Chapter 2
Basic Optics

2.1 Introduction

In this chapter we will discuss the basic concepts associated with polarization, diffraction, and interference of a light wave. The concepts developed in this chapter will be used in the rest of the book. For more details on these basic concepts, the reader may refer to Born and Wolf (1999), Jenkins and White (1981), Ghatak (2009), Ghatak and Thyagarajan (1989), and Tolansky (1955).

2.2 The Wave Equation

All electromagnetic phenomena can be said to follow from Maxwell's equations. For a charge-free homogeneous, isotropic dielectric, Maxwell's equations simplify to

$$\nabla.\mathbf{E} = 0 \tag{2.1}$$

$$\nabla.\mathbf{H} = 0 \tag{2.2}$$

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \tag{2.3}$$

and

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t} \tag{2.4}$$

where ε and μ represent the dielectric permittivity and the magnetic permeability of the medium and **E** and **H** represent the electric field and magnetic field, respectively. For most dielectrics, the magnetic permeability of the medium is almost equal to that of vacuum, i.e.,

$$\mu = \mu_0 = 4\pi \times 10^{-7} \ \mathrm{N\,C^{-2}\,s^2}$$

If we take the curl of Eq. (2.3), we would obtain

$$\text{curl (curl E)} = -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H} = -\varepsilon\mu \frac{\partial^2 \mathbf{E}}{\partial t^2} \tag{2.5}$$

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where we have used Eq. (2.4). Now, the operator $\nabla^2\mathbf{E}$ is *defined* by the following equation:

$$\nabla^2\mathbf{E} \equiv \text{grad}\,(\mathbf{div}\,\mathbf{E}) - \text{curl}\,(\text{curl E}) \tag{2.6}$$

Using Cartesian coordinates, one can easily show that

$$\left(\nabla^2\mathbf{E}\right)_x = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \text{div}\,(\text{grad}\,E_x)$$

i.e., a Cartesian component of $\nabla^2\mathbf{E}$ is the div grad of the Cartesian component.[1] Thus, using

$$\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla.\mathbf{E}) - \nabla^2\,\mathbf{E}$$

we obtain

$$\nabla\,(\nabla.\mathbf{E}) - \nabla^2\,\mathbf{E} = -\varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{2.7}$$

or

$$\nabla^2\,\mathbf{E} = \varepsilon\mu\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{2.8}$$

where we have used the equation $\nabla.\mathbf{E} = 0$ [see Eq. (2.1)]. Equation (2.8) is known as the three-dimensional wave equation and each Cartesian component of **E** satisfies the scalar wave equation:

$$\nabla^2\psi = \varepsilon\mu\frac{\partial^2\psi}{\partial t^2} \tag{2.9}$$

In a similar manner, one can derive the wave equation satisfied by **H**

$$\nabla^2\,\mathbf{H} = \varepsilon\mu\frac{\partial^2\mathbf{H}}{\partial t^2} \tag{2.10}$$

For plane waves (propagating in the direction of **k**), the electric and magnetic fields can be written in the form

$$\mathbf{E} = \mathbf{E}_0\exp[i(\omega t - \mathbf{k.r})] \tag{2.11}$$

and

$$\mathbf{H} = \mathbf{H}_0\exp[i(\omega t - \mathbf{k.r})] \tag{2.12}$$

[1] However, $(\mathbf{E})_r \neq$ div grad E_r

where $\mathbf{E_0}$ and $\mathbf{H_0}$ are space- and time-independent vectors; but may, in general, be complex. If we substitute Eq. (2.11) in Eq. (2.8), we would readily get

$$\frac{\omega^2}{k^2} = \frac{1}{\varepsilon\mu}$$

where

$$k^2 = k_x^2 + k_y^2 + k_z^2$$

Thus the velocity of propagation (v) of the wave is given by

$$v = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon\mu}} \tag{2.13}$$

In free space

$$\varepsilon = \varepsilon_0 = 8.8542 \times 10^{-12}\,\mathrm{C^2\,N^{-1}m^{-2}} \quad \text{and}\ \mu = \mu_0 = 4\pi \times 10^{-7}\mathrm{N\,C^{-2}\,s^2} \tag{2.14}$$

so that

$$\begin{aligned} v = c = \frac{1}{\sqrt{\varepsilon_0\mu_0}} &= \frac{1}{\sqrt{8.8542 \times 10^{-12} \times 4\pi \times 10^{-7}}} \\ &= 2.99794 \times 10^8 \mathrm{m\ s^{-1}} \end{aligned} \tag{2.15}$$

which is the velocity of light in free space. In a dielectric characterized by the dielectric permittivity ε, the velocity of propagation (v) of the wave will be

$$v = \frac{c}{n} \tag{2.16}$$

where

$$n = \sqrt{\frac{\varepsilon}{\varepsilon_0}} \tag{2.17}$$

is known as the refractive index of the medium. Now, if we substitute the plane wave solution [Eq. (2.11)] in the equation $\nabla.\mathbf{E} = 0$, we would obtain

$$i[k_x E_{0x} + k_y E_{0y} + k_z E_{0z}]\ \exp[i(\omega t - \mathbf{k.r})] = 0$$

implying

$$\mathbf{k.E} = 0 \tag{2.18}$$

Similarly the equation $\nabla.\mathbf{H} = 0$ would give us

$$\mathbf{k.H} = 0 \tag{2.19}$$

Equations (2.18) and (2.19) tell us that $\mathbf{E}$ and $\mathbf{H}$ are at right angles to $\mathbf{k}$; thus the waves are transverse in nature. Further, if we substitute the plane wave solutions

[Eqs. (2.11) and (2.12)] in Eqs. (2.3) and (2.4), we would obtain

$$\mathbf{H} = \frac{\mathbf{k} \times \mathbf{E}}{\omega\mu} \quad \text{and} \quad \mathbf{E} = \frac{\mathbf{H} \times \mathbf{k}}{\omega\,\varepsilon} \tag{2.20}$$

Thus **E**, **H**, and **k** are all at right angles to each other. Either of the above equations will give

$$E_0 = \eta\, H_0 \tag{2.21}$$

where η is known as the intrinsic impedance of the medium given by

$$\eta = \frac{k}{\omega\,\varepsilon} = \frac{\omega\,\mu}{k} = \sqrt{\frac{\mu}{\varepsilon}} = \eta_0\sqrt{\frac{\varepsilon_0}{\varepsilon}} \tag{2.22}$$

and

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \approx 377\ \Omega$$

is known as the impedance of free space. In writing Eq. (2.22) we have assumed $\mu = \mu_0 = 4\pi \times 10^{-7}\ \ \mathrm{N\,C^{-2}\,s^2}$. The (time-averaged) energy density associated with a propagating electromagnetic wave is given by

$$<u> = \frac{1}{2}\varepsilon\, E_0^2 \tag{2.23}$$

In the SI system, the units of u will be J m^{-3}. In the above equation, E_0 represents the amplitude of the electric field. The intensity I of the beam (which represents the energy crossing an unit area per unit time) will be given by

$$I = <u>\, v$$

where v represents the velocity of the wave. Thus

$$I = \frac{1}{2}\,\varepsilon\, v\, E_0^2 = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu_0}}\, E_0^2 \tag{2.24}$$

Example 2.1 Consider a 5 mW He–Ne laser beam having a beam diameter of 4 mm propagating in air. Thus

$$I = \frac{5 \times 10^{-3}}{\pi\left(2 \times 10^{-3}\right)^2} \approx 400 \quad \mathrm{J\,m^{-2}s^{-1}}$$

Since

$$I = \frac{1}{2}\varepsilon_0\, c\, E_0^2 \Rightarrow E_0 = \sqrt{\frac{2I}{\varepsilon_0\, c}}$$

we get

$$E_0 = \sqrt{\frac{2 \times 400}{\left(8.854 \times 10^{-12}\right) \times \left(3 \times 10^8\right)}} \approx 550 \quad \mathrm{V\,m^{-1}}$$

2.3 Linearly Polarized Waves

As shown above, associated with a plane electromagnetic wave there is an electric field **E** and a magnetic field **H** which are at right angles to each other. For a linearly polarized plane electromagnetic wave propagating in the x-direction (in a uniform isotropic medium), the electric and magnetic fields can be written in the form (see Fig. 2.1)

$$E_y = E_0 \cos(\omega t - kx),\ E_z = 0,\ E_x = 0 \tag{2.25}$$

and

$$H_x = 0,\ H_y = 0,\ H_z = H_0 \cos(\omega t - kx) \tag{2.26}$$

Since the longitudinal components E_x and H_x are zero, the wave is said to be a transverse wave. Also, since the electric field oscillates in the y-direction, Eqs. (2.25) and (2.26) describe what is usually referred to as a y-polarized wave. The direction of propagation is along the vector ($\mathbf{E} \times \mathbf{H}$) which in this case is along the x-axis.

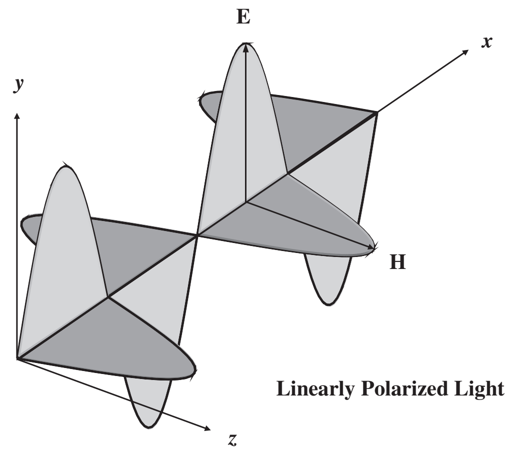


Fig. 2.1 A y-polarized electromagnetic wave propagating in the x-direction

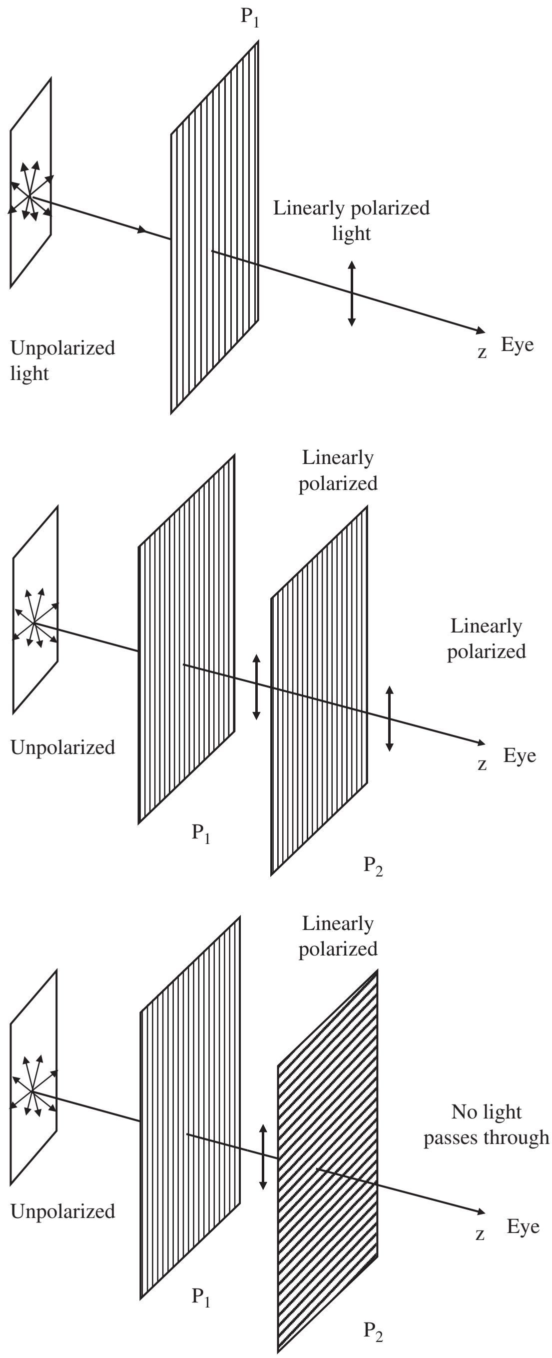


Fig. 2.2 If an ordinary light beam is allowed to fall on a Polaroid, then the emerging beam will be linearly polarized along the pass axis of the Polaroid. If we place another Polaroid P_2, then the intensity of the transmitted light will depend on the relative orientation of P_2 with respect to P_1

For a z-polarized plane wave (propagating in the $+x$-direction), the corresponding fields would be given by

$$E_x = 0,\ E_y = 0, \quad E_z = E_0 \cos(\omega t - kx), \tag{2.27}$$

and

$$H_x = 0,\ H_y = -H_0 \cos(\omega t - kx)\ ,\ H_z = 0 \tag{2.28}$$

An ordinary light beam, like the one coming from a sodium lamp or from the sun, is unpolarized (or randomly polarized), because its electric vector (on a plane transverse to the direction of propagation) keeps changing its direction in a random manner as shown in Fig. 2.2. If we allow the unpolarized beam to fall on a piece of Polaroid sheet then the beam emerging from the Polaroid will be linearly polarized. In Fig. 2.2 the lines shown on the Polaroid represent what is referred to as the "pass axis" of the Polaroid, i.e., the Polaroid absorbs the electric field perpendicular to its pass axis. Polaroid sheets are extensively used for producing linearly polarized light beams. As an interesting corollary, we may note that if a second Polaroid (whose pass axis is at right angles to the pass axis of the first Polaroid) is placed immediately after the first Polaroid, then no light will come through it; the Polaroids are said to be in a "crossed position" (see Fig. 2.2c).

2.4 Circularly and Elliptically Polarized Waves

We can superpose two plane waves of equal amplitudes, one polarized in the y-direction and the other polarized in the z-direction, with a phase difference of $\pi/2$ between them:

$$\mathbf{E_1} = E_0\,\hat{\mathbf{y}}\,\cos(\omega t - kx),$$

$$\mathbf{E_2} = E_0\,\hat{\mathbf{z}}\,\cos\left(\omega t - kx + \frac{\pi}{2}\right), \tag{2.29}$$

The resultant electric field is given by

$$\mathbf{E} = E_0\,\hat{\mathbf{y}}\,\cos(\omega t - kx) - E_0\,\hat{\mathbf{z}}\,\sin(\omega t - kx) \tag{2.30}$$

which describes a *left circularly polarized* (usually abbreviated as LCP) wave. At any particular value of x, the tip of the $\mathbf{E}$-vector, with increasing time t, can easily be shown to rotate on the circumference of a circle like a left-handed screw. For example, at $x = 0$ the y and z components of the electric vector are given by

$$E_y = E_0 \cos \omega t,\ E_z = -E_0 \sin \omega t \tag{2.31}$$

thus the tip of the electric vector rotates on a circle in the anti-clockwise direction (see Fig. 2.3) and therefore it is said to represent an LCP beam. When propagating in air or in any isotropic medium, the state of polarization (SOP) is maintained,

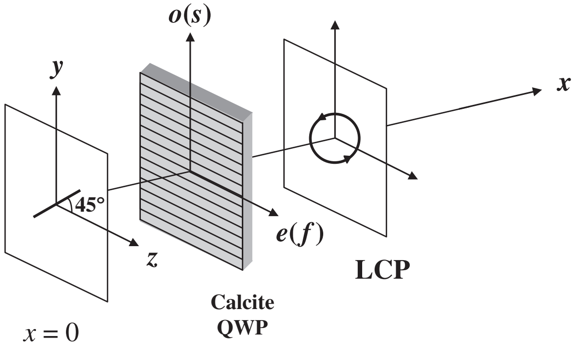


Fig. 2.3 A linearly polarized beam making an angle 45° with the *z*-axis gets converted to an LCP after propagating through a calcite Quarter Wave Plate (usually abbreviated as QWP); the optic axis in the QWP is along the *z*-direction as shown by lines parallel to the *z*-axis

i.e., a linearly polarized beam will remain linearly polarized; similarly, *right circularly polarized* (usually abbreviated as RCP) beam will remain RCP. In general, the superposition of two beams with arbitrary amplitudes and phase

$$E_y = E_0 \cos(\omega t - kx) \text{ and } E_z = E_1 \cos(\omega t - kx + \phi) \tag{2.32}$$

will represent an elliptically polarized beam.

How to obtain a circularly polarized beam? If a linearly polarized beam is passed through a properly oriented quarter wave plate we obtain a circularly polarized beam (see, e.g., Ghatak and Thyagarajan 1989). Crystals such as calcite and quartz are called anisotropic crystals and are characterized by two refractive indices, namely ordinary refractive index n_o and extraordinary refractive index n_e. Inside a crystal-like calcite, there is a preferred direction (known as the optic axis of the crystal); we will assume the crystal to be cut in a way so that the optic axis is parallel to one of the surfaces. In Fig. 2.3 we have assumed the *z*-axis to be along the optic axis. If the incident beam is *y*-polarized the beam will propagate as (what is known as) an ordinary wave with velocity (c/n_o). On the other hand, if the incident beam is *z*-polarized the beam will propagate as (what is known as) an extraordinary wave with velocity (c/n_e). For any other state of polarization of the incident beam, both the extraordinary and the ordinary components will be present. For a crystal-like calcite $n_e < n_o$ and the e-wave will travel faster than the o-wave; this is shown by putting s (slow) and f (fast) inside the parenthesis in Fig. 2.3. Let the electric vector (of amplitude E_0) associated with the incident-polarized beam make an angle ϕ with the *z*-axis; in Fig. 2.3, ϕ has been shown to be equal to 45°. Such a beam can be assumed to be a superposition of two linearly polarized beams (vibrating in phase), polarized along the *y*- and *z*-directions with amplitudes $E_0 \sin\phi$ and $E_0 \cos\phi$, respectively. The *y* component (whose amplitude is $E_0 \sin\phi$) passes through as an ordinary beam propagating with velocity c/n_o and the *z* component (whose amplitude is $E_0 \cos\phi$) passes through as an extraordinary beam propagating with velocity c/n_e; thus

$$E_y = E_0 \sin\phi \cos(\omega t - k_o x) = E_0 \sin\phi \cos\left(\omega t - \frac{2\pi}{\lambda_0} n_o x\right) \tag{2.33}$$

and

$$E_z = E_0 \cos\phi \cos(\omega t - k_e x) = E_0 \cos\phi \cos\left(\omega t - \frac{2\pi}{\lambda_0} n_e x\right) \tag{2.34}$$

where λ_0 is the free-space wavelength given by

$$\lambda_0 = \frac{2\pi c}{\omega} \tag{2.35}$$

Since $n_e \neq n_o$, the two beams will propagate with different velocities and, as such, when they come out of the crystal, they will not be in phase. Consequently, the emergent beam (which will be a superposition of these two beams) will be, in general, elliptically polarized. If the thickness of the crystal (denoted by d) is such that the phase difference produced is $\pi/2$, i.e.,

$$\frac{2\pi}{\lambda_0} d(n_o - n_e) = \frac{\pi}{2} \tag{2.36}$$

we have what is known as a quarter wave plate. Obviously, the thickness d of the quarter wave plate will depend on λ_0. For calcite, at λ_0= 5893 Å (at 18°C)

$$n_o = 1.65836, \; n_e = 1.48641$$

and for this wavelength the thickness of the quarter wave plate will be given by

$$d = \frac{5893 \times 10^{-8}}{4 \times 0.17195} \text{ cm} \approx 0.000857 \text{ mm}$$

If we put two identical quarter wave plates one after the other we will have what is known as a half-wave plate and the phase difference introduced will be π. Such a plate is used to change the orientation of an input linearly polarized wave.

2.5 The Diffraction Integral

In order to consider the propagation of an electromagnetic wave in an infinitely extended (isotropic) medium, we start with the scalar wave equation [see Eq. (2.9)]:

$$\nabla^2 \psi = \varepsilon \mu_0 \frac{\partial^2 \psi}{\partial t^2} \tag{2.37}$$

We assume the time dependence of the form $e^{i\omega t}$ and write

$$\psi = U(x, y, z)\, e^{i\omega t} \tag{2.38}$$

to obtain

$$\nabla^2 U + k^2 U = 0 \tag{2.39}$$

where

$$k = \omega\sqrt{\varepsilon\mu_0} = \frac{\omega}{v} \tag{2.40}$$

and U represents one of the Cartesian components of the electric field. The solution of Eq. (2.39) can be written as

$$U\ (x, y, z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F\left(k_x,\ k_y\right)\ e^{-i\left(k_x x + k_y y + k_z z\right)}\ \mathrm{d}k_x \mathrm{d}k_y \tag{2.41}$$

where

$$k_z = \pm\sqrt{k^2 - k_x^2 - k_y^2} \tag{2.42}$$

For waves making small angles with the z-axis we may write

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2} \quad \approx k\left[1 - \frac{k_x^2 + k_y^2}{2\,k^2}\right]$$

Thus

$$U\ (x, y,\ z) = e^{-ikz} \iint \quad F\left(k_x, k_y\right) \exp\left[-i\left(k_x x + k_y y - \frac{k_x^2 + k_y^2}{2\,k} z\right)\right] \mathrm{d}k_x \mathrm{d}k_y \tag{2.43}$$

and the field distribution on the plane $z = 0$ will be given by

$$U\ (x, y,\ z\ = 0) = \iint \quad F\left(k_x, k_y\ \right)\ e^{-i(k_x x + k_y y)}\ \mathrm{d}k_x \mathrm{d}k_y \tag{2.44}$$

Thus $U\ (x, y,\ z = 0)$ is the Fourier transform of $F\left(k_x,\ k_y\right)$. The inverse transform will give us

$$F\left(k_x,\ k_y\ \right) = \frac{1}{(2\pi)^2} \iint\ U(x', y', 0) \quad e^{i(k_x\, x' + k_y\, y')}\ \mathrm{d}x'\mathrm{d}y' \tag{2.45}$$

Substituting the above expression for $F\left(k_x,\ k_y\right)$ in Eq. (2.43), we get

$$U\,(x, y, z) = \frac{e^{-ikz}}{4\pi^2} \iint\ U(x', y', 0) \quad I_1 I_2\ \ \mathrm{d}x'\mathrm{d}y'$$

where

$$\begin{aligned} I_1 &= \int_{-\infty}^{+\infty} \exp\left[ik_x\left(x' - x\right)\right] \exp\left[\frac{ik_x^2}{2\,k}z\right] \mathrm{d}k_x \\ &= \sqrt{\frac{i4\pi^2}{\lambda z}} \exp\left[-\frac{ik\left(x' - x\right)^2}{2z}\right] \end{aligned} \tag{2.46}$$

and we have used the following integral

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2+\beta x}\,\mathrm{d}x = \sqrt{\frac{\pi}{\alpha}} \exp\left[\frac{\beta^2}{4\alpha}\right] \tag{2.47}$$

Similarly

$$\begin{aligned} I_2 &= \int_{-\infty}^{+\infty} \exp\left[ik_y\left(y' - y\right)\right] \exp\left[\frac{ik_y^2}{2k}z\right] \mathrm{d}k_y \\ &= \sqrt{\frac{i4\pi^2}{\lambda z}} \exp\left[-\frac{ik\left(y' - y\right)^2}{2z}\right] \end{aligned} \tag{2.48}$$

Thus

$$u\,(x, y, z) = \frac{i}{\lambda z}e^{-ikz} \iint u\left(x', y', 0\right) \exp\left[-\frac{ik}{2z}\left\{\left(x - x'\right)^2 + \left(y - y'\right)^2\right\}\right] \mathrm{d}x'\mathrm{d}y' \tag{2.49}$$

The above equation (known as the diffraction integral) represents the diffraction pattern in the Fresnel approximation. If we know the field $u(x,y)$ on a plane referred to as $z= 0$, then Eq. (2.49) helps us to calculate the field generated in any plane z. The field changes as it propagates due to diffraction effects.

2.6 Diffraction of a Gaussian Beam

A beam coming out of a laser can be well approximated by a Gaussian distribution of electric field amplitude. We consider a Gaussian beam propagating along the z-direction whose amplitude distribution on the plane $z= 0$ is given by

$$u(x, y, 0) = A \exp\left[-\frac{x^2 + y^2}{w_0^2}\right] \tag{2.50}$$

implying that the phase front is plane at $z= 0$. From the above equation it follows that at a distance w_0 from the z-axis, the amplitude falls by a factor $1/e$ (i.e., the intensity reduces by a factor $1/e^2$). This quantity w_0 is called the *spot size* of the beam. If we substitute Eq. (2.50) in Eq. (2.49) and use Eq. (2.47) to carry out the integration, we would obtain

$$u\,(x,\ y,\ z) \approx \frac{A}{(1 - i\,\gamma)} \quad \exp\left[-\frac{x^2 + y^2}{w^2\,(z)}\right]\, e^{-i\,\Phi} \tag{2.51}$$

where

$$\gamma = \frac{\lambda\, z}{\pi\, w_0^2} \tag{2.52}$$

$$w\,(z) = w_0\sqrt{1 + \gamma^2} = w_0\sqrt{1 + \frac{\lambda^2\, z^2}{\pi^2\, w_0^4}} \tag{2.53}$$

$$\Phi = k\,z + \frac{k}{2\,R\,(z)}\ \left(x^2 + y^2\right) \tag{2.54}$$

$$R(z) \equiv z\left(1 + \frac{1}{\gamma^2}\right) = z\left[1 + \frac{\pi^2\, w_0^4}{\lambda^2\, z^2}\right] \tag{2.55}$$

Thus the intensity distribution varies with z according to the following equation:

$$I\,(x,\ y,\ z) = \frac{I_0}{1 + \gamma^2}\,\exp\left[-\frac{2\,\left(x^2 + y^2\right)}{w^2(z)}\right] \tag{2.56}$$

which shows that the transverse intensity distribution remains Gaussian with the beamwidth increasing with z which essentially implies diffraction divergence. As can be seen from Eq. (2.53), for small values of z, the width increases quadratically with z but for values of $z >>> {w_0}^2/\lambda$, we obtain

$$w\,(z)\ \approx\ w_0\frac{\lambda\, z}{\pi\, w_0^2}\ = \frac{\lambda\, z}{\pi\, w_0} \tag{2.57}$$

which shows that the width increases linearly with z. This is the Fraunhofer region of diffraction. We define the diffraction angle as

$$\tan\,\theta = \frac{w(z)}{z}\ \approx\ \frac{\lambda}{\pi\, w_0} \tag{2.58}$$

showing that the rate of increase in the width is proportional to the wavelength and inversely proportional to the initial width of the beam. In order to get some numerical values we assume λ= 0.5 μm. Then, for w_0= 1 mm

$$2\theta \approx 0.018^\circ \quad \text{and} \quad w \approx 1.59\,\text{mm} \quad \text{at} \quad z = 10\,\text{m}$$

Similarly, for w_0= 0.25 mm,

$$2\theta \approx 0.073^\circ \text{ and} \quad w \approx 6.37\,\text{mm} \quad \text{at} \quad z = 10\,\text{m}$$

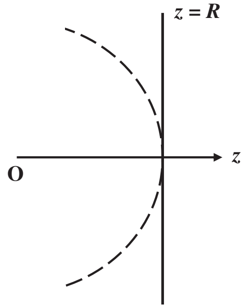


Fig. 2.4 A spherical wave diverging from the point O. The *dashed curve* represents a section of the spherical wavefront at a distance R from the source

Notice that θ increases with decrease in w_0 (smaller the size of the aperture, greater the diffraction). Further, for a given value of w_0, the diffraction effects decrease with λ. From Eq. (2.51) one can readily show that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I(x, y, z)\ \mathrm{d}x\mathrm{d}y = \frac{\pi\, w_0^2}{2}\ I_0$$

which is independent of z. This is to be expected, as the total energy crossing the entire x–y plane will not change with z.

Now, for a spherical wave *diverging* from the origin, the field distribution is given by

$$u \sim \frac{1}{r}\ e^{-ikr} \tag{2.59}$$

On the plane $z= R$ (see Fig. 2.4)

$$\begin{aligned} r &= \left[x^2 + y^2 + R^2\right]^{1/2} \\ &= R\left[1 + \frac{x^2 + y^2}{R^2}\right]^{1/2} \\ &\approx\ R + \frac{x^2 + y^2}{2\,R} \end{aligned} \tag{2.60}$$

where we have assumed $|x|\,,\ |y| << R$. Thus on the plane $z= R$, the phase distribution (corresponding to a diverging spherical wave of radius R) would be given by

$$e^{-ikr} \approx e^{-ikR}\ \ e^{-\frac{ik}{2R}\left(x^2+y^2\right)} \tag{2.61}$$

From the above equation it follows that a phase variation of the type

$$\exp\left[-i\frac{k}{2R}\left(x^2 + y^2\right)\right] \tag{2.62}$$

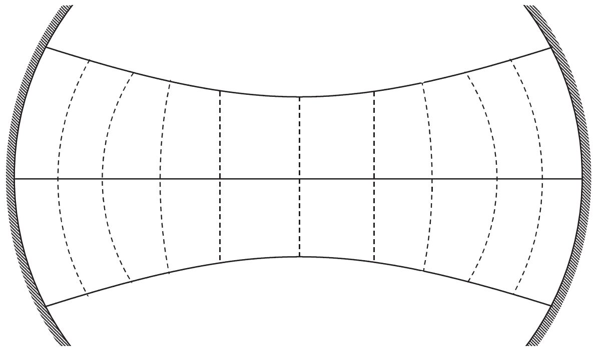


Fig. 2.5 Diffraction divergence of a Gaussian beam whose phase front is plane at $z = 0$. The *dashed curves* represent the phase fronts

(on the x–y plane) represents a *diverging* spherical wave of radius R. If we compare the above expression with Eqs. (2.59) and (2.60) we see that as the Gaussian beam propagates, the phase front curvature changes and we obtain the following approximate expression for the radius of curvature of the phase front at any value z:

$$R(z) \approx z\left(1 + \frac{\pi^2 w_0^4}{\lambda^2 z^2}\right) \tag{2.63}$$

Thus as the beam propagates, the phase front which was plane at $z = 0$ becomes curved. In Fig. 2.5 we have shown a Gaussian beam resonating between two identical spherical mirrors of radius R; the plane $z = 0$, where the phase front is plane and the beam has the minimum spot size, is referred to as the waist of the Gaussian beam. For the beam to resonate, the phase front must have a radius of curvature equal to R on the mirrors. For this to happen we must have

$$R \approx \frac{d}{2}\left(1 + \frac{4\pi^2 w_0^4}{\lambda^2 d^2}\right) \tag{2.64}$$

where d is the distance between the two mirrors. We will discuss more details about the optical resonators in Chapter 7.

It should be mentioned that although in the derivation of Eq. (2.51) we have assumed z to be large, Eq. (2.51) does give the correct field distribution even at z=0.

2.7 Intensity Distribution at the Back Focal Plane of a Lens

If a truncated plane wave of diameter $2a$ propagating along the z-axis is incident on a converging lens of focal length f (see Fig. 2.6a), the intensity distribution on the back focal plane is given by (see, e.g., Born and Wolf (1999))

$$I = I_0 \left[\frac{2J_1(v)}{v}\right]^2 \tag{2.65}$$

where

$$v = \frac{2\pi a}{\lambda f} r, \tag{2.66}$$

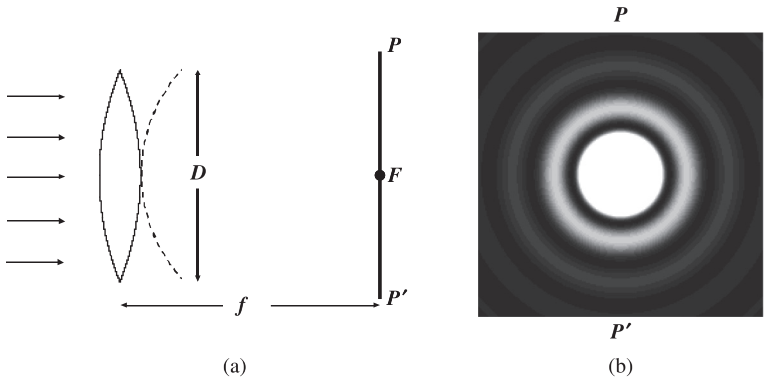


Fig. 2.6 (**a**) Plane wave falling on a converging lens gets focused at the focus of the lens. (**b**) The Airy pattern formed at the focus of the lens

I_0 is the intensity at the axial point F and r is the distance from the point F on the focal plane. Equation (2.65) describes the well-known Airy pattern (see Fig. 2.6b). The intensity is zero at the zeroes of the Bessel function $J_1(v)$ and $J_1(v) = 0$ when v= 3.832, 7.016, 10.174,. . ..

About 84% of the light energy is contained within the first dark ring and about 7% of light energy is contained in the annular region between the first two dark rings, etc., the first two dark rings occurring at

$$v = 3.832 \quad \text{and} \quad 7.016$$

2.8 Two-Beam Interference

Whenever two waves superpose, one obtains what is known as the interference pattern. In this section, we will consider the interference pattern produced by waves emanating from two point sources. As is well known, a stationary interference pattern is observed when the two interfering waves maintain a constant phase difference. For light waves, due to the very process of emission, one cannot observe a stationary interference pattern between the waves emanating from two independent sources, although interference does take place. Thus one tries to derive the interfering waves from a single wave so that a definite phase relationship is maintained all through.

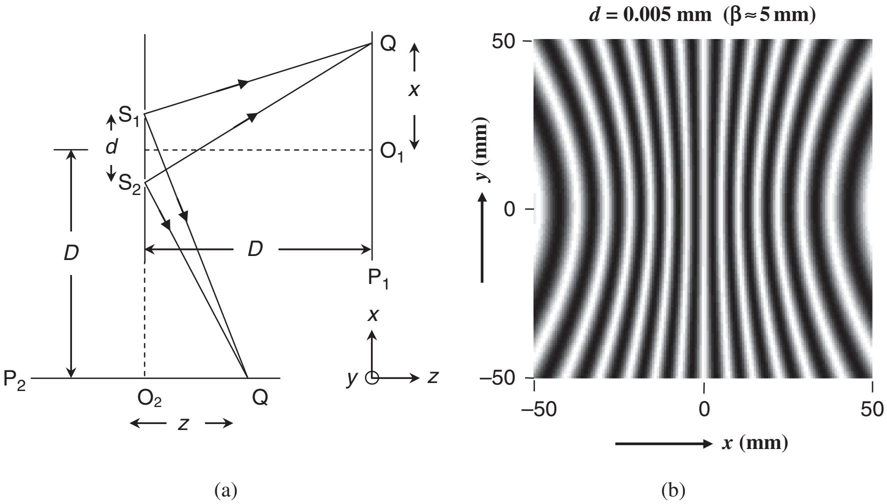


Fig. 2.7 (**a**) Waves emanating from two point sources interfere to produce interference fringes shown in Fig. 2.7 (**b**)

Let S_1 and S_2 represent two coherent point sources emitting waves of wavelength λ (see Fig. 2.7a). We wish to determine the interference pattern on the photographic plate P_1; the interference pattern on the photographic plate P_2 is discussed in Problem 2.11. The intensity distribution is given by

$$I = 4I_0 \cos^2 \delta/2 \tag{2.67}$$

where I_0 is the intensity produced by either of the waves independently and

$$\delta = \frac{2\pi}{\lambda}\Delta \tag{2.68}$$

where

$$\Delta = S_1Q - S_2Q \tag{2.69}$$

represents the path difference between the two interfering waves. Thus, when

$$\delta = 2n\pi \Rightarrow \Delta = S_1Q - S_2Q = n\lambda, \; n = 0, 1, 2, \ldots\ldots \text{(Bright Fringe)} \tag{2.70}$$

we will have a bright fringe, and when

$$\delta = (2n+1)\pi \Rightarrow \Delta = S_1Q - S_2Q = (n + \frac{1}{2})\lambda, \; n = 0, 1, 2, \ldots\ldots \text{(Dark Fringe)} \tag{2.71}$$

we will have a dark fringe. Using simple geometry one can show that the locus of the points (on the plane P_1) such that $S_1Q \sim S_2Q = \Delta$ is a hyperbola, given by

$$(d^2 - \Delta^2)x^2 - \Delta^2 y^2 = \Delta^2 \left[D^2 + \frac{1}{4}\left(d^2 - \Delta^2\right) \right] \tag{2.72}$$

Now,

$$\Delta = 0 \Rightarrow x = 0$$

which represents the central bright fringe. Equation (2.72) can be written in the form (see, e.g., Ghatak (2009))

$$x = \sqrt{\frac{\Delta^2}{d^2 - \Delta^2}} \left[y^2 + D^2 + \frac{1}{4}\left(d^2 - \Delta^2\right) \right]^{1/2} \tag{2.73}$$

For values of y such that

$$y^2 << D^2 \tag{2.74}$$

the loci are straight lines parallel to the y-axis and one obtains straight line fringes as shown in Fig.2.7b. The corresponding fringe width would be

$$\beta = \frac{\lambda D}{d} \tag{2.75}$$

Thus for D= 50 cm, d= 0.05 cm, and λ= 6000 Å, we get β= 0.06 cm.

2.9 Multiple Reflections from a Plane Parallel Film

We next consider the incidence of a plane wave on a plate of thickness h (and of refractive index n_2) surrounded by a medium of refractive index n_1 as shown in Fig. 2.8; [the Fabry–Perot interferometer consists of two partially reflecting mirrors (separated by a fixed distance h) placed in air so that $n_1 = n_2 = 1$].

Let A_0 be the (complex) amplitude of the incident wave. The wave will undergo multiple reflections at the two interfaces as shown in Fig. 2.8a. Let r_1 and t_1 represent the amplitude reflection and transmission coefficients when the wave is incident from n_1 toward n_2 and let r_2 and t_2 represent the corresponding coefficients when the wave is incident from n_2 toward n_1. Thus the amplitude of the successive reflected waves will be

$$A_0\, r_1\,,\; A_0\, t_1\, r_2\, t_2\, e^{i\,\delta},\; A_0\, t_1\, r_2^3\, e^{2\,i\,\delta}, \ldots\ldots$$

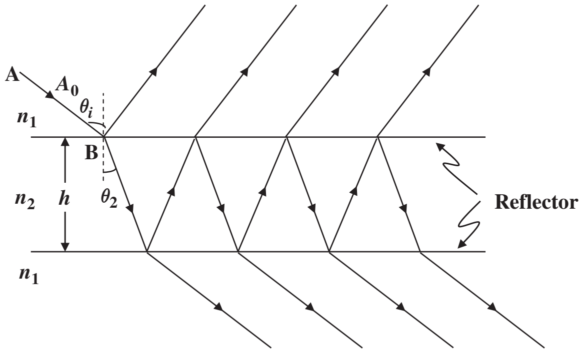


Fig. 2.8 Reflection and transmission of a beam of amplitude A_0 incident at a angle θ_i on a film of refractive index n_2 and thickness h

where

$$\delta = \frac{2\pi}{\lambda_0}\,\Delta = \frac{4\pi\, n_2\, h\, \cos\theta_2}{\lambda_0} \tag{2.76}$$

represents the phase difference (between two successive waves emanating from the plate) due to the additional path traversed by the beam in the film, and in Eq. (2.76), θ_2 is the angle of refraction *inside* the film (of refractive index n_2), h the film thickness, and λ_0 is the free-space wavelength. Thus the resultant (complex) amplitude of the reflected wave will be

$$\begin{aligned} A_r &= A_0 \left[r_1 + t_1\, t_2\, r_2\, e^{i\delta}\left(1 + r_2^2\, e^{i\delta} + r_2^4\, e^{2i\delta} + \ldots.\right)\right] \\ &= A_0 \left[r_1 + \frac{t_1\, t_2\, r_2\, e^{i\delta}}{1 - r_2^2\, e^{i\delta}}\right] \end{aligned} \tag{2.77}$$

Now, if the reflectors are lossless, the reflectivity and the transmittivity at each interface are given by

$$R = r_1^2 = r_2^2$$

$$\tau = t_1\, t_2 = 1 - R$$

[We are reserving the symbol T for the transmittivity of the Fabry–Perot etalon]. Thus

$$\frac{A_r}{A_0} = r_1 \left[1 - \frac{(1-R)\, e^{i\delta}}{1 - R\, e^{i\delta}}\right] \tag{2.78}$$

where we have used the fact that $r_2 = -\, r_1$. Thus the reflectivity of the Fabry–Perot etalon is given by

$$
\begin{aligned}
P = \left|\frac{A_r}{A_0}\right|^2 &= R \,.\, \left|\frac{1 - e^{i\delta}}{1 - R\, e^{i\delta}}\right|^2 \\
&= R\, \frac{(1-\cos\delta)^2 + \sin^2\delta}{(1 - R\cos\delta)^2 + R^2 \sin^2\delta} \\
&= \frac{4R\sin^2\frac{\delta}{2}}{(1-R)^2 + 4R\sin^2\frac{\delta}{2}}
\end{aligned}
$$

or

$$
P = \frac{F\sin^2\frac{\delta}{2}}{1 + F\sin^2\frac{\delta}{2}} \tag{2.79}
$$

where

$$
F = \frac{4R}{(1-R)^2} \tag{2.80}
$$

is called the coefficient of Finesse. One can immediately see that when $R << 1$, F is small and the reflectivity is proportional to $\sin^2 \delta/2$. The same intensity distribution is obtained in the two-beam interference pattern; we may mention here that we have obtained $\sin^2 \delta/2$ instead of $\cos^2 \delta/2$ because of the additional phase change of π in one of the reflected beams.

Similarly, the amplitude of the successive transmitted waves will be

$$
A_0\, t_1\, t_2\,,\; A_0\, t_1\, t_2\, r_2^2\, e^{i\delta},\; A_0\, t_1\, t_2\, r_2^4\, e^{2i\delta}, \ldots
$$

where, without any loss of generality, we have assumed the first transmitted wave to have zero phase. Thus the resultant amplitude of the transmitted wave will be given by

$$
\begin{aligned}
A_t &= A_0\, t_1\, t_2 \left[1 + r_2^2\, e^{i\delta} + r_2^4\, e^{2i\delta} + \ldots.\right] \\
&= A_0\, \frac{t_1\, t_2}{1 - r_2^2\, e^{i\delta}} = A_0\, \frac{1 - R}{1 - R\, e^{i\delta}}
\end{aligned}
$$

Thus the transmittivity T of the film is given by

$$
T = \left|\frac{A_t}{A_0}\right|^2 = \frac{(1-R)^2}{(1 - R\cos\delta)^2 + R^2\sin^2\delta}
$$

or

$$
T = \frac{1}{1 + F\sin^2\frac{\delta}{2}} \tag{2.81}
$$

It is immediately seen that the reflectivity and the transmittivity of the Fabry–Perot etalon add up to unity. Further,

$$T = 1$$

when

$$\delta = 2\,m\,\pi \quad , \quad m = 1, 2, 3, \ldots . \tag{2.82}$$

In Fig. 2.9 we have plotted the transmittivity as a function of δ for different values of F. In order to get an estimate of the width of the transmission resources, let

$$T = \frac{1}{2} \quad \text{for} \quad \delta = 2\,m\,\pi \;\pm\; \frac{\Delta\,\delta}{2}$$

Thus

$$F\,\sin^2\frac{\Delta\,\delta}{4} = 1 \tag{2.83}$$

The quantity $\Delta\delta$ represents the FWHM (full width at half maximum). In almost all cases, $\Delta\delta <<< 1$ and therefore, to a very good approximation, it is given by

$$\Delta\,\delta \;\approx\; \frac{4}{\sqrt{F}} = \frac{2\,(1-R)}{\sqrt{R}} \tag{2.84}$$

Thus the transmission resonances become sharper as the value of F increases (see Fig. 2.9).

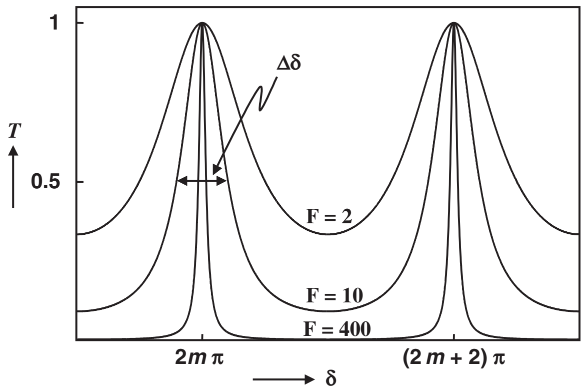


Fig. 2.9 The transmittivity of a Fabry–Perot etalon as a function of d for different values of F; the value of m is usually large. The transmission resonances become sharper as we increase the value of F. The FWHM (Full Width at Half Maximum) is denoted by $\Delta\delta$

2.10 Modes of the Fabry–Perot Cavity

We consider a polychromatic beam incident normally ($\theta_2 = 0$) on a Fabry–Perot cavity with air between the reflecting plates ($n_2 = 1$) – see Fig. 2.8. Equations (2.76) and (2.82) tell us that transmission resonance will occur whenever the incident frequency satisfies the following equation:

$$\nu = \nu_m = m\frac{c}{2h} \tag{2.85}$$

where m is an integer. The above equation represents the different (longitudinal) modes of the (Fabry–Perot) cavity. For $h = 10$ cm, the frequency spacing of two adjacent modes would be given by

$$\delta\nu = \frac{c}{2h} = 1500 \text{ MHz}$$

For an incident beam having a central frequency of

$$\nu = \nu_0 = 6 \times 10^{14} \text{ Hz}$$

and a spectral width[2] of 7000 MHz the output beam will have frequencies

$$\nu_0, \nu_0 \pm \delta\nu \text{ and } \nu_0 \pm 2\,\delta\nu$$

as shown in Fig. 2.10. One can readily calculate that the five lines correspond to

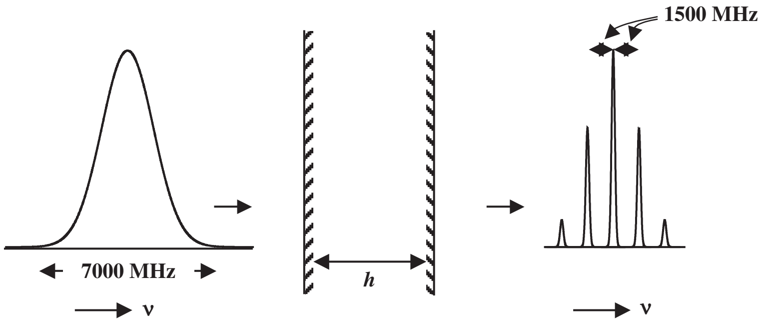


Fig. 2.10 A beam having a spectral width of about 7000 MHz (around $\nu_0 = 6 \times 10^{14}$ Hz) is incident normally on a Fabry–Perot etalon with $h = 10$ cm and $n_2 = 1$. The output has five narrow spectral lines

[2]For $\nu_0 = 6 \times 10^{14}$ Hz, $\lambda_0 = 5000$ Å and a spectral width of 7000 MHz would imply $\left|\frac{\Delta\lambda_0}{\lambda_0}\right| = \frac{\Delta\nu}{\nu_0} = \frac{7\times 10^9}{6\times 10^{14}} \approx 1.2\times 10^{-5}$ giving $\Delta\lambda_0 \approx 0.06$ Å. Thus a frequency spectral width of 7000 MHz (around $\nu_0 = 6 \times 10^{14}$ Hz) implies a wavelength spread of only 0.06 Å.

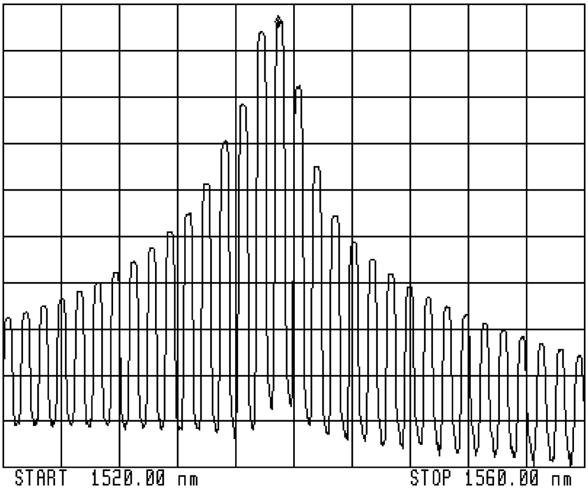


Fig. 2.11 Typical output spectrum of a Fabry–Perot multi longitudinal mode (MLM) laser diode; the wavelength spacing between two modes is about 1.25 nm

$$m = 399998, 399999, 400000, 400001, \text{ and } 400002$$

Figure 2.11 shows a typical output of a multilongitudinal (MLM) laser diode.

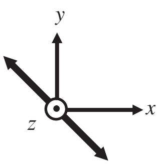
Problems

Problem 2.1 The electric field components of a plane electromagnetic wave are

$$E_x = -3E_0 \sin(\omega t - kz); \quad E_y = E_0 \sin(\omega t - kz)$$

Plot the resultant field at various values of time and show that it describes a linearly polarized wave.

Solution The beam will be linearly polarized



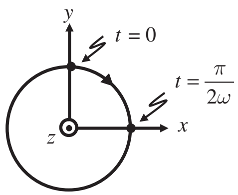
Problem 2.2 The electric field components of a plane electromagnetic wave are

$$E_x = E_0 \sin(\omega t + kz); \quad E_y = E_0 \cos(\omega t + kz)$$

Show that it describes a left circularly polarized wave.

Solution Propagation along the + z-direction (coming out of the page). At $z = 0$

$$E_x = E_0 \sin \omega t;\ E_y = E_0 \ \cos \omega t$$
$$\Rightarrow E_x^2 + E_y^2 = E_0^2 \ \Rightarrow \ \text{Circularly polarized}$$



Since propagation is along the +z-axis, i.e., coming out of the page, we have an LCP wave.

Problem 2.3 The electric field components of a plane electromagnetic wave are

$$E_x = -2E_0 \ \cos(\omega t + kz); \quad E_y = E_0 \ \sin(\omega t + kz)$$

Show that it describes a right elliptically polarized wave.

Problem 2.4 In Fig. 2.3 if we replace the quarter wave plate by a (calcite) half-wave plate, what will be the state of polarization of the output beam?

Problem 2.5 For calcite, at $\lambda_0 = 5893$ Å (at 18°C) n_o= 1.65836, $n_e = 1.48641$. The thickness of the corresponding QWP is 0.000857 mm (see Section 2.4). If in Fig. 2.3 the wavelength of the incident linearly polarized beam is changed to 6328 Å determine the state of polarization of the output beam.

Problem 2.6 A left circularly polarized beam is incident on a calcite half-wave plate. Show that the emergent beam will be right circularly polarized.

Problem 2.7 A 3 mW laser beam ($\lambda_0 \approx 6328$ Å) is incident on the eye. On the retina, it forms a circular spot of radius of about 20 μm. Calculate approximately the intensity on the retina.

Solution Area of the focused spot $A = \pi \left(20 \times 10^{-6}\right)^2 \approx 1.3 \times 10^{-9}\ \text{m}^2$. On the retina, the intensity will be approximately given by

$$I \approx \frac{P}{A} \approx \frac{3 \times 10^{-3}\ \text{W}}{1.3 \times 10^{-9}\ \text{m}^2} \approx 2.3 \times 10^6\ \text{W/m}^2$$

Problem 2.8 Consider a Gaussian beam propagating along the z-direction whose phase front is plane at $z = 0$ [see Eq. (2.50)]. The spot size of the beam at $z = 0$, w_0 is 0.3 mm. Calculate (a) the spot size and (b) the radius of curvature of the phase front at $z = 60$ cm. Assume $\lambda_0 \approx 6328$ Å.

[Ans : (a) $w\,(z = 60\,\text{cm}) \approx 0.84$ mm (b) $R\,(z = 60\,\text{cm}) \approx 93.3$ cm].

Problem 2.9 In continuation of the previous problem, show that for a simple resonator consisting of a plane mirror and a spherical mirror (of radius of curvature 93.3 cm) separated by 60 cm, the spot size of the beam at the plane mirror would be 0.3 mm.

Problem 2.10 Consider a He–Ne laser beam (with $\lambda_0 \approx 6328$ Å) incident on a circular aperture of radius 0.02 cm. Calculate the radii of the first two dark rings of the Airy pattern produced at the focal plane of a convex lens of focal length 20 cm.

Solution The radius of the first dark ring would be [see Eq. (2.66)]

$$r_1 \approx \frac{3.832 \times 6.328 \times 10^{-5} \times 20}{2\pi \times 0.02} \approx 0.039 \text{ cm}$$

Similarly, the radius of the second dark ring is

$$r_2 \approx \frac{7.016 \times 6.328 \times 10^{-5} \times 20}{2\pi \times 0.02} \approx 0.071 \text{ cm}$$

Problem 2.11 Consider two coherent point sources S_1 and S_2 emitting waves of wavelength λ (see Fig. 2.7a). Show that the interference pattern on a plane normal to the line joining S_1 and S_2 will consist of concentric circular fringes.

Problem 2.12 Consider a light beam of all frequencies lying between $\nu = \nu_0 = 5.0 \times 10^{14}$ Hz to $\nu = 5.00002 \times 10^{14}$ Hz incident normally on a Fabry–Perot interferometer (see Fig. 2.10) with $R = 0.95$, $n_0 = 1$, and $d = 25$ cm. Calculate the frequencies (in the above frequency range) and the corresponding mode number which will correspond to transmission resonances.

Solution Transmission resonances occur at
$\nu = \nu_m = m\frac{c}{2d} = m\frac{3\times10^{10}}{2\times25} = (6 \times 10^8\, m)$ Hz
For $\nu = \nu_0 = 5 \times 10^{14}$ Hz; $m = \frac{5\times10^{14}}{6\times10^8} = 833333.3$
Since m is not an integer the frequency ν_0 does not correspond to a mode.

For $m = 833334$, $\quad \nu = 5.000004 \times 10^{14}$ Hz $= \nu_0 + 400$ MHz

For $m = 833335$, $\quad \nu = 5.000010 \times 10^{14}$ Hz $= \nu_0 + 1000$ MHz

For $m = 833336$, $\quad \nu = 5.000016 \times 10^{14}$ Hz $= \nu_0 + 1600$ MHz

Finally, for $m = 833337$, $\quad \nu = 5.000022 \times 10^{14}$Hz $= \nu_0 + 2200$ MHz which is beyond the given range.

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