

Chapter 2

Principles of Interference

2.1 Coherence of Optical Waves

The word ‘coherence’ describes the ability of radiation to produce interference phenomena and the notion of coherence is defined by the correlation properties between the various quantities of an optical field. The optical coherence is related to the various forms of the correlations of the random processes (Born and Wolf 1984; Mandel and Wolf and Wolf 1995). The interference phenomena stems from the principle of superposition, which states that the resultant displacement (at a particular point) produced by two or more waves is the vector sum of the displacements produced by each one of the disturbances. It reveals the correlations between light waves. The degree of correlation that exists between the fluctuations in two light waves determines the interference effects arising when the beams are superposed. The correlated fluctuation can be partially or completely coherent. A polychromatic point source on the sky produces a fringe packet as a function of an applied path length difference. This fringe packet has an extent referred to as the coherence length,

$$l_c = \frac{c}{\Delta\nu} = \frac{\lambda^2}{\Delta\lambda} = c\tau_c, \quad (2.1)$$

where $\Delta\nu$ is the effective spectral width, c the speed of light, and

$$\tau_c = \frac{\lambda^2}{c\Delta\lambda} \sim \frac{1}{\Delta\nu}, \quad (2.2)$$

the coherence time.

The coherence time is defined as the maximum transit time difference for good visibility, \mathcal{V} of the fringes. It is pertinent to note that the visibility, a dimensionless number lying between zero and one, is the attribute of interference fringes describing the contrast between the bright and dark interference regions. Mathematically, it is a complex (phasor) quantity whose magnitude quantifies the fringe contrast, and phase describes the position of the fringes with respect to a phase center. The coherence time is the evolution time of the complex amplitude; a factor less than unity affects the degree of coherence. In order to keep the time correlation close to

unity, the delay τ must be limited to a small fraction of the temporal width or coherence time. The visibility falls to approximately zero when the pathlength difference exceeds the coherence length, l_c , which measures the maximum path difference for which the fringes are still observable. Or equivalently, the visibility approaches zero when the relative time delay exceeds the coherence time. An interferometer based on amplitude division, is generally used to measure the temporal coherence of a source, where it compares a light wave with itself at different moments in time, τ_c ; this type of coherence properties encode the spectral content of the intensity distribution. The classical Michelson interferometer is an example of such a method, while the spatial coherence describes the correlation between two light waves at different points in space, for example the Young's double slit experiment. An astronomical interferometer measures spatio-temporal coherence properties of the radiation emerging from a celestial source. The measured interferometric signal depends on both structural and spectral contents of the celestial body concerned.

2.1.1 Interference of Partially Coherent Beams

The Fig. 2.1 is a sketch of a Young's set up where the wave field is produced by an extended polychromatic source. In this, a screen (A) with pin-holes at positions $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ separated by, h , is placed at a distance, d , from the source, S . The complex disturbance produced at a receiving point, $P(\mathbf{r})$, on a second screen (B) situated at distances s_1 and s_2 , respectively, from the pin-holes $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$, is expressed as,

$$U(\mathbf{r}, t) = K_1 U(\mathbf{r}_1, t - t_1) + K_2 U(\mathbf{r}_2, t - t_2), \quad (2.3)$$

where $U(\mathbf{r}, t)$ represents the associated analytical signal, $t_1 = s_1/c$ and $t_2 = s_2/c$ the transit time of the light from $P_1(\mathbf{r}_1)$ to $P(\mathbf{r})$, and from $P_2(\mathbf{r}_2)$ to $P(\mathbf{r})$, respectively, and $K_{j=1,2}$ the constant factors that depend on the size of the openings and on

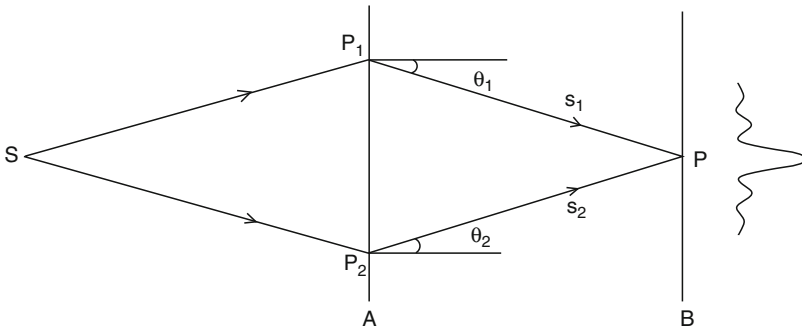


Fig. 2.1 Coherence of the two holes $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ illuminated by an extended source σ

the geometry of the arrangement, i.e., the angle of incident and diffraction at $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$; the refractive index of the medium between the screens are assumed to be unity.

The complex valued constants are given by,

$$K_1 \simeq \int_{P_1} \frac{\chi(\theta_1)}{i\bar{\lambda}s_1} dS_1, \quad (2.4a)$$

$$K_2 \simeq \int_{P_2} \frac{\chi(\theta_2)}{i\bar{\lambda}s_2} dS_2, \quad (2.4b)$$

in which $\theta_{(j=1,2)}$ are angles indicated in Fig. 2.1 and χ the inclination factor.

These factors K_1 and K_2 are inversely proportional to s_1 and s_2 . Since the secondary wavelets from $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ are out of phase with the primary wave by a quarter of a period, K_1 and K_2 are imaginary numbers. If the pin-holes are small and the diffracted fields are considered to be uniform, the values $|K_j|$ satisfy $K_1^* K_2 = K_1 K_2^* = K_1 K_2$. The diffracted fields are approximately uniform, that is, K_1 and K_2 do not depend on θ_1 and θ_2 . In order to derive the intensity of the light at $P(\mathbf{r})$, by neglecting the polarization effects, one may assume that the averaging time is effectively infinite which is valid assumption for true thermal light. The observable intensity (or power), $I(\mathbf{r}, t)$ at $P(\mathbf{r})$ is defined by the formula,

$$I(\mathbf{r}, t) = \langle U(\mathbf{r}, t) U^*(\mathbf{r}, t) \rangle. \quad (2.5)$$

It follows from (2.3, 2.5),

$$\begin{aligned} I(\mathbf{r}, t) = & |K_1|^2 \langle |U(\mathbf{r}_1, t - t_1)|^2 \rangle + |K_2|^2 \langle |U(\mathbf{r}_2, t - t_2)|^2 \rangle \\ & + K_1 K_2^* \langle U(\mathbf{r}_1, t - t_1) U^*(\mathbf{r}_2, t - t_2) \rangle \\ & + K_1^* K_2 \langle U(\mathbf{r}_2, t - t_2) U^*(\mathbf{r}_1, t - t_1) \rangle. \end{aligned} \quad (2.6)$$

The field was assumed to be stationary. One may shift the origin of time in all these expressions. Therefore,

$$\langle U(\mathbf{r}_1, t - t_1) U^*(\mathbf{r}_1, t - t_1) \rangle = \langle U(\mathbf{r}_1, t) U^*(\mathbf{r}_1, t) \rangle = I(\mathbf{r}_1, t), \quad (2.7)$$

$$\langle U(\mathbf{r}_2, t - t_2) U^*(\mathbf{r}_2, t - t_2) \rangle = \langle U(\mathbf{r}_2, t) U^*(\mathbf{r}_2, t) \rangle = I(\mathbf{r}_2, t), \quad (2.8)$$

and if one sets $\tau = t_2 - t_1$, then

$$\langle U(\mathbf{r}_1, t - t_1) U^*(\mathbf{r}_2, t - t_2) \rangle = \langle U(\mathbf{r}_1, t + \tau) U^*(\mathbf{r}_2, t) \rangle, \quad (2.9a)$$

$$\langle U^*(\mathbf{r}_1, t - t_1) U(\mathbf{r}_2, t - t_2) \rangle = \langle U^*(\mathbf{r}_1, t + \tau) U(\mathbf{r}_2, t) \rangle. \quad (2.9b)$$

The coherence function (Zernike 1938) relates the wave fields received at two neighboring (which is considered to as relative to the distance from the source) points in space. Equation (2.9) is called the mutual coherence function and represents a temporal complex cross-correlation between the functions $U(\mathbf{r}_1, t - t_1)$ and

$U(\mathbf{r}_2, t - t_2)$ during the time interval T . Such a function depends on the time delay, $\tau [= (s_2 - s_1)/c]$, as well as on the separation between \mathcal{P}_1 and \mathcal{P}_2 . In a stationary field, Γ depends on t_1 and t_2 only through the difference $t_1 - t_2 = \tau$; therefore, by defining the mutual coherence function, $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, as,

$$\begin{aligned}\Gamma(\mathbf{r}_1, \mathbf{r}_2; t_1, t_2) &= \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) \\ &= \langle U(\mathbf{r}_1, t + \tau) U^*(\mathbf{r}_2, t) \rangle.\end{aligned}\quad (2.10)$$

At a point where both the points coincide, the self-coherence reduces to ordinary intensity. When $\tau = 0$,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0) = I(\mathbf{r}_1, t), \quad (2.11a)$$

$$\Gamma(\mathbf{r}_2, \mathbf{r}_2, 0) = I(\mathbf{r}_2, t). \quad (2.11b)$$

To note, the self-coherence function characterizes the interference effects given by the following equation,

$$\begin{aligned}\Gamma(\mathbf{r}, \tau) &= \langle U^*(\mathbf{r}, t) U(\mathbf{r}, t + \tau) \rangle = |A(\mathbf{r}, t)|^2 e^{-i2\pi\nu\tau} \\ &= 4 \int_0^\infty \hat{\Gamma}(\mathbf{r}, \mathbf{r}, \nu) e^{-i2\pi\nu\tau} d\nu,\end{aligned}\quad (2.12)$$

where $\hat{\Gamma}(\mathbf{r}, \nu)$ is the power spectrum of the complex light field, $\omega = 2\pi\nu$, and ν the frequency. By denoting $I_j(\mathbf{r}, t) = |K_j|^2 \langle |U(\mathbf{r}_j, t - t_j)|^2 \rangle$, in which $j = 1, 2$, one derives the intensity at $\mathcal{P}(\mathbf{r})$,

$$I(\mathbf{r}, t) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2|K_1 K_2| \Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)]. \quad (2.13)$$

The term $|K_1|^2 I_1(\mathbf{r}_1, t)$ is the intensity observed at $\mathcal{P}(\mathbf{r})$ when the pin-hole at $\mathcal{P}_1(\mathbf{r}_1)$ alone is opened ($K_2 = 0$) and the term $|K_2|^2 I_2(\mathbf{r}_2, t)$ has similar interpretation. These intensities may be denoted as, $I_1(\mathbf{r}, t)$ and $I_2(\mathbf{r}, t)$ respectively, i.e.,

$$I_1(\mathbf{r}, t) = |K_1|^2 I_1(\mathbf{r}, t) = |K_1|^2 \Gamma(\mathbf{r}_1, \mathbf{r}_1, 0), \quad (2.14a)$$

$$I_2(\mathbf{r}, t) = |K_2|^2 I_2(\mathbf{r}, t) = |K_2|^2 \Gamma(\mathbf{r}_2, \mathbf{r}_2, 0). \quad (2.14b)$$

The complex degree of (mutual) coherence of the light vibrations, $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, of an observed source is a normalized correlation function between the complex fields at the points $\mathcal{P}_1(\mathbf{r}_1)$ and $\mathcal{P}_2(\mathbf{r}_2)$,

$$\begin{aligned}\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) &= \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0)} \sqrt{\Gamma(\mathbf{r}_2, \mathbf{r}_2, 0)}} \\ &= \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)}{\sqrt{I_1(\mathbf{r}, t)} \sqrt{I_2(\mathbf{r}, t)}}.\end{aligned}\quad (2.15)$$

To note, The normalized form of $\Gamma(\mathbf{r}, \tau)$, known as the complex degree of self-coherence describing the correlation of vibrations at a fixed and a variable point of the light is given by,

$$\gamma(\mathbf{r}, \tau) = \frac{\Gamma(\mathbf{r}, \tau)}{\Gamma(\mathbf{r}, 0)} = \frac{\int_0^\infty \hat{\Gamma}(\mathbf{r}, \mathbf{r}, \nu) e^{-i2\pi\nu\tau} d\nu}{\int_0^\infty \hat{\Gamma}(\mathbf{r}, \mathbf{r}, \nu) d\nu}, \quad (2.16)$$

in which $\hat{\Gamma}(\mathbf{r}, \nu)$ is referred to as the self-spectral density of the two beams.

The complex degree of coherence, $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, measures both the spatial and the temporal coherence and is characterized by the following properties:

- it is a function with a maximum value at the origin for $\tau = 0$,
- the degree of coherence of the vibration is given by the Cauchy-Schwarz's inequality,¹ $0 \leq |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \leq 1$, and
- the modulus of the complex degree of coherence is proportional to the contrast or visibility of the interference fringes, therefore, by measuring this one may obtain information about the quality of the source of the interference system.

In general, two light beams are not correlated but the correlation term, $U(\mathbf{r}_1, t)U^*(\mathbf{r}_2, t)$, takes on significant values for a short period of time and $\langle U(\mathbf{r}_1, t)U^*(\mathbf{r}_2, t) \rangle = 0$. Time variations of $U(\mathbf{r}, t)$ are statistical in nature (Mandel and Wolf 1995). Hence, one seeks a statistical description of the field (correlations) as the field is due to a partially coherent source. Depending upon the correlations between the phasor amplitudes at different object points, one would expect a definite correlation between the two points of the field. The effect of $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|$ is to reduce the visibility of the fringes. There are three following operating regimes for the interference system as a function of the value of $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|$:

$$|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \begin{cases} = 1 & \text{completely coherent,} \\ < 1 & \text{partially coherent,} \\ = 0 & \text{incoherent superposition.} \end{cases} \quad (2.17)$$

In the first case, the system is operating in the coherent limit, and the vibrations at $P_1(\mathbf{r}_1)$ and $P_1(\mathbf{r}_1)$ may said to be coherent. In the second case, the vibrations are said to be partially coherent and the source operates with partial degree of coherence. In the case of interferometry, two different polychromatic self-luminous point sources are often considered to be spatially coherent if the fringe packets produced on a detector fall in the same scanning region of the applied path length difference. This coherence is known as partial coherence, which applies to extended sources.

¹ The modulus of the inner product of two vectors is smaller than or equal to the product of the norms of these vectors, i.e., for x, y in an inner space $|\langle x, y \rangle| \leq |x| |y|$ and the equality holds if $\{x, y\}$ is a linearly dependent set.

Such a coherence is a property of two waves whose relative phase undergoes random fluctuations which are not enough to make the wave completely incoherent. The third case describes that the system operates in the incoherent limit and the superposition of the two beams do not give rise to any interference effect.

The (2.13) for $I(\mathbf{r}, t)$ can be recast in,

$$I(\mathbf{r}, t) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}\Re[\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)]. \quad (2.18)$$

This (2.18) is the general interference law for stationary optical fields. In order to determine the light intensity at $\mathbf{P}(\mathbf{r})$ when the two light waves are superposed, the intensity of each beam and the value of the real term, $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, of the complex degree of coherence must be available.

2.1.2 Source and Visibility

The intensity at the point of superposition varies between maxima which exceeds the sum of the intensities in the beams and minima, which may be zero. The exact pattern depends on the wavelength of the light, the angular size of the source, and the separation between the apertures or slits. The resolution of an interferometer depends on the separation between the slits and is dictated by the spacing between the maxima, which is known as the fringe angular spacing. If one of the apertures is closer to the light source by a half of a wavelength, a crest in one light beam corresponds to a trough in the other beam; hence, these two light waves cancel each other, making the source disappear. The light source reappears and disappears every time the delay between the apertures is a multiple of the wave period. If the apertures are kept sufficiently wide, the source is resolved. When the light emitted by a source σ is quasi-monochromatic with a mean frequency $\bar{\nu}$, it has a spectral range, $\Delta\nu \ll \bar{\nu}$; hence, the complex degree of coherence turns out to be,

$$\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| e^{i[\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - 2\pi\bar{\nu}\tau]}, \quad (2.19)$$

with

$$\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) = 2\pi\bar{\nu}\tau + \arg[\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)]. \quad (2.20)$$

To note, the normalized complex degree of self-coherence is given by,

$$\gamma(\mathbf{r}, \tau) = |\gamma(\mathbf{r}, \tau)| e^{i[\Phi(\mathbf{r}, \tau) - 2\pi\bar{\nu}\tau]},$$

where $\Phi(\mathbf{r}, \tau) = 2\pi\bar{\nu}\tau + \arg[\gamma(\mathbf{r}, \tau)]$.

If $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| = 0$, (2.18) becomes,

$$I(\mathbf{r}, t) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t), \quad (2.21)$$

in which $I_1(\mathbf{r}, t)$, $I_2(\mathbf{r}, t)$ are the intensities of the light in each arm of the interferometer. When the vibrations add up in intensity at $\mathbf{P}(\mathbf{r})$, the vibrations are said to be incoherent. The intensity at a point, $\mathbf{P}(\mathbf{r})$, in the case of $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| = 1$ is,

$$I(\mathbf{r}, t) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)} \times |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \cos [\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta], \quad (2.22)$$

where

$$\delta = 2\pi \bar{\nu} \tau = \frac{2\pi(s_2 - s_1)}{\bar{\lambda}} = \bar{\kappa}(s_2 - s_1),$$

with $\bar{\kappa} = 2\pi \bar{\nu}/c = 2\pi/\bar{\lambda}$, $\bar{\kappa}$ as the mean wave number, $\bar{\nu}$ the mean frequency, $\bar{\nu}$ the mean wavelength, and the argument $\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta$, in which δ arises from the path difference, s_1 and s_2 two optical path lengths, and $\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau)$ contains information about the source.

The visibility function, \mathcal{V} , is related to brightness morphology that an interferometer measures and indicates the extent to which a source is resolved on the baseline used. It is the modulus of the degree of coherence, $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, at the spatial frequency vector, $\mathbf{u}(= u, v)$, i.e., (projected) baseline vectors.² The spatial frequency coordinates depend on the separation of the telescopes projected in the direction of observation,

$$u_{ij} = \frac{B_{x_0}}{\lambda} = \frac{\kappa B_{x_0}}{2\pi}; \quad (2.23a)$$

$$v_{ij} = \frac{B_{y_0}}{\lambda} = \frac{\kappa B_{y_0}}{2\pi}. \quad (2.23b)$$

where \mathbf{B} is the baseline vector and λ the wavelength of observation.

Both \mathbf{B} and λ are linear quantities and must be expressed in the same units. The orientation of fringes is normal to the baseline vector and the phase of the fringe pattern is equal to the Fourier phase of the same spatial frequency component. To note, a spatial fringe pattern is a 2-D signal as a function of the image coordinates with fringes along one coordinate enveloped by an Airy disk, while a temporal fringe pattern is a 1-D signal as a function of OPD enveloped by the Fourier transform of the spectrum. Figure 2.2 depicts the snapshots and the corresponding intensity curves of two beam interference fringes obtained from a circular source with various spacings of pin-holes. As the source becomes small compared to the fringe spacing, the visibility approaches unity, while the latter approaches zero as the former becomes very small compared to the source; source extent is equal to the fringe spacing or multiples thereof.

² The projected baseline is defined as the projection of the baseline vector onto the plane, which is perpendicular to the line-of-sight of the source.

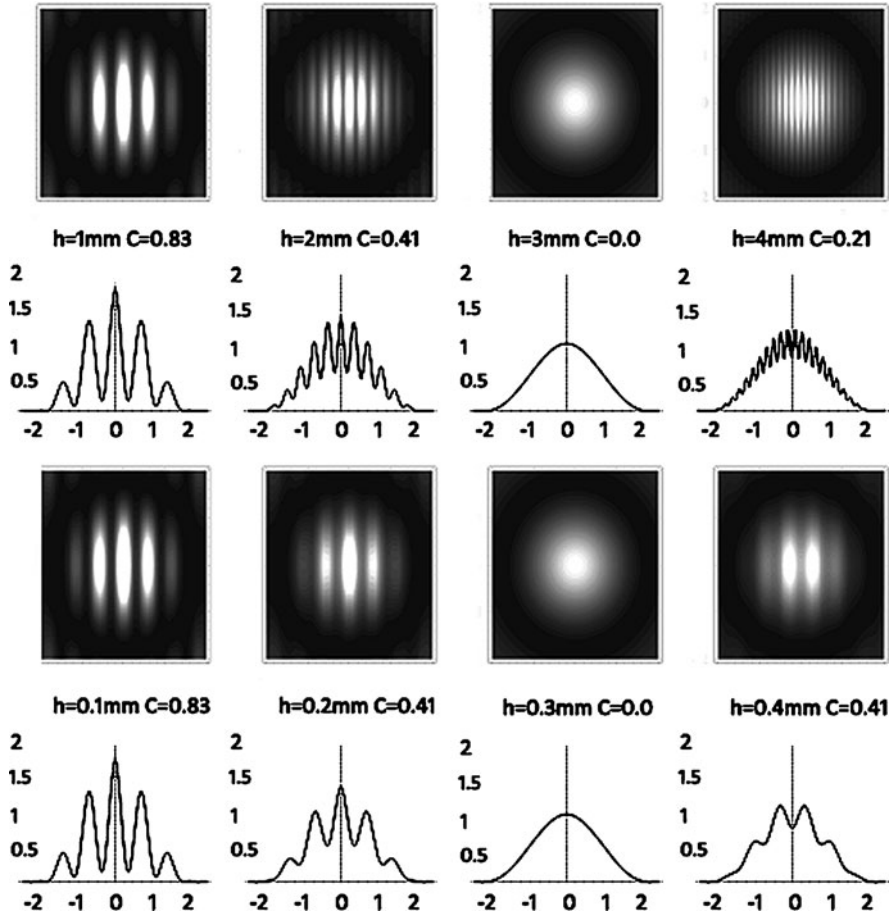


Fig. 2.2 Illustration of the influence on the spatial coherence (on the contrast C) of h , the distance between the holes, and a , the semi-diameter of a circular source, in the Young's two-hole experiment, assuming $d = 1$ m (distance of the source) and a wavelength of 600 nm. For a between 0.1 and 0.4 mm (top simulation), $h = 1$ mm; for h between 1 and 4 mm (bottom simulation), $a = 0.1$ mm. The corresponding intensity curves of the two beam interference fringes obtained from the circular source changes with the various spacings of holes and the size of the holes, being zero when ' $a \times h = 0.3$ ' (constructive fringe for ' $a \times h < 0.3$ ' and destructive fringe for $0.3 < a \times h < 0.6$). Courtesy: Luc Damé

The function, $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, can be approximated to $\gamma(\mathbf{r}_1, \mathbf{r}_2, 0)e^{-i2\pi\bar{\nu}\tau}$ for $\tau \ll \tau_c$. The exponential term is nearly constant and $\gamma(\mathbf{r}_1, \mathbf{r}_2, 0)$, measures the spatial coherence. Let $\Phi(\mathbf{r}_1, \mathbf{r}_2)$, be the argument of $\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$, thus,

$$I(\mathbf{r}_1, \mathbf{r}_2, \tau) = I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)} \times \Re \left[|\gamma(\mathbf{r}_1, \mathbf{r}_2, 0)| e^{i[\Phi(\mathbf{r}_1, \mathbf{r}_2) - 2\pi\bar{\nu}\tau]} \right]. \quad (2.24)$$

The (2.24) illustrates the Young's experiment. For a quasi-monochromatic optical source, $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|$ is nearly constant (amplitude of fringe envelope) and the phase, $\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau)$, when is considered as functions of τ , varies slowly compared to $\cos 2\pi\bar{\nu}\tau$ and $\sin 2\pi\bar{\nu}\tau$. If the openings at $\mathcal{P}_1(\mathbf{r}_1)$ and $\mathcal{P}_2(\mathbf{r}_2)$ are sufficiently small, the intensity distributions of the light in the vicinity of \mathcal{P} consist of an almost uniform background (no intensity envelope due to diffraction by a finite aperture) $I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t)$ in which a sinusoidal intensity distribution is superimposed with constant amplitude $2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|$. The intensity maxima and minima near $\mathcal{P}(\mathbf{r})$ are, therefore given by,

$$\begin{aligned} I_{max} &= I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|, \\ I_{min} &= I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) - 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|. \end{aligned} \quad (2.25)$$

The modulus of the fringe visibility is estimated as the ratio of high frequency to low frequency energy in the average spectral density. In the vicinity of zero path length difference, the visibility of the fringes $\mathcal{V}(\mathbf{r})$, at a point $\mathcal{P}(\mathbf{r})$ in terms of the intensity of the two beams and of their degree of coherence is estimated as,

$$\begin{aligned} \mathcal{V}(\mathbf{r}) &= \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \\ &= |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \frac{2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}}{I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t)}, \end{aligned} \quad (2.26)$$

in which τ is a possible delay between the signals transmitted by the two sub-apertures of the interferometer; for a zero time shift τ such a correlation normally yields the maximum value, and if $\tau \ll 1/\Delta\nu$, the visibility turns out to be, $\mathcal{V}(\mathbf{r}) = |\gamma(\mathbf{r}_1, \mathbf{r}_2, 0)|$.

For the special case that both openings transmit an equal power and $I_1(\mathbf{r}, t) = I_2(\mathbf{r}, t)$, the visibility function $\mathcal{V} = |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|$ equals the modulus of the complex degree of coherence, with secondary sources at \mathcal{P}_1 and \mathcal{P}_2 . Although this expression provides a contrast as a function of the location $\mathcal{P}(\mathbf{r})$, it should be noted that this contrast is not a function of the size and shape of the collectors. According to the Schwarz's inequality, one finds $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| \leq 1$, hence from (2.24), it is derived as,

$$I(\mathbf{r}, t) = \begin{cases} [I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t)]^2 & \text{if } \cos [\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta] = 1, \\ [I_1(\mathbf{r}, t) - I_2(\mathbf{r}, t)]^2 & \text{if } \cos [\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta] = -1. \end{cases} \quad (2.27)$$

The resulting amplitude at $\mathcal{P}(\mathbf{r})$ in certain cases is equal to the sum of the amplitudes (vibration in phase). It can also be equal to the difference of the amplitudes (vibrations in anti-phase). The (2.24) can be recast in the form,

$$\begin{aligned} I(\mathbf{r}, t) &= \left[I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)} \cos [\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta] \right] \\ &\quad \times |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| + (1 - |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|) [I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t)]. \end{aligned} \quad (2.28)$$

The first term arises from coherent superposition of the two beams of intensities $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|I_1(\mathbf{r}, t)$ and $|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|I_2(\mathbf{r}, t)$ and of relative phase difference $\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau) - \delta$ and the second term arises from incoherent superposition of the two beams of intensities, $[1 - |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|]I_1(\mathbf{r}, t)$ and $[1 - |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|]I_2(\mathbf{r}, t)$. Thus, the light which reaches $\mathcal{P}(\mathbf{r})$ from all other pin-holes may be regarded as mixture of coherent and incoherent light beams, with intensity in the ratio,

$$\frac{I_{coh}}{I_{incoh}} = \frac{|\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|}{1 - |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|}, \quad (2.29a)$$

$$\frac{I_{coh}}{I_{tot}} = |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)|, \quad (2.29b)$$

where $I_{tot} = I_{coh} + I_{incoh}$.

The first zero of the intensity pattern for an interferometer is given by,

$$\theta_{int} = \frac{\lambda}{2B} \quad \text{rad.} \quad (2.30)$$

2.1.3 Power-spectral Density of the Light Beam

The power-spectral density refers to the power of a signal or time series as a function of a frequency variable associated with a stationary stochastic process³ having units of power per frequency (watts per Hz). It is given by the Wiener–Khintchine theorem (see Appendix B). The power-spectral densities are statistical measures, which can be estimated from real data by averaging over the results from many measurements. They can be specified as one-sided functions of positive frequencies, or as two-sided functions of positive and negative frequencies. Optical power densities are, in general, one-sided and can be measured with an optical spectrum analyzer. In the case of ergodicity, temporal coherence is found to be linked to the power spectral density of the source. Following (1.21), one may write,

$$\begin{aligned} \int_{-\infty}^{\infty} U_T^{(r)}(\mathbf{r}, t + \tau) U_T^{(r)}(\mathbf{r}, t) dt &= \int_{-\infty}^{\infty} U_T^{(r)}(\mathbf{r}, t) dt \\ &\times \left[\int_{-\infty}^{\infty} \widehat{U}_T^{(r)}(\mathbf{r}, \nu) e^{-i2\pi\nu(t+\tau)} d\nu \right] \\ &= \int_{-\infty}^{\infty} \widehat{U}_T^{(r)}(\mathbf{r}, \nu) e^{-i2\pi\nu\tau} d\nu \\ &\times \left[\int_{-\infty}^{\infty} U_T^{(r)}(\mathbf{r}, t) e^{-i2\pi\nu t} dt \right]. \end{aligned} \quad (2.31)$$

³ A stochastic process, concerns a sequence of events governed by probability theorem. It is a family of interdependent random variables that evolve over time (Papoulis 1984); there is some indeterminacy in its future evolution described by probability distributions. Some basic types of stochastic processes include (1) Markov processes, (2) Poisson processes (for instance, radioactive decay), and (3) time series, with the index variable referring to time.

After dividing both sides by $2T$, (2.31) can be recast in,

$$\frac{1}{2T} \int_{-\infty}^{\infty} U_T^{(r)}(\mathbf{r}, t + \tau) U_T^{(r)}(\mathbf{r}, t) dt = \frac{1}{2T} \int_{-\infty}^{\infty} \widehat{U}_T^{(r)}(\mathbf{r}, \nu) \widehat{U}_T^{(r)*}(\mathbf{r}, \nu) e^{-i2\pi\nu\tau} d\nu. \quad (2.32)$$

The smoothing operation is applied to the right hand side quantity by taking ensemble average over the ensemble of the random function $U^{(r)}$,

$$\langle U^{(r)}(\mathbf{r}, t + \tau) U^{(r)}(\mathbf{r}, t) \rangle = \int_{-\infty}^{\infty} \widehat{\Gamma}(\mathbf{r}, \nu) e^{-i2\pi\nu\tau} d\nu, \quad (2.33)$$

in which the function $\widehat{\Gamma}(\mathbf{r}, \nu)$ known as the power-spectral density of the optical disturbance, and is given by,

$$\widehat{\Gamma}(\mathbf{r}, \nu) = \lim_{T \rightarrow \infty} \left[\frac{\widehat{U}_T(\mathbf{r}, \nu) \widehat{U}_T^*(\mathbf{r}, \nu)}{2T} \right]. \quad (2.34)$$

To note, for an analytical signal $U(t)$ associated with $U^{(r)}(\mathbf{r}, t)$, the self-coherence function, $\Gamma(\mathbf{r}, \tau)$ (see 2.12). A similar procedure may be followed in the case of the spatial coherence,

$$\int_{-\infty}^{\infty} U_T^{(r)}(\mathbf{r}_1, t + \tau) U_T^{(r)}(\mathbf{r}_2, t) dt = \int_{-\infty}^{\infty} \widehat{U}_T^{(r)}(\mathbf{r}_1, \nu) \widehat{U}_T^{(r)*}(\mathbf{r}_2, \nu) e^{-i2\pi\nu\tau} d\nu. \quad (2.35)$$

After applying the smoothing operation to the right hand side quantity, the ensemble average is given by,

$$\langle U^{(r)}(\mathbf{r}_1, t + \tau) U^{(r)}(\mathbf{r}_2, t) \rangle = \int_{-\infty}^{\infty} \widehat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) e^{-i2\pi\nu\tau} d\nu, \quad (2.36)$$

in which the mutual spectral density or cross-spectral density of the light vibrations at $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$, $\widehat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is given by,

$$\widehat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) = \lim_{T \rightarrow \infty} \left[\frac{\widehat{U}_T(\mathbf{r}_1, \nu) \widehat{U}_T^*(\mathbf{r}_2, \nu)}{2T} \right]. \quad (2.37)$$

According to the Hermitian property, $\widehat{\Gamma}(\mathbf{r}_2, \mathbf{r}_1, \nu) = \widehat{\Gamma}^*(\mathbf{r}_1, \mathbf{r}_2, \nu)$; hence, the cross-spectral density function or the cross power spectrum, $\widehat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)$ of the light vibrations at points \mathbf{r}_1 and \mathbf{r}_2 is defined by,

$$\langle \widehat{U}^*(\mathbf{r}_1, \nu) \widehat{U}(\mathbf{r}_2, \nu') \rangle = \widehat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) \delta'(\nu - \nu'), \quad (2.38)$$

in which δ' is the Dirac delta function and the ensemble average $\langle \rangle$ is taken over the different realizations of the field.

The (2.37) implies that the cross-spectral density function, $\hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)$, is a measure of the correlation between the spectral amplitudes of any specific frequency component of the light disturbances at the points $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$. When these points, coincide, the cross-spectral density function, $\hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)$ turns out to be a function of one frequency, thus represents the spectral density of the light, $\hat{\Gamma}_p(\mathbf{r}, \nu)$. For a signal represented by $U(\mathbf{r}, t)$ and associated with $U^{(r)}(\mathbf{r}, t)$, the spectral representation of the mutual coherence function can be derived following (1.21 and 2.36),

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = 4 \int_0^\infty \hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) e^{-i2\pi\nu\tau} d\nu. \quad (2.39)$$

If $\Gamma^{(r)}(\mathbf{r}_1, \mathbf{r}_2)$ and $\Gamma^{(i)}(\mathbf{r}_1, \mathbf{r}_2)$ denote its real and imaginary parts, i.e.,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) = \Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)] + \Im [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)], \quad (2.40)$$

the relation between them can be connected by a Hilbert transform (Papoulis 1968). Thus, the correlation between the real and imaginary part of $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is given by,

$$\Im [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)] = \frac{1}{\pi} P \int_{-\infty}^\infty \frac{\Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau')]}{\tau' - \tau} d\tau', \quad (2.41)$$

$$\Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)] = -\frac{1}{\pi} P \int_{-\infty}^\infty \frac{\Im [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau')]}{\tau' - \tau} d\tau', \quad (2.42)$$

where P stands for the Cauchy principal value at $t = \tau$.

The coherence function, $|\Gamma(\mathbf{r}_1, \mathbf{r}_2)|$ considered as a function of τ , is the envelope of $\Re[\Gamma(\mathbf{r}_1, \mathbf{r}_2)]$,

$$\begin{aligned} \Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)] &= 2 \langle U^{(r)}(\mathbf{r}_1, t + \tau) U^{(r)}(\mathbf{r}_2, t) \rangle \\ &= 2 \int_{-\infty}^\infty \hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) e^{-i2\pi\nu\tau} d\nu, \end{aligned} \quad (2.43)$$

and $|\gamma(\mathbf{r}_1, \mathbf{r}_2)|$ is the envelope of the real correlation factor,

$$\begin{aligned} \Re [\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)] &= \frac{\Re [\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)]}{[\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0) \Gamma(\mathbf{r}_2, \mathbf{r}_2, 0)]^{1/2}} \\ &= \frac{\langle U^{(r)}(\mathbf{r}_1, t + \tau) U^{(r)}(\mathbf{r}_2, t) \rangle}{\left[\langle U^{(r)2}(\mathbf{r}_1, t) \rangle \langle U^{(r)2}(\mathbf{r}_2, t) \rangle \right]^{1/2}}. \end{aligned} \quad (2.44)$$

The (2.44) shows that the real part of $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ is equal to twice the cross correlation function of the real functions $U^{(r)}(\mathbf{r}_1, t)$ and $U^{(r)}(\mathbf{r}_2, t)$.

2.1.4 Mutual Intensity

Under quasi-monochromatic condition, the time delay, τ that is introduced between the interfering beams must also be small compared to the coherence time τ_c of the light, i.e., $\tau \ll \tau_c$. Thus, by invoking (2.15, 2.19, and 2.39) one obtains,

$$\begin{aligned} |\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| e^{i\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau)} &= \sqrt{I_1(\mathbf{r}, t) I_2(\mathbf{r}, t)} |\gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)| e^{i\Phi(\mathbf{r}_1, \mathbf{r}_2, \tau)} \\ &= 4 \int_0^\infty \hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu) e^{-i2\pi(\nu - \bar{\nu})\tau} d\nu. \end{aligned} \quad (2.45)$$

If the time delay is so small that $|(\nu - \bar{\nu})\tau| \ll 1$ for all frequencies for which $|\hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)|$ is appreciable, i.e., if $|\tau| \ll 1/\Delta\nu$, a small error is introduced if the exponential term of the integrand in (2.45) is replaced by unity. The mutual coherence function, as well as the complex degree of coherence are expressed as,

$$\begin{aligned} \Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) &\simeq |J(\mathbf{r}_1, \mathbf{r}_2)| e^{i(\Psi(\mathbf{r}_1, \mathbf{r}_2) - 2\pi\bar{\nu}\tau)} \\ &= J(\mathbf{r}_1, \mathbf{r}_2) e^{-i2\pi\bar{\nu}\tau}, \end{aligned} \quad (2.46a)$$

$$\begin{aligned} \gamma(\mathbf{r}_1, \mathbf{r}_2, \tau) &\simeq |\mu(\mathbf{r}_1, \mathbf{r}_2)| e^{i(\Psi(\mathbf{r}_1, \mathbf{r}_2) - 2\pi\bar{\nu}\tau)} \\ &= \mu(\mathbf{r}_1, \mathbf{r}_2) e^{-i2\pi\bar{\nu}\tau}, \end{aligned} \quad (2.46b)$$

with

$$\begin{aligned} J(\mathbf{r}_1, \mathbf{r}_2) &= \Gamma(\mathbf{r}_1, \mathbf{r}_2, 0) = \langle U(\mathbf{r}_1, t) U^*(\mathbf{r}_2, t) \rangle \\ &= \langle A_1(\mathbf{r}_1, t) A_2^*(\mathbf{r}_2, t) \rangle. \end{aligned} \quad (2.47)$$

as the mutual intensity of the light at the apertures $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$,

$$\begin{aligned} \mu(\mathbf{r}_1, \mathbf{r}_2) &= \gamma(\mathbf{r}_1, \mathbf{r}_2, 0) = \frac{\Gamma(\mathbf{r}_1, \mathbf{r}_2, 0)}{\sqrt{\Gamma(\mathbf{r}_1, \mathbf{r}_1, 0)} \sqrt{\Gamma(\mathbf{r}_2, \mathbf{r}_2, 0)}} \\ &= \frac{J(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{J(\mathbf{r}_1, \mathbf{r}_1)} \sqrt{J(\mathbf{r}_2, \mathbf{r}_2)}} = \frac{J(\mathbf{r}_1, \mathbf{r}_2)}{\sqrt{I_1(\mathbf{r}, t)} \sqrt{I_2(\mathbf{r}, t)}}, \end{aligned} \quad (2.48)$$

the complex coherence factor of the light, and

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \Phi(\mathbf{r}_1, \mathbf{r}_2, 0) = \arg \{ \gamma(\mathbf{r}_1, \mathbf{r}_2, 0) \} = \arg \{ \mu(\mathbf{r}_1, \mathbf{r}_2) \}, \quad (2.49)$$

the phase.

The mutual intensity, $J(\mathbf{r}_1, \mathbf{r}_2)$ is regarded as a phasor amplitude of a spatial sinusoidal fringe, while $\mu(\mathbf{r}_1, \mathbf{r}_2)$ is a normalized version of $J(\mathbf{r}_1, \mathbf{r}_2)$ having the property, $0 \leq |\mu(\mathbf{r}_1, \mathbf{r}_2)| \leq 1$. If the coherence length of the light is much greater

than the maximum pathlength difference (for vacuum) encountered in passage from the source to the interference region of interest, i.e., $\Delta v \ll \bar{v}$, i.e.,

$$|\Delta\varphi| = |s_2 - s_1| = \frac{\bar{\lambda}}{2\pi} \delta \ll \frac{\lambda^2}{\Delta\lambda}, \quad (2.50)$$

the interference law (2.24) can be recast in,

$$I(\mathbf{r}, t) \sim I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t) + 2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}|\mu(\mathbf{r}_1, \mathbf{r}_2)|\cos(\Psi(\mathbf{r}_1, \mathbf{r}_2) - \delta). \quad (2.51)$$

Equation (2.51) represents the basic formula of the quasi-monochromatic theory of partial coherence. The complex coherence factor, $\mu(\mathbf{r}_1, \mathbf{r}_2)$ is often called the complex visibility; it fulfills the condition $|\mu| \leq 1$. Although complex, it can be measured from recorded intensities by varying τ . Given the positions of the secondary point sources, the complex visibility holds information about the light source, which is valid only if the assumptions of quasi-chromaticity and point-like collectors or pin-holes are satisfied. When $I_1(\mathbf{r}, t)$ and $I_2(\mathbf{r}, t)$ are constant in the case of a quasi-monochromatic light, the observed interference pattern has constant visibility and constant phase across the observation region. The visibility in terms of the complex coherence factor is,

$$v = \begin{cases} \frac{2\sqrt{I_1(\mathbf{r}, t)}\sqrt{I_2(\mathbf{r}, t)}}{I_1(\mathbf{r}, t) + I_2(\mathbf{r}, t)}\mu(\mathbf{r}_1, \mathbf{r}_2) & \text{when } I_1(\mathbf{r}, t) \neq I_2(\mathbf{r}, t), \\ \mu(\mathbf{r}_1, \mathbf{r}_2) & \text{otherwise.} \end{cases} \quad (2.52)$$

Such a complex coherence factor is usually called visibility by the astronomers. If the complex coherent factor, $\mu(\mathbf{r}_1, \mathbf{r}_2)$, turns out to be zero, the fringes vanish and the two lights are known to be mutually incoherent, and in the case of $\mu(\mathbf{r}_1, \mathbf{r}_2)$ being 1, the two waves are called mutually coherent. For an intermediate value of $\mu(\mathbf{r}_1, \mathbf{r}_2)$, the two waves are partially coherent. The spectral degree of coherence, $\hat{\mu}(\mathbf{r}_1, \mathbf{r}_2, \nu)$, at frequency ν of the light at the points, $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$, is given by,

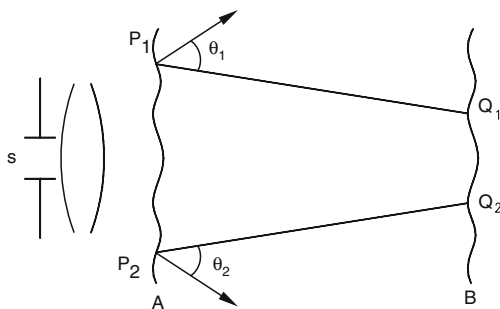
$$\hat{\mu}(\mathbf{r}_1, \mathbf{r}_2, \nu) = \frac{\hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_2, \nu)}{\sqrt{\hat{\Gamma}(\mathbf{r}_1, \mathbf{r}_1, \nu)}\sqrt{\hat{\Gamma}(\mathbf{r}_2, \mathbf{r}_2, \nu)}}. \quad (2.53)$$

The term $\hat{\mu}(\mathbf{r}_1, \mathbf{r}_2, \nu)$ is also referred as the complex degree of spatial (or spectral) coherence at frequency ν (Mandel and Wolf and Wolf 1976, 1995).

2.1.5 Propagation of Mutual Coherence

In order to derive the propagation laws for the cross-spectral density function and for the mutual coherence function, the well known Huygens–Fresnel principle for propagation of monochromatic light is elucidated to begin with. The notations relating to the laws for these functions are illustrated in Fig. 2.3. Assume that a wavefront with

Fig. 2.3 Geometry for propagation laws for the cross-spectral density and for the mutual coherence



arbitrary coherence properties emerges from an optical system lying on a surface, A, which intercepts the beam from an extended source σ and propagates to a surface, B. Knowing the values for the mutual coherence function, $\Gamma(\mathbf{r}_1, \mathbf{r}_2, \tau)$ (see 2.10), all points $P(\mathbf{r}_1)$ and $P(\mathbf{r}_2)$ on the surface A, one may determine the values of this coherence function,

$$\Gamma(\mathbf{r}'_1, \mathbf{r}'_2, \tau) = \langle U(\mathbf{r}'_1, t + \tau) U^*(\mathbf{r}'_2, t) \rangle, \quad (2.54)$$

for all points at $Q(\mathbf{r}'_1)$ and $Q(\mathbf{r}'_2)$, on the surface B. Here, $U(\mathbf{r}'_1, t)$ and $U(\mathbf{r}'_2, t)$ are the respective complex light disturbances at $Q_1(\mathbf{r}'_1)$, and $Q_2(\mathbf{r}'_2)$, and $U(\mathbf{r}'_1, \nu)$ and $U(\mathbf{r}'_2, \nu)$ the corresponding spectral amplitudes.

The refractive index of the medium between A and B is considered to be unity. It is found that to a good approximation, the coherence functions depend on time delay τ only through a harmonic term (see 2.46a, b). Following the Huygens–Fresnel principle (see Sect. 1.4.1), the complex amplitudes at the points $Q_1(\mathbf{r}'_1)$ and $Q_2(\mathbf{r}'_2)$, are expressed in terms of the complex amplitudes at all points on the surface B,

$$U(\mathbf{r}'_1, t + \tau) = \int_A \frac{\chi_1(\kappa)}{s_1} U\left(\mathbf{r}_1, t + \tau - \frac{s_1}{c}\right) dS_1, \quad (2.55a)$$

$$U^*(\mathbf{r}'_2, t) = \int_A \frac{\chi_2^*(\kappa)}{s_2} U^*\left(\mathbf{r}_2, t - \frac{s_2}{c}\right) dS_2, \quad (2.55b)$$

in which χ_1 and χ_2 are the inclination factors, s_1 and s_2 the distances P_1Q_1 and P_2Q_2 respectively.

If the effective spectral range of the light is sufficiently small, one may replace these factors by $\bar{\chi}_1 = \chi_1(\bar{\kappa})$, $\bar{\chi}_2 = \chi_2(\bar{\kappa})$, where $\bar{\kappa} = 2\pi\bar{\nu}/c$, and $\bar{\nu}$ denotes the mean frequency of the light. By plugging (2.55) into (2.10), the mutual coherence function on the surface B is given by,

$$\Gamma(\bar{\mathbf{r}}'_1, \bar{\mathbf{r}}'_2, \tau) = \int_A \int_A \frac{\chi_1}{s_1} \frac{\chi_2^*}{s_2} \Gamma\left(\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2, \tau - \frac{s_2 - s_1}{c}\right) dS_1 dS_2. \quad (2.56)$$

The (2.56) represents the propagation law for the mutual coherence function at points $Q_1(\mathbf{r}'_1)$ and $Q_2(\mathbf{r}'_2)$ of the surface B in terms of the mutual intensity at all pairs of points on the surface A. Under quasi-monochromatic conditions, one may write,

$$\Gamma(\vec{r}_1, \vec{r}_2, \tau - \frac{s_2 - s_1}{c}) = J(\vec{r}_1, \vec{r}_2) e^{i\vec{k}(s_2 - s_1)}. \quad (2.57)$$

The (2.56) can be recast as,

$$J(\vec{r}'_1, \vec{r}'_2) = \int_A \int_A \frac{\chi_1}{s_1} \frac{\chi_2^*}{s_2} J(\vec{r}_1, \vec{r}_2) e^{i\vec{k}(s_2 - s_1)} dS_1 dS_2. \quad (2.58)$$

Equation (2.58) relates the general law for propagation of mutual intensity. When the two points $Q_1(\vec{r}'_1)$ and $Q_2(\vec{r}'_2)$ coincide at $Q(\vec{r}')$ and $\tau = 0$, the intensity distribution on the surface B is deduced as,

$$I(\mathbf{r}') = \int_A \int_A \frac{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}}{s_1 s_2} \gamma\left(\mathbf{r}_1, \mathbf{r}_2, \frac{s_2 - s_1}{c}\right) \chi_1 \chi_2^* dS_1 dS_2, \quad (2.59)$$

in which γ is the correlation function (see 2.15).

The (2.59) expresses the intensity at an arbitrary point $P(\mathbf{r}')$ as the sum of surface contributions from all pairs of elements of the arbitrary surface A.

2.2 Van Cittert–Zernike Theorem

The van Cittert–Zernike theorem (van Cittert 1934; Zernike 1938) deals with the field correlations generated by an extended incoherent quasi-monochromatic source. It relates the complex visibility function of the fringes to a unique Fourier component of the impinging brightness distribution. The modulus of the complex degree of coherence in a plane illuminated by an incoherent quasi-monochromatic source is equal to the modulus of the normalized spatial Fourier transform of its brightness distribution.

An extended polychromatic source is considered to be a mosaic of point sources, and hence the response of a telescope or an interferometer can be taken as the summation of the response functions to all these sources. Any two elements of such a source are assumed to be uncorrelated. This source, σ , produces an electromagnetic wave field, represented by the analytic signal, $S(\mathbf{r}', t)$, which is a function of position \mathbf{S} and time t . Since it is a scalar quantity, it neglects polarization. This source may be divided into elements $d\sigma_1, d\sigma_2, \dots$ centered on points S_1, S_2, \dots , which are mutually incoherent, and of linear dimensions small compared to the mean wavelength λ . If $U(\mathbf{r}_{m1}, t)$ and $U(\mathbf{r}_{m2}, t)$ are the complex disturbances at $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ due to the element $d\sigma$, the total disturbances at these points are,

$$U(\mathbf{r}_1, t) = \sum_m U(\mathbf{r}_{m1}, t), \quad U(\mathbf{r}_2, t) = \sum_m U(\mathbf{r}_{m2}, t). \quad (2.60)$$

The mutual intensity, $J(\mathbf{r}_1, \mathbf{r}_2)$, at $P_1(\mathbf{r}_1)$ and $P_2(\mathbf{r}_2)$ is given by,

$$J(\mathbf{r}_1, \mathbf{r}_2) = \langle U(\mathbf{r}_1, t) U^*(\mathbf{r}_2, t) \rangle \\ = \sum_m \langle U(\mathbf{r}_{m1}, t) U^*(\mathbf{r}_{m2}, t) \rangle + \sum_{m \neq n} \sum_n \langle U(\mathbf{r}_{m1}, t) U^*(\mathbf{r}_{n2}, t) \rangle. \quad (2.61)$$

The term $\langle U(\mathbf{r}_1, t) U^*(\mathbf{r}_2, t) \rangle$ is a measure of spatial coherence of the radiation and is proportional to the correlation coefficient between $U(\mathbf{r}_1, t)$ and $U^*(\mathbf{r}_2, t)$.

The light vibrations arising from different elements of the source may be assumed to be statistically independent (mutually incoherent) and of the zero mean value, so that

$$\begin{aligned} \langle U(\mathbf{r}_{m1}, t) U^*(\mathbf{r}_{n2}, t) \rangle &= \langle U(\mathbf{r}_{m1}, t) \rangle \langle U^*(\mathbf{r}_{n2}, t) \rangle \\ &= 0; \quad m \neq n. \end{aligned} \quad (2.62)$$

The geometrical factors required for derivation of the van Cittert–Zernike theorem are depicted in Fig. 2.4. The field points, $P_1(\mathbf{r}'_1)$ and $P_2(\mathbf{r}'_2)$, on a screen A are illuminated by a self-luminous extended but unresolved quasi-monochromatic source, σ . Here, one assumes that the medium between the source and the screen is homogeneous, and the linear dimensions of the source are small compared to the distance OO' between the source and screen. The complex disturbance due to element $d\sigma_m$ at a point $P_{j=1,2}$ in the screen is,

$$U_{mj}(t) = A_m \left(t - \frac{s_{mj}}{c} \right) \frac{e^{-i2\pi\bar{\nu}(t-s_{mj}/c)}}{s_{mj}}, \quad (2.63)$$

where the strength and phase of the radiation coming from element $d\sigma_m$ are characterized by $|A_m|$ and $\arg(A_m)$, respectively, and s_{mj} is the distance from the element $d\sigma_m$ to the point P_j .

The mutual coherence function (see 2.10) of P_1 and P_2 turns out to be,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, 0) = \sum_m \left\langle A_m(t) A_m^*(t) \right\rangle \frac{e^{i2\pi\bar{\nu}(s_{m1}-s_{m2})/c}}{s_{m1} s_{m2}}. \quad (2.64)$$

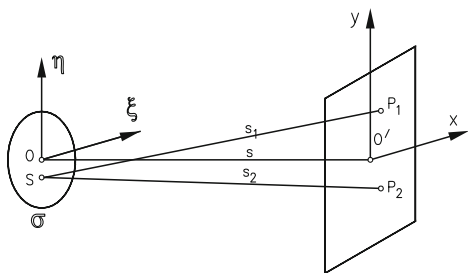


Fig. 2.4 Calculation of degree of coherence of $P_1(\mathbf{r}'_1)$ and $P_2(\mathbf{r}'_2)$ illuminated by σ

Considering a source with a total number of elements so large that it can be regarded as continuous, the sum in (2.64) is replaced by the integral,

$$\Gamma(\mathbf{r}_1, \mathbf{r}_2, 0) = \int_{\sigma} I(\mathbf{r}) \frac{e^{i\bar{k}(s_1-s_2)}}{s_1 s_2} dS. \quad (2.65)$$

in which s_1 and s_2 are the distances of $P_1(\mathbf{r}'_1)$ and $P_2(\mathbf{r}'_2)$ from a typical point $S(\mathbf{r})$ on the source, respectively, $\bar{k} = 2\pi/\bar{\lambda}$ the wave number, and $I(\mathbf{r})$ the intensity per unit area of the source.

According to (2.48) and (2.65), the complex degree of coherence $\mu(\mathbf{r}_1, \mathbf{r}_2)$ is,

$$\mu(\mathbf{r}'_1, \mathbf{r}'_2) = \frac{1}{\sqrt{I(\mathbf{r}_1)I(\mathbf{r}_2)}} \left(\frac{\bar{k}}{2\pi} \right)^2 \int_{\sigma} \frac{1}{s_1 s_2} I(\mathbf{r}) e^{i\bar{k}(s_1-s_2)} dS, \quad (2.66)$$

where

$$I(\mathbf{r}_j) = J(\mathbf{r}'_j, \mathbf{r}'_j) = \left(\frac{\bar{k}}{2\pi} \right)^2 \int_{\sigma} \frac{I(\mathbf{r})}{s_j^2} dS, \quad (2.67)$$

in which $j = 1, 2$ and $I(\mathbf{r}_j)$ the corresponding intensities at $P_j(\mathbf{r}'_j)$.

The (2.66) is the result of van Cittert–Zernike theorem. It expresses the complex degree of coherence at two fixed points, $P_1(\mathbf{r}'_1)$ and $P_2(\mathbf{r}'_2)$ in the field illuminated by an extended quasi-monochromatic source in terms of the intensity distribution $I(\mathbf{r})$ across the source and the intensity $I(\mathbf{r}_1)$ and $I(\mathbf{r}_2)$ at the corresponding points, $P_1(\mathbf{r}'_1)$ and $P_2(\mathbf{r}'_2)$.

Let (ξ, η) be the coordinates of the source plane, $S(\mathbf{r})$, referred to axes at O ,

$$\begin{aligned} s_1 &= \sqrt{s^2 + (x_1 - \xi)^2 + (y_1 - \eta)^2} \simeq s + \frac{(x_1 - \xi)^2 + (y_1 - \eta)^2}{2s}, \\ s_2 &= \sqrt{s^2 + (x_2 - \xi)^2 + (y_2 - \eta)^2} \simeq s + \frac{(x_2 - \xi)^2 + (y_2 - \eta)^2}{2s}, \end{aligned} \quad (2.68)$$

where (x_j, y_j) are the coordinates in the observation plane, and the term in x_j/s , y_j/s , ξ/s , and η/s are retained.

On setting,

$$p = \frac{(x_1 - x_2)}{s}, \quad q = \frac{(y_1 - y_2)}{s}, \quad (2.69a)$$

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \bar{k} \left[\frac{(x_1^2 + y_1^2) - (x_2^2 + y_2^2)}{2s} \right]. \quad (2.69b)$$

The quantity $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ represents the phase difference $2\pi(OP_1 - OP_2)/\bar{\lambda}$ and may be neglected if $(OP_1 - OP_2) \ll \bar{\lambda}$. By normalizing, the van Cittert–Zernike theorem (2.66) yields,

$$\mu(\mathbf{r}'_1, \mathbf{r}'_2) = e^{i\Psi(\mathbf{r}_1, \mathbf{r}_2)} \frac{\iint_{-\infty}^{\infty} I(\xi, \eta) e^{-i\bar{k}(p\xi + q\eta)} d\xi d\eta}{\iint_{-\infty}^{\infty} I(\xi, \eta) d\xi d\eta}. \quad (2.70)$$

The (2.70) states that for an incoherent circular source, the complex coherence factor far from the source is equal to the normalized Fourier transform of its brightness distribution.

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