

$$\begin{aligned} M(\theta, \tau) &= \sum_i M(\psi_i, \psi_i) M(\theta, \tau) \sum_j M(\psi_j, \psi_j) \\ &= \sum_{i,j} (\tau, \psi_j)(\psi_i, \theta) M(\psi_i, \psi_j) \end{aligned}$$

which shows that all the $M(\theta, \tau)$ are in the linear span of the $M(\psi_i, \psi_j)$. On the other hand the rule for multiplication for the symbols, namely,

$$M(\psi_i, \psi_j) M(\psi_k, \psi_m) = \delta(j, k) M(\psi_i, \psi_m)$$

shows that this linear span is a *subalgebra* of \mathcal{M} . Thus

$$\mathcal{M} = \sum_{i,j} \mathbf{C} \cdot M(\psi_i, \psi_j)$$

showing that

$$\dim(\mathcal{M}) \leq N^2.$$

This finishes the proof. We have thus obtained

Theorem (Schwinger). *The measurement algebra is naturally isomorphic to $L(\mathcal{H})$ via the map*

$$M(\psi, \theta) \longmapsto L_{\psi \otimes \theta}$$

In this isomorphism \dagger goes over to the adjoint.

One should note that the canonical isomorphism

$$M(\psi, \theta) \longmapsto L_{\psi \otimes \theta}$$

is essentially the “leap of imagination” of Schwinger that identifies $M(\psi, \theta)$ with $\psi \otimes \theta$ interpreted as an element of $L(\mathcal{H})$, namely $L_{\psi \otimes \theta}$. One should also note that the measurement algebra is spanned by the N^2 symbols associated to *any* orthonormal frame. We have thus completed our program of giving a full physical interpretation of the complex quantum algebra and its involution.

The above discussion does not go very much beyond what Schwinger did. Accardi carried out a much more penetrating analysis in [17]. I shall now give a brief description of some of Accardi’s ideas and results.

The work of Accardi

The starting point for Accardi is a not necessarily finite dimensional \ast -algebra \mathbf{A} over the reals \mathbf{R} with center \mathbf{K} , and certain additional axioms. If \mathbf{A} is such an algebra, a *projection* in \mathbf{A} is an element X such that $X = X^* = X^2$. A *partition of unity* is a family (A_a) of projections such that $A_a \neq 0$ and $\sum_a A_a = 1$; such a family is *maximal* if the linear span of the A_a (which is an abelian subalgebra) is maximal in \mathbf{A} , namely, if $B \in \mathbf{A}$, $BA_a = A_a B$ for all a implies that $B = \sum c_a A_a$ for suitable $c_a \in \mathbf{K}$. \mathbf{A} is a *Schwinger algebra* if there is an integer n and a set T of maximal partitions of unity,

$$(A(x)) = (A_a(x))_{1 \leq a \leq n} \quad (x \in T)$$

such that \mathbf{A} is generated by the $A_a(x) (1 \leq a \leq n, x \in T)$ over \mathbf{K} . The integer n occurring in the definitions could be ∞ . \mathbf{A} is a *Heisenberg algebra* if, first of all, it is a Schwinger algebra with $|T| = 2$, and, writing the generating set as $(A_a), (B_b)$, the $A_a B_b$ is a basis for \mathbf{A} over \mathbf{K} . One can then show from the axioms that in a Schwinger algebra \mathbf{A} we have

$$A_a(x) B_b(y) A_a(x) = p_{ab}(x, y) A_a(x) \quad (1 \leq a, b \leq n, x, y \in T),$$

where $p_{ab}(x, y) \in \mathbf{K}$, $p_{ab}(x, y) \geq 0$ with

$$p_{ab}(x, y) = p_{ba}(y, x), \quad \sum_a p_{ab}(x, y) = \sum_b p_{ab}(x, y) = 1$$

so that the matrix $P(x, y) = (p_{ab}(x, y))$ is *doubly stochastic*. It is called the *transition matrix* between $A(x)$ and $B(y)$. The additional axioms I mentioned

earlier are needed to guarantee the positivity of the $p_{ab}(x, y)$. Here positive elements of \mathbf{A} are those which are sums of elements of the form A^*A .

Accardi's main concerns are the following: (a) given a doubly stochastic matrix with entries in \mathbf{K} , when does there exist a Heisenberg algebra with this transition matrix? (b) given a family of transition matrices $P(x, y)_{x, y \in T}$, when does there exist a Schwinger algebra with these transition matrices? (c) How do the conventional Hilbert space models fit into this context? For (a) he finds, under suitable assumptions on \mathbf{K} and the genericity condition that $p_{ab} > 0$ for all a, b , that the necessary and sufficient condition is that there is a unitary \mathbf{K} -valued matrix $U = (u_{ab})$ such that $p_{ab} = u_{ab}^* u_{ab} = |u_{ab}|^2$ for all a, b . For (b) his condition (always assuming that the transition matrices are generic in the above sense) is that one should be able to choose unitary $U(x, y)$ generating the $P(x, y)$ such that for all triples $x, y, z \in T$, we have (what he calls the Schrödinger equations) $U(x, z) = U(x, y)U(y, z)$. The Hilbert space models are then constructed assuming that $\mathbf{K} = \mathbf{C}$, from these unitary matrices. The beautiful aspect of Accardi's theory is that the *probability amplitude* u_{ab} *appears in a natural way, and transition probabilities are the absolute squares* $|u_{ab}|^2$ *of these amplitudes, a philosophy that is the corner stone of physics from Dirac to Schwinger to Feynman.*

We shall now outline another approach to obtain at least some of the results of Accardi. We shall work from the very beginning with *complex* algebras equipped with an involution $*$ ($*$ -algebras). The involution is conjugate linear and reverses the order of multiplication. Let then \mathbf{A} be a *finite dimensional* $*$ -algebra of dimension N . We denote by ℓ (resp. r) the left (resp. right) regular representation of \mathbf{A} ; it is faithful. Let

$$t(x) = \text{Tr } \ell(x).$$

Then

$$t(xy) = t(yx)$$

and

$$\langle x, y \rangle = t(xy^*)$$

defines a hermitian form on \mathbf{A} . Let \mathbf{A} be a Heisenberg algebra with respect to two maximal partitions of unity, $(A_a), (B_b)$ generating \mathbf{A} over \mathbf{C} . Then, following Accardi, we have the *structure constants* γ_{ab}^{cd} of the algebra \mathbf{A} , defined by

$$B_b A_a = \sum_{cd} \gamma_{ab}^{cd} A_c B_d$$

where a, b, c, d etc go from 1 to n . From this formula we get

$$A_a B_b A_a = \sum_d \gamma_{ab}^{ad} A_a B_d$$

so that, since $\sum_d B_d = 1$, we have

$$\sum_d p_{ab} A_a B_d = \sum_d \gamma_{ab}^{ad} A_a B_d$$

giving

$$\gamma_{ab}^{ad} = p_{ab} \quad (1 \leq d \leq n).$$

We shall now compute explicitly $\langle x, x \rangle = t(x^* x)$ for

$$x = \sum_{ab} x_{ab} A_a B_b.$$

Then

$$x^* x = \sum_{abcd} \bar{x}_{ab} x_{cd} B_b A_a A_c B_d = \sum_{abd} \bar{x}_{ab} x_{ad} B_b A_a B_d.$$

Hence, as $t(B_b A_a B_d) = t(A_a B_d B_b) = \delta_{bd} t(B_b A_a)$, we have

$$t(x^* x) = \sum_{abd} \bar{x}_{ab} x_{ad} t(B_b A_a B_d) = \sum_{ab} |x_{ab}|^2 t(B_b A_a).$$

To calculate $t(B_b A_a)$ we note that $\ell(B_b A_a)$ kills $A_c B_d$ if $c \neq a$, and

$$\ell(B_b A_a)(A_a B_d) = B_b A_a B_d = \sum_{uv} \gamma_{ab}^{uv} A_u B_v B_d = \sum_u \gamma_{ab}^{ud} A_u B_d.$$

Hence

$$t(B_b A_a) = \sum_d \gamma_{ab}^{ad} = n p_{ab}.$$

Thus

$$t(x^* x) = n \sum_{ab} p_{ab} |x_{ab}|^2, \quad x = \sum_{ab} x_{ab} A_a B_b.$$

So the Schwinger-Accardi condition

$$p_{ab} \geq 0$$

is equivalent to the invariant condition that $\langle x, x \rangle$ is nonnegative definite. The genericity condition that

$$p_{ab} > 0$$

is equivalent to the invariant condition that $\langle x, x \rangle$ is *positive definite*.

We thus define a *premeasurement algebra* as a finite dimensional $*$ -algebra whose natural hermitian form is nonnegative. A *$*$ -representation* of \mathbf{A} is a representation ρ of \mathbf{A} in a finite dimensional Hilbert space such that $\rho(x^*) = \rho(x)^*$ for all $x \in \mathbf{A}$. A *measurement algebra* \mathbf{A} is a premeasurement algebra whose natural hermitian form is strictly positive definite. Our aim now is to show that measurement algebras are nothing more than direct sums of full matrix algebras. The results follow simply from the classical theory of finite dimensional algebras (Wedderburn) and so we will be brief.

Suppose that \mathbf{A} is a measurement algebra. From $t(xy^*) = t(y^*x)$ we get $\langle x, y \rangle = \langle y^*, x^* \rangle$, and from $t(xyz) = t(yzx)$ we get

$$\langle ax, y \rangle = \langle x, a^*y \rangle, \quad \langle xa, y \rangle = \langle x, ya^* \rangle.$$

In other words, if we view \mathbf{A} as a Hilbert space with scalar product $\langle x, y \rangle$, both the left and the right regular representations are $*$ -representations. If M is an ideal (resp. left, right, two-sided), so in M^\perp . Moreover, the action of \mathbf{A} on M (resp. left, right, or two-sided) is a $*$ -representation. So we can write \mathbf{A} as a direct sum of minimal two-sided ideals M_i .

Let M be a two-sided ideal of \mathbf{A} . Then M^\perp is also a two-sided ideal and $\mathbf{A} = M \oplus M^\perp$. Let P be the orthogonal projection from \mathbf{A} to M and let $P1 = e \in M$. Since P commutes with left and right multiplications, we have $P = \ell(e) = r(e)$; moreover, as $P = P^* = P^2$ we see that e is in the center of \mathbf{A} and $e = e^* = e^2$. If now $x \in M$, we have $x = ex$ so that, taking adjoints, $x^* = x^*e$, showing that $x^* \in M$. Hence M is a $*$ -ideal, i.e., it is closed under $*$. In particular we may regard M as a nondegenerate $*$ -algebra with e as its unit element.

Consider the orthogonal decomposition $\mathbf{A} = \oplus_i M_i$ into minimal two-sided ideals. Let us fix i and let us take a *minimal* left ideal $N \subset M_i$. Then ℓ acts on N as an irreducible $*$ -representation. Hence, ℓ is a $*$ -homomorphism of \mathbf{A} onto the $*$ -algebra $\text{End}(N)$. Now for $x \in M_i$, $y \in M_i^\perp$, xy lies in $M_i \cap M_i^\perp$ so that $xy = 0$. Hence M_i^\perp is contained in the kernel K of ℓ acting on N . But $K \neq \mathbf{A}$ and so, as M_i is minimal, $K \cap M_i = 0$, showing that $K = M_i^\perp$. We may thus regard the left regular representation of M_i acting on N as a $*$ -isomorphism of M_i with $\text{End}(N)$. We thus see that a measurement algebra is a direct sum of matrix $*$ -algebras. The converse is trivial. If the algebra is *simple*, or if it is central, i.e., its center is \mathbf{C} , then it is a full matrix $*$ -algebra.

Since it is obvious that that direct sums of full matrix algebras are Schwinger algebras, we have justified our definition of measurement algebras.

The generic Heisenberg algebras of Accardi are simple measurement algebras. If M is a nonzero two-sided ideal and contains the element $x = \sum_{ab} x_{ab} A_a B_b$ where $x_{cd} \neq 0$, then $A_c x B_d = x_{cd} A_c B_d \in M$ so that $A_c B_d \in M$. Hence $A_c B_d A_c = p_{cd} A_c \in M$, showing that $A_c \in M$, hence $B_b A_c B_b \in M$, hence finally that $B_b \in M$. Summing over b we get $1 \in M$. Hence for these $*$ -algebras we have a *Hilbert space model* for \mathbf{A} .

Suppose \mathbf{A} is a full matrix $*$ -algebra. If (A_a) is a maximal partition of unity, we have an ON basis of M , say (e_a) such that A_a are the orthogonal projection $M \rightarrow \mathbb{C}e_a$. If we have a family $(A_a(x))_{x \in T}$, we have a family of ON bases $(e_a(x))_{x \in T}$ and it is clear that $p_{ab}(x, y) = |u_{ab}(x, y)|^2$ where $U(x, y) = (u_{ab}(x, y))$ is unitary with $u_{ab}(x, y) = \langle e_a(x), e_b(y) \rangle$. Then

$$U(x, x) = I, U(x, y)U(y, z) = U(x, z) \quad (x, y, z \in T).$$

Conversely, suppose we are given a family $(U(x, y))_{x, y \in T}$ of unitary matrices satisfying these relations. We select an n -dimensional Hilbert space M and an ON basis $(e_a(x_0))$ of it for a fixed $x_0 \in T$ and define

$$e_a(x) = U(x_0, x)^{-1} e_a(x_0).$$

As $U(x, y) = U(x_0, x)^{-1} U(x_0, y)$ we find that $U(x, y)e_a(y) = e_a(x)$ and $p_{ab}(x, y) = |\langle e_a(x), e_b(y) \rangle|^2$. This completes the treatment of Accardi's theory in the finite dimensional nondegenerate case.

If \mathbf{A} is only a premeasurement algebra, $\langle x, x \rangle$ is only nonnegative. The null vectors form a two-sided ideal \mathbf{N} and \mathbf{A}/\mathbf{N} is a measurement algebra. The above analysis applies to it.

It may be of interest to extend this analysis to the infinite dimensional context that Accardi considers and compare his treatment to the limits of finite dimensional measurement algebras that Schwinger treats in his papers¹⁴.

Let us now conclude by discussing briefly a class of finite models. They are constructed out of finite abelian groups and go back to Weyl and Schwinger^{6,19}. Let G be a finite abelian group and let \widehat{G} be its dual group. The Hilbert space of the quantum system is $\mathcal{H} = L^2(G)$, the space of all complex functions on G with the scalar product

$$i(f, g) = \frac{1}{|G|} \sum_{x \in G} f(x) \overline{g(x)}$$

Then there are two sets of orthonormal bases for \mathcal{H} , the basis of delta functions $(|G|^{1/2}\delta_x)_{x \in G}$ and $(\chi)_{\chi \in \widehat{G}}$. Let A and B be two observables with distinct sets of values $(a_x)_{x \in G}$ and $(b_\chi)_{\chi \in \widehat{G}}$ and corresponding eigenvectors as the two bases. Then

$$|(|G|^{1/2}\delta_x, \chi)|^2 = \frac{1}{N}$$

so that

$$M(a_x)M(b_\chi)M(a_x) = \frac{1}{N}M(a_x)$$

In other words, in a state in which A has a sharp value, when we make a measurement of B and then see what is the A -value is after the B -measurement, we find that all the A -values are equally likely. Thus all information about the first A -measurement has been erased by the B -measurement. The relationship between A and B is symmetrical and it is clear why we should regard them as complementary to each other. The simplest case is when $G = \widehat{G} = \mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$, the additive group of integers mod N ; this was the case studied by Weyl and Schwinger. It is obvious that we have a nondegenerate Heisenberg algebra here.

2.4 Weyl–Moyal algebra and the Moyal bracket

We now turn to the remaining questions raised earlier, namely, the construction of the quantum algebra for quantum systems of finitely many degrees of freedom, its uniqueness, and its dependence on \hbar . The key to this investigation is the *Weyl quantization map*. Its construction goes back to his great 1927 paper, explicated later in his famous book [6]. Weyl was unable to prove the uniqueness of the quantum algebra given the relations (2) and this was done by von Neumann and Stone independently [22, 23]. I shall refer to their works in more detail later on.

Weyl’s projective representation and its uniqueness

Before I start explaining the work of Weyl which is a great landmark in the subject, it is of interest to quote from Schwinger’s recapitulation of his own work on measurement algebras in [16]:

... Very early in my study of physics, Weyl became one of my gods. I use the word “god” rather than, say, “outstanding teacher” for the ways of gods are mysterious, inscrutable, and beyond the comprehension of ordinary mortals. . .

Already, even at the very inception of quantum theory [5], Weyl had made the suggestion that the Heisenberg–Dirac commutation rules (2) be replaced by the commutation rules between the corresponding *unitary operator groups*

$$(e^{iaq_r}), \quad (e^{ibp_s})$$

Let

$$(5) \quad U(a) = e^{ia \cdot q}, \quad V(b) = e^{ib \cdot p} \quad (a, b \in \mathbf{R}^k)$$

where

$$a \cdot q = a_1 q_1 + \cdots + a_k q_k, \quad b \cdot p = b_1 p_1 + \cdots + b_k p_k$$

Then the relations (2) are formally equivalent to

$$(6) \quad U(a)V(b) = e^{-iha \cdot b} V(b)U(a)$$

The commutation rules (6) have the advantage of dealing with unitary operators which are better behaved than the *unbounded* operators that must represent p and q .

The essential originality of Weyl consisted in considering not just U and V by themselves but to *treat them both together*. To this end Weyl started with the map

$$(a, b) \longmapsto U(a)V(b)$$

Although the $U(a)$ and $V(b)$ are unitary representations of \mathbf{R}^k , the fact that U and V do not commute shows that this map is *not* a unitary representation of \mathbf{R}^{2k} . But it is a *projective unitary representation*, i.e., a unitary representation up to phase factors, or equivalently, a *homomorphism* of \mathbf{R}^{2k} into the projective unitary group of the Hilbert space of the system. This is quite appropriate since it is the structure of this projective space with its probability functional $|(\psi, \theta)|^2$ that determines the structure of quantum mechanics.

Let me recall some basic facts about projective (unitary) representations. Given a separable locally compact topological group G , a projective representation of G is a map

$$L : x \longmapsto L_x$$

of G into the unitary operators of a Hilbert space such that the induced map into the projective unitary group is a continuous homomorphism. In down-to-earth language this is the same as saying that

$$L_x L_y = \lambda(x, y) L_{xy} \quad (x, y \in G)$$

where

$$\lambda(x, y) \in T$$

T being the multiplicative group of complex numbers of absolute value 1. The continuity assumptions are equivalent to requiring that $x \mapsto (L_x \psi, \theta)$ is Borel for all ψ, θ in the Hilbert space. The function $\lambda : G \times G \rightarrow T$ is called the *multiplier* of the projective representation. The map of G into the projective unitary group of the Hilbert space is unchanged if we replace L_x by $a(x)L_x$ for any borel map $A : G \rightarrow T$; then λ changes to $x, y \mapsto \lambda(x, y)a(x)a(y)/a(xy)$.

Following von Neumann²³ we consider the normalized map

$$(7) \quad W : (a, b) \mapsto e^{(i\hbar/2)a \cdot b} U(a) V(b)$$

which is also a projective representation of \mathbf{R}^{2k} . This normalization is also implicit in Weyl's 1927 paper (see formula immediately before (52) of that paper). A simple calculation shows that

$$(8) \quad W(a, b)W(a', b') = m(a, b : a', b')W(a + a', b + b')$$

where, m , the so-called *multiplier* of the projective unitary representation W , is given by

$$(9) \quad m(a, b : a', b') = e^{(i\hbar/2)(a' \cdot b - a \cdot b')}$$

Weyl postulated that *quantum kinematics is described completely by the projective unitary representation W* . It is clear from this that the infinitesimal generators Q_r, P_r of the unitary groups

$$U(0, \dots, t, \dots), \quad V(0, \dots, t, \dots) \quad (t \text{ is in the } r^{\text{th}} \text{ place})$$

satisfy the commutation rules (2). He also showed that if one takes *any projective unitary representation* of a real vector space V of finite dimension d and suppose that it is *faithful and irreducible*, then its multiplier m is

necessarily of the form

$$m(x : y) = e^{(i\hbar/2)\beta(x;y)} \quad (x, y \in V)$$

where β is a *symplectic form* on $V \times V$. It then follows that $d = 2k$ is even and one can choose a linear isomorphism of V with $\mathbf{R}^k \times \mathbf{R}^k$ such that (identifying V with $\mathbf{R}^k \times \mathbf{R}^k$)

$$\beta(a, b : a', b') = a' \cdot b - a \cdot b'$$

This means that one can restrict attention to projective unitary representations W with multiplier (9).

This was Weyl's formulation of *matrix mechanics*. A special choice of W can now be introduced, the so-called *Schrödinger model*. Here the Hilbert space \mathcal{H} is $L^2(\mathbf{R}^k)$ and $W = W_0$ is defined by

$$(10) \quad (W_0(a, b)\psi)(q) = e^{(i\hbar/2)a \cdot b} e^{ia \cdot q} \psi(q + \hbar b) \quad (q \in \mathbf{R}^k)$$

Weyl then formulated *wave mechanics* as the kinematics determined by the representation W_0 and raised the problem of equivalence of wave and matrix mechanics in the following form: is every irreducible projective unitary representation of $\mathbf{R}^k \times \mathbf{R}^k$ with multiplier m as in (9) unitarily equivalent to W_0 ? It is of course quite easy to check that W_0 is irreducible.

In this form Weyl was not able to prove the uniqueness. But he made a remarkable observation, namely, that if one replaces the space $\mathbf{R}^k \times \mathbf{R}^k$ by the *finite group* $\mathbf{Z}_k \times \mathbf{Z}_k$ where \mathbf{Z}_k is the additive group $\mathbf{Z}/k\mathbf{Z}$ of intergers mod k , and m by

$$m_k(a, b : a', b') = e^{2\pi i(a' \cdot b - a \cdot b')/k}$$

then the same problem of uniqueness can be formulated in this context. He then showed in [6] that the answer to the uniqueness question in the finite context is affirmative. He also suggested that in the limit as $k \rightarrow \infty$ it is possible to view the Schrödinger model as the limit of these finite quantum systems, so that the uniqueness in the general case is very plausible.

Direct proofs of the uniqueness of Weyl kinematics were constructed by Stone [22] and Von Neumann [23] very soon after Weyl's paper [6] was published. They established uniqueness in the following strong form: *any projective unitary representation of $\mathbf{R}^k \times \mathbf{R}^k$ with multiplier m defined by*

(9) is unitarily equivalent to the direct sum of a number of copies of the Schrödinger model W_0 . Subsequently Mackey formulated and proved the uniqueness theorem in the context of any separable locally compact abelian group, and this theorem had far-reaching consequences [24].

Weyl quantization map

The concept of a quantization map is of course fundamental because the quantum Hamiltonian is obtained by applying this map to the classical Hamiltonian. Preferring to work with bounded operators, Weyl defined the quantization map Q_h on a space of sufficiently nice functions on $\mathbf{R}^k \times \mathbf{R}^k$. He required that

$$Q_h : e^{ia \cdot q} e^{ib \cdot p} \longmapsto W(a, b) = e^{(i\hbar/2)a \cdot b} U(a) V(b)$$

and then extended it by linearity. Using Fourier analysis this means that Q_h is well defined for all functions on the algebra \mathcal{F} which are *Fourier transforms of integrable functions of the dual variables* a, b . More precisely, let $f = \mathbf{F}g$ where \mathbf{F} is the Fourier transform map:

$$f(q, p) = (\mathbf{F}g)(q, p) = \int e^{i(a \cdot q + b \cdot p)} g(a, b) da db.$$

Then $Q_h(f) = W(g)$:

$$(11) \quad \begin{aligned} f &= \mathbf{F}g \\ Q_h(f) &= \int g(a, b) W(a, b) da db. \end{aligned}$$

Here da etc refers to the *self-dual* Lebesgue measure on \mathbf{R}^k which is $(2\pi)^{-k/2}$ times the standard Lebesgue measure.

The Weyl quantization map is thus

$$(12) \quad Q_h : f \longmapsto W(\mathbf{F}^{-1}f) \quad (f \in \mathcal{F})$$

Weyl actually did not insist that the Fourier integrals be convergent but be defined only in some distributional sense. This allowed him to calculate formally the quantization of *polynomial functions* in q, p . He found that the quantization of a polynomial is obtained by replacing q_r, p_r by the corresponding quantum mechanical operators Q_r, P_r , and replacing any monomial in the q_r, p_r by the *averages of the corresponding products in all possible*

orderings (the Weyl ordering)[23]:

$$Q_h(f) =: f(q, p) :$$

Furthermore, it is easy to see that

$$W(a, b)^\dagger = W(a, b)^{-1} = W(-a, -b)$$

so that

$$\int g(a, b)W(a, b)dadb \text{ is self adjoint } \iff g(a, b) = g^{\text{conj}}(-a, -b)$$

which leads to

$$Q_h(f) \text{ is self adjoint } \iff f \text{ is real}$$

If we make additional restrictions on f , for instance that f be smooth and rapidly decreasing, i.e., that f is in the *Schwartz space*, then $Q_h(f)$ is an integral operator of trace class with kernel

$$(13) \quad K_h(f)(q, q') = \widehat{g}_1\left((\hbar/2)(q + q'), q' - q\right)$$

where \widehat{g}_1 is the partial Fourier transform of $g = \mathbf{F}^{-1}f$ in the first variable.

Going back to W let me recall that if A is a locally compact abelian group and w is a unitary representation of A , then the map

$$g \longmapsto w(g) := \int w(x)g(x)dx$$

is a $*$ -homomorphism of the *convolution algebra* $L^1(A)$ into the algebra of bounded operators in the Hilbert space of w with the property that

$$w(g^*) = w(g)^\dagger \quad g^*(x) = g(-x)^{\text{conj}}$$

If now w is only a projective unitary representation with multiplier δ , then Weyl observed that

$$g \longmapsto w(g) := \int g(x)dx$$

is still a $*$ -homomorphism on $L^1(A)_\delta$ where $L^1(A)_\delta$ is the *twisted convolution algebra* associated to δ , i.e., the underlying vector space of $L^1(A)_\delta$ is $L^1(A)$ with the twisted convolution defined by

$$\begin{aligned}
 (14) \quad (g_1 *_{\delta} g_2)(x) &= \int g_1(y) g_2(x-y) \delta(y, x-y) dy \\
 &= \int g_1(x-y) g_2(y) \delta(x-y, y) dy,
 \end{aligned}$$

while the $*$ -property corresponds to

$$w(g^*) = w(g)^{\dagger} \quad g^*(x) = g(-x)^{\text{conj}} \delta(x, -x)$$

Let us now specialize this to the case when

$$A = \mathbf{R}^k \times \mathbf{R}^k, \delta = m$$

and write

$$g_1 *_{\hbar} g_2 = g_1 *_{\hbar} g_2$$

which is justifiable since m is determined uniquely by \hbar . Then

$$(15) \quad (g_1 *_{\hbar} g_2)(a, b) = \int g_1(a', b') g_2(a-a', b-b') e^{(i\hbar/2)(a \cdot b' - a' \cdot b)} da' db'$$

Thus

$$(16) \quad Q(\mathbf{F}g_1)Q(\mathbf{F}g_2) = Q(\mathbf{F}g), \quad g = g_1 *_{\hbar} g_2$$

or

$$Q(f_1)Q(f_2) = Q(f)$$

where

$$f(q, p) = \int \widehat{f}_1(a', b') \widehat{f}_2(a-a', b-b') e^{(i\hbar/2)(a \cdot b' - a' \cdot b) - i(a \cdot q + b \cdot p)} da' db' da db$$

with

$$\widehat{g}(a, b) = \int g(q, p) e^{i(a \cdot q + b \cdot p)} dq dp$$

Weyl quantization on an abelian group

The completely group theoretic nature of his quantization of the classical phase space, which is so obvious in the above discussion, led Weyl to a far-reaching generalization of the process of quantization. He formulated in his book [6] (pp. 275–276) the general principle of quantization as follows.

The kinematical structure of a physical system is expressed by an irreducible Abelian group of unitary ray rotations in system space. The real elements of the algebra of this group are the physical quantities of the system; the representation of the abstract group by rotations of system space associates with each such quantity a definite hermitian form which “represents” it.

Here “group of ray rotations” is Weyl’s term for a subgroup of the projective unitary group and so is the same as a projective unitary representation. Replacing the classical phase space \mathbf{R}^{2k} by an abelian group A we can formulate Weyl’s principle of quantization as follows. Let \mathcal{F} be an algebra of complex-valued functions on a separable locally compact abelian group A . One need not be very specific as to how \mathcal{F} is to be restricted but as in most of Weyl’s examples we take \mathcal{F} to a subalgebra of the algebra of functions on A which are the *Fourier transforms* of integrable functions on the dual group \hat{A} . Then *quantization is a linear and faithful map*

$$Q_h : f \longmapsto Q_h(f) \quad (f \in \mathcal{F})$$

from \mathcal{F} into the algebra of bounded operators on a Hilbert space \mathcal{H} , the Hilbert space of the quantum system. If \mathcal{F}_h is the image of \mathcal{F} by Q_h , he viewed \mathcal{F}_h as the quantization of \mathcal{F} . The map Q_h was required to satisfy the self adjoint property, namely,

$$Q_h(f^{\text{conj}}) = Q_h(f)^{\dagger} \quad (f \in \mathcal{F})$$

This means that \mathcal{F}_h has an involution, namely the adjoint operation † , and that the quantum observable that corresponds to a real classical observable f is a self adjoint element of \mathcal{F}_h .

Although Weyl treated only the cases of a vector space or a finite group, one can clearly work with *any* separable locally compact abelian group. This was done in my paper [25]. Let me now sketch briefly the main results.

For any separable locally compact abelian group G let $Z^2(G)$ be the commutative group of Borel 2-cocycles with values in the circle group, $B^2(G)$ for the subgroup of trivial cocycles, and $H^2(G) = Z^2(G)/B^2(G)$ for the quotient group of multipliers. For any $m \in Z^2(G)$ we can give $L^1(G)$ the structure of a Banach algebra by defining the *twisted convolution* by

$$(f_1 *_m f_2)(x) = \int_A f_1(y) f_2(x - y) m(y, x - y) dy \quad (x \in G)$$

This algebra structure depends only on the cohomology class $[m]$ of m , namely on the image of m in $H^2(G)$; and is commutative if and only if m is trivial, in which case it is isomorphic to the usual convolution algebra structure on $L^1(G)$. Let $C^2(G)$ be the group of *bicharacters* of G , $S^2(G)$ the subgroup of symmetric ones and $\Lambda^2(G)$ the subgroup of alternating ones ($m(x, x) = 1$, $m(x, y)m(y, x) = 1$). One has $C^2(G) \subset Z^2(G)$ and if G has no 2-torsion (this is equivalent to the dual group \widehat{G} having no 2-torsion) then $C^2(G)$ maps onto $H^2(G)$ with kernel $S^2(G)$ while $C^2(G) = S^2(G) \oplus \Lambda^2(G)$ so that under the natural map of $\Lambda^2(G) \rightarrow Z^2(G)$ we have

$$\Lambda^2(G) \simeq H^2(G) \quad (G \text{ has no 2-torsion}).$$

Let now A be a separable locally compact abelian group with character group \widehat{A} . Let $\mathcal{W}(A)$ be the space of all functions on A which are in $L^1(A)$ and whose Fourier transforms are in $L^1(\widehat{A})$. (It is possible to work with the larger space of functions which are Fourier transforms of elements of $L^1(\widehat{A})$ but $\mathcal{W}(A)$ is preferable as it is mapped isomorphically onto $\mathcal{W}(\widehat{A})$ by the Fourier transform, so that it is self dual.) It is easy to see that $\mathcal{W}(A)$ and $\mathcal{W}(\widehat{A})$ are both closed under convolution, multiplication, translations by group elements, and multiplications by characters; moreover, if h, k are continuous functions on A (resp. \widehat{A}), then $h * k$ is in $\mathcal{W}(A)$ (resp. $\mathcal{W}(\widehat{A})$). Then, for any $\widehat{m} \in Z^2(\widehat{A})$ $\mathcal{W}(A)$ has the structure of an associative algebra given by

$$f_1 \cdot_{\widehat{m}} f_2 = g, \quad \widehat{g} = \widehat{f}_1 *_{\widehat{m}} \widehat{f}_2$$

The structure of this algebra depends only on the cohomology class of \widehat{m} , and is abelian if and only if \widehat{m} is trivial, in which case it is just ordinary multiplication. If A has no 2-torsion, we can take $\widehat{m} \in \Lambda^2(\widehat{A})$, the $*$ -operation is $f \mapsto f^{\text{conj}}$ on $\mathcal{W}(A)$, and the trivial case corresponds to $\widehat{m} = 1$. The *generic case* is when \widehat{m} is *symplectic*: this means that (in the general case when \widehat{m} is not necessarily a bicharacter) that

$$\frac{\widehat{m}(x, y)}{\widehat{m}(y, x)},$$

which is always a bicharacter, induces an isomorphism of A with \widehat{A} (when A has no 2-torsion and \widehat{m} is in $\Lambda^2(\widehat{A})$, this is the case when \widehat{m} itself induces an isomorphism of A with \widehat{A}). In this case it is appropriate to call $\mathcal{W}(A)$ the *Weyl algebra* of A .

We have already seen that when $A = \mathbf{R}^{2k}$ the symplectic multiplier leads to the usual Weyl–Heisenberg representations. Weyl also discussed the finite case in [6]. However recent work in physics suggests that it may be of interest to consider situations where the real manifolds are replaced by p -adic ones which arise in number theory. This leads to what I call the *arithmetic Weyl algebras*. An example of these is as follows [25]. Let K be a *nonarchimedean local field* of characteristic $\neq 2$, let V be a vector space of finite *even* dimension over K , β a symplectic bilinear form on $V \times V$, and let χ be a nontrivial additive character on K which we use to define the Fourier transform on the space $\mathcal{S}(V)$ of Schwartz–Bruhat functions on V . Then $\mathcal{S}(V)$ is the arithmetic Weyl algebra with multiplication defined by

$$(f_1 \cdot_\chi f_2)(x) = \int \widehat{f}_1(\eta) \widehat{f}_2(\xi - \eta) \chi(\beta(\eta - x, \xi)) d\eta d\xi$$

The theory of quantum systems over local and global fields is very interesting but this is not the place to go into them.

Weyl–Moyal algebra and the Moyal bracket

Weyl’s ideas were taken up more than twenty years later in 1949 by Moyal. In a remarkable paper [26] Moyal made a deep study of Weyl’s quantization map. Moyal’s motivation was to reintroduce the classical phase space in quantum mechanics by identifying the quantum algebra with an algebra of functions of the “non commuting quantities q_r, p_s ”. He did this by pulling back the operator product via the Weyl quantization map to a product operation on the space of functions on the classical phase space. The resulting algebra is the *Weyl–Moyal algebra*. This is of course nothing other than the Fourier transform of Weyl’s twisted convolution and so we call it the *Weyl–Moyal product*. *Unlike Weyl who did not investigate the Poisson structure on the classical function space when it is quantized, Moyal went further and calculated the operator commutator in terms of this product and obtained what we now call the Moyal bracket on the classical function space $\mathcal{W}(\mathbf{R}^k \times \mathbf{R}^k)$.*

The Weyl–Moyal product

$$f = f_1 \cdot_\hbar f_2$$

is defined by first convolving the Fourier transforms in the twisted manner and then applying the inverse Fourier transform:

(17)

$$f(q, p) = \int \widehat{f}_1(a', b') \widehat{f}_2(a - a', b - b') e^{(i\hbar/2)(a \cdot b' - a' \cdot b) - i(a \cdot q + b \cdot p)} da' db' dadb.$$

Then

(18)

$$Q_\hbar(f_1 \cdot_\hbar f_2) = Q_\hbar(f_1) Q_\hbar(f_2), \quad Q_\hbar(f_1 \cdot_\hbar f_2 - f_2 \cdot_\hbar f_1) = [Q_\hbar(f_1), Q_\hbar(f_2)].$$

Since the map $f \mapsto Q_\hbar(f)$ is injective, it follows from (17), (18), and the associativity of operator multiplication that

$$(19) \quad f_1, f_2 \mapsto f_1 \cdot_\hbar f_2, \quad f_1, f_2 \mapsto f_1 \cdot_\hbar f_2 - f_2 \cdot_\hbar f_1$$

define, respectively, the structures of associative and Lie algebras on the classical function space $\mathcal{W}(\mathbf{R}^k \times \mathbf{R}^k)$.

To explore the dependence of the algebra defined by (17) on \hbar , one may proceed by expanding the exponential

$$(20) \quad e^{(i\hbar/2)(a \cdot b' - a' \cdot b)}$$

as a series in powers of \hbar . From now on we shall assume that the classical function space is $\mathcal{S}(\mathbf{R}^k \times \mathbf{R}^k)$, the algebra (under both multiplication and convolution) of the *Schwartz functions* in q, p . The terms of the series can then be explicitly evaluated. For instance, the constant term is $(f_1 f_2)(q, p)$, while the next term becomes, after some calculation,

$$(21) \quad \frac{i\hbar}{2} P(f_1, f_2)(q, p)$$

where $P(f_1, f_2) = \{f_1, f_2\}$ is the Poisson Bracket. Thus

$$(22) \quad f_1 \cdot_\hbar f_2 = f_1 f_2 + \frac{i\hbar}{2} P(f_1, f_2) + O(\hbar^2).$$

We define the *Moyal Bracket* of f_1 and f_2 by

$$(23) \quad [f_1, f_2]_\hbar = f_1 \cdot_\hbar f_2 - f_2 \cdot_\hbar f_1.$$

Then

$$(24) \quad [f_1, f_2]_\hbar = i\hbar \{f_1, f_2\} + O(\hbar^2).$$

This is just the Dirac prescription but with the essential refinement that $[f_1, f_2]_h$ is not exactly equal to $i\hbar\{f_1, f_2\}$ but is only equal to it modulo \hbar^2 . We have thus constructed the quantum algebra associated to a general classical system, giving meaning *ex post facto* to Dirac's profound and remarkable insight. We shall see presently that for the canonical variables the Dirac and Moyal brackets are the same, and the same is true for angular momentum observables and certain obvious generalizations of them.

A similar calculation gives us all the higher order terms as well. To make it transparent we introduce two sets of variables

$$q^{(1)}, p^{(1)}, q^{(2)}, p^{(2)}$$

and the associated Fourier transform variables

$$a^{(1)}, b^{(1)}, a^{(2)}, b^{(2)}.$$

We then have the differential operator

$$\Pi = \sum_{1 \leq j \leq k} \left(\frac{\partial}{\partial q_j^{(1)}} \otimes \frac{\partial}{\partial p_j^{(2)}} - \frac{\partial}{\partial p_j^{(1)}} \otimes \frac{\partial}{\partial q_j^{(2)}} \right)$$

and let $\hat{\Pi}$ be the induced operator on the Fourier transform space via the transform. Then $\hat{\Pi}$ is multiplication by the polynomial

$$(25) \quad \hat{\Pi} = \sum_{1 \leq j \leq k} \left(-a_j^{(1)} b_j^{(2)} + b_j^{(1)} a_j^{(2)} \right).$$

Finally, let Res be the operator from $\mathcal{S}(\mathbf{R}^{2k} \times \mathbf{R}^{2k})$ to $\mathcal{S}(\mathbf{R}^{2k})$ which is the restriction to the diagonal:

$$\text{Res}(f)(q, p) = f(q, p, q, p).$$

Then

$$P = \text{Res} \circ \Pi(f_1 \otimes f_2).$$

We define the bidifferential operator P_N by

$$(26) \quad P_N : f_1, f_2 \longmapsto (\text{Res} \circ \Pi^N)(f_1 \otimes f_2).$$

Then the general term of the expansion of (17) in powers of \hbar is

$$(27) \quad \frac{(i\hbar/2)^N}{N!} P_N(f_1, f_2)$$

We thus have the formal expansion of (17):

$$(28) \quad [f_1, f_2]_h = 2 \sum_{M=0}^{\infty} \frac{(i\hbar/2)^{2M+1}}{(2M+1)!} P_{2M+1}(f_1, f_2)$$

Symbolically we write this as

$$(29) \quad f_1 \cdot_h f_2 = e^{(i\hbar/2)P}, \quad [f_1, f_2]_h = \frac{2}{\hbar} \sin \frac{\hbar}{2} P$$

which go back to Moyal's paper [26].

Let us summarize this discussion. Let \mathcal{S} be the Schwartz space of the classical phase space \mathbf{R}^{2k} . Then \mathcal{S} is a commutative algebra under multiplication. For any real value of the parameter \hbar , the definition

$$f_1 \cdot_h f_2 = e^{(i\hbar/2)P}$$

interpreted by (17) converts \mathcal{S} into an associative algebra denoted by \mathcal{S}_h . Then \mathcal{S}_h is a family of noncommutative associative algebras which have \mathcal{S} as their limit when $\hbar \rightarrow 0$. Moreover, under the commutator product, \mathcal{S}_h becomes a Lie algebra whose Lie bracket becomes the Poisson Bracket on \mathcal{S} in the limit when $\hbar \rightarrow 0$:

$$(30) \quad [f_1, f_2]_h := f_1 \cdot_h f_2 - f_2 \cdot_h f_1 = i\hbar \{f_1, f_2\} + O(\hbar^3)$$

It is natural to ask if this product structure can be defined for larger algebras than \mathcal{S} . This can be done at least *formally* by treating \hbar as a *formal parameter*. Let us write $t = i\hbar/2$, treating t as a formal parameter and define

$$(31) \quad f_1 \cdot_t f_2 = e^{tP} = f_1 f_2 + \sum_{r \geq 1} \frac{t^r}{r!} P_r(f_1, f_2),$$

where the series is treated formally. It then makes sense *even when the f_i are in $C^\infty(\mathbf{R}^{2k})$* because the P_r are bidifferential operators. The associative law persists in the formal sense because the identities that have to be established for associativity are differential operator identities that are true for all differentiable functions once they are true for Schwartz functions. Thus (31)

defines the structure of an associative algebra in the formal sense on $C^\infty(\mathbf{R}^{2k})$. We shall make this more precise in the next section.

The quantum algebra as a limit of finite dimensional quantum algebras

This idea, as we have mentioned earlier, was due to Weyl and taken up by Schwinger later [6, 19]. Kinematically this was done in great generality in [20]. To understand the limiting process dynamically we consider, for simplicity, motion on a line where the potential is big at infinity (it is enough that $V(q)/\log|q| \rightarrow \infty$ as $|q| \rightarrow \infty$) so that the energy spectrum is discrete, as in the case of the harmonic oscillator. Indeed, taking N to be an odd natural number, one can identify \mathbf{Z}_N with the finite grid $X(N) \subset \mathbf{R}$:

$$X(N) = \{r\varepsilon_N \mid r = 0, \pm 1, \pm 2, \dots, \pm N^0\}$$

where

$$\varepsilon_N = \left(\frac{2\pi}{N}\right)^{1/2}, \quad N = 2N^0 + 1.$$

The Hilbert space of the system is now $L^2(X(N))$; the position operator q_N is multiplication by the coordinate; the momentum operator p_N is the Fourier transform of q_N , the Fourier transform being the one defined for the finite group $\mathbf{Z}(N) = \mathbf{Z}_{\varepsilon_N}/N \cdot \mathbf{Z}_{\varepsilon_N}$ with which $X(N)$ can be identified. The Hamiltonian is

$$H_N = \frac{1}{2}p_N^2 + V(q_N).$$

In the limit when $N \rightarrow \infty$ this system goes over to the usual system on the real line with the Hamiltonian

$$H = \frac{1}{2}p^2 + V(q).$$

An invariant way to formulate this limit is to ask that

$$\mathrm{Tr}(f(H_N)) \longrightarrow \mathrm{Tr}(f(H)) \quad (N \rightarrow \infty)$$

for a large class of functions f for which $f(H)$ is of trace class. See [6, 19, 20, 21] where this is done in arbitrary dimension and for functions of the form e^{-sx} of the real variable x , leading to the result that the dynamical operators in imaginary time converge in a very strong manner:

$$\|\exp(-sH_N) - \exp(-sH)\|_1 \rightarrow 0 \quad (N \rightarrow \infty)$$

where $\|\cdot\|_1$ is the trace norm in the Banach space of trace class operators.

2.5 Quantum algebras over phase space

The Weyl–Moyal theory may be summarized by saying that quantum mechanics is essentially equivalent to imposing on the classical *commutative algebra* of functions on the phase space *a family of non commutative algebra structures* depending on \hbar such that (22)–(24) are satisfied, thus giving the Dirac insight a very satisfactory interpretation. Depending upon the choice of the classical algebra, the quantum parameter \hbar can be analytic or formal. The Hilbert space of quantum mechanics now arises as the space on which these non commutative algebras are represented by operators.

Mathematicians will recognize in the relations (22)–(31) the fact that the quantum algebra may be viewed as a *deformation* of the classical algebra with \hbar as the deformation parameter. If I am not mistaken, this point of view was pioneered by (the late) Moshe Flato and his collaborators and colleagues, F. Bayen, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, and H. Basart, in a series of seminal papers [27, 28, 29, 30]. Although deformations of associative and Lie algebras had been an established field, the application of this subject to the foundations of quantum theory was new and was first done by them. I shall now try to describe briefly some of the main results in this development which is still very active. This has fitted well with the recent fascination among theoretical physicists for non commutative geometry and so there is a great following for this type of work.

The calculations involving the Weyl–Moyal product and the Moyal bracket may be viewed as describing *a class of deformations of \mathcal{S} or $C^\infty(\mathbf{R}^{2k})$, viewed as a Poisson algebra*. To explain this point of view let us begin with the notion of deformation of associative algebras first. We take a naive point of view.

Analytical deformations

Let \mathcal{A} be an associative algebra over \mathbf{C} . Intuitively one speaks of a *deformation of \mathcal{A}* when one is given a family of associative algebras \mathcal{A}_t defined for small values of a parameter t whose underlying vector spaces coincide with

that of \mathcal{A} such that item (i) if we write

$$f, g \mapsto f \cdot_t g$$

for the product in \mathcal{A}_t , then the product map is nice in some sense (smooth, analytic, etc)

(ii) For $t = 0$, $f \cdot_t g$ reduces to fg , the product in \mathcal{A} .

t is called the *deformation parameter*.

Formal deformations

Here one views t as an *indeterminate* and passes to $\mathcal{A}[[t]]$ which is the algebra of formal power series

$$f = a_0 + a_1 t + a_2 t^2 + \dots \quad (a_i \in \mathcal{A})$$

Clearly the product

$$f, g \mapsto fg$$

in \mathcal{A} extends to a product

$$\sum f_p t^p, \sum g_q t^q \mapsto \sum k_r t^r$$

in $\mathcal{A}[[t]]$ where

$$k_r = \sum_{p, q \geq 0, p+q=r} f_p g_q$$

We denote by

$$\mathcal{A}[[t]]_0$$

the algebra thus obtained. A *formal deformation* of \mathcal{A} is now defined as an associative algebra whose underlying space is $\mathcal{A}[[t]]$ and whose multiplication

$$f, g \mapsto f \cdot_t g \quad (f, g \in \mathcal{A}[[t]])$$

is such that when $t = 0$ it reduces to the algebra $\mathcal{A}[[t]]_0$. If we write

$$f \cdot_t g = fg + tC_1(f, g) + t^2C_2(f, g) + \dots \quad (f, g \in \mathcal{A})$$

where

$$C_j : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A} \quad (C_0 = \text{id})$$

are bilinear maps, then the associativity requirement means that the C_j satisfy the identities

$$\sum_{r \geq 0, s \geq 0, r+s=n} C_r(a, C_s(b, c)) = \sum_{r \geq 0, s \geq 0, r+s=n} C_r(C_s(a, b), c)$$

Conversely, if the $C_j : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ are bilinear maps satisfying these identities, the definition of product for two elements of \mathcal{A} extends to an associative algebra structure on the vector space $\mathcal{A}[[t]]$ that coincides with the algebra structure $\mathcal{A}[[t]]_0$ when $t = 0$. The reason for introducing formal deformations is that they are algebraic objects and so may be studied with more ease than analytical deformations. Moreover, any analytical deformation in which the product is analytic in the parameter gives rise to a formal deformation. However *true quantization requires analytical deformations and not formal deformations, so that the formal deformations become more remote from the physical point of view*. In particular, if in any given context we establish the existence of a formal deformation, there still remains the problem of making analytical sense for these formal deformations for at least a large subalgebra of the original algebra, if the deformation is to be given a physical interpretation, for instance, to obtain a representation in a Hilbert space. We call such deformations *quantizations*.

Considerable work has been done in the directions suggested by the above remarks. Quantizations of symplectic and Poisson manifolds have been obtained. In the most important case of \mathbf{R}^{2k} as the phase space it was shown by Moshe Flato and his collaborators that the Weyl-Moyal deformation is *essentially the only one*, thus exhibiting quantum mechanics as the *only possible deformation of classical mechanics*. In other words the Dirac quantum algebra is the only one possible. In my opinion this uniqueness theorem is the most fundamental foundational result in making transparent the transition from the classical to the quantum algebra.

2.6 Moshe Flato remembered

The idea that the Weyl-Moyal algebras should be treated in the broader framework of deformations of Poisson algebras was pioneered by Moshe Flato and his collaborators F. Bayen, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer. Soon others such as De Wilde, Le Conte, Fedesov, Deligne, and Kontsevich

took up this theme and went very far with it. It is unfortunate that Moshe passed away at the peak of his creative powers. I want to pen a few lines in which I try to give a glimpse of the mercurial personality of this remarkable man.

We first met in Liege in the mid 1970s in a beautiful conference on representation theory and fundamental physics that he organized. After that we always spent time together during his visits to UCLA, and once in Dijon when he invited me for a talk in his seminar. He told me once that had we met earlier we could have done much work together. When the AMS and the International Press published my Selected Papers, I sent him a copy with a letter recalling to him this remark. But unfortunately it reached him when he was already in a coma.

He had a very broad view of physics and expressed his preferences openly without worrying whether they were politically correct. He was that rare person who was very strict with the big people and very gentle with younger and less established scientists. To illustrate his kindness to young people the following incident would suffice. I had a student, Luca Guerrini, working on deformations of Lie algebras under me and after he finished his thesis I asked him to write up his results and submit it to the *Letters in Mathematical Physics*. The submission was rejected but I noticed in the rejection letter a tiny ray of hope: it was the comment that the paper does not explain why anyone would or should be interested in the questions studied in the submission. I told Luca that I will help him write an introduction and resubmit without any other change. This was done and the paper got accepted without requiring any modifications almost immediately. Some months later I was in Paris and met Moshe just for a few hours. The first thing he told me was “don’t think I did not know that you wrote that intro.” It turned out that this was the last time that I would see him for he died a few months later. A few months earlier he had agreed to be the mentor for Charlie Conley, another student of mine. Moshe suggested new lines of investigations and put Charlie in touch with people who were doing such things. This changed Charlie’s entire scientific life in a dramatic manner and launched him on an extremely successful research career.

When Moshe passed away, there was going on a work shop on representation theory at UCLA, organized by Chris Fronsdal and Joe Wolf. I gave a eulogy for Moshe at that workshop, which was later published in one of the volumes of the Proceedings of the Conference that was organized in his memory (by Daniel Sternheimer and others). I reproduce it here.

Moshe Flato (1937–1998)¹

Moshe Flato died on Friday, Nov 27, 1998, in Paris, of complications from a massive cerebral haemorrhage he suffered a couple of days earlier. His shocking and untimely death is a tremendous loss to the world of mathematical physics, and even more so, to his family and large circle of friends and admirers. He was buried in Tel-Aviv.

Moshe's work covers a large spectrum of high energy physics, straddling diverse aspects of quantum field theory and elementary particles. His interest was always in fundamental questions. He was constantly searching for a conceptual understanding of the strange world of elementary particles, and explored many new ideas and ways of thinking about them. He created a school in Dijon and Paris and shaped the intellectual developments of a huge number of young scientists, some of whom became his closest friends, such as Daniel Sternheimer and Jacques Simon.

I would like to make some very brief comments on his work. In quantum mechanics there is a famous uniqueness theorem due to von Neumann and Stone, which says that any system of operators satisfying the Heisenberg–Weyl commutation rules is isomorphic to the standard system which is called the Schrödinger model. The content of this theorem is that there is only one way to quantize a classical system with the specified commutation rules, namely through canonical quantization. In a famous paper published in the mid 1970's, Moshe, and his collaborators Bayen, Fronsdal, Lichnerowicz, and Sternheimer, proved a wonderful variant of this uniqueness in a very different context. Their idea was to think of quantization as a deformation of the Poisson algebra of classical observables. Canonical quantization furnishes such a deformation, the Moyal–Weyl deformation. They proved that this is essentially the only deformation of the classical Poisson algebra. In my opinion this uniqueness theorem lies at a much greater depth than the von Neumann–Stone theorem and may be thought of as the definitive clarification of the questions that were raised first by Hermann Weyl in 1927 regarding the foundations of quantum physics. The ideas coming out of this paper revolutionized quantization theory, made it possible to think of quantization in contexts where canonical quantization does not make sense (curved space-time, constrained mechanics, etc), and led inevitably to the theory of quantum

¹ Talk delivered on Dec 5, 1998, in the Lie group workshop held at UCLA on Dec 5–6, 1998.

groups and quantum homogeneous spaces. Recent work on q -quantum mechanics by Finkelstein and others would not have been possible without this seminal achievement. During the 1980's, Moshe, in collaboration with Fronsdal, made a deep analysis of the representation theory of the conformal group. Their theory of the so called singletons, which was very much ahead of its time, has recently come to the forefront in high energy physics.

Throughout his life he was deeply interested in formulating and understanding the nonlinear aspects of quantum field theory. He called this the "nonlinear program" and wrote many papers on this theme, with Simon and others. His recent work in collaboration with Simon and Taflin is a technical tour de force on the Cauchy problem and scattering theory for the coupled Maxwell and Dirac equations that arise in quantum electrodynamics.

He loved the subject of mathematical physics deeply and was very unhappy that (in those days) there was no forum where important and interesting discoveries in mathematical physics could be published quickly. The physicists had several such journals, but it was often very difficult to get papers published in them because of lack of sympathy of the physics community toward major discoveries that were more mathematical and did not lead to immediate physical interpretations. Moshe understood this situation and with characteristic energy and enthusiasm founded, in 1977, the *Letters in Mathematical Physics* precisely to correct this anomaly. The *Letters* has become one of the premier journals in mathematical physics. If you look through its pages you will find all the major developments in the last two decades represented there when they first came up. There is no question that he was the moving spirit behind the *Letters*. This is one achievement that will live long after him.

But bigger than all of his achievements in science is the fact that he was a decent man. Behind his rough exterior lay a heart of extraordinary generosity. He was lavish in giving his time to others. I learned a lot of physics by talking to him and reading his papers. He was especially generous to young people, understood their difficulties at the start of their scientific careers, and helped them in every possible way. He treated his friends royally, and influenced their lives in ways that cannot be enumerated. He was one of the few men I have known of whom it could be said that he was larger than life (a comment also made by one of the young scientists who was deeply influenced by Moshe).

As one gets older, one finds invariably that one cannot do all the things that one did when one was young, and certainly not at the depth that was reached in youth. The natural tendency to compensate for this is to reduce

one's ambitions and do what one can to advance science. Moshe scorned this way of living. He pushed himself and others around him without any slackening and lived on the creative edge all the time, right up to the moment of his death. According to the Hindu view of life this is one of the noblest ways to live and die.

When I learned of Moshe's death I was reminded of Isaac Stern's words on the occasion of David Oistrakh's death, when he referred to Oistrakh as a golden man. Moshe was a golden man.

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