# Chapter 2
# Discrete-Time Markov Models

## 2.1 Discrete-Time Markov Chains

Consider a system that is observed at times $0, 1, 2, \ldots$. Let $X_n$ be the state of the system at time $n$ for $n = 0, 1, 2, \ldots$. Suppose we are currently at time $n = 10$. That is, we have observed $X_0, X_1, \ldots, X_{10}$. The question is: can we predict, in a probabilistic way, the state of the system at time 11? In general, $X_{11}$ depends (in a possibly random fashion) on $X_0, X_1, \ldots, X_{10}$. Considerable simplification occurs if, given the complete history $X_0, X_1, \ldots, X_{10}$, the next state $X_{11}$ depends only upon $X_{10}$. That is, as far as predicting $X_{11}$ is concerned, the knowledge of $X_0, X_1, \ldots, X_9$ is redundant if $X_{10}$ is known. If the system has this property at all times $n$ (and not just at $n = 10$), it is said to have a *Markov property.* (This is in honor of Andrey Markov, who, in the 1900s, first studied the stochastic processes with this property.) We start with a formal definition below.

**Definition 2.1.** (Markov Chain). A stochastic process $\{X_n, n \geq 0\}$ on state space $S$ is said to be a discrete-time Markov chain (DTMC) if, for all $i$ and $j$ in $S$,

$$\mathsf{P}(X_{n+1} = j | X_n = i, X_{n-1}, \ldots, X_0) = \mathsf{P}(X_{n+1} = j | X_n = i). \tag{2.1}$$

A DTMC $\{X_n, n \geq 0\}$ is said to be time homogeneous if, for all $n = 0, 1, \ldots,$

$$\mathsf{P}(X_{n+1} = j | X_n = i) = \mathsf{P}(X_1 = j | X_0 = i). \tag{2.2}$$

Note that (2.1) implies that the conditional probability on the left-hand side is the same no matter what values $X_0, X_1, \ldots, X_{n-1}$ take. Sometimes this property is described in words as follows: given the present state of the system (namely $X_n$), the future state of the DTMC (namely $X_{n+1}$) is independent of its past (namely $X_0, X_1, \ldots, X_{n-1}$). The quantity $\mathsf{P}(X_{n+1}{=}j | X_n = i)$ is called a *one-step transition probability* of the DTMC at time $n$. Equation (2.2) implies that, for time-homogeneous DTMCs, the one-step transition probability depends on $i$ and $j$ but is the same at all times $n$; hence the terminology *time homogeneous.*

In this chapter we shall consider only time-homogeneous DTMCs with *finite* state space $S = \{1, 2, \ldots, N\}$. We shall always mean time-homogeneous DTMC

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when we say DTMC. For such DTMCs, we introduce a shorthand notation for the one-step transition probability:

$$p_{i,j} = \mathsf{P}(X_{n+1} = j | X_n = i), \quad i, j = 1, 2, \ldots, N. \tag{2.3}$$

Note the absence of $n$ in the notation. This is because the right-hand side is independent of $n$ for time-homogeneous DTMCs. Note that there are $N^2$ one-step transition probabilities $p_{i,j}$. It is convenient to arrange them in an $N \times N$ matrix form as shown below:

$$P = \begin{bmatrix} p_{1,1} & p_{1,2} & p_{1,3} & \cdots & p_{1,N} \\ p_{2,1} & p_{2,2} & p_{2,3} & \cdots & p_{2,N} \\ p_{3,1} & p_{3,2} & p_{3,3} & \cdots & p_{3,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{N,1} & p_{N,2} & p_{N,3} & \cdots & p_{N,N} \end{bmatrix}. \tag{2.4}$$

The matrix $P$ in the equation above is called the *one-step transition probability matrix*, or transition matrix for short, of the DTMC. Note that the rows correspond to the starting state and the columns correspond to the ending state of a transition. Thus the probability of going from state 2 to state 3 in one step is stored in row number 2 and column number 3.

The information about the transition probabilities can also be represented in a graphical fashion by constructing a *transition diagram* of the DTMC. A transition diagram is a directed graph with $N$ nodes, one node for each state of the DTMC. There is a directed arc going from node $i$ to node $j$ in the graph if $p_{i,j}$ is positive; in this case, the value of $p_{i,j}$ is written next to the arc for easy reference. We can use the transition diagram as a tool to visualize the dynamics of the DTMC as follows. Imagine a particle on a given node, say $i$, at time $n$. At time $n+1$, the particle moves to node 2 with probability $p_{i,2}$, node 3 with probability $p_{i,3}$, etc. $X_n$ can then be thought of as the position (node index) of the particle at time $n$.

*Example 2.1.* (Transition Matrix and Transition Diagram). Suppose $\{X_n, n \geq 0\}$ is a DTMC with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} .20 & .30 & .50 \\ .10 & .00 & .90 \\ .55 & .00 & .45 \end{bmatrix}. \tag{2.5}$$

If the DTMC is in state 3 at time 17, what is the probability that it will be in state 1 at time 18? The required probability is $p_{3,1}$ and is given by the element in the third row and the first column of the matrix $P$. Hence the answer is .55.

If the DTMC is in state 2 at time 9, what is the probability that it will be in state 3 at time 10? The required probability can be read from the element in the second row and third column of $P$. It is $p_{2,3} = .90$.

The transition diagram for this DTMC is shown in Figure 2.1. Note that it has no arc from node 3 to node 2, representing the fact that $p_{3,2} = 0$. ∎

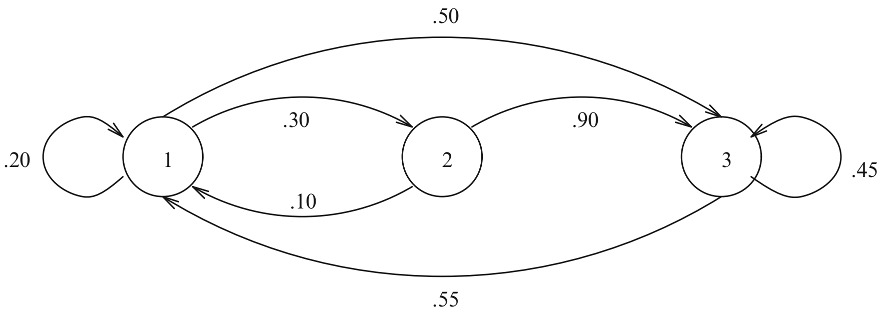


**Fig. 2.1** Transition diagram of the DTMC in Example 2.1.

Next we present two main characteristics of a transition probability matrix in the following theorem.

**Theorem 2.1.** (Properties of a Transition Probability Matrix). *Let* $P = [p_{i,j}]$ *be an* $N \times N$ *transition probability matrix of a DTMC* $\{X_n, n \geq 0\}$ *with state space* $S = \{1, 2, \ldots, N\}$. *Then*

1. $p_{i,j} \geq 0,\ 1 \leq i, j \leq N;$
2. $\sum_{j=1}^{N} p_{i,j} = 1,\ 1 \leq i \leq N.$

*Proof.* The nonnegativity of $p_{i,j}$ follows since it is a (conditional) probability. To prove the second assertion, we have

$$\sum_{j=1}^{N} p_{i,j} = \sum_{j=1}^{N} \mathsf{P}(X_{n+1} = j | X_n = i)$$
$$= \mathsf{P}(X_{n+1} \in S | X_n = i). \tag{2.6}$$

Since $X_{n+1}$ must take some value in the state space $S$, regardless of the value of $X_n$, it follows that the last quantity is 1. Hence the theorem follows. ∎

Any square matrix possessing the two properties of the theorem above is called a *stochastic matrix* and can be thought of as a transition probability matrix of a DTMC.

## 2.2 Examples of Markov Models

Discrete-time Markov chains appear as appropriate models in many fields: biological systems, inventory systems, queueing systems, computer systems, telecommunication systems, manufacturing systems, manpower systems, economic systems, and so on. The following examples will give some evidence of this diversity. In each of these examples, we derive the transition probability matrix for the appropriate DTMC.

*Example 2.2.* (Machine Reliability). The Depend-On-Us company manufactures a machine that is either up or down. If it is up at the beginning of a day, then it is up at the beginning of the next day with probability .98 (regardless of the history of the machine), or it fails with probability .02. Once the machine goes down, the company sends a repair person to repair it. If the machine is down at the beginning of a day, it is down at the beginning of the next day with probability .03 (regardless of the history of the machine), or the repair is completed and the machine is up with probability .97. A repaired machine is as good as new. Model the evolution of the machine as a DTMC.

Let $X_n$ be the state of the machine at the beginning of day $n$, defined as follows:

$$X_n = \begin{cases} 0 \text{ if the machine is down at the beginning of day } n, \\ 1 \text{ if the machine is up at the beginning of day } n. \end{cases}$$

The description of the system shows that $\{X_n, n \geq 0\}$ is a DTMC with state space $\{0, 1\}$ and the transition probability matrix

$$P = \begin{bmatrix} .03 & .97 \\ .02 & .98 \end{bmatrix}. \tag{2.7}$$

Now suppose the company maintains two such machines that are identical and behave independently of each other, and each has its own repair person. Let $Y_n$ be the number of machines in the "up" state at the beginning of day $n$. Is $\{Y_n, n \geq 0\}$ a DTMC?

First we identify the state space of $\{Y_n, n \geq 0\}$ to be $\{0, 1, 2\}$. Next we see if the Markov property holds; that is, we check if $\mathsf{P}(Y_{n+1} = j | Y_n = i, Y_{n-1}, \ldots, Y_0)$ depends only on $i$ and $j$ for $i, j = 0, 1, 2$. For example, consider the case $Y_n = i = 1$ and $Y_{n+1} = j = 0$. Thus, one machine is up (and one down) at time $n$. Since both machines are identical, it does not matter which is up and which is down. In order to move to state 0 at time $n + 1$, the down machine must stay down and the up machine must go down at the beginning of the next day. Since the machines are independent, the probability of this happening is $.03 * .02$, independent of the history of the two machines. Hence we get

$$\mathsf{P}(Y_{n+1} = 0 | Y_n = 1, Y_{n-1}, \ldots, Y_0) = .03 * .02 = .0006 = p_{1,0}.$$

Proceeding in this fashion, we construct the following transition probability matrix:

$$P = \begin{bmatrix} .0009 & .0582 & .9409 \\ .0006 & .0488 & .9506 \\ .0004 & .0392 & .9604 \end{bmatrix}. \; \blacksquare \tag{2.8}$$

*Example 2.3.* (Weather Model). The weather in the city of Heavenly is classified as sunny, cloudy, or rainy. Suppose that tomorrow's weather depends only on today's weather as follows: if it is sunny today, it is cloudy tomorrow with probability .3 and rainy with probability .2; if it is cloudy today, it is sunny tomorrow with probability .5 and rainy with probability .3; and finally, if it is rainy today, it is sunny tomorrow with probability .4 and cloudy with probability .5. Model the weather process as a DTMC.

Let $X_n$ be the weather conditions in Heavenly on day $n$, defined as follows:

$$X_n = \begin{cases} 1 & \text{if it is sunny on day } n, \\ 2 & \text{if it is cloudy on day } n, \\ 3 & \text{if it is rainy on day } n. \end{cases}$$

Then we are told that $\{X_n, n \geq 0\}$ is a DTMC with state space $\{1, 2, 3\}$. We next compute its transition matrix. We are given that $p_{1,2} = .3$ and $p_{1,3} = .2$. We are not explicitly given $p_{1,1}$. We use

$$p_{1,1} + p_{1,2} + p_{1,3} = 1$$

to obtain $p_{1,1} = .5$. Similarly, we can obtain $p_{2,2}$ and $p_{3,3}$. This yields the transition probability matrix

$$P = \begin{bmatrix} .50 & .30 & .20 \\ .50 & .20 & .30 \\ .40 & .50 & .10 \end{bmatrix}. \quad \blacksquare \tag{2.9}$$

*Example 2.4.* (Inventory System). Computers-R-Us stocks a wide variety of PCs for retail sale. It is open for business Monday through Friday 8:00 a.m. to 5:00 p.m. It uses the following operating policy to control the inventory of PCs. At 5:00 p.m. Friday, the store clerk checks to see how many PCs are still in stock. If the number is less than two, then he orders enough PCs to bring the total in stock up to five at the beginning of the business day Monday. If the number in stock is two or more, no action is taken. The demand for the PCs during the week is a Poisson random variable with mean 3. Any demand that cannot be immediately satisfied is lost. Develop a stochastic model of the inventory of the PCs at Computers-R-Us.

Let $X_n$ be the number of PCs in stock at Computers-R-Us at 8:00 a.m. Monday of the $n$th week. Let $D_n$ be the number of PCs demanded during the $n$th week. Then the number of PCs left in the store at the end of the week is $\max(X_n - D_n, 0)$. If $X_n - D_n \geq 2$, then there are two or more PCs left in the store at 5:00 p.m. Friday of the $n$th week. Hence no more PCs will be ordered that weekend, and we will have $X_{n+1} = X_n - D_n$. On the other hand, if $X_n - D_n \leq 1$, there are 1 or 0 PCs left in the store at the end of the week. Hence enough will be ordered over the weekend so that $X_{n+1} = 5$. Putting these observations together, we get

$$X_{n+1} = \begin{cases} X_n - D_n & \text{if } X_n - D_n \geq 2, \\ 5 & \text{if } X_n - D_n \leq 1. \end{cases}$$

**Table 2.1** Demand distribution for Example 2.4.

| $k \rightarrow$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $\mathsf{P}(D_n = k)$ | .0498 | .1494 | .2240 | .2240 | .1680 |
| $\mathsf{P}(D_n \geq k)$ | 1.000 | .9502 | .8008 | .5768 | .3528 |

It then follows that the state space of $\{X_n, n \geq 0\}$ is $\{2, 3, 4, 5\}$. Now assume that the demands from week to week are independent of each other and the inventory in the store. Then we can show that $\{X_n, n \geq 0\}$ is a DTMC. We have, for $j = 2, 3, 4$ and $i = j, j+1, \ldots, 5$,

$$\begin{aligned}\mathsf{P}(X_{n+1} &= j | X_n = i, X_{n-1}, \ldots, X_0) \\ &= \mathsf{P}(X_n - D_n = j | X_n = i, X_{n-1}, \ldots, X_0) \\ &= \mathsf{P}(i - D_n = j | X_n = i, X_{n-1}, \ldots, X_0) \\ &= \mathsf{P}(D_n = i - j).\end{aligned}$$

Similarly,

$$\mathsf{P}(X_{n+1} = 5 | X_n = i) = \mathsf{P}(X_n - D_n \leq 1 | X_n = i) = \mathsf{P}(D_n \geq i - 1)$$

for $i = 2, 3, 4$. Using the fact that $D_n$ is a P(3) random variable, we get Table 2.1. Using the data in this table, we can compute the transition probability matrix of the DTMC $\{X_n, n \geq 0\}$ as follows:

$$P = \begin{bmatrix} .0498 & 0 & 0 & .9502 \\ .1494 & .0498 & 0 & .8008 \\ .2240 & .1494 & .0498 & .5768 \\ .2240 & .2240 & .1494 & .4026 \end{bmatrix}. \tag{2.10}$$

Note the entry for $p_{5,5}$. The inventory can move from 5 to 5 if the demand is either 0 or at least 4. Hence we have

$$p_{5,5} = \mathsf{P}(D_n = 0) + \mathsf{P}(D_n \geq 4). \quad \blacksquare$$

*Example 2.5.* (Manufacturing). The Gadgets-R-Us company has a manufacturing setup consisting of two distinct machines, each producing one component per hour. Each component can be tested instantly to be identified as defective or nondefective. Let $a_i$ be the probability that a component produced by machine $i$ is nondefective, $i = 1, 2$. (Obviously $1-a_i$ is the probability that a component produced by machine $i$ is defective.) The defective components are discarded and the nondefective components produced by each machine are stored in two separate bins. When a component is present in each bin, the two are instantly assembled together and shipped out. Each bin can hold at most two components. When a bin is full, the corresponding machine is turned off. It is turned on again when the bin has space for at least one component. Model this system by a DTMC.

Let $A_n$ be the number of components in the bin of machine 1 and let $B_n$ be the number of components in the bin of machine 2 at the end of hour $n$, after accounting for the production and assembly during the hour. The bin-size restrictions imply that $0 \le A_n, B_n \le 2$. Note that both bins cannot be nonempty simultaneously since assembly is instantaneous. Thus, $A_n > 0$ implies that $B_n = 0$, and $B_n > 0$ implies that $A_n = 0$. Let $X_n = A_n - B_n$. Then $X_n$ can take values in $S = \{-2, -1, 0, 1, 2\}$. Note that we can recover $A_n$ and $B_n$ from $X_n$ as follows:

$$X_n \le 0 \Rightarrow A_n = 0; B_n = -X_n,$$
$$X_n \ge 0 \Rightarrow A_n = X_n; B_n = 0.$$

If the successive components are independent of each other, then $\{X_n, n \ge 0\}$ is a DTMC. For example, suppose the history $\{X_0, \ldots, X_{n-1}\}$ is known and $X_n = -1$. That is, the bin for machine 1 is empty, while the bin for machine 2 has one component in it. During the next hour, each machine will produce one new component. If both of these components are nondefective, then they will be assembled instantaneously. If both are defective, they will be discarded. In either case, $X_{n+1}$ will remain $-1$. If the newly produced component 1 is nondefective and the newly produced component 2 is defective, then the new component 2 will be discarded and the old component from bin 2 will be assembled with the newly produced nondefective component 1 and shipped out. Hence $X_{n+1}$ will become 0. If the new component 1 is defective while new component 2 is nondefective, $X_{n+1}$ will become $-2$. Bin 2 will now be full, and machine 2 will be turned off. In the next hour, only machine 1 will produce an item. From this analysis, we get

$$\mathsf{P}(X_{n+1} = j | X_n = -1, X_{n-1}, \ldots, X_0) = \begin{cases} a_1(1-a_2) & \text{if } j = 0, \\ a_1 a_2 + (1-a_1)(1-a_2) & \text{if } j = -1, \\ (1-a_1)a_2 & \text{if } j = -2, \\ 0 & \text{otherwise.} \end{cases}$$

Proceeding in this fashion, we can show that $\{X_n, n \ge 0\}$ is a DTMC on $\{-2, -1, 0, 1, 2\}$ with the following transition probability matrix (we use $a = (1-a_1)a_2, b = a_1 a_2 + (1-a_1)(1-a_2)$, and $c = a_1(1-a_2)$ for compactness):

$$P = \begin{bmatrix} 1-a_1 & a_1 & 0 & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a_2 & 1-a_2 \end{bmatrix}. \blacksquare \tag{2.11}$$

*Example 2.6.* (Manpower Planning). Paper Pushers, Inc., is an insurance company that employs 100 workers organized into four grades, labeled 1, 2, 3, and 4. For the sake of simplicity, we shall assume that workers may get promoted from one grade to another, or leave the company, only at the beginning of a week. A worker in grade 1 at the beginning of a week gets promoted to grade 2 with probability .03, leaves the company with probability .02, or continues in the same grade at the beginning of the next week. A worker in grade 2 at the beginning of a week gets promoted to grade 3 with probability .01, leaves the company with probability .008, or continues in the same grade at the beginning of the next week. A worker in grade 3 at the beginning of a week gets promoted to grade 4 with probability .005, leaves the company with probability .02, or continues in the same grade at the beginning of the next week. A worker in grade 4 at the beginning of a week either leaves the company with probability .01 or continues in the same grade at the beginning of the next week. If a worker leaves the company, he is instantly replaced by a new one in grade 1. Model the worker movement in this company using a DTMC.

We shall assume that all worker promotions are decided in an independent manner. This simplifies our model considerably. Instead of keeping track of all 100 workers, we keep track of a single worker, say worker number $k$, where $k = 1, 2, \ldots, 100$. We think of $k$ as a worker ID, and if and when this worker leaves the company, it gets assigned to the new replacement. Let $X_n^k$ be the grade that the $k$th worker is in at the beginning of the $n$th week. Now, if we assume that the worker promotions are determined independently of the history of the worker so far (meaning the time spent in a given grade does not affect one's chances of promotion), we see that, for $k = 1, 2, \ldots, 100$, $\{X_n^k, n \geq 0\}$ is a DTMC with state space $\{1, 2, 3, 4\}$. We illustrate the computation of the transition probabilities with an example. Suppose $X_n^k = 3$; i.e., employee number $k$ is in grade 3 at the beginning of week $n$. If he gets promoted (which happens with probability .005), we see that $X_{n+1}^k = 4$. Hence $\mathsf{P}(X_{n+1}^k = 4 | X_n^k = 3) = .005$. If he leaves the company (which happens with probability .02), he is replaced by a new employee in grade 1 carrying the ID $k$, making $X_{n+1}^k = 1$. Hence $\mathsf{P}(X_{n+1}^k = 1 | X_n^k = 3) = .02$. With the remaining probability, .975, the employee continues in the same grade in the next week, making $X_{n+1}^k = 3$. Hence $\mathsf{P}(X_{n+1}^k = 3 | X_n^k = 3) = .975$. Proceeding in a similar way, we obtain the following transition probability matrix:

$$P = \begin{bmatrix} .9700 & .0300 & 0 & 0 \\ .0080 & .9820 & .0100 & 0 \\ .0200 & 0 & .9750 & .0050 \\ .0100 & 0 & 0 & .9900 \end{bmatrix}. \tag{2.12}$$

Note that the 100 DTMCs $\{X_n^1, n \geq 0\}$ through $\{X_n^{100}, n \geq 0\}$ are independent of each other and have the same transition probability matrix. ∎

*Example 2.7.* (Stock Market). The common stock of the Gadgets-R-Us company is traded in the stock market. The chief financial officer of Gadgets-R-Us buys and sells the stock in his own company so that the price never drops below \$2 and never goes above \$10. For simplicity, we assume that $X_n$, the stock price at the end of day $n$, takes only integer values; i.e., the state space of $\{X_n, n \geq 0\}$ is $\{2, 3, \ldots, 9, 10\}$. Let $I_{n+1}$ be the potential movement in the stock price on day $n + 1$ in the absence of any intervention from the chief financial officer. Thus, we have

$$X_{n+1} = \begin{cases} 2 & \text{if } X_n + I_{n+1} \le 2, \\ X_n + I_{n+1} & \text{if } 2 < X_n + I_{n+1} < 10, \\ 10 & \text{if } X_n + I_{n+1} \ge 10. \end{cases}$$

Statistical analysis of past data suggests that the potential movements $\{I_n, n \ge 1\}$ form an iid sequence of random variables with common pmf given by

$$\mathsf{P}(I_n = k) = .2, \; k = -2, -1, 0, 1, 2.$$

This implies that $\{X_n, n \ge 0\}$ is a DTMC on $\{2, 3, \ldots, 9, 10\}$. We illustrate the computation of the transition probabilities with three cases:

$$\begin{aligned} \mathsf{P}(X_{n+1} = 2 | X_n = 3) &= \mathsf{P}(X_n + I_{n+1} \le 2 | X_n = 3) \\ &= \mathsf{P}(I_{n+1} \le -1) \\ &= .4, \end{aligned}$$

$$\begin{aligned} \mathsf{P}(X_{n+1} = 6 | X_n = 5) &= \mathsf{P}(X_n + I_{n+1} = 6 | X_n = 5) \\ &= \mathsf{P}(I_{n+1} = 1) \\ &= .2, \end{aligned}$$

$$\begin{aligned} \mathsf{P}(X_{n+1} = 10 | X_n = 10) &= \mathsf{P}(X_n + I_{n+1} \ge 10 | X_n = 10) \\ &= \mathsf{P}(I_{n+1} \ge 0) \\ &= .6. \end{aligned}$$

Proceeding in this fashion, we get the following transition probability matrix for the DTMC $\{X_n, n \ge 0\}$:

$$P = \begin{bmatrix} .6 & .2 & .2 & 0 & 0 & 0 & 0 & 0 & 0 \\ .4 & .2 & .2 & .2 & 0 & 0 & 0 & 0 & 0 \\ .2 & .2 & .2 & .2 & .2 & 0 & 0 & 0 & 0 \\ 0 & .2 & .2 & .2 & .2 & .2 & 0 & 0 & 0 \\ 0 & 0 & .2 & .2 & .2 & .2 & .2 & 0 & 0 \\ 0 & 0 & 0 & .2 & .2 & .2 & .2 & .2 & 0 \\ 0 & 0 & 0 & 0 & .2 & .2 & .2 & .2 & .2 \\ 0 & 0 & 0 & 0 & 0 & .2 & .2 & .2 & .4 \\ 0 & 0 & 0 & 0 & 0 & 0 & .2 & .2 & .6 \end{bmatrix}. \; \blacksquare \qquad (2.13)$$

*Example 2.8.* (Telecommunications). The Tel-All Switch Corporation manufactures switching equipment for communication networks. Communication networks move data from switch to switch at lightning-fast speed in the form of packets; i.e., strings of zeros and ones (called bits). The Tel-All switches handle data packets of constant length; i.e., the same number of bits in each packet. At a conceptual level, we can think of the switch as a storage device where packets arrive from network users according to a random process. They are stored in a buffer with the capacity to store $K$ packets and are removed from the buffer one-by-one according to a pre-specified protocol. Under one such protocol, time is slotted into intervals of fixed length, say a microsecond. If there is a packet in the buffer at the beginning of a slot, it is removed instantaneously. If there is no packet at the beginning of a slot, no packet is removed during the slot even if more packets arrive during the slot. If a packet arrives during a slot and there is no space for it, it is discarded. Model this as a DTMC.

Let $A_n$ be the number of packets that arrive at the switch during the $n$th slot. (Some of these may be discarded.) Let $X_n$ be the number of packets in the buffer at the end of the $n$th slot. Now, if $X_n = 0$, then there are no packets available for transmission at the beginning of the $(n+1)$st slot. Hence all the packets that arrive during that slot, namely $A_{n+1}$, are in the buffer at the end of that slot unless $A_{n+1} > K$, in which case the buffer is full at the end of the $(n+1)$st slot. Hence $X_{n+1} = \min\{A_{n+1}, K\}$. If $X_n > 0$, one packet is removed at the beginning of the $(n+1)$st slot and $A_{n+1}$ packets are added during that slot, subject to capacity limitations. Combining these cases, we get

$$X_{n+1} = \begin{cases} \min\{A_{n+1}, K\} & \text{if } X_n = 0, \\ \min\{X_n + A_{n+1} - 1, K\} & \text{if } 0 < X_n \le K. \end{cases}$$

Assume that $\{A_n, n \ge 1\}$ is a sequence of iid random variables with common pmf

$$\mathsf{P}(A_n = k) = a_k,\ k \ge 0.$$

Under this assumption, $\{X_n, n \ge 0\}$ is a DTMC on state space $\{0, 1, 2, \dots, K\}$. The transition probabilities can be computed as follows. For $0 \le j < K$,

$$\begin{aligned} \mathsf{P}(X_{n+1} = j | X_n = 0) &= \mathsf{P}(\min\{A_{n+1}, K\} = j | X_n = 0) \\ &= \mathsf{P}(A_{n+1} = j) \\ &= a_j, \\ \mathsf{P}(X_{n+1} = K | X_n = 0) &= \mathsf{P}(\min\{A_{n+1}, K\} = K | X_n = 0) \\ &= \mathsf{P}(A_{n+1} \ge K) \\ &= \sum_{k=K}^{\infty} a_k. \end{aligned}$$

Similarly, for $1 \le i \le K$ and $i - 1 \le j < K$,

$$\begin{aligned} \mathsf{P}(X_{n+1} = j | X_n = i) &= \mathsf{P}(\min\{X_n + A_{n+1} - 1, K\} = j | X_n = i) \\ &= \mathsf{P}(A_{n+1} = j - i + 1) \\ &= a_{j-i+1}. \end{aligned}$$

Finally, for $1 \le i \le K$,

$$\begin{aligned}\mathsf{P}(X_{n+1} = K|X_n = i) &= \mathsf{P}(\min\{X_n + A_{n+1} - 1, K\} = K|X_n = i)\\ &= \mathsf{P}(A_{n+1} \ge K - i + 1)\\ &= \sum_{k=K-i+1}^{\infty} a_k.\end{aligned}$$

Combining all these cases and using the notation

$$b_j = \sum_{k=j}^{\infty} a_k,$$

we get the transition probability matrix

$$P = \begin{bmatrix} a_0 & a_1 & \cdots & a_{K-1} & b_K \\ a_0 & a_1 & \cdots & a_{K-1} & b_K \\ 0 & a_0 & \cdots & a_{K-2} & b_{K-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_0 & b_1 \end{bmatrix}. \blacksquare \tag{2.14}$$

Armed with the collection of examples above, we set out to "analyze" them in the next section.

## 2.3 Transient Distributions

Let $\{X_n, n \ge 0\}$ be a time-homogeneous DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$ and initial distribution $a = [a_1, \ldots, a_N]$, where

$$a_i = \mathsf{P}(X_0 = i),\ 1 \le i \le N.$$

In this section, we concentrate on the *transient distribution*; i.e., the distribution of $X_n$ for a fixed $n \ge 0$. In other words, we are interested in $\mathsf{P}(X_n = j)$ for all $j \in S$ and $n \ge 0$. We have

$$\begin{aligned}\mathsf{P}(X_n = j) &= \sum_{i=1}^{N} \mathsf{P}(X_n = j|X_0 = i)\mathsf{P}(X_0 = i)\\ &= \sum_{i=1}^{N} a_i \mathsf{P}(X_n = j|X_0 = i).\end{aligned} \tag{2.15}$$

Thus it suffices to study the conditional probability $\mathsf{P}(X_n = j|X_0 = i)$. This quantity is called the *n-step transition probability* of the DTMC. We use the notation

$$a_j^{(n)} = \mathsf{P}(X_n = j), \tag{2.16}$$

$$p_{i,j}^{(n)} = \mathsf{P}(X_n = j|X_0 = i). \tag{2.17}$$

Analogous to the one-step transition probability matrix $P = [p_{i,j}]$, we build an *n-step transition probability matrix* as follows:

$$P^{(n)} = \begin{bmatrix} p_{1,1}^{(n)} & p_{1,2}^{(n)} & p_{1,3}^{(n)} & \cdots & p_{1,N}^{(n)} \\ p_{2,1}^{(n)} & p_{2,2}^{(n)} & p_{2,3}^{(n)} & \cdots & p_{2,N}^{(n)} \\ p_{3,1}^{(n)} & p_{3,2}^{(n)} & p_{3,3}^{(n)} & \cdots & p_{3,N}^{(n)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_{N,1}^{(n)} & p_{N,2}^{(n)} & p_{N,3}^{(n)} & \cdots & p_{N,N}^{(n)} \end{bmatrix}. \tag{2.18}$$

We discuss two cases, $P^{(0)}$ and $P^{(1)}$, below. We have

$$p_{i,j}^{(0)} = \mathsf{P}(X_0 = j|X_0 = i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

This implies that

$$P^{(0)} = I,$$

an $N \times N$ identity matrix. From (2.17) and (2.3), we see that

$$p_{i,j}^{(1)} = \mathsf{P}(X_1 = j|X_0 = i) = p_{i,j}. \tag{2.19}$$

Hence, from (2.19), (2.4), and (2.18), we get

$$P^{(1)} = P.$$

Now, construct the transient pmf vector

$$a^{(n)} = \left[a_1^{(n)}, a_2^{(n)}, \ldots, a_N^{(n)}\right], \tag{2.20}$$

so that (2.15) can be written in matrix form as

$$a^{(n)} = a * P^{(n)}. \tag{2.21}$$

Note that

$$a^{(0)} = a * P^{(0)} = a * I = a,$$

the initial distribution of the DTMC.

In this section, we develop methods of computing the $n$-step transition probability matrix $P^{(n)}$. The main result is given in the following theorem.

**Theorem 2.2.** ($n$-Step Transition Probability Matrix).

$$P^{(n)} = P^n, \tag{2.22}$$

*where $P^n$ is the nth power of the matrix $P$.*

*Proof.* Since $P^0 = I$ and $P^1 = P$, the theorem is true for $n = 0, 1$. Hence, let $n \geq 2$. We have

$$\begin{aligned}
p_{i,j}^{(n)} &= \mathsf{P}(X_n = j | X_0 = i) \\
&= \sum_{k=1}^{N} \mathsf{P}(X_n = j | X_{n-1} = k, X_0 = i) \mathsf{P}(X_{n-1} = k | X_0 = i) \\
&= \sum_{k=1}^{N} p_{i,k}^{(n-1)} \mathsf{P}(X_n = j | X_{n-1} = k, X_0 = i) \text{ (from (2.17))} \\
&= \sum_{k=1}^{N} p_{i,k}^{(n-1)} \mathsf{P}(X_n = j | X_{n-1} = k) \text{ (due to the Markov property)} \\
&= \sum_{k=1}^{N} p_{i,k}^{(n-1)} \mathsf{P}(X_1 = j | X_0 = k) \text{ (due to time homogeneity)} \\
&= \sum_{k=1}^{N} p_{i,k}^{(n-1)} p_{k,j}. 
\end{aligned} \tag{2.23}$$

The last sum can be recognized as a matrix multiplication operation, and the equation above, which is valid for all $1 \leq i, j \leq N$, can be written in a more succinct fashion in matrix terminology as

$$P^{(n)} = P^{(n-1)} * P. \tag{2.24}$$

Using the equation above for $n = 2$, we get

$$P^{(2)} = P^{(1)} * P = P * P.$$

Similarly, for $n = 3$, we get

$$P^{(3)} = P^{(2)} * P = P * P * P.$$

In general, using $P^n$ as the $n$th power of the matrix $P$, we get (2.22). ∎

From the theorem above, we get the following two corollaries.

**Corollary 2.1.**

$$a^{(n)} = a * P^n.$$

*Proof.* This is left to the reader as Conceptual Problem 2.5. ∎

**Corollary 2.2.**

$$\begin{aligned}\mathsf{P}(X_{n+m} = j | X_n = i, X_{n-1}, \dots, X_0) \\ = \mathsf{P}(X_{n+m} = j | X_n = i) = p_{ij}^{(m)}.\end{aligned}$$

*Proof.* Use induction on $m$. ∎

A more general version of (2.23) is given in the following theorem.

**Theorem 2.3.** (Chapman–Kolmogorov Equation). *The $n$-step transition probabilities satisfy the following equation, called the Chapman–Kolmogorov equation:*

$$p_{i,j}^{(n+m)} = \sum_{k=1}^{N} p_{i,k}^{(n)} p_{k,j}^{(m)}. \tag{2.25}$$

*Proof.*

$$\begin{aligned}\mathsf{P}(X_{n+m} = j | X_0 = i) &= \sum_{k=1}^{N} \mathsf{P}(X_{n+m} = j | X_n = k, X_0 = i)\mathsf{P}(X_n = k | X_0 = i) \\ &= \sum_{k=1}^{N} \mathsf{P}(X_{n+m} = j | X_n = k)\mathsf{P}(X_n = k | X_0 = i) \\ &\quad \text{(due to Corollary 2.2)} \\ &= \sum_{k=1}^{N} \mathsf{P}(X_m = j | X_0 = k)\mathsf{P}(X_n = k | X_0 = i) \\ &\quad \text{(due to time homogeneity)} \\ &= \sum_{k=1}^{N} \mathsf{P}(X_n = k | X_0 = i)\mathsf{P}(X_m = j | X_0 = k) \\ &= \sum_{k=1}^{N} p_{i,k}^{(n)} p_{k,j}^{(m)}. \quad ∎\end{aligned}$$

In matrix form, (2.25) can be expressed as

$$P^{(n+m)} = P^{(n)} * P^{(m)}. \tag{2.26}$$

Interchanging the roles of $n$ and $m$, we get

$$P^{(n+m)} = P^{(m)} * P^{(n)}.$$

The equations above imply that the matrices $P^{(n)}$ and $P^{(m)}$ commute for all $n$ and $m$. This is an unusual property for matrices. Theorem 2.2 makes it especially easy to compute the transient distributions in DTMCs since it reduces the computations to matrix powers and multiplication. Several matrix-oriented computer packages are available that make these computations easy to perform.

*Example 2.9.* Consider the three-state DTMC with the transition matrix as given in Example 2.1 and the initial distribution $a = [.1\ .4\ .5]$. Compute the pmf of $X_3$.

Using Corollary 2.1, we see that the pmf of $X_3$ can be computed as

$$\begin{aligned} a^{(3)} &= a * P^3 \\ &= [.1\ .4\ .5] * \begin{bmatrix} .20 & .30 & .50 \\ .10 & .00 & .90 \\ .55 & .00 & .45 \end{bmatrix}^3 \\ &= [0.3580\ 0.1258\ 0.5162]. \quad \blacksquare \end{aligned}$$

*Example 2.10.* (Weather Model). Mr. and Mrs. Happy have planned to celebrate their 25th wedding anniversary in Honeymooners' Paradise, a popular resort in the city of Heavenly. Counting today as the first day, they are supposed to be there on the seventh and eighth days. They are thinking about buying vacation insurance that promises to reimburse them for the entire vacation package cost of \$2500 if it rains on both days, and nothing is reimbursed otherwise. The insurance costs \$200. Suppose the weather in the city of Heavenly changes according to the model in Example 2.3. Assuming that it is sunny today in the city of Heavenly, should Mr. and Mrs. Happy buy the insurance?

Let $R$ be the reimbursement the couple gets from the insurance company. Letting $X_n$ be the weather on the $n$th day, we see that $X_1 = 1$ and

$$R = \begin{cases} 2500 & \text{if } X_7 = X_8 = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Hence

$$\begin{aligned} \mathsf{E}(R) &= 2500\mathsf{P}(X_7 = X_8 = 3 | X_1 = 1) \\ &= 2500\mathsf{P}(X_8 = 3 | X_7 = 3, X_1 = 1)\mathsf{P}(X_7 = 3 | X_1 = 1) \\ &= 2500\mathsf{P}(X_1 = 3 | X_0 = 3)\mathsf{P}(X_6 = 3 | X_0 = 1) \\ &= (2500)(.10) \left[ \begin{bmatrix} .50 & .30 & .20 \\ .50 & .20 & .30 \\ .40 & .50 & .10 \end{bmatrix}^6 \right]_{1,3} \\ &= 52.52. \end{aligned}$$

Since this is less than \$200, the insurance is not worth it. $\blacksquare$

*Example 2.11.* (Manufacturing). Consider the manufacturing operation described in Example 2.5. Suppose that both bins are empty at time 0 at the beginning of an 8-hour shift. What is the probability that both bins are empty at the end of the 8-hour shift? (Assume $a_1 = .99$ and $a_2 = .995$.)

Let $\{X_n, n \geq 0\}$ be the DTMC described in Example 2.5. We see that we are interested in $p^{(8)}_{0,0}$. Using the data given above, we see that the transition probability matrix is given by

$$P = \begin{bmatrix} .0100 & .9900 & 0 & 0 & 0 \\ .00995 & .9851 & .00495 & 0 & 0 \\ 0 & .00995 & .9851 & .00495 & 0 \\ 0 & 0 & .00995 & .9851 & .00495 \\ 0 & 0 & 0 & .9950 & .0050 \end{bmatrix}. \tag{2.27}$$

Computing $P^8$, we get

$$p^{(8)}_{0,0} = .88938.$$

What is the probability that machine 2 is shut down at the end of the shift? It is given by

$$p^{(8)}_{0,-2} = .0006533. \blacksquare$$

*Example 2.12.* (Telecommunications). Consider the model of the Tel-All data switch as described in Example 2.8. Let $Y_n$ be the number of packets *lost* during the $n$th time slot. Show how to compute $\mathsf{E}(Y_n)$ assuming that the buffer is initially empty. Tabulate $\mathsf{E}(Y_n)$ as a function of $n$ if the buffer size is seven packets and the number of packets that arrive at the switch during one time slot is a Poisson random variable with mean 1, packet arrivals in successive time slots being iid.

Let $X_n$ be the number of packets in the buffer at the end of the $n$th time slot. Then $\{X_n, n \geq 0\}$ is a DTMC as described in Example 2.8. Let $A_n$ be the number of packet arrivals during the $n$th slot. Then

$$Y_{n+1} = \begin{cases} \max\{0, A_{n+1} - K\} & \text{if } X_n = 0, \\ \max\{0, X_n - 1 + A_{n+1} - K\} & \text{if } X_n > 0. \end{cases}$$

Hence

$$\begin{aligned} \mathsf{E}(Y_{n+1}) = {} & \mathsf{E}(\max\{0, A_{n+1} - K\}) p^{(n)}_{0,0} \\ & + \sum_{k=1}^{K} \mathsf{E}(\max\{0, k - 1 + A_{n+1} - K\}) p^{(n)}_{0,k}. \end{aligned}$$

Using $a_r = \mathsf{P}(A_n = r)$, we get

$$\begin{aligned}\mathsf{E}(Y_{n+1}) = p_{0,0}^{(n)} \sum_{r=K}^{\infty} (r-K)a_r \\ + \sum_{k=1}^{K} p_{0,k}^{(n)} \sum_{r=K+1-k}^{\infty} (r-K-1+k)a_r.\end{aligned} \tag{2.28}$$

We are given that $A_n$ is P(1). The pmf and complementary cdf of $A_n$ are as given in Table 2.2.

Using the data given there, and using the analysis of Example 2.8, we see that $\{X_n, n \geq 0\}$ is a DTMC with state space $\{0, 1, \ldots, 7\}$ and transition probability matrix

$$P = \begin{bmatrix} .3679 & .3679 & .1839 & .0613 & .0153 & .0031 & .0005 & .0001 \\ .3679 & .3679 & .1839 & .0613 & .0153 & .0031 & .0005 & .0001 \\ 0 & .3679 & .3679 & .1839 & .0613 & .0153 & .0031 & .0006 \\ 0 & 0 & .3679 & .3679 & .1839 & .0613 & .0153 & .0037 \\ 0 & 0 & 0 & .3679 & .3679 & .1839 & .0613 & .0190 \\ 0 & 0 & 0 & 0 & .3679 & .3679 & .1839 & .0803 \\ 0 & 0 & 0 & 0 & 0 & .3679 & .3679 & .2642 \\ 0 & 0 & 0 & 0 & 0 & 0 & .3679 & .6321 \end{bmatrix}. \tag{2.29}$$

The matrix $P^n$ is computed numerically from $P$, and the $n$-step transition probabilities $p_{0,k}^{(n)}$ are obtained from it by using Theorem 2.2. These are then used in (2.28) to compute the expected number of packets lost in each slot. The final answer is given in Table 2.3.

Note that even if the switch removes one packet per time slot and one packet arrives at the switch on average per time slot, the losses are not zero. This is due to the randomness in the arrival process. Another interesting feature to note is that, as $n$ becomes large, $\mathsf{E}(Y_n)$ seems to tend to a limiting value. We present more on this in Section 2.5. ∎

**Table 2.2** Data for Example 2.12.

| $r \rightarrow$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| $\mathsf{P}(A_n = r)$ | .3679 | .3679 | .1839 | .0613 | .0153 | .0031 | .0005 | .0001 |
| $\mathsf{P}(A_n \geq r)$ | 1.000 | .6321 | .2642 | .0803 | .0190 | .0037 | .0006 | .0001 |

**Table 2.3** Expected packet loss in Example 2.12.

| $n \rightarrow$ | 1 | 2 | 5 | 10 | 20 | 30 | 40 | 80 |
|---|---|---|---|---|---|---|---|---|
| $\mathsf{E}(Y_n)$ | .0000 | .0003 | .0063 | .0249 | .0505 | .0612 | .0654 | .0681 |

## 2.4 Occupancy Times

Let $\{X_n, n \geq 0\}$ be a time-homogeneous DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$ and initial distribution $a = [a_1, \ldots, a_N]$. In this section, we study occupancy times; i.e., the expected amount of time the DTMC spends in a given state during a given interval of time. Since the DTMC undergoes one transition per unit time, the occupancy time is the same as the expected number of times it visits a given state in a finite number of transitions. We define this quantity formally below.

Let $N_j(n)$ be the number of times the DTMC visits state $j$ over the time span $\{0, 1, \ldots, n\}$, and let

$$m_{i,j}(n) = \mathsf{E}(N_j(n)|X_0 = i).$$

The quantity $m_{i,j}(n)$ is called the *occupancy time* up to $n$ of state $j$ starting from state $i$. Let

$$M(n) = \begin{bmatrix} m_{1,1}(n) & m_{1,2}(n) & m_{1,3}(n) & \cdots & m_{1,N}(n) \\ m_{2,1}(n) & m_{2,2}(n) & m_{2,3}(n) & \cdots & m_{2,N}(n) \\ m_{3,1}(n) & m_{3,2}(n) & m_{3,3}(n) & \cdots & m_{3,N}(n) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{N,1}(n) & m_{N,2}(n) & m_{N,3}(n) & \cdots & m_{N,N}(n) \end{bmatrix} \tag{2.30}$$

be the *occupancy times matrix*. The next theorem gives a simple method of computing the occupancy times.

**Theorem 2.4.** (Occupancy Times). *Let $\{X_n, n \geq 0\}$ be a time-homogeneous DTMC on state space $S = \{1, 2, \ldots, N\}$, with transition probability matrix $P$. The occupancy times matrix is given by*

$$M(n) = \sum_{r=0}^{n} P^r. \tag{2.31}$$

*Proof.* Fix $i$ and $j$. Let $Z_n = 1$ if $X_n = j$ and $Z_n = 0$ if $X_n \neq j$. Then,

$$N_j(n) = Z_0 + Z_1 + \cdots + Z_n.$$

Hence

$$\begin{aligned} m_{i,j}(n) &= \mathsf{E}(N_j(n)|X_0 = i) \\ &= \mathsf{E}(Z_0 + Z_1 + \cdots + Z_n|X_0 = i) \\ &= \sum_{r=0}^{n} \mathsf{E}(Z_r|X_0 = i) \end{aligned}$$

$$= \sum_{r=0}^{n} \mathsf{P}(Z_r = 1 | X_0 = i)$$
$$= \sum_{r=0}^{n} \mathsf{P}(X_r = j | X_0 = i)$$
$$= \sum_{r=0}^{n} p_{i,j}^{(r)}. \tag{2.32}$$

We get (2.31) by writing the equation above in matrix form. ■

*Example 2.13.* (Three-State DTMC). Consider the DTMC in Example 2.1. Compute the occupancy time matrix $M(10)$.

We have, from Theorem 2.4,

$$M(10) = \sum_{r=0}^{10} \begin{bmatrix} .20 & .30 & .50 \\ .10 & .00 & .90 \\ .55 & .00 & .45 \end{bmatrix}^r$$
$$= \begin{bmatrix} 4.5317 & 1.2484 & 5.2198 \\ 3.5553 & 1.9555 & 5.4892 \\ 3.8583 & 1.0464 & 6.0953 \end{bmatrix}.$$

Thus the expected number of visits to state 1 starting from state 1 over the first ten transitions is 4.5317. Note that this includes the visit at time 0. ■

*Example 2.14.* (Weather Model). Consider the three-state weather model described in Example 2.3. Suppose it is sunny in Heavenly today. Compute the expected number of rainy days in the week starting today.

Let $X_n$ be the weather in Heavenly on the $n$th day. Then, from Example 2.3, $\{X_n, n \geq 0\}$ is a DTMC with transition matrix $P$ given in (2.9). The required quantity is given by $m_{1,3}(6)$ (why 6 and not 7?). The occupancy matrix $M(6)$ is given by

$$M(6) = \sum_{r=0}^{6} \begin{bmatrix} .5 & .3 & .2 \\ .5 & .2 & .3 \\ .4 & .5 & .1 \end{bmatrix}^r$$
$$= \begin{bmatrix} 3.8960 & 1.8538 & 1.2503 \\ 2.8876 & 2.7781 & 1.3343 \\ 2.8036 & 2.0218 & 2.1747 \end{bmatrix}.$$

Hence $m_{1,3}(6) = 1.25$. ■

## 2.5 Limiting Behavior

Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. In this section, we study the limiting behavior of $X_n$ as $n$ tends to infinity. We start with the most obvious question:

*Does the pmf of $X_n$ approach a limit as n tends to infinity?*

If it does, we call it the *limiting or steady-state distribution* and denote it by

$$\pi = [\pi_1, \pi_2, \ldots, \pi_N], \tag{2.33}$$

where

$$\pi_j = \lim_{n\to\infty} \mathsf{P}(X_n = j), \; j \in S. \tag{2.34}$$

The next question is a natural follow-up:

*If the limiting distribution exists, is it unique?*

This question makes sense since it is conceivable that the limit may depend upon the starting state, or the initial distribution of the DTMC. Finally, the question of practical importance is:

*If there is a unique limiting distribution, how do we compute it?*

It so happens that the answers to the first two questions are complex but the answer to the last question is easy. Hence we give that first in the following theorem.

**Theorem 2.5.** (Limiting Distributions). *If a limiting distribution $\pi$ exists, it satisfies*

$$\pi_j = \sum_{i=1}^{N} \pi_i p_{i,j}, \; j \in S, \tag{2.35}$$

*and*

$$\sum_{j=1}^{N} \pi_j = 1. \tag{2.36}$$

*Proof.* Conditioning on $X_n$ and using the Law of Total Probability, we get

$$\mathsf{P}(X_{n+1} = j) = \sum_{i=1}^{N} \mathsf{P}(X_n = i) p_{i,j}, \; j \in S. \tag{2.37}$$

Now, let $n$ tend to infinity on both the right- and left-hand sides. Then, assuming that the limiting distribution exists, we see that

$$\lim_{n\to\infty} \mathsf{P}(X_n = j) = \lim_{n\to\infty} \mathsf{P}(X_{n+1} = j) = \pi_j.$$

Substituting in (2.37), we get (2.35). Equation (2.36) follows since $\pi$ is a pmf. ∎

Equations (2.35) can be written in matrix form as

$$\pi = \pi P \tag{2.38}$$

and are called *the balance equations* or *the steady-state equations*. Equation (2.36) is called the *normalizing equation*. We illustrate Theorem 2.5 with an example.

*Example 2.15.* (A DTMC with a Unique Limiting Distribution). Suppose $\{X_n, n \geq 0\}$ is a DTMC with state space $\{1, 2, 3\}$ and the following transition matrix (see Example 2.1):

$$P = \begin{bmatrix} .20 & .30 & .50 \\ .10 & .00 & .90 \\ .55 & .00 & .45 \end{bmatrix}.$$

We give below the $n$-step transition probability matrix for various values of $n$:

$$n = 2\colon P^2 = \begin{bmatrix} .3450 & .0600 & .5950 \\ .5150 & .0300 & .4550 \\ .3575 & .1650 & .4775 \end{bmatrix},$$

$$n = 4\colon P^4 = \begin{bmatrix} .3626 & .1207 & .5167 \\ .3558 & .1069 & .5373 \\ .3790 & .1052 & .5158 \end{bmatrix},$$

$$n = 10\colon P^{10} = \begin{bmatrix} .3704 & .1111 & .5185 \\ .3703 & .1111 & .5186 \\ .3704 & .1111 & .5185 \end{bmatrix},$$

$$n \geq 11\colon P^n = \begin{bmatrix} .3704 & .1111 & .5185 \\ .3704 & .1111 & .5185 \\ .3704 & .1111 & .5185 \end{bmatrix},$$

From this we see that the pmf of $X_n$ approaches

$$\pi = [.3704, .1111, .5185].$$

It can be checked that $\pi$ satisfies (2.35) and (2.36). Furthermore, all the rows of $P^n$ are the same in the limit, implying that the limiting distribution of $X_n$ is the same regardless of the initial distribution. ∎

*Example 2.16.* (A DTMC with No Limiting Distribution). Consider a DTMC $\{X_n, n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ .10 & 0 & .90 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.39}$$

We can check numerically that

$$P^{2n} = \begin{bmatrix} .1000 & 0 & .9000 \\ 0 & 1.0000 & 0 \\ .1000 & 0 & .9000 \end{bmatrix}, \; n \ge 1,$$

$$P^{2n-1} = \begin{bmatrix} 0 & 1.0000 & 0 \\ .1000 & 0 & .9000 \\ 0 & 1.0000 & 0 \end{bmatrix}, \; n \ge 1.$$

Let $a$ be the initial distribution of the DTMC. We see that the pmf of $X_n, n \ge 1$, is $[.1(a_1 + a_3), a_2, .9(a_1 + a_3)]$ if $n$ is even and $[.1a_2, a_1 + a_3, .9a_2]$ if $n$ is odd. Thus the pmf of $X_n$ does not approach a limit. It fluctuates between two pmfs that depend on the initial distribution. The DTMC has no limiting distribution. ∎

Although there is no limiting distribution for the DTMC of the example above, we can still solve the balance equations and the normalizing equation. The solution is unique and is given by [.05 .50 .45]. To see what this solution means, suppose the initial distribution of the DTMC is [.05 .50 .45]. Then, using (2.37) for $n = 0$, we can show that the pmf of $X_1$ is also given by [.05 .50 .45]. Proceeding this way, we see that the pmf of $X_n$ is [.05 .50 .45] for all $n \ge 0$. Thus the pmf of $X_n$ remains the same for all $n$ if the initial distribution is chosen to be [.05 .50 .45]. We call this initial distribution a *stationary distribution*. Formally, we have the following definition.

**Definition 2.2.** (Stationary Distribution). A distribution

$$\pi^* = [\pi_1^*, \pi_2^*, \ldots, \pi_N^*] \tag{2.40}$$

is called a stationary distribution if

$$\begin{aligned} \mathsf{P}(X_0 = i) = \pi_i^* \text{ for all } 1 \le i \le N \Rightarrow \\ \mathsf{P}(X_n = i) = \pi_i^* \text{ for all } 1 \le i \le N, \text{and } n \ge 0. \end{aligned}$$

The questions about the limiting distribution (namely existence, uniqueness, and method of computation) can be asked about the stationary distribution as well. We have a slightly stronger result for the stationary distribution, as given in the following theorem.

**Theorem 2.6.** (Stationary Distributions). $\pi^* = [\pi_1^*, \pi_2^*, \ldots, \pi_N^*]$ *is a stationary distribution if and only if it satisfies*

$$\pi_j^* = \sum_{i=1}^{N} \pi_i^* p_{i,j}, \; j \in S, \tag{2.41}$$

*and*

$$\sum_{j=1}^{N} \pi_j^* = 1. \tag{2.42}$$

*Proof.* First suppose $\pi^*$ is a stationary distribution. This implies that if

$$\mathsf{P}(X_0 = j) = \pi_j^*, \quad j \in S,$$

then

$$\mathsf{P}(X_1 = j) = \pi_j^*, \quad j \in S.$$

But, using $n = 0$ in (2.37), we have

$$\mathsf{P}(X_1 = j) = \sum_{i=1}^{N} \mathsf{P}(X_0 = i) p_{i,j}.$$

Substituting the first two equations into the last, we get (2.41). Equation (2.42) holds because $\pi^*$ is a pmf.

Now suppose $\pi^*$ satisfies (2.41) and (2.42). Suppose

$$\mathsf{P}(X_0 = j) = \pi_j^*, \; j \in S.$$

Then, from (2.37), we have

$$\begin{aligned}
\mathsf{P}(X_1 = j) &= \sum_{i=1}^{N} \mathsf{P}(X_0 = i) p_{i,j} \\
&= \sum_{i=1}^{N} \pi_i^* p_{i,j} \\
&= \pi_j^* \text{ due to (2.41)}.
\end{aligned}$$

Thus the pmf of $X_1$ is $\pi^*$. Using (2.37) repeatedly, we can show that the pmf of $X_n$ is $\pi^*$ for all $n \geq 0$. Hence $\pi^*$ is a stationary distribution. ∎

Theorem 2.6 implies that, if there is a solution to (2.41) and (2.42), it is a stationary distribution. Also, note that the stationary distribution $\pi^*$ satisfies the same balance equations and normalizing equation as the limiting distribution $\pi$. This yields the following corollary.

**Corollary 2.3.** *A limiting distribution, when it exists, is also a stationary distribution.*

*Proof.* Let $\pi$ be a limiting distribution. Then, from Theorem 2.5, it satisfies (2.35) and (2.36). But these are the same as (2.41) and (2.42). Hence, from Theorem 2.6, $\pi^* = \pi$ is a stationary distribution. ∎

*Example 2.17.* Using Theorem 2.6 or Corollary 2.3, we see that $\pi^* = [.3704, .1111, .5185]$ is a (unique) limiting, as well as stationary distribution to the DTMC of Example 2.15. Similarly, $\pi^* = [.05,\ .50,\ .45]$ is a stationary distribution for the DTMC in Example 2.16, but there is no limiting distribution for this DTMC. ∎

The next example shows that the limiting or stationary distributions need not be unique.

*Example 2.18.* (A DTMC with Multiple Limiting and Stationary Distributions). Consider a DTMC $\{X_n, n \geq 0\}$ with state space $\{1, 2, 3\}$ and transition matrix

$$P = \begin{bmatrix} .20 & .80 & 0 \\ .10 & .90 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.43}$$

Computing the matrix powers $P^n$ for increasing values of $n$, we get

$$\lim_{n\to\infty} P^n = \begin{bmatrix} .1111 & .8889 & 0 \\ .1111 & .8889 & 0 \\ 0 & 0 & 1.0000 \end{bmatrix}.$$

This implies that the limiting distribution exists. Now let the initial distribution be $a = [a_1, a_2, a_3]$, and define

$$\begin{aligned} \pi_1 &= .1111(a_1 + a_2), \\ \pi_2 &= .8889(a_1 + a_2), \\ \pi_3 &= a_3. \end{aligned}$$

We see that $\pi$ is a limiting distribution of $\{X_n, n \geq 0\}$. Thus the limiting distribution exists but is not unique. It depends on the initial distribution. From Corollary 2.3, it follows that any of the limiting distributions is also a stationary distribution of this DTMC. ∎

It should be clear by now that we need to find a normalized solution (i.e., a solution satisfying the normalizing equation) to the balance equations in order to study the limiting behavior of the DTMC. There is another important interpretation of the normalized solution to the balance equations, as discussed below.

Let $N_j(n)$ be the number of times the DTMC visits state $j$ over the time span $\{0, 1, \ldots, n\}$. We studied the expected value of this quantity in Section 2.4. The *occupancy* of state $j$ is defined as

$$\hat{\pi}_j = \lim_{n\to\infty} \frac{\mathsf{E}(N_j(n)|X_0 = i)}{n + 1}. \tag{2.44}$$

Thus occupancy of state $j$ is the same as the long-run fraction of the time the DTMC spends in state $j$. The next theorem shows that the *occupancy distribution*

$$\hat{\pi} = [\hat{\pi}_1, \hat{\pi}_2, \ldots, \hat{\pi}_N],$$

if it exists, satisfies the same balance and normalizing equations.

**Theorem 2.7.** (Occupancy Distribution). *If the occupancy distribution $\hat{\pi}$ exists, it satisfies*

$$\hat{\pi}_j = \sum_{i=1}^{N} \hat{\pi}_i \, p_{i,j}, \; j \in S, \tag{2.45}$$

*and*

$$\sum_{j=1}^{N} \hat{\pi}_j = 1. \tag{2.46}$$

*Proof.* From Theorem 2.4, we have

$$\mathsf{E}(N_j(n)|X_0 = i) = m_{i,j}(n) = \sum_{r=0}^{n} p_{i,j}^{(r)}.$$

Hence,

$$\begin{aligned}
\frac{\mathsf{E}(N_j(n)|X_0 = i)}{n+1} &= \frac{1}{n+1} \sum_{r=0}^{n} p_{i,j}^{(r)} \\
&= \frac{1}{n+1} \left( p_{i,j}^{(0)} + \sum_{r=1}^{n} p_{i,j}^{(r)} \right) \\
&= \frac{1}{n+1} \left( p_{i,j}^{(0)} + \sum_{r=1}^{n} \sum_{k=1}^{N} p_{i,k}^{(r-1)} p_{k,j} \right) \\
&\quad \text{(using Chapman–Kolmogorov equations)} \\
&= \frac{1}{n+1} \left( p_{i,j}^{(0)} + \sum_{k=1}^{N} \sum_{r=1}^{n} p_{i,k}^{(r-1)} p_{k,j} \right) \\
&= \frac{1}{n+1} (p_{i,j}^{(0)}) + \frac{n}{n+1} \sum_{k=1}^{N} \frac{1}{n} \left( \sum_{r=0}^{n-1} p_{i,k}^{(r)} p_{k,j} \right).
\end{aligned}$$

Now, let $n$ tend to $\infty$,

$$\begin{aligned}
\lim_{n\to\infty} \frac{\mathsf{E}(N_j(n)|X_0 = i)}{n+1} &= \lim_{n\to\infty} \frac{1}{n+1} (p_{i,j}^{(0)}) \\
&\quad + \lim_{n\to\infty} \left( \frac{n}{n+1} \right) \sum_{k=1}^{N} \lim_{n\to\infty} \frac{1}{n} \left( \sum_{r=0}^{n-1} p_{i,k}^{(r)} \right) p_{k,j}.
\end{aligned}$$

Assuming the limits exist and using (2.44) and (2.32), we get

$$\hat{\pi}_j = \sum_{k=1}^{N} \hat{\pi}_k p_{k,j},$$

which is (2.45). The normalization equation (2.46) follows because

$$\sum_{j=1}^{N} N_j(n) = n + 1. \blacksquare$$

Thus the normalized solution of the balance equations can have as many as three interpretations: limiting distribution, stationary distribution, or occupancy distribution. The questions are: Will there always be a solution? Will it be unique? When can this solution be interpreted as a limiting distribution, stationary distribution, or occupancy distribution? Although these questions can be answered strictly in terms of solutions of linear systems of equations, it is more useful to develop the answers in terms of the DTMC framework. That is what we do below.

**Definition 2.3.** (Irreducible DTMC). A DTMC $\{X_n, n \geq 0\}$ on state space $S = \{1, 2, \ldots, N\}$ is said to be irreducible if, for every $i$ and $j$ in $S$, there is a $k > 0$ such that

$$\mathsf{P}(X_k = j | X_0 = i) > 0. \tag{2.47}$$

A DTMC that is not irreducible is called reducible.

Note that the condition in (2.47) holds if and only if it is possible to go from any state $i$ to any state $j$ in the DTMC in one or more steps or, alternatively, there is a directed path from any node $i$ to any node $j$ in the transition diagram of the DTMC. It is in general easy to check if the DTMC is irreducible.

*Example 2.19.* (Irreducible DTMCs).

(a) The DTMC of Example 2.15 is irreducible since the DTMC can visit any state from any other state in two or fewer steps.
(b) The DTMC of Example 2.3 is irreducible since it can go from any state to any other state in one step.
(c) The five-state DTMC of Example 2.5 is irreducible since it can go from any state to any other state in four steps or less.
(d) The nine-state DTMC of Example 2.7 is irreducible since it can go from any state to any other state in seven steps or less.
(e) The $(K + 1)$-state DTMC of Example 2.8 is irreducible since it can go from any state to any other state in $K$ steps or less. $\blacksquare$

*Example 2.20.* (Reducible DTMCs). The DTMC of Example 2.18 is reducible since this DTMC cannot visit state 3 from state 1 or 2. $\blacksquare$

The usefulness of the concept of irreducibility arises from the following two theorems, whose proofs are beyond the scope of this book.

**Theorem 2.8.** (Unique Stationary Distribution). *A finite-state irreducible DTMC has a unique stationary distribution; i.e., there is a unique normalized solution to the balance equation.*

**Theorem 2.9.** (Unique Occupancy Distribution). *A finite-state irreducible DTMC has a unique occupancy distribution and is equal to the stationary distribution.*

Next we introduce the concept of periodicity. This will help us decide when the limiting distribution exists.

**Definition 2.4.** (Periodicity). Let $\{X_n, n \geq 0\}$ be an irreducible DTMC on state space $S = \{1, 2, \ldots, N\}$, and let $d$ be the largest integer such that

$$\mathsf{P}(X_n = i | X_0 = i) > 0 \Rightarrow n \text{ is an integer multiple of } d \tag{2.48}$$

for all $i \in S$. The DTMC is said to be periodic with period $d$ if $d > 1$ and aperiodic if $d = 1$.

A DTMC with period $d$ can return to its starting state only at times $d$, $2d, 3d, \ldots$. It is an interesting fact of irreducible DTMCs that it is sufficient to find the largest $d$ satisfying (2.48) for any one state $i \in S$. All other states are guaranteed to produce the same $d$. This makes it easy to establish the periodicity of an irreducible DTMC. In particular, if $p_{i,i} > 0$ for any $i \in S$ for an irreducible DTMC, then $d$ must be 1 and the DTMC must be aperiodic!

Periodicity is also easy to spot from the transition diagrams. First, define a directed cycle in the transition diagram as a directed path from any node to itself. If all the directed cycles in the transition diagram of the DTMC are multiples of some integer $d$ and this is the largest such integer, then this is the $d$ of the definition above.

*Example 2.21.* (Aperiodic DTMCs). All the irreducible DTMCs mentioned in Example 2.19 are aperiodic since each of them has at least one state $i$ with $p_{i,i} > 0$. ■

*Example 2.22.* (Periodic DTMCs).

(a) Consider a DTMC on state space $\{1, 2\}$ with the transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{2.49}$$

This DTMC is periodic with period 2.

(b) Consider a DTMC on state space $\{1, 2, 3\}$ with the transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.50}$$

This DTMC is periodic with period 3.

(c) Consider a DTMC on state space $\{1, 2, 3\}$ with the transition matrix

$$P = \begin{bmatrix} 0 & 1 & 0 \\ .5 & 0 & .5 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.51}$$

This DTMC is periodic with period 2. ■

The usefulness of the concept of irreducibility arises from the following main theorem, whose proof is beyond the scope of this book.

**Theorem 2.10.** (Unique Limiting Distribution). *A finite-state irreducible aperiodic DTMC has a unique limiting distribution.*

Theorem 2.10, along with Theorem 2.5, shows that the (unique) limiting distribution of an irreducible aperiodic DTMC is given by the solution to (2.35) and (2.36). From Corollary 2.3, this is also the stationary distribution of the DTMC, and from Theorem 2.9 this is also the occupancy distribution of the DTMC.

We shall restrict ourselves to irreducible and aperiodic DTMCs in our study of limiting behavior. The limiting behavior of periodic and/or reducible DTMCs is more involved. For example, the pmf of $X_n$ eventually cycles with period $d$ if $\{X_n, n \geq 0\}$ is an irreducible periodic DTMC with period $d$. The stationary/limiting distribution of a reducible DTMC is not unique and depends on the initial state of the DTMC. We refer the reader to an advanced text for a more complete discussion of these cases.

We end this section with several examples.

*Example 2.23.* (Three-State DTMC). Consider the DTMC of Example 2.15. This DTMC is irreducible and aperiodic. Hence the limiting distribution, the stationary distribution, and the occupancy distribution all exist and are given by the unique solution to

$$[\pi_1 \; \pi_2 \; \pi_3] = [\pi_1 \; \pi_2 \; \pi_3] * \begin{bmatrix} .20 & .30 & .50 \\ .10 & .00 & .90 \\ .55 & .00 & .45 \end{bmatrix}$$

and the normalizing equation

$$\pi_1 + \pi_2 + \pi_3 = 1.$$

Note that although we have four equations in three unknowns, one of the balance equations is redundant. Solving the equations above simultaneously yields

$$\pi_1 = .3704, \pi_2 = .1111, \pi_3 = .5185.$$

This matches our answer in Example 2.15. Thus we have

$$\pi = \pi^* = \hat{\pi} = [.3704,\ .1111,\ .5185].$$

Now consider the three-state DTMC from Example 2.16. This DTMC is irreducible but periodic. Hence there is no limiting distribution. However, the stationary distribution exists and is given by the solution to

$$[\pi_1^* \ \pi_2^* \ \pi_3^*] = [\pi_1^* \ \pi_2^* \ \pi_3^*] * \begin{bmatrix} 0 & 1 & 0 \\ .10 & 0 & .90 \\ 0 & 1 & 0 \end{bmatrix}$$

and

$$\pi_1^* + \pi_2^* + \pi_3^* = 1.$$

The solution is

$$[\pi_1^* \ \pi_2^* \ \pi_3^*] = [.0500, \ .5000, \ .4500].$$

This matches with the numerical analysis presented in Example 2.16. Since the DTMC is irreducible, the occupancy distribution is also given by $\hat{\pi} = \pi^*$. Thus the DTMC spends 45% of the time in state 3 in the long run. ∎

*Example 2.24.* (Telecommunications). Consider the DTMC model of the Tel-All data switch described in Example 2.12.

(a) Compute the long-run fraction of the time that the buffer is full.
Let $X_n$ be the number of packets in the buffer at the beginning of the $n$th time slot. Then $\{X_n, n \geq 0\}$ is a DTMC on state space $\{0, 1, \ldots, 7\}$ with transition probability matrix $P$ given in (2.29). We want to compute the long-run fraction of the time the buffer is full; i.e., the occupancy of state 7. Since this is an irreducible aperiodic DTMC, the occupancy distribution exists and is given by the solution to

$$\hat{\pi} = \hat{\pi} * P$$

and

$$\sum_{i=0}^{7} \hat{\pi}_i = 1.$$

The solution is given by

$$\hat{\pi} = [.0681, \ .1171, \ .1331, \ .1361, \ .1364, \ .1364, \ .1364, \ .1364].$$

The occupancy of state 7 is .1364. Hence the buffer is full 13.64% of the time.

(b) Compute the expected number of packets waiting in the buffer in steady state.
Note that the DTMC has a limiting distribution and is given by $\pi = \hat{\pi}$. Hence the expected number of packets in the buffer in steady state is given by

$$\lim_{n\to\infty} \mathsf{E}(X_n) = \sum_{i=0}^{7} i\pi_i = 3.7924.$$

Thus the buffer is a little more than half full on average in steady state. ∎

*Example 2.25.* (Manufacturing). Consider the manufacturing operation of the Gadgets-R-Us company as described in Example 2.11. Compute the long-run fraction of the time that both the machines are operating.

Let $\{X_n, n \geq 0\}$ be the DTMC described in Example 2.5 with state space $\{-2, -1, 0, 1, 2\}$ and the transition probability matrix given in (2.27). We are interested in computing the long-run fraction of the time that the DTMC spends in states $-1, 0, 1$. This is an irreducible and aperiodic DTMC. Hence the occupancy distribution exists and is given by the solution to

$$\hat{\pi} = \hat{\pi} * \begin{bmatrix} .0100 & .9900 & 0 & 0 & 0 \\ .00995 & .9851 & .00495 & 0 & 0 \\ 0 & .00995 & .9851 & .00495 & 0 \\ 0 & 0 & .00995 & .9851 & .00495 \\ 0 & 0 & 0 & .9950 & .0050 \end{bmatrix}$$

and

$$\sum_{i=-2}^{2} \hat{\pi}_i = 1.$$

Solving, we get

$$\hat{\pi} = [.0057,\ .5694,\ .2833,\ .1409,\ .0007].$$

Hence the long-run fraction of the time that both machines are working is given by

$$\hat{\pi}_{-1} + \hat{\pi}_0 + \hat{\pi}_1 = 0.9936. \quad \blacksquare$$

## 2.6 Cost Models

Recall the inventory model of Example 1.3(c), where we were interested in computing the total cost of carrying the inventory over 10 weeks. In this section, we develop methods of computing such costs. We start with a simple cost model first.

Let $X_n$ be the state of a system at time $n$. Assume that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{1, 2, \ldots, N\}$ with transition probability matrix $P$. Suppose the system incurs a random cost of $C(i)$ dollars every time it visits state $i$. Let $c(i) = \mathsf{E}(C(i))$ be the expected cost incurred at every visit to state $i$. Although we think of $c(i)$ as a cost per visit, it need not be so. It may be any other quantity, like reward per visit, loss per visit, profit per visit, etc. We shall consider two cost-performance measures in the two subsections below.

### *2.6.1 Expected Total Cost over a Finite Horizon*

In this subsection, we shall develop methods of computing *expected total cost* (ETC) up to a given finite time $n$, called the horizon. The actual cost incurred at time $r$ is $C(X_r)$. Hence the actual total cost up to time $n$ is given by

$$\sum_{r=0}^{n} C(X_r),$$

and the ETC is given by

$$\mathsf{E}\left(\sum_{r=0}^{n} C(X_r)\right).$$

For $1 \le i \le N$, define

$$g(i,n) = \mathsf{E}\left(\sum_{r=0}^{n} C(X_r)\middle| X_0 = i\right) \tag{2.52}$$

as the ETC up to time $n$ starting from state $i$. Next, let

$$c = \begin{bmatrix} c(1) \\ c(2) \\ \vdots \\ c(N) \end{bmatrix}$$

and

$$g(n) = \begin{bmatrix} g(1,n) \\ g(2,n) \\ \vdots \\ g(N,n) \end{bmatrix}.$$

Let $M(n)$ be the occupancy time matrix of the DTMC as defined in (2.30). The next theorem gives a method of computing $g(n)$ in terms of $M(n)$.

**Theorem 2.11.** (ETC: Finite Horizon).

$$g(n) = M(n) * c. \tag{2.53}$$

*Proof.* We have

$$\begin{aligned} g(i,n) &= \mathsf{E}\left(\sum_{r=0}^{n} C(X_r)\middle| X_0 = i\right) \\ &= \sum_{r=0}^{n} \sum_{j=1}^{N} \mathsf{E}(C(X_r)|X_r = j)\mathsf{P}(X_r = j|X_0 = i) \end{aligned}$$

$$= \sum_{r=0}^{n} \sum_{j=1}^{N} c(j) p_{i,j}^{(r)}$$

$$= \sum_{j=1}^{N} \left[ \sum_{r=0}^{n} p_{i,j}^{(r)} \right] c(j)$$

$$= \sum_{j=1}^{N} m_{i,j}(n) c(j), \tag{2.54}$$

where the last equation follows from (2.32). This yields (2.53) in matrix form. ∎

We illustrate the theorem with several examples.

*Example 2.26.* (Manufacturing). Consider the manufacturing model of Example 2.11. Assume that both bins are empty at the beginning of a shift. Compute the expected total number of assembled units produced during an 8-hour shift.

Let $\{X_n, n \geq 0\}$ be the DTMC described in Example 2.5. The transition probability matrix is given by (see (2.27))

$$P = \begin{bmatrix} .0100 & .9900 & 0 & 0 & 0 \\ .00995 & .9851 & .00495 & 0 & 0 \\ 0 & .00995 & .9851 & .00495 & 0 \\ 0 & 0 & .00995 & .9851 & .00495 \\ 0 & 0 & 0 & .9950 & .0050 \end{bmatrix}. \tag{2.55}$$

Recall that $a_i$ is the probability that a component produced by machine $i$ is nondefective, $i = 1, 2$. Let $c(i)$ be the expected number of assembled units produced in 1 hour if the DTMC is in state $i$ at the beginning of the hour. (Note that $c(i)$ as defined here is not a cost but can be treated as such!) Thus, if $i = 0$, both the bins are empty and a unit is assembled in the next hour if both machines produce nondefective components. Hence the expected number of assembled units produced per visit to state 0 is $a_1 a_2 = .99 * .995 = .98505$. A similar analysis for other states yields

$$c(-2) = .99, c(-1) = .99, c(1) = .995, c(2) = .995.$$

We want to compute $g(0, 7)$. (Note that the production during the eighth hour is counted as production at time 7.) Using Theorem 2.4 and (2.53), we get

$$g(7) = \begin{bmatrix} 7.9195 \\ 7.9194 \\ 7.8830 \\ 7.9573 \\ 7.9580 \end{bmatrix}.$$

Hence the expected production during an 8-hour shift starting with both bins empty is 7.8830 units. If there were no defectives, the production would be 8 units. Thus the loss due to defective production is .1170 units on this shift! ∎

*Example 2.27.* (Inventory Systems). Consider the DTMC model of the inventory system as described in Example 2.4. Suppose the store buys the PCs for \$1500 and sells them for \$1750. The weekly storage cost is \$50 per PC that is in the store at the beginning of the week. Compute the net revenue the store expects to get over the next 10 weeks, assuming that it begins with five PCs in stock at the beginning of the week.

Following Example 2.4, let $X_n$ be the number of PCs in the store at the beginning of the $n$th week. $\{X_n, n \geq 0\}$ is a DTMC on state space $\{2, 3, 4, 5\}$ with the transition probability matrix given in (2.10). We are given $X_0 = 5$. If there are $i$ PCs at the beginning of the $n$th week, the expected storage cost during that week is $50i$. Let $D_n$ be the demand during the $n$th week. Then the expected number of PCs sold during the $n$th week is $\mathsf{E}(\min(i, D_n))$. Hence the expected net revenue is given as

$$c(i) = -50i + (1750 - 1500)\mathsf{E}(\min(i, D_n)), \; 2 \leq i \leq 5.$$

Computing the expectations above, we get

$$c = \begin{bmatrix} 337.7662 \\ 431.9686 \\ 470.1607 \\ 466.3449 \end{bmatrix}.$$

Note that the expected total net revenue over the next $n$ weeks, starting in state $i$, is given by $g(i, n-1)$. Hence we need to compute $g(5, 9)$. Using Theorem 2.4 and (2.53), we get

$$g(9) = \begin{bmatrix} 4298.65 \\ 4381.17 \\ 4409.41 \\ 4404.37 \end{bmatrix}.$$

Hence the expected total net revenue over the next 10 weeks, starting with five PCs, is \$4404.37. Note that the figure is higher if the initial inventory is 4! This is the result of storage costs. ∎

### 2.6.2 *Long-Run Expected Cost Per Unit Time*

The ETC $g(i, n)$ computed in the previous subsection tends to $\infty$ as $n$ tends to $\infty$ in many examples. In such cases, it makes more sense to compute the expected long-run cost rate, defined as

$$g(i) = \lim_{n \to \infty} \frac{g(i, n)}{n + 1}.$$

The following theorem shows that this long-run cost rate is independent of $i$ when the DTMC is irreducible and gives an easy method of computing it.

**Theorem 2.12.** (Long-Run Cost Rate). *Suppose* $\{X_n, n \geq 0\}$ *is an irreducible DTMC with occupancy distribution* $\hat{\pi}$*. Then*

$$g = g(i) = \sum_{j=1}^{N} \hat{\pi}_j c(j). \tag{2.56}$$

*Proof.* From Theorem 2.9, we have

$$\lim_{n\to\infty} \frac{m_{i,j}(n)}{n+1} = \hat{\pi}_j.$$

Using this and Theorem 2.11, we get

$$\begin{aligned} g(i) &= \lim_{n\to\infty} \frac{g(i,n)}{n+1} \\ &= \lim_{n\to\infty} \frac{1}{n+1} \sum_{j=1}^{N} m_{i,j}(n) c(j) \\ &= \sum_{j=1}^{N} \left[ \lim_{n\to\infty} \frac{m_{i,j}(n)}{n+1} \right] c(j) \\ &= \sum_{j=1}^{N} \hat{\pi}_j c(j). \end{aligned}$$

This yields the desired result. ■

The theorem is intuitive: in the long run, among all the visits to all the states, $\hat{\pi}_j$ is the fraction of the visits made by the DTMC to state $j$. The DTMC incurs a cost of $c(j)$ dollars for every visit to state $j$. Hence the expected cost per visit in the long run must be $\sum c(j)\hat{\pi}_j$. We can use Theorem 2.7 to compute the occupancy distribution $\hat{\pi}$. We illustrate this with two examples below.

*Example 2.28.* (Manpower Planning). Consider the manpower planning model of Paper Pushers Insurance Company, Inc., as described in Example 2.6. Suppose the company has 70 employees and this level does not change with time. Suppose the per person weekly payroll expenses are \$400 for grade 1, \$600 for grade 2, \$800 for grade 3, and \$1000 for grade 4. Compute the long-run weekly payroll expenses for the company.

We shall compute the long-run weekly payroll expenses for each employee slot and multiply that figure by 70 to get the final answer since all employees behave identically. The grade of an employee evolves according to a DTMC with state space $\{1, 2, 3, 4\}$ and transition probability matrix as given in (2.12). Since this is an irreducible DTMC, the unique occupancy distribution is obtained using (2.45) and (2.46) as

$$\hat{\pi} = [.2715,\ .4546,\ .1826,\ .0913].$$

The cost vector is

$$c = [400\ 600\ 800\ 1000]'.$$

Hence the long-run weekly payroll expense for a single employee is

$$\sum_{j=1}^{4} \hat{\pi}_j c(j) = 618.7185.$$

For the 70 employees, we get as the total weekly payroll expense $\$70 * 618.7185 =$ \$43,310.29. ■

*Example 2.29.* (Telecommunications). Consider the model of the data switch as described in Examples 2.8 and 2.12. Compute the long-run packet-loss rate if the parameters of the problem are as in Example 2.12.

Let $c(i)$ be the expected number of packets lost during the $(n+1)$st slot if there were $i$ packets in the buffer at the end of the $n$th slot. Following the analysis of Example 2.12, we get

$$\begin{aligned} c(i) &= \sum_{r=K}^{\infty} (r-K)a_r \text{ if } i = 0 \\ &= \sum_{r=K+1-i}^{\infty} (r-K-1+i)a_r \text{ if } 0 < i \le K, \end{aligned}$$

where $a_r$ is the probability that a Poisson random variable with parameter 1 takes a value $r$. Evaluating these sums, we get

$$c = [.0000,\ .0000,\ .0001,\ .0007,\ .0043,\ .0233,\ .1036,\ .3679].$$

The occupancy distribution of this DTMC has already been computed in Example 2.24. It is given by

$$\hat{\pi} = [.0681,\ .1171,\ .1331,\ .1361,\ .1364,\ .1364,\ .1364,\ .1364].$$

Hence the long-run rate of packet loss per slot is

$$\sum_{j=0}^{7} \hat{\pi}_j c(j) = .0682.$$

Since the arrival rate of packets is one packet per slot, this implies that the loss fraction is 6.82%. This is too high in practical applications. This loss can be reduced by either increasing the buffer size or reducing the input packet rate. Note that the expected number of packets lost during the $n$th slot, as computed in Example 2.12, was .0681 for $n = 80$. This agrees quite well with the long-run loss rate computed in this example. ∎

## 2.7 First-Passage Times

We saw in Example 1.3(a) that one of the questions of interest in weather prediction was "How long will the current heat wave last?" If the heat wave is declared to be over when the temperature falls below 90°F, the problem can be formulated as "When will the stochastic process representing the temperature visit a state below 90°F?" Questions of this sort lead us to study the *first-passage time*; i.e., the random time at which a stochastic process "first passes into" a given subset of the state space. In this section, we study the first-passage times in DTMCs.

Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. We shall first study a simple case, first-passage time into state $N$, defined as

$$T = \min\{n \geq 0 : X_n = N\}. \tag{2.57}$$

Note that $T$ is *not* the minimum number of steps in which the DTMC can reach state $N$. It is the (random) number of steps that it takes to actually visit state $N$. Typically $T$ can take values in $\{0, 1, 2, 3, \ldots\}$. We shall study the expected value of this random variable in detail below.

Let

$$m_i = \mathsf{E}(T|X_0 = i). \tag{2.58}$$

Clearly, $m_N = 0$. The next theorem gives a method of computing $m_i, 1 \leq i \leq N-1$.

**Theorem 2.13.** (Expected First-Passage Times). $\{m_i, 1 \leq i \leq N-1\}$ *satisfy*

$$m_i = 1 + \sum_{j=1}^{N-1} p_{i,j} m_j, \; 1 \leq i \leq N-1. \tag{2.59}$$

*Proof.* We condition on $X_1$. Suppose $X_0 = i$ and $X_1 = j$. If $j = N$, then $T = 1$, and if $j \neq N$, then the DTMC has already spent one time unit to go to state $j$ and the expected time from then on to reach state $N$ is now given by $m_j$. Hence we get

$$\mathsf{E}(T|X_0 = i, X_1 = j) = \begin{cases} 1 & \text{if } j = N, \\ 1 + m_j & \text{if } j \neq N. \end{cases}$$

Unconditioning with respect to $X_1$ yields

$$\begin{aligned}
m_i &= \mathsf{E}(T|X_0 = i) \\
&= \sum_{j=1}^{N} \mathsf{E}(T|X_0 = i, X_1 = j)\mathsf{P}(X_1 = j|X_0 = i) \\
&= \sum_{j=1}^{N-1} (1 + m_j)p_{i,j} + (1)(p_{i,N}) \\
&= \sum_{j=1}^{N} (1)p_{i,j} + \sum_{j=1}^{N-1} p_{i,j}m_j \\
&= 1 + \sum_{j=1}^{N-1} p_{i,j}m_j
\end{aligned}$$

as desired. ∎

The following examples illustrate the theorem above.

*Example 2.30.* (Machine Reliability). Consider the machine shop with two independent machines as described by the three-state DTMC $\{Y_n, n \geq 0\}$ in Example 2.2. Suppose both machines are up at time 0. Compute the expected time until both machines are down for the first time.

Let $Y_n$ be the number of machines in the "up" state at the beginning of day $n$. From Example 2.2, we see that $\{Y_n, n \geq 0\}$ is a DTMC with state space $\{0, 1, 2\}$ and transition probability matrix given by

$$P = \begin{bmatrix} .0009 & .0582 & .9409 \\ .0006 & .0488 & .9506 \\ .0004 & .0392 & .9604 \end{bmatrix}. \tag{2.60}$$

Let $T$ be the first-passage time into state 0 (both machines down). We are interested in $m_2 = \mathsf{E}(T|Y_0 = 2)$. Equations (2.59) become

$$\begin{aligned}
m_2 &= 1 + .9604m_2 + .0392m_1, \\
m_1 &= 1 + .9506m_2 + .0488m_1.
\end{aligned}$$

Solving simultaneously, we get

$$m_1 = 2451 \text{ days}, m_2 = 2451.5 \text{ days}.$$

Thus the expected time until both machines are down is $2451.5/365 = 6.71$ years! ∎

*Example 2.31.* (Manpower Planning). Consider the manpower model of Example 2.6. Compute the expected amount of time a new recruit spends with the company.

Note that the new recruit starts in grade 1. Let $X_n$ be the grade of the new recruit at the beginning of the $n$th week. If the new recruit has left the company by the beginning of the $n$th week, we set $X_n = 0$. Then, using the data in Example 2.6, we see that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{0, 1, 2, 3, 4\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ .020 & .950 & .030 & 0 & 0 \\ .008 & 0 & .982 & .010 & 0 \\ .020 & 0 & 0 & .975 & .005 \\ .010 & 0 & 0 & 0 & .990 \end{bmatrix}. \tag{2.61}$$

Note that state 0 is absorbing since once the new recruit leaves the company the problem is finished. Let $T$ be the first-passage time into state 0. We are interested in $m_1 = \mathsf{E}(T|X_0 = 1)$. Equations (2.59) can be written as follows:

$$\begin{aligned} m_1 &= 1 + .950m_1 + .030m_2, \\ m_2 &= 1 + .982m_2 + .010m_3, \\ m_3 &= 1 + .975m_3 + .005m_4, \\ m_4 &= 1 + .990m_4. \end{aligned}$$

Solving simultaneously, we get

$$m_1 = 73.33, m_2 = 88.89, m_3 = 60, m_4 = 100.$$

Thus the new recruit stays with the company for 73.33 weeks, or about 1.4 years. ■

So far we have dealt with a first-passage time into a single state. What if we are interested in a first-passage time into a set of states? We consider such a case next.

Let $A$ be a subset of states in the state space, and define

$$T = \min\{n \geq 0 : X_n \in A\}. \tag{2.62}$$

Theorem 2.13 can be easily extended to the case of the first-passage time defined above. Let $m_i(A)$ be the expected time to reach the set $A$ starting from state $i$. Clearly, $m_i(A) = 0$ if $i \in A$. Following the same argument as in the proof of Theorem 2.13, we can show that

$$m_i(A) = 1 + \sum_{j \notin A} p_{i,j} m_j(A), \; i \notin A. \tag{2.63}$$

In matrix form, the equations above can be written as

$$m(A) = e + P(A)m(A), \tag{2.64}$$

where $m(A)$ is a column vector $[m_i(A)]_{i \notin A}$, $e$ is a column vector of ones, and $P(A) = [p_{i,j}]_{i,j \notin A}$ is a submatrix of $P$. A matrix language package can be used to solve this equation easily. We illustrate this with an example.

*Example 2.32.* (Stock Market). Consider the model of stock movement as described in Example 2.7. Suppose Mr. Jones buys the stock when it is trading for \$5 and decides to sell it as soon as it trades at or above \$8. What is the expected amount of time that Mr. Jones will end up holding the stock?

Let $X_n$ be the value of the stock in dollars at the end of the $n$th day. From Example 2.7, $\{X_n, n \geq 0\}$ is a DTMC on state space $\{2, 3, \ldots, 9, 10\}$. We are given that $X_0 = 5$. Mr. Jones will sell the stock as soon as $X_n$ is 8 or 9 or 10. Thus we are interested in the first-passage time $T$ into the set $A = \{8, 9, 10\}$, in particular in $m_5(A)$. Equations (2.64) are

$$\begin{bmatrix} m_2(A) \\ m_3(A) \\ m_4(A) \\ m_5(A) \\ m_6(A) \\ m_7(A) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} .6 & .2 & .2 & 0 & 0 & 0 \\ .4 & .2 & .2 & .2 & 0 & 0 \\ .2 & .2 & .2 & .2 & .2 & 0 \\ 0 & .2 & .2 & .2 & .2 & .2 \\ 0 & 0 & .2 & .2 & .2 & .2 \\ 0 & 0 & 0 & .2 & .2 & .2 \end{bmatrix} \begin{bmatrix} m_2(A) \\ m_3(A) \\ m_4(A) \\ m_5(A) \\ m_6(A) \\ m_7(A) \end{bmatrix}.$$

Solving the equation above, we get

$$\begin{bmatrix} m_2(A) \\ m_3(A) \\ m_4(A) \\ m_5(A) \\ m_6(A) \\ m_7(A) \end{bmatrix} = \begin{bmatrix} 24.7070 \\ 23.3516 \\ 21.0623 \\ 17.9304 \\ 13.2601 \\ 9.0476 \end{bmatrix}.$$

Thus the expected time for the stock to reach \$8 or more, starting from \$5, is about 18 days.
■

*Example 2.33.* (Gambler's Ruin). Two gamblers, A and B, bet on successive independent tosses of a coin that lands heads up with probability $p$. If the coin turns up heads, gambler A wins a dollar from gambler B, and if the coin turns up tails, gambler B wins a dollar from gambler A. Thus the total number of dollars among the two gamblers stays fixed, say $N$. The game stops as soon as either gambler is ruined; i.e., is left with no money! Compute the expected duration of the game, assuming that the game stops as soon as one of the two gamblers is ruined. Assume the initial fortune of gambler A is $i$.

Let $X_n$ be the amount of money gambler A has after the $n$th toss. If $X_n = 0$, then gambler A is ruined and the game stops. If $X_n = N$, then gambler B is ruined and the game stops. Otherwise the game continues. We have

$$X_{n+1} = \begin{cases} X_n & \text{if } X_n \text{ is 0 or } N, \\ X_n + 1 & \text{if } 0 < X_n < N \text{ and the coin turns up heads,} \\ X_n - 1 & \text{if } 0 < X_n < N \text{ and the coin turns up tails.} \end{cases}$$

Since the successive coin tosses are independent, we see that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{0, 1, \ldots, N-1, N\}$ with the following transition probability matrix (with $q = 1 - p$):

$$P = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ q & 0 & p & \cdots & 0 & 0 & 0 \\ 0 & q & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & p & 0 \\ 0 & 0 & 0 & \cdots & q & 0 & p \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}. \tag{2.65}$$

The game ends when the DTMC visits state 0 or $N$. Thus we are interested in $m_i(A)$, where $A = \{0, N\}$. Equations (2.63) are

$$\begin{aligned} m_0(A) &= 0, \\ m_i(A) &= 1 + q m_{i-1}(A) + p m_{i+1}(A), \quad 1 \leq i \leq N-1, \\ m_N(A) &= 0. \end{aligned}$$

We leave it to the reader to verify that the solution, whose derivation is rather tedious, is given by

$$m_i(A) = \begin{cases} \dfrac{i}{q-p} - \dfrac{N}{q-p} \cdot \dfrac{1-(q/p)^i}{1-(q/p)^N} & \text{if } q \neq p, \\ (N-i)(i) & \text{if } q = p. \end{cases} \tag{2.66}$$

∎

## 2.8 Case Study: Passport Credit Card Company

This case study, is inspired by a paper by P. E. Pfeifer and R. L. Carraway (2000).

Passport is a consumer credit card company that has a large number of customers (or accounts). These customers charge some of their purchases on their Passport cards. The charges made in one month are due by the end of the next month. If a customer fails to make the minimum payment in a given month, the company flags the account as delinquent. The company keeps track of the payment history of each customer so that it can identify customers who are likely to default on their obligations and not pay their debt to the company.

Here we describe the simplest method by which passport tracks its accounts. A customer is said to be in state (or delinquency stage) $k$ if he or she has missed making the minimum payment for the last $k$ consecutive months. A customer in state $k$ has four possible futures: make a minimum payment (or more) and move to stage 0, make no payment (or less than the minimum payment) and move to stage $k+1$, default by declaring bankruptcy, thus moving to stage $D$, or the company can cancel the customer's card and terminate the account, in which case the customer moves to stage $C$. Currently the company has a simple policy: it terminates an account as soon as it misses seven minimum payments in a row and writes off the remaining outstanding balance on that account as a loss.

To make the discussion above more precise, let $p_k$ be the probability that a customer in state $k$ fails to make the minimum payment in the current period and thus moves to state $k+1$. Let $q_k$ be the probability that a customer in state $k$ declares bankruptcy in the current period and thus moves to state $D$. Also, let $b_k$ be the average outstanding balance of a customer in state $k$.

From its experience with its customers over the years, the company has estimated the parameters above for $0 \leq k \leq 6$ as given in Table 2.4. Note that the company has no data for $k > 6$ since it terminates an account as soon as it misses the seventh payment in a row.

We will build stochastic models using DTMCs to help the management of Passport analyze the performance of this policy in a rational way. First we assume that the state of an account changes in a Markov fashion. Also, when a customer account is terminated or the customer declares bankruptcy, we shall simply replace that account with an active one, so that the number of accounts does not change. This is the same modeling trick we used in the manpower planning model of Example 2.6.

Let $X_n$ be the state of a particular customer account at time $n$ (i.e., during the $n$th month). When the customer goes bankrupt or the account is closed, we start a new account in state 0. Thus $\{X_n, n \geq 0\}$ is a stochastic process on state space $S = \{0, 1, 2, 3, 4, 5, 6\}$. We assume that it is a DTMC. In this case, the dynamics of $\{X_n, n \geq 0\}$ are given by

$$X_{n+1} = \begin{cases} X_n + 1 & \text{if the customer misses the minimum payment in the } n\text{th} \\ & \text{month} \\ 0 & \text{if the customer makes the minimum payment in the } n\text{th} \\ & \text{month, declares bankruptcy, or the account is terminated.} \end{cases}$$

**Table 2.4** Data for Passport account holders.

| $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $p_k$ | .033 | .048 | .090 | .165 | .212 | .287 | .329 |
| $q_k$ | .030 | .021 | .037 | .052 | .075 | .135 | .182 |
| $b_k$ | 1243.78 | 2090.33 | 2615.16 | 3073.13 | 3502.99 | 3905.77 | 4280.26 |

With the interpretation above, we see that $\{X_n, n \geq 0\}$ is a DTMC on state space $\{0, 1, 2, 3, 4, 5, 6\}$ with the following transition probabilities:

$$p_{k,k+1} = \mathsf{P}(X_{n+1} = k+1 | X_n = k) = p_k, \quad 0 \leq k \leq 5,$$
$$p_{k,0} = 1 - p_k, \quad 0 \leq k \leq 6.$$

Using the data from Table 2.4, we get the following transition probability matrix:

$$P = \begin{bmatrix} .967 & .033 & 0 & 0 & 0 & 0 & 0 \\ .952 & 0 & .048 & 0 & 0 & 0 & 0 \\ .910 & 0 & 0 & .090 & 0 & 0 & 0 \\ .835 & 0 & 0 & 0 & .165 & 0 & 0 \\ .788 & 0 & 0 & 0 & 0 & .212 & 0 \\ .713 & 0 & 0 & 0 & 0 & 0 & .287 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{2.67}$$

We are now ready to analyze the current policy ($P_c$), which terminates an account as soon as it misses the seventh minimum payment in a row. To analyze the performance of the policy, we need a performance measure. Although one can devise many performance measures, here we concentrate on the expected annual loss due to bankruptcies and account closures. Let $l_k$ be the expected loss from an account in state $k$ in one month. Now, for $0 \leq k \leq 6$, a customer in state $k$ declares bankruptcy with probability $q_k$, and that leads to a loss of $b_k$, the outstanding balance in that account. Additionally, in state 6, a customer fails to make the minimum payment with probability $p_6$, in which case the account is terminated with probability 1 and creates a loss of $b_6$. Thus

$$l_k = \begin{cases} q_k b_k & \text{if } 0 \leq k \leq 5, \\ (p_6 + q_6) b_6 & \text{if } k = 6. \end{cases}$$

Let $l = [l_0, l_1, \cdots, l_6]$. Using the data in Table 2.4, we get

$$l = [37.31 \quad 43.90 \quad 96.76 \quad 159.80 \quad 262.72 \quad 527.28 \quad 779.01].$$

Now note that the transition probability matrix $P$ of (2.67) is irreducible and aperiodic. From Theorem 2.10 we see that the (unique) limiting distribution of such a DTMC exists and is given as the solution to (2.35) and (2.36). Solving these, we get

$$\pi = [0.9664 \quad 0.0319 \quad 0.0015 \quad 0.0001 \quad 0.0000 \quad 0.0000 \quad 0.0000].$$

Hence the long-run net loss per account is given by

$$L_a = \sum_{k=0}^{6} \pi_k l_k = 37.6417$$

dollars per month. Of course, the company must generate far more than this in revenue on average from each account to stay in business.

Now Passport has been approached by a debt collection agency, We Mean Business, or WMB for short. If a customer declares bankruptcy, Passport loses the entire outstanding balance as before. However, if a customer does not declare bankruptcy, the company can decide to terminate the account and turn it over to the WMB company. If Passport decides to do this, WMB pays Passport 75% of the current outstanding balance on that account. When an account is turned over to WMB, it collects the outstanding balance on the account from the account holder by (barely) legal means. Passport also has to pay WMB an annual retainer fee of \$50,000 for this service. Passport management wants to decide if they should hire WMB and, if they do, when they should turn over an account to them.

Passport management has several possible policy options if it decides to retain WMB's services. We study six such policy options, denoted by $P_m$ $(2 \le m \le 7)$. Under $P_m$, Passport turns over an account to WMB as soon as it misses $m$ minimum payments in a row. Clearly, under $P_m$, $\{X_n, n \ge 0\}$ is a DTMC with state space $\{0, 1, \cdots, m-1\}$. The expected cost $l_k$ in state $k$ is the same as under the policy $P_c$ for $0 \le k \le m-2$. In state $m-1$, we have

$$l_{m-1} = .25 p_{m-1} b_{m-1} + q_{m-1} b_{m-1}.$$

We give the main steps in the performance evaluation of $P_2$. In this case, $\{X_n, n \ge 0\}$ is a DTMC on state space $\{0, 1\}$ with transition probability matrix

$$P = \begin{bmatrix} .967 & .033 \\ 1 & 0 \end{bmatrix} \tag{2.68}$$

and limiting distribution

$$\pi = [0.96805 \quad 0.03195].$$

The expected loss vector is

$$l = [37.3134 \quad 94.06485].$$

Hence the long-run net loss per account is given by

$$L_a = \sum_{k=0}^{1} \pi_k l_k = 38.3250$$

dollars per month.

**Table 2.5** Annual losses per account for different policies.

| Policy | Annual Loss $/Year |
|---|---|
| $P_2$ | 459.9005 |
| $P_3$ | 452.4540 |
| $P_4$ | 451.7845 |
| $P_5$ | 451.6870 |
| $P_6$ | 451.6809 |
| $P_7$ | 451.6828 |
| $P_c$ | 451.7003 |

Similar analyses can be done for the other policies. Table 2.5 gives the annual loss rate for all of them. For comparison, we have also included the annual loss rate of the current policy $P_c$.

It is clear that among the policies above it is best to follow the policy $P_6$; that is, turn over the account to WMB as soon as the account holder fails to make six minimum payments in a row. This policy saves Passport

$$451.7003 - 451.6809 = .0194$$

dollars per year per account over the current Passport policy $P_c$. Since Passport also has to pay the annual fee of $50,000/year, the services of WMB are worth it if the number of Passport accounts is at least

$$50{,}000/.0194 = 2{,}577{,}319.$$

We are told that Passport has 14 million accounts, which is much larger than the number above. So our analysis suggests that the management of Passport should hire WMB and follow policy $P_6$. This will save Passport

$$.0194 * 14{,}000{,}000 - 50{,}000 = 221{,}600$$

dollars per year.

At this point, we should see what assumptions we have made that may not be accurate. First of all, what we have presented here is an enormously simplified version of the actual problem faced by a credit card company. We have assumed that all accounts are stochastically similar and independent. Both these assumptions are patently untrue. In practice, the company will classify the accounts into different classes so that accounts within a class are similar. The independence assumption might be invalidated if a large fraction of the account holders work in a particular sector of the economy (such as real estate), and if that sector suffers, then the bankruptcy rates can be affected by the health of that sector. The Markov nature of account evolution is another assumption that may or may not hold. This can be validated by further statistical analysis. Another important assumption is that the data in Table 2.5 remain unaffected by the termination policy that Passport follows. This is probably not true, and there is no easy way to verify it short of implementing a new policy and studying how customer behavior changes in response. One needs to be aware of all such pitfalls before trusting the results of such an exercise.

We end this section with the Matlab function that we used to do the computations to produce Table 2.5.

```
*************************
function rpa = consumercreditcase(p,q,b,m,r)
%consumer credit case study
%p(k) = probability that the customer in state k makes a minimum payment
%q(k) = probability that the customer in state k declares bankruptcy
%b(k) = average outstanding balance owed by the customer in state k
%m = the account is terminated as soon as it misses m payments in a row
%r = WMB buys an account in state k for r*b(k) 0 ≤ r ≤ 1
%Set r = 0 if WMB is not being used
%Output: rpa = annual expected loss from a single account
l = zeros(1,m); %l(k) = loss in state k for k = 1:m
l(k) = q(k)*b(k);
end;
l(m) = l(m) + p(m)*(1-r)*b(m);
P = zeros(m,m);
for k = 1:m-1
P(k,k+1) = p(k);
P(k,1) = 1-p(k);
end
P(m,1) = 1;
P100 = P^100;
piv = P100(1,:);
rpa = 12*piv*l(1:m)' %annual loss per account
```

## 2.9 Problems

### CONCEPTUAL PROBLEMS

**2.1.** Let $\{X_n, n \geq 0\}$ be a time-homogeneous DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. Then, for $i_0, i_1, \ldots, i_{k-1}, i_k \in S$, show that

$$\mathsf{P}(X_1 = i_1, \ldots, X_{k-1} = i_{k-1}, X_k = i_k | X_0 = i_0) = p_{i_0,i_1} \cdots p_{i_{k-1},i_k}.$$

**2.2.** Let $\{X_n, n \geq 0\}$ be a time-homogeneous DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. Prove or disprove by counterexample

$$\mathsf{P}(X_1 = i, X_2 = j, X_3 = k) = \mathsf{P}(X_2 = i, X_3 = j, X_4 = k).$$

**2.3.** Consider the machine reliability model of Example 2.2. Now suppose that there are three independent and identically behaving machines in the shop. If a machine is up at the beginning of a day, it stays up at the beginning of the next day with probability $p$, and if it is down at the beginning of a day, it stays down at the beginning of the next day with probability $q$, where $0 < p, q < 1$ are fixed numbers. Let $X_n$ be the number of working machines at the beginning of the $n$th day. Show that $\{X_n, n \geq 0\}$ is a DTMC, and display its transition probability matrix.

**2.4.** Let $P$ be an $N \times N$ stochastic matrix. Using the probabilistic interpretation, show that $P^n$ is also a stochastic matrix.

**2.5.** Prove Corollaries 2.1 and 2.2.

**2.6.** Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. Let $Y_n = X_{2n}, n \geq 0$. Show that $\{Y_n, n \geq 0\}$ is a DTMC on $S$ with transition matrix $P^2$.

**2.7.** Let $\{X_n, n \geq 0\}$ be a DTMC on state space $S = \{1, 2, \ldots, N\}$ with transition probability matrix $P$. Suppose $X_0 = i$ with probability 1. The sojourn time $T_i$ of the DTMC in state $i$ is said to be $k$ if $\{X_0 = X_1 = \cdots = X_{k-1} = i, X_k \neq i\}$. Show that $T_i$ is a $\mathrm{G}(1 - p_{i,i})$ random variable.

**2.8.** Consider a machine that works as follows. If it is up at the beginning of a day, it stays up at the beginning of the next day with probability $p$ and fails with probability $1 - p$. It takes exactly 2 days for the repairs, at the end of which the machine is as good as new. Let $X_n$ be the state of the machine at the beginning of day $n$, where the state is 0 if the machine has just failed, 1 if 1 day's worth of repair work is done on it, and 2 if it is up. Show that $\{X_n, n \geq 0\}$ is a DTMC, and display its transition probability matrix.

**2.9.** Items arrive at a machine shop in a deterministic fashion at a rate of one per minute. Each item is tested before it is loaded onto the machine. An item is found to be nondefective with probability $p$ and defective with probability $1 - p$. If an item is found defective, it is discarded. Otherwise, it is loaded onto the machine. The machine takes exactly 1 minute to process the item, after which it is ready to process the next one. Let $X_n$ be 0 if the machine is idle at the beginning of the $n$th minute and 1 if it is starting the processing of an item. Show that $\{X_n, n \geq 0\}$ is a DTMC, and display its transition probability matrix.

**2.10.** Consider the system of Conceptual Problem 2.9. Now suppose the machine can process two items simultaneously. However, it takes 2 minutes to complete the processing. There is a bin in front of the machine where there is room to store two nondefective items. As soon as there are two items in the bin, they are loaded onto the machine and the machine starts processing them. Model this system as a DTMC.

**2.11.** Consider the system of Conceptual Problem 2.10. However, now suppose that the machine starts working on whatever items are waiting in the bin when it becomes idle. It takes 2 minutes to complete the processing whether the machine is processing one or two items. Processing on a new item cannot start unless the machine is idle. Model this as a DTMC.

**2.12.** The weather at a resort city is either sunny or rainy. The weather tomorrow depends on the weather today and yesterday as follows. If it was sunny yesterday and today, it will be sunny tomorrow with probability .9. If it was rainy yesterday but sunny today, it will be sunny tomorrow with probability .8. If it was sunny yesterday but rainy today, it will be sunny tomorrow with probability .7. If it was rainy yesterday and today, it will be sunny tomorrow with probability .6. Define today's state of the system as the pair (weather yesterday, weather today). Model this system as a DTMC, making appropriate independence assumptions.

**2.13.** Consider the following weather model. The weather normally behaves as in Example 2.3. However, when the cloudy spell lasts for two or more days, it continues to be cloudy for another day with probability .8 or turns rainy with probability .2. Develop a four-state DTMC model to describe this behavior.

**2.14.** $N$ points, labeled $1, 2, \ldots, N$, are arranged clockwise on a circle and a particle moves on it as follows. If the particle is on point $i$ at time $n$, it moves one step in clockwise fashion with probability $p$ or one step in counterclockwise fashion with probability $1 - p$ to move to a new point at time $n + 1$. Let $X_n$ be the position of the particle at time $n$. Show that $\{X_n, n \geq 0\}$ is a DTMC, and display its transition probability matrix.

**2.15.** Let $\{X_n, n \geq 0\}$ be an irreducible DTMC on state space $\{1, 2, \ldots, N\}$. Let $u_i$ be the probability that the DTMC visits state 1 before it visits state $N$, starting from state $i$. Using the first-step analysis, show that

$$
\begin{aligned}
u_1 &= 1, \\
u_i &= \textstyle\sum_{j=1}^{N} p_{i,j} u_j, \ 2 \leq i \leq N - 1, \\
u_N &= 0.
\end{aligned}
$$

**2.16.** A total of $N$ balls are put in two urns, so that initially urn A has $i$ balls and urn B has $N - i$ balls. At each step, one ball is chosen at random from the $N$ balls. If it is from urn A, it is moved to urn B, and vice versa. Let $X_n$ be the number of balls in urn A after $n$ steps. Show that $\{X_n, n \geq 0\}$ is a DTMC, assuming that the successive random drawings of the balls are independent. Display the transition probability matrix of the DTMC.

**2.17.** The following selection procedure is used to select one of two brands, say A and B, of infrared light bulbs. Suppose the brand A light bulb life-times are iid exp($\lambda$) random variables and brand B light bulb lifetimes are iid exp($\mu$) random variables. One light bulb from each brand is turned on simultaneously, and the experiment ends when one of the two light bulbs fails. Brand A wins a point if the brand A light bulb outlasts brand B, and vice versa. (The probability that the bulbs fail simultaneously is zero.) The experiment is repeated with new light bulbs until one of the brands accumulates five points more than the other, and that brand is selected as the better brand. Let $X_n$ be the number of points for brand A minus the number of points accumulated by brand B after $n$ experiments. Show that $\{X_n, n \geq 0\}$ is a DTMC, and display its transition probability matrix. (*Hint*: Once $X_n$ takes a value of 5 or $-5$, it stays there forever.)

**2.18.** Let $\{X_n, n \geq 0\}$ be a DTMC on state space $\{1, 2, \ldots, N\}$. Suppose it incurs a cost of $c(i)$ dollars every time it visits state $i$. Let $g(i)$ be the total expected cost incurred by the DTMC until it visits state $N$ starting from state $i$. Derive the following equations:

$$\begin{aligned} g(N) &= 0, \\ g(i) &= c(i) + \textstyle\sum_{j=1}^{N} p_{i,j} g(j), 1 \leq j \leq N-1. \end{aligned}$$

**2.19.** Another useful cost model is when the system incurs a random cost of $C(i, j)$ dollars whenever it undergoes a transition from state $i$ to $j$ in one step. This model is called the *cost per transition model*. Define

$$c(i) = \sum_{j=1}^{N} \mathsf{E}(C(i, j)) p_{i,j}, 1 \leq i \leq N.$$

Show that $g(i, T)$, the total cost over a finite horizon $T$, under this cost model satisfies Theorem 2.11 with $c(i)$ as defined above. Also show that $g(i)$, the long-run cost rate, satisfies Theorem 2.12.

## COMPUTATIONAL PROBLEMS

**2.1.** Consider the telecommunications model of Example 2.12. Suppose the buffer is full at the end of the third time slot. Compute the expected number of packets in the buffer at the end of the fifth time slot.

**2.2.** Consider the DTMC in Conceptual Problem 2.14 with $N = 5$. Suppose the particle starts on point 1. Compute the probability distribution of its position at time 3.

**2.3.** Consider the stock market model of Example 2.7. Suppose Mr. BigShot has bought 100 shares of stock at \$5 per share. Compute the expected net change in the value of his investment in 5 days.

**2.4.** Mr. Smith is a coffee addict. He keeps switching between three brands of coffee, say A, B, and C, from week to week according to a DTMC with the following transition probability matrix:

$$P = \begin{bmatrix} .10 & .30 & .60 \\ .10 & .50 & .40 \\ .30 & .20 & .50 \end{bmatrix}. \tag{2.69}$$

If he is using brand A this week (i.e., week 1), what is the probability distribution of the brand he will be using in week 10?

**2.5.** Consider the telecommunications model of Example 2.8. Suppose the buffer is full at the beginning. Compute the expected number of packets in the buffer at time $n$ for $n = 1, 2, 5$, and 10, assuming that the buffer size is 10 and that the number of packets arriving during one time slot is a binomial random variable with parameters (5, .2).

**2.6.** Consider the machine described in Conceptual Problem 2.8. Suppose the machine is initially up. Compute the probability that the machine is up at times $n = 5, 10, 15$, and 20. (Assume $p = .95$.)

**2.7.** Consider Paper Pushers Insurance Company, Inc., of Example 2.6. Suppose it has 100 employees at the beginning of week 1, distributed as follows: 50 in grade 1, 25 in grade 2, 15 in grade 3, and 10 in grade 4. If employees behave independently of each other, compute the expected number of employees in each grade at the beginning of week 4.

**2.8.** Consider the machine reliability model in Example 2.2 with one machine. Suppose the machine is up at the beginning of day 0. Compute the probability that the state of the machine at the beginning of the next three days is up, down, down, in that order.

**2.9.** Consider the machine reliability model in Example 2.2 with two machines. Suppose both machines are up at the beginning of day 0. Compute the probability that the number of working machines at the beginning of the next three days is two, one, and two, in that order.

**2.10.** Consider the weather model of Example 2.3. Compute the probability that once the weather becomes sunny, the sunny spell lasts for at least 3 days.

**2.11.** Compute the expected length of a rainy spell in the weather model of Example 2.3.

**2.12.** Consider the inventory system of Example 2.4 with a starting inventory of five PCs on a Monday. Compute the probability that the inventory trajectory over the next four Mondays is as follows: 4, 2, 5, and 3.

**2.13.** Consider the inventory system of Example 2.4 with a starting inventory of five PCs on a Monday. Compute the probability that an order is placed at the end of the first week for more PCs.

**2.14.** Consider the manufacturing model of Example 2.5. Suppose both bins are empty at time 0. Compute the probability that both bins stay empty at times $n = 1, 2$, and then machine 1 is shut down at time $n = 4$.

**2.15.** Compute the occupancy matrix $M(10)$ for the DTMCs with transition matrices as given below:

(a)

$$P = \begin{bmatrix} .10 & .30 & .20 & .40 \\ .10 & .30 & .40 & .20 \\ .30 & .30 & .10 & .30 \\ .15 & .25 & .35 & .25 \end{bmatrix},$$

(b)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

(c)

$$P = \begin{bmatrix} .10 & 0 & .90 & 0 \\ 0 & .30 & 0 & .70 \\ .30 & 0 & .70 & 0 \\ 0 & .25 & 0 & .75 \end{bmatrix},$$

(d)

$$P = \begin{bmatrix} .10 & .30 & 0 & .60 \\ .10 & .30 & 0 & .60 \\ .30 & .10 & .10 & .50 \\ .5 & .25 & 0 & .25 \end{bmatrix}.$$

**2.16.** Consider the inventory system of Example 2.4. Compute the occupancy matrix $M(52)$. Using this, compute the expected number of weeks that Computers-R-Us starts with a full inventory (i.e., five PCs) during a year given that it started the first week of the year with an inventory of five PCs.

**2.17.** Consider the manufacturing model of Example 2.11. Suppose that at time 0 there is one item in bin 1 and bin 2 is empty. Compute the expected amount of time that machine 1 is turned off during an 8-hour shift.

**2.18.** Consider the telecommunications model of Example 2.12. Suppose the buffer is empty at time 0. Compute the expected number of slots that have an empty buffer at the end during the next 50 slots.

**2.19.** Classify the DTMCs with the transition matrices given in Computational Problem 2.15 as irreducible or reducible.

**2.20.** Classify the irreducible DTMCs with the transition matrices given below as periodic or aperiodic:

(a)

$$P = \begin{bmatrix} .10 & .30 & .20 & .40 \\ .10 & .30 & .40 & .20 \\ .30 & .10 & .10 & .50 \\ .15 & .25 & .35 & .25 \end{bmatrix},$$

(b)

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

(c)

$$P = \begin{bmatrix} 0 & .20 & .30 & .50 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

(d)

$$P = \begin{bmatrix} 0 & 0 & .40 & .60 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

**2.21.** Compute a normalized solution to the balance equations for the DTMC in Computational Problem 2.20(a). When possible, compute:

1. the limiting distribution;
2. the stationary distribution;
3. the occupancy distribution.

**2.22.** Do Computational Problem 2.21 for Computational Problem 2.20(b).

**2.23.** Do Computational Problem 2.21 for Computational Problem 2.20(c).

**2.24.** Do Computational Problem 2.21 for Computational Problem 2.20(d).

**2.25.** Consider the DTMC of Computational Problem 2.5. Compute:

1. the long-run fraction of the time that the buffer is full;
2. the expected number of packets in the buffer in the long run.

**2.26.** Consider Computational Problem 2.7. Compute the expected number of employees in each grade in steady state.

**2.27.** Consider the weather model of Conceptual Problem 2.12. Compute the long-run fraction of days that are rainy.

**2.28.** Consider the weather model of Conceptual Problem 2.13. Compute the long-run fraction of days that are sunny.

**2.29.** What fraction of the time does the coffee addict of Computational Problem 2.4 consume brand A coffee?

**2.30.** Consider the machine described in Conceptual Problem 2.8. What is the long-run fraction of the time that this machine is up? (Assume $p = .90$.)

**2.31.** Consider the manufacturing model of Example 2.11. Compute the expected number of components in bins A and B in steady state.

**2.32.** Consider the stock market model of Example 2.7. What fraction of the time does the chief financial officer have to interfere in the stock market to control the price of the stock?

**2.33.** Consider the single-machine production system of Conceptual Problem 2.10. Compute the expected number of items processed by the machine in 10 minutes, assuming that the bin is empty and the machine is idle to begin with. (Assume $p = .95$.)

**2.34.** Do Computational Problem 2.33 for the production system of Conceptual Problem 2.11. (Assume $p = .95$.)

**2.35.** Which one of the two production systems described in Conceptual Problems 2.10 and 2.11 has a higher per minute rate of production in steady state?

**2.36.** Consider the three-machine workshop described in Conceptual Problem 2.3. Suppose each working machine produces revenue of \$500 per day, while repairs cost \$300 per day per machine. What is the net rate of revenue per day in steady state? (*Hint*: Can we consider the problem with one machine to obtain the answer for three machines?)

**2.37.** Consider the inventory system of Example 2.27. Compute the long-run expected cost per day of operating this system.

**2.38.** Consider the manufacturing system of Example 2.11. Compute the expected number of assemblies produced per hour in steady state.

**2.39.** (Computational Problem 2.38 continued). What will be the increase in the production rate (in number of assemblies per hour) if we provide bins of capacity 3 to the two machines in Example 2.11?

**2.40.** Compute the long-run expected number of packets transmitted per unit time by the data switch of Example 2.12. How is this connected to the packet-loss rate computed in Example 2.29?

**2.41.** Consider the brand-switching model of Computational Problem 2.4. Suppose the per pound cost of coffee is \$6, \$8, and \$15 for brands A, B, and C, respectively. Assuming Mr. Smith consumes one pound of coffee per week, what is his long-run expected coffee expense per week?

**2.42.** Compute the expected time to go from state 1 to 4 in the DTMCs of Computational Problems 2.20(a) and (c).

**2.43.** Compute the expected time to go from state 1 to 4 in the DTMCs of Computational Problems 2.20(b) and (d).

**2.44.** Consider the selection procedure of Conceptual Problem 2.17. Suppose the mean lifetime of Brand A light bulbs is 1, while that of Brand B light bulbs is 1.25. Compute the expected number of experiments done before the selection procedure ends. (*Hint*: Use the Gambler's ruin model of Example 2.33.)

**2.45.** Consider the DTMC model of the data switch described in Example 2.12. Suppose the buffer is full to begin with. Compute the expected amount of time (counted in number of time slots) before the buffer becomes empty.

**2.46.** Do Computational Problem 2.45 for the data buffer described in Computational Problem 2.5.

**2.47.** Consider the manufacturing model of Example 2.11. Compute the expected time (in hours) before one of the two machines is shut down, assuming that both bins are empty at time 0.

**Case Study Problems.** You may use the Matlab program of Section 2.8 to solve the following problems.

**2.48.** Suppose Passport has decided not to employ the services of WMB. However, this has generated discussion within the company about whether it should terminate accounts earlier. Let $T_m$ $(1 \leq m \leq 7)$ be the policy of terminating the account as soon as it misses $m$ payments in a row. Which policy should Passport follow?

**2.49.** Consider the current policy $P_c$. One of the managers wants to see if it would help to alert the customers of their impending account termination in a more dire form by a phone call when the customer has missed six minimum payments in a row. This will cost a dollar per call. The manager estimates that this will decrease the missed payment probability from the current $p_6 = .329$ to .250. Is this policy cost-effective?

**2.50.** The company has observed over the past year that the downturn in the economy has increased the bankruptcy rate by 50%. In this changed environment, should Passport engage the services of WMB? When should it turn over the accounts to WMB?

**2.51.** Passport has been approached by another collection agency, which is willing to work with no annual service contract fee. However, it pays only 60% of the outstanding balance of any account turned over to them. Is this option better than hiring WMB?

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