

Definitions and Basic Properties of Extended Riemann–Stieltjes Integrals

2.1 Regulated and Interval Functions

Regulated functions

Let X be a Banach space, and let J be a nonempty interval in \mathbb{R} , which may be bounded or unbounded, and open or closed at either end. Recall that an interval is called nondegenerate if it has nonempty interior or equivalently contains more than one point. Let \bar{J} be the closure of J in the extended real line $[-\infty, \infty]$. A function f on J with values in X is called *regulated on J* , just as for real-valued functions in Chapter 1, if the right limit $f(t+) := \lim_{s \downarrow t} f(s)$ exists in X for $t \in \bar{J}$ not equal to the right endpoint of J , and if the left limit $f(t-) := \lim_{s \uparrow t} f(s)$ exists in X for $t \in \bar{J}$ not equal to the left endpoint of J . If $J = \{a\} = [a, a]$ is a singleton these conditions hold vacuously and we say that f is regulated. The class of all regulated functions on J with values in X will be denoted by $\mathcal{R}(J; X)$.

Let $a < b$ throughout this paragraph (defining quantities f_- , f_+ , and Δ). For a regulated function f on $J = \llbracket a, b \rrbracket$, define a function $f_-^{(a)}(t) := f(t-)$ for $t \in (a, b]$ or $f(a)$ if $t = a \in J$. Similarly define $f_+^{(b)}(t) := f(t+)$ for $t \in \llbracket a, b)$ or $f(b)$ if $t = b \in J$. Define $\Delta^+ f$ on J by $(\Delta^+ f)(t) := f(t+) - f(t)$, called the right jump of f at t , for all $t \in J$ except the right endpoint. Similarly, define a function $\Delta^- f$ on J by $(\Delta^- f)(t) := f(t) - f(t-)$, called the left jump of f at t , for all $t \in J$ except the left endpoint. Also, let $\Delta^\pm f := \Delta^+ f + \Delta^- f$, called the two-sided jump, on the interior of J . For a regulated function f on $J = \llbracket a, b \rrbracket$, let $\Delta_J^+ f(t) := \Delta^+ f(t)$ if $t \in \llbracket a, b)$, or 0 if $t = a \notin J$ or $t = b$; $\Delta_J^- f(t) := \Delta^- f(t)$ if $t \in (a, b]$, or 0 if $t = b \notin J$ or $t = a$; and

$$\begin{aligned} \Delta_J^\pm f(t) &:= \Delta_J^- f(t) + \Delta_J^+ f(t) \\ &= \begin{cases} \Delta^+ f(a) & \text{if } t = a \in J, \text{ or } 0 \text{ if } t = a \notin J; \\ \Delta^\pm f(t) & \text{if } t \in (a, b); \\ \Delta^- f(b) & \text{if } t = b \in J, \text{ or } 0 \text{ if } t = b \notin J. \end{cases} \end{aligned} \quad (2.1)$$

The definition of Δ_J^\pm is made so that some later formulas will not need to have special endpoint terms.

A *step function* from an interval J into a Banach space X is a finite sum $\sum_{j=1}^m 1_{A(j)} x_j$, where $x_j \in X$ and $A(j)$ are intervals, some of which may be singletons. Clearly a step function is regulated. The next fact shows that a function is regulated if and only if it can be approximated uniformly by step functions.

Theorem 2.1. *Let X be a Banach space and $-\infty < a < b < +\infty$. The following properties are equivalent for a function $f: J := \llbracket a, b \rrbracket \rightarrow X$:*

- (a) $f \in \mathcal{R}(\llbracket a, b \rrbracket; X)$;
- (b) for each $\epsilon > 0$, there exists a Young interval partition $\{(t_{i-1}, t_i)\}_{i=1}^n$ of $\llbracket a, b \rrbracket$ such that $\text{Osc}(f; (t_{i-1}, t_i)) < \epsilon$ for each $i \in \{1, \dots, n\}$;
- (c) f is a uniform limit of step functions.

Proof. (a) \Rightarrow (b). Let $\epsilon > 0$. By definition of $\mathcal{R}(J; X)$, if $J = (a, b]$, $f(a+)$ exists, and if $J = \llbracket a, b \rrbracket$, $f(b-)$ exists. Thus there exist $a_1 > a$ and $b_1 < b$ such that $\text{Osc}(f; (a, a_1)) < \epsilon$ and $\text{Osc}(f; (b_1, b)) < \epsilon$. So it suffices to consider the case that $\llbracket a, b \rrbracket$ is a closed, bounded interval $[a, b]$ (specifically, $[a_1, b_1]$). For each $s \in (a, b)$, choose $\delta_s > 0$ such that $A_s := (s - \delta_s, s + \delta_s) \subset [a, b]$ and the oscillation of f over the open intervals $(s - \delta_s, s)$ and $(s, s + \delta_s)$ is less than ϵ . For the endpoints, choose δ_a and $\delta_b \in (0, b - a)$ such that $\text{Osc}(f; (a, a + \delta_a)) < \epsilon$ and $\text{Osc}(f; (b - \delta_b, b)) < \epsilon$. Letting $A_a := [a, a + \delta_a)$ and $A_b := (b - \delta_b, b]$, the sets $\{A_s: s \in [a, b]\}$ form a cover of the compact interval $[a, b]$ by relatively open sets. Therefore there is a finite subcover $A_{s_0}, A_{s_1}, \dots, A_{s_m}$ of $[a, b]$ with $a = s_0 < s_1 < \dots < s_m = b$. Take $t_0 := s_0$, $t_1 \in A_{s_0} \cap A_{s_1} \cap (s_0, s_1)$, $t_2 := s_1, \dots, t_{2m-2} := s_{m-1}$, $t_{2m-1} \in A_{s_{m-1}} \cap A_{s_m} \cap (s_{m-1}, s_m)$, and $t_{2m} := s_m$. Thus (b) holds with $n = 2m + 1$.

(b) \Rightarrow (c). Given $\epsilon > 0$, choose a Young interval partition $\{(t_{i-1}, t_i)\}_{i=1}^n$ of J as in (b). Define a step function f_ϵ on J by $f_\epsilon(t) := f(s_i)$ with $s_i \in (t_{i-1}, t_i)$ if $t \in (t_{i-1}, t_i)$ for some $i \in \{1, \dots, n\}$ and $f_\epsilon(t_i) := f(t_i)$ for $i \in \{0, \dots, n\}$ if $t_i \in J$. Then $\|f_\epsilon(t) - f(t)\| < \epsilon$ for each $t \in J$. Since ϵ is arbitrary, (c) follows.

(c) \Rightarrow (a). Given $\epsilon > 0$, choose a step function f_ϵ such that $\|f_\epsilon(t) - f(t)\| < \epsilon$ for each $t \in J$. Then for any $s, t \in J$, $\|f(t) - f(s)\| < 2\epsilon + \|f_\epsilon(t) - f_\epsilon(s)\|$. Since f_ϵ is regulated, the right side can be made arbitrarily small for all s, t close enough from the left or right to any given point of J , proving (a). The proof of Theorem 2.1 is complete. \square

The following is an easy consequence of the preceding theorem.

Corollary 2.2. *If f is regulated on $\llbracket a, b \rrbracket$ then f is bounded, Borel measurable, and for each $\epsilon > 0$,*

$$\text{card} \left\{ u \in \llbracket a, b \rrbracket : \text{either } \left\| \Delta_{\llbracket a, b \rrbracket}^+ f(u) \right\| > \epsilon \text{ or } \left\| \Delta_{\llbracket a, b \rrbracket}^- f(u) \right\| > \epsilon \right\} < \infty.$$

Proof. By implication (a) \Rightarrow (b) of Theorem 2.1, there is a Young interval partition $\{(t_{i-1}, t_i)\}_{i=1}^n$ of $\llbracket a, b \rrbracket$ such that $\text{Osc}(f; (t_{i-1}, t_i)) \leq 1$ for each $i = 1, \dots, n$. Let $t \in \llbracket a, b \rrbracket$ and let i be such that $t = t_i$ or $t \in (t_{i-1}, t_i)$. Then $\|f(t)\| \leq 1 + \max_{1 \leq i \leq n} \|f(t_{i-1}+)\| + \max_{0 \leq i \leq n} \|f(t_i)\|$, proving the first part of the conclusion. The third part also follows from Theorem 2.1(b) since each jump can be approximated arbitrarily closely by increments of f . It then follows that f is continuous except on a countable set, and so it is Borel measurable, completing the proof. \square

Interval functions

Let X be a Banach space with norm $\|\cdot\|$, let J be a nonempty interval in \mathbb{R} , possibly unbounded, and let $\mathfrak{J}(J)$ be the class of all subintervals of J . Any function $\mu: \mathfrak{J}(J) \rightarrow X$ will be called an *interval function on J* . The class of all X -valued interval functions on J is denoted by $\mathcal{I}(J; X)$, and it is denoted by $\mathcal{I}(J)$ if $X = \mathbb{R}$. An interval function μ on J will be called *additive* if $\mu(A \cup B) = \mu(A) + \mu(B)$ whenever $A, B \in \mathfrak{J}(J)$ are disjoint and $A \cup B \in \mathfrak{J}(J)$. If μ is an additive interval function then clearly $\mu(\emptyset) = 0$. An additive interval function μ on J is uniquely determined by its restriction to the class $\mathfrak{J}_{os}(J)$, i.e. the class of all open subintervals and singletons of J .

For intervals as for other sets, $A_n \uparrow A$ will mean $A_1 \subset A_2 \subset \dots$ and $\bigcup_{n=1}^{\infty} A_n = A$, while $A_n \downarrow A$ will mean $A_1 \supset A_2 \supset \dots$ and $\bigcap_{n=1}^{\infty} A_n = A$, and $A_n \rightarrow A$ will mean $1_{A_n}(t) \rightarrow 1_A(t)$ as $n \rightarrow \infty$ for all $t \in \mathbb{R}$.

Definition 2.3. 1. An interval function μ on J will be called *upper continuous* if $\mu(A_n) \rightarrow \mu(A)$ for any $A, A_1, A_2, \dots \in \mathfrak{J}(J)$ such that $A_n \downarrow A$.

2. An interval function μ on J will be called *upper continuous at \emptyset* if $\mu(A_n) \rightarrow \mu(\emptyset)$ for any $A_1, A_2, \dots \in \mathfrak{J}(J)$ such that $A_n \downarrow \emptyset$.

If J is a singleton then the conditions of Definition 2.3 hold vacuously and each interval function on J is upper continuous as well as upper continuous at \emptyset .

In this section we establish relations between the class of additive upper continuous interval functions on J and classes of regulated functions on J . But first, here is an example of an interval function which is not upper continuous.

Example 2.4. Let J be a nondegenerate interval. The Banach space of all real-valued and bounded functions on J with the supremum norm is denoted by $\ell^\infty(J)$. Recall that for Lebesgue measure λ on J , $L^\infty(J, \lambda)$ is the Banach space of λ -equivalence classes of λ -essentially bounded functions with the essential supremum norm. For an interval $A \subset J$, let $\mu(A) := 1_A$. Then for 1_A as a member of $\ell^\infty(J)$ or of $L^\infty(J, \lambda)$, μ is an additive interval function on J , but not upper continuous at \emptyset .

For a regulated function h on $\llbracket a, b \rrbracket$ with values in X there is a corresponding additive interval function $\mu_h := \mu_{h, \llbracket a, b \rrbracket}$ on $\llbracket a, b \rrbracket$ defined by

$$\begin{aligned}\mu_{h, \llbracket a, b \rrbracket}((u, v)) &:= h(v-) - h(u+) \text{ for } a \leq u < v \leq b, \\ \mu_{h, \llbracket a, b \rrbracket}(\{u\}) &:= \Delta_{\llbracket a, b \rrbracket}^{\pm} h(u) \text{ for } u \in \llbracket a, b \rrbracket,\end{aligned}\tag{2.2}$$

if $a < b$, and $\mu_{h, [a, a]}(\{a\}) := \mu_{h, [a, a]}(\emptyset) := 0$ if $a = b$. For $a < b$ we have $\mu_{h, (a, b]}(\{b\}) = \Delta_{(a, b]}^{\pm} h(b) = \Delta^{-} h(b)$. It follows that if h on $(a, b]$ is right-continuous on (a, b) then for $a \leq c < d \leq b$ we have $\mu_h((c, d]) = h(d) - h(c)$. Thus the definition (2.2) of μ_h just given extends the one given earlier for the definition (1.6) of the Lebesgue–Stieltjes integral.

Notice that μ_h does not depend on the values of h at its jump points (points where it has a non-zero left or right jump) in (a, b) . In the converse direction, for any interval function μ on $\llbracket a, b \rrbracket$, define two functions $R_{\mu, a}$ and $L_{\mu, a}$ on $\llbracket a, b \rrbracket$ by

$$\begin{aligned}R_{\mu, a}(t) &:= \begin{cases} \mu(\emptyset) & \text{if } t = a \in \llbracket a, b \rrbracket \\ \mu(\llbracket a, t \rrbracket) & \text{if } t \in (a, b], \end{cases} \\ L_{\mu, a}(t) &:= \begin{cases} \mu(\llbracket a, t \rrbracket) & \text{if } t \in \llbracket a, b \rrbracket \\ \mu(\llbracket a, b \rrbracket) & \text{if } t = b \in \llbracket a, b \rrbracket. \end{cases}\end{aligned}\tag{2.3}$$

(Recall that if μ is additive, $\mu(\emptyset) = 0$.) If μ is upper continuous then $R_{\mu, a}$ and $L_{\mu, a}$ are both regulated point functions on $\llbracket a, b \rrbracket$, as will be shown in Proposition 2.6 when $a < b$. The converse is not true as the following shows:

Example 2.5. Let $a < b$, $\mu([u, v)) := \mu((u, v]) := 0$ and $\mu([u, v]) := 1$ for $a \leq u \leq v \leq b$, and $\mu((u, v)) := -1$ if also $u < v$. Then μ is an additive interval function on $[a, b]$, $R_{\mu, a} = 1_{(a, b]}$ and $L_{\mu, a} = 1_{\{b\}}$ are regulated, but μ is not upper continuous at \emptyset . For $h = R_{\mu, a}$ or $L_{\mu, a}$, $\mu \neq \mu_h$.

For $x \in J$, we will say that the singleton $\{x\}$ is an *atom* of μ if $\mu(\{x\}) \neq 0$. The following gives a characterization of additive upper continuous interval functions on J . In particular, such interval functions cannot have more than countably many atoms.

Proposition 2.6. *Let J be a nondegenerate interval and let $\mu \in \mathcal{I}(J; X)$ be additive. The following five statements are equivalent:*

- (a) μ is upper continuous;
- (b) μ is upper continuous at \emptyset ;
- (c) $\mu(A_n) \rightarrow 0$ whenever open intervals $A_n \downarrow \emptyset$, and

$$\text{card}\{u \in J : \|\mu(\{u\})\| > \epsilon\} < \infty \quad \text{for all } \epsilon > 0;\tag{2.4}$$

- (d) $\mu(A_n) \rightarrow \mu(A)$ whenever intervals $A_n \uparrow A$;
- (e) $\mu(A_n) \rightarrow \mu(A)$ whenever intervals $A_n \rightarrow A \neq \emptyset$.

If $J = \llbracket a, b \rrbracket$ then the above statements are equivalent to each of the following two statements:

- (f) $R_{\mu, a}$ is regulated, $R_{\mu, a}(x-) = \mu(\llbracket a, x \rrbracket)$ for $x \in (a, b]$, and $R_{\mu, a}(x+) = \mu(\llbracket a, x \rrbracket)$ for $x \in \llbracket a, b \rrbracket$;

(g) $L_{\mu,a}$ is regulated, $L_{\mu,a}(x-) = \mu(\llbracket a, x))$ for $x \in (a, b]$, and $L_{\mu,a}(x+) = \mu(\llbracket a, x])$ for $x \in \llbracket a, b)$.

Proof. (a) \Leftrightarrow (b). Clearly, (a) implies (b). For (b) \Rightarrow (a), let intervals $A_n \downarrow A$. Then $A_n = B_n \cup A \cup C_n$ for intervals $B_n \prec A \prec C_n$, with $B_n \downarrow \emptyset$ and $C_n \downarrow \emptyset$. So $\mu(A_n) \rightarrow \mu(A)$ by additivity.

(b) \Rightarrow (c). The first part of (c) is clear. For the second part, suppose there exist $\epsilon > 0$ and an infinite sequence $\{u_j\}$ of different points of J such that $\|\mu(\{u_j\})\| > \epsilon$ for all j . Then there are a $u \in \bar{J}$ and a subsequence $\{u_{j'}\}$ such that either $u_{j'} \downarrow u$ or $u_{j'} \uparrow u$ as $j' \rightarrow \infty$. In the first case, by additivity, $\mu(\{u_{j'}\}) = \mu((u, u_{j'}]) - \mu((u, u_{j'})) \rightarrow 0$ as $j' \rightarrow \infty$, because both $(u, u_{j'}] \downarrow \emptyset$ and $(u, u_{j'}) \downarrow \emptyset$. This contradiction proves (2.4) in the case $u_{j'} \downarrow u$. The proof in the case $u_{j'} \uparrow u$ is symmetric.

For (c) \Rightarrow (b), let intervals $A_n \downarrow \emptyset$. Then for some $u \in \bar{J}$, for all sufficiently large n , either A_n is left-open at u , or A_n is right-open at u . Using additivity, in each of the two cases $\mu(A_n) \rightarrow 0$ follows by (c).

For (b) \Rightarrow (e), let intervals $A_n \rightarrow A \neq \emptyset$. If $A = (u, v)$ then $\{u\} \prec A_n \prec \{v\}$ for n large enough. For such n , there are intervals C_n and D_n with $\{u\} \prec C_n \prec A_n \prec D_n \prec \{v\}$ and $C_n \cup A_n \cup D_n = A$. Also for such n , we have either $C_n = \emptyset$ or $C_n = (u, \cdot]$, and either $D_n = \emptyset$ or $D_n = [\cdot, v)$. Clearly $C_n \rightarrow \emptyset$. If $N := \{n : C_n \neq \emptyset\}$ is infinite, there is a function $j \mapsto n(j)$ onto N such that $C_{n(j)} \downarrow \emptyset$. Thus $\mu(C_n) \rightarrow 0$. Similarly $\mu(D_n) \rightarrow 0$. Therefore $\mu(A_n) = \mu(A) - \mu(C_n) - \mu(D_n) \rightarrow \mu(A)$. Similar arguments apply to other cases $A = (u, v]$, $[u, v)$, or $[u, v]$.

Clearly, (e) implies (d). For (d) \Rightarrow (b), let intervals $A_n \downarrow \emptyset$. Then $A_1 = B_n \cup A_n \cup C_n$ for some intervals B_n, C_n with $B_n \prec A_n \prec C_n$, and $B_n \uparrow B$, $C_n \uparrow C$ for some intervals B, C with $A_1 = B \cup C$. Since μ is additive, $\mu(A_n) \rightarrow 0$.

(b) \Leftrightarrow (f) \Leftrightarrow (g). The implications (b) \Rightarrow (f) and (b) \Rightarrow (g) are clear. We prove (f) \Rightarrow (b) only, because the proof of (g) \Rightarrow (b) is similar. Let intervals $A_n \downarrow \emptyset$. Then for some u and all sufficiently large n , either A_n is left-open at $u \in \llbracket a, b)$, or A_n is right-open at $u \in (a, b]$. By assumption, in the first case we have $\lim_n \mu(A_n) = R_{\mu,a}(u+) - R_{\mu,a}(u) = 0$ and in the second case we have $\lim_n \mu(A_n) = R_{\mu,a}(u-) - R_{\mu,a}(u-) = 0$. The proof of Proposition 2.6 is complete. \square

The class of all additive and upper continuous functions in $\mathcal{I}(J; X)$ will be denoted by $\mathcal{AI}(J; X)$. For the next two theorems, recall the definition (2.2) of the interval function μ_h corresponding to a regulated function h .

Theorem 2.7. *For any regulated function h on a nonempty interval $\llbracket a, b]$, the interval function μ_h is in $\mathcal{AI}(\llbracket a, b]; X)$, and the map $h \mapsto \mu_h$ is linear.*

Proof. We can assume that $a < b$. Additivity is immediate from the definition of μ_h , as is linearity of $h \mapsto \mu_h$. For upper continuity it is enough to prove statement (c) of Proposition 2.6. Let open intervals $A_n \downarrow \emptyset$. Thus there exist

$\{u, v_n: n \geq 1\} \subset \llbracket a, b \rrbracket$ such that for all large enough n , either $A_n = (v_n, u)$ with $v_n \uparrow u$ or $A_n = (u, v_n)$ with $v_n \downarrow u$. Then $\mu_h(A_n) = h(u-) - h(v_n+) \rightarrow 0$ as $n \rightarrow \infty$ in the first case and $\mu_h(A_n) = h(v_n-) - h(u+) \rightarrow 0$ as $n \rightarrow \infty$ in the second case. Since $\mu(\{t\}) = \Delta_{\llbracket a, b \rrbracket}^\pm h(t)$ for $t \in \llbracket a, b \rrbracket$, $\text{card}\{t \in \llbracket a, b \rrbracket: \|\mu(\{t\})\| > \epsilon\} < \infty$ for all $\epsilon > 0$ by Corollary 2.2. Thus μ is upper continuous by Proposition 2.6, and Theorem 2.7 is proved. \square

Theorem 2.8. *For an additive $\mu \in \mathcal{I}(\llbracket a, b \rrbracket; X)$ with $a < b$, the following statements are equivalent:*

- (a) μ is upper continuous;
- (b) $\mu = \mu_h$ on $\mathfrak{I}[\llbracket a, b \rrbracket]$ for $h = R_{\mu, a}$;
- (b') $\mu = \mu_h$ on $\mathfrak{I}[\llbracket a, b \rrbracket]$ for $h = L_{\mu, a}$;
- (c) $\mu = \mu_h$ on $\mathfrak{I}[\llbracket a, b \rrbracket]$ for some $h \in \mathcal{R}(\llbracket a, b \rrbracket; X)$ with $h(a) = 0$ if $a \in \llbracket a, b \rrbracket$ and $h(a+) = 0$ otherwise.

Proof. (a) \Rightarrow (b): suppose that an additive interval function μ on $\llbracket a, b \rrbracket$ is upper continuous. Let $h := R_{\mu, a}$. Then by (a) \Leftrightarrow (f) of Proposition 2.6, and by the definition of $R_{\mu, a}$, h is regulated and right-continuous on (a, b) , and $\mu_h = \mu$ on all intervals $\llbracket a, t \rrbracket$ and $\llbracket a, t \rrbracket$, $t \in \llbracket a, b \rrbracket$. Since μ and μ_h are additive, $\mu = \mu_h$ on $\mathfrak{I}[\llbracket a, b \rrbracket]$. A proof that (a) \Rightarrow (b') is similar.

The implications (b) \Rightarrow (c) and (b') \Rightarrow (c) are immediate. The implication (c) \Rightarrow (a) follows from Theorem 2.7, proving the theorem. \square

By the preceding theorem, for each $\mu \in \mathcal{AI}(\llbracket a, b \rrbracket; X)$ with $a < b$ there is a regulated function h on $\llbracket a, b \rrbracket$ such that $\mu = \mu_h$. The following will be used to clarify differences between all such h .

Definition 2.9. For any intervals $A \subset J$ and Banach space $(X, \|\cdot\|)$, $c_0(A) := c_0(A, J) := c_0(A, J; X)$ is the set of functions $f: J \rightarrow X$ such that for some sequence $\{t_n\}_{n=1}^\infty \subset A$, $f(t_n) \rightarrow 0$ as $n \rightarrow \infty$ and $f(t) = 0$ if $t \neq t_n$ for all n .

Notice that if $a < b$ and $f \in c_0((a, b), [a, b]; X)$ then f is regulated on $[a, b]$ with $f_-^{(a)} \equiv f_+^{(b)} \equiv 0$ on $[a, b]$.

Proposition 2.10. *Let $g, h \in \mathcal{R}([a, b]; X)$. Then $\mu_g = \mu_h$ on $\mathfrak{I}[a, b]$ if and only if for some constant c and $\psi \in c_0((a, b)) = c_0((a, b), [a, b]; X)$, $g - h = c + \psi$ on $[a, b]$. Moreover, if $\mu_g = \mu_h$ on $\mathfrak{I}[a, b]$ and $g(a) = h(a)$ then $c = 0$.*

Proof. We can assume that $a < b$. Since $\psi(s+) = \psi(t-) = 0$ for $a \leq s < t \leq b$, the “if” part holds. For the converse implication suppose that $\mu_g = \mu_h$. Let D_g be the set of all points $t \in (a, b)$ such that either $\Delta^-g(t) \neq 0$ or $\Delta^+g(t) \neq 0$. Define D_h similarly. Let $D := D_g \cup D_h$ and let $c := g(a) - h(a)$. Define ψ on $[a, b]$ by $\psi(t) := g(t) - h(t) - c$ for $t \in D$ and 0 elsewhere. Since $\mu_g = \mu_h$ on $\mathfrak{I}[a, b]$, $g(u-) = c + h(u-)$ and $g(v+) = c + h(v+)$ for $a \leq v < u \leq b$. Thus

$g - h = c$ on $[a, b] \setminus D$ and $g - h = c + \psi$ on D . By Corollary 2.2, $\psi \in c_0((a, b))$, completing the proof of the converse implication. \square

The regulated function h in Theorem 2.8(c) can be chosen uniquely if it satisfies additional properties. For $a < b$, let $\mathcal{D}([a, b]; X)$ be the set of all $h \in \mathcal{R}([a, b]; X)$ such that h is right-continuous on (a, b) and either $h(a) = 0$ if $a \in [a, b]$ or $h(a+) = 0$ if $a \notin [a, b]$. Thus $h \in \mathcal{D}([a, b]; X)$ need not be right-continuous at a , just as $R_{\mu,a}$ is not if $\{a\}$ is an atom for $\mu \in \mathcal{AI}([a, b]; X)$.

Corollary 2.11. *Let $a < b$ and let X be a Banach space. The mappings*

$$\mathcal{D}([a, b]; X) \ni h \mapsto \mu := \mu_h \in \mathcal{AI}([a, b]; X) \quad (2.5)$$

and

$$\mathcal{AI}([a, b]; X) \ni \mu \mapsto h := R_{\mu,a} \in \mathcal{D}([a, b]; X) \quad (2.6)$$

are one-to-one linear operators between the vector spaces $\mathcal{AI}([a, b]; X)$ and $\mathcal{D}([a, b]; X)$. Moreover, the two mappings are inverses of each other.

Proof. If $h \in \mathcal{D}([a, b]; X)$ then $\mu_h \in \mathcal{AI}([a, b]; X)$ by Theorem 2.8(c) \Rightarrow (a). If $\mu_g = \mu_h$ on $\mathcal{I}[a, b]$ for some $g, h \in \mathcal{D}([a, b]; X)$ then $h(x) - g(x) = \mu_h([a, x]) - \mu_g([a, x])$ for each $x \in (a, b]$, and so $g \equiv h$ on $[a, b]$. Thus the mapping (2.5) is one-to-one. It clearly is linear.

If $\mu \in \mathcal{AI}([a, b]; X)$ then $R_{\mu,a} \in \mathcal{D}([a, b]; X)$ by Proposition 2.6(f). By the same proposition, if $R_{\mu,a} = R_{\nu,a}$ on $[a, b]$ for some $\mu, \nu \in \mathcal{AI}([a, b]; X)$ then $\mu = \nu$ on $\mathcal{I}[a, b]$. Thus the mapping (2.6) also is one-to-one and linear, proving the first part of the conclusion.

To prove the second part of the conclusion, first let $h \in \mathcal{D}([a, b]; X)$. By (2.3), (2.2), and Proposition 2.6(f), if $t \in (a, b]$ then we have $R_{\mu_h,a}(t) = h(t)$. Also, if $a \in [a, b]$ then $R_{\mu_h,a}(a) = 0 = h(a)$. Thus the composition of (2.5) with (2.6) maps $h \in \mathcal{D}([a, b]; X)$ into itself. Since both maps are one-to-one and (2.6) is onto, the proof of the corollary is complete. \square

Next is a property of an upper continuous additive interval function similar to the property in Theorem 2.1(b) for regulated functions. For an interval function μ on $[a, b]$ and an interval $J \subset [a, b]$, let

$$\text{Osc}(\mu; J) := \sup \{ \|\mu(A)\| : A \in \mathcal{I}[a, b], A \subset J \}. \quad (2.7)$$

In the case $J = [a, b]$, we also write $\|\mu\|_{\text{sup}} := \text{Osc}(\mu; [a, b])$.

Corollary 2.12. *Let X be a Banach space, $\mu \in \mathcal{AI}([a, b]; X)$ with $a < b$ and $\epsilon > 0$. There exists a Young interval partition $\{(t_{i-1}, t_i)\}_{i=1}^n$ of $[a, b]$ such that $\text{Osc}(\mu; (t_{i-1}, t_i)) \leq \epsilon$ for each $i = 1, \dots, n$. In particular, μ is bounded.*

Proof. Let $h := R_{\mu,a}$ be the function from $\llbracket a, b \rrbracket$ to X defined by (2.3). By implication (a) \Rightarrow (f) of Proposition 2.6, h is a regulated function on $\llbracket a, b \rrbracket$, and so by implication (a) \Rightarrow (b) of Theorem 2.1, there exists a Young interval partition $\{(t_{i-1}, t_i)\}_{i=1}^n$ of $\llbracket a, b \rrbracket$ such that $\text{Osc}(h; (t_{i-1}, t_i)) < \epsilon$ for each $i = 1, \dots, n$. By implication (a) \Rightarrow (b) of Theorem 2.8, for an interval $A \subset (t_{i-1}, t_i)$, $\mu(A) = \mu_h(A)$, and so $\|\mu(A)\| \leq \epsilon$, proving the first part of the conclusion. The second part follows similarly because h is bounded by Corollary 2.2, completing the proof of the corollary.

2.2 Riemann–Stieltjes Integrals

Suppose that the basic assumption (1.14) holds. Let f, h be functions defined on an interval $[a, b]$ with $-\infty < a \leq b < +\infty$ and having values in X, Y , respectively. If $a < b$, given a tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, b]$, define the *Riemann–Stieltjes sum* $S_{RS}(\tau) = S_{RS}(f, dh; \tau)$ based on τ by

$$S_{RS}(f, dh; \tau) := \sum_{i=1}^n f(s_i) \cdot [h(t_i) - h(t_{i-1})].$$

The *Riemann–Stieltjes* or *RS* integral $(RS) \int_a^b f \cdot dh$ is defined to be 0 if $a = b$ and otherwise is defined if the limit exists in $(Z, \|\cdot\|)$ as

$$(RS) \int_a^b f \cdot dh := \lim_{|\tau| \downarrow 0} S_{RS}(f, dh; \tau). \quad (2.8)$$

The *refinement Riemann–Stieltjes* or *RRS* integral $(RRS) \int_a^b f \cdot dh$ is defined to be 0 if $a = b$ and otherwise is defined as

$$(RRS) \int_a^b f \cdot dh := \lim_{\tau} S_{RS}(f, dh; \tau) \quad (2.9)$$

provided the limit exists in the refinement sense, that is, $(RRS) \int_a^b f \cdot dh := A$ if for every $\epsilon > 0$ there is a point partition λ of $[a, b]$ such that for every tagged partition $\tau = (\kappa, \xi)$ such that κ is a refinement of λ , $\|S_{RS}(f, dh; \tau) - A\| < \epsilon$.

Integrals $\int_a^b dh \cdot f$ for both the *RS* and *RRS* integrals are defined symmetrically via (1.15). If $a < b$, the Riemann–Stieltjes sum based on a tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, b]$, with the integrand and integrator interchanged, will be denoted by

$$S_{RS}(dh, f; \tau) := \sum_{i=1}^n [h(t_i) - h(t_{i-1})] \cdot f(s_i).$$

Then the integral $\int_a^b dh \cdot f$ is defined in the sense of *RS* or *RRS* if and only if the limit (2.8) or (2.9), respectively, exists with $S_{RS}(f, dh; \tau)$ replaced by $S_{RS}(dh, f; \tau)$.

Proposition 2.13. *The refinement Riemann–Stieltjes integral extends the Riemann–Stieltjes integral.*

Proof. For $a = b$, both integrals are 0. For $a < b$, and a partition λ , a refinement κ of λ has mesh $|\kappa| \leq |\lambda|$, so the conclusion holds. \square

Example 2.14. Let f be a real-valued function on $[a, b]$ and let $\ell_c := 1_{[c, b]}$ be the indicator function of $[c, b]$ for some $a < c < b$. For any tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, b]$, we have $S_{RS}(f, d\ell_c; \tau) = f(s_i)$ for $t_{i-1} < c \leq t_i$ and $s_i \in [t_{i-1}, t_i]$. The integral $(RS) \int_a^b f d\ell_c$ exists and equals $f(c)$ if and only if f is continuous at c . Taking c to be a partition point, it follows that the integral $(RRS) \int_a^b f d\ell_c$ exists and equals $f(c)$ if and only if f is left-continuous at c . Therefore $(RRS) \int_a^b 1_{[a, c]} d\ell_c$ exists and equals 1, while the same integral in the Riemann–Stieltjes sense does not exist. On the other hand, the integral $(RRS) \int_a^b \ell_c d\ell_c$ also does not exist.

Two functions f, h on an interval $[a, b]$ will be said not to have *common discontinuities* if for each $t \in [a, b]$, at least one of f and h is continuous at t . Also, f and h will be said to have no *common one-sided discontinuities* on $[a, b]$ if at least one of f and h is left-continuous at each $t \in (a, b]$ and at least one is right-continuous at each $t \in [a, b)$.

Proposition 2.15. *Assuming (1.14), for any functions f and h from $[a, b]$ into X and Y , respectively, we have:*

- (a) *If $(RS) \int_a^b f \cdot dh$ exists then f and h have no common discontinuities;*
- (b) *If $(RRS) \int_a^b f \cdot dh$ exists then f and h have no common one-sided discontinuities.*

Proof. For (a) suppose that the integral $(RS) \int_a^b f \cdot dh$ exists. For each $t \in [a, b]$ and a partition $\kappa = \{x_i\}_{i=1}^n$ of $[a, b]$, there is an index i such that $t \in [x_{i-1}, x_i]$, where we can take i to be unique for partitions with arbitrarily small mesh. Subtracting two Riemann–Stieltjes sums based on the partition κ with small enough mesh $|\kappa|$ and with all terms equal except for the i th term, one can conclude that $\|[f(y'_i) - f(y''_i)] \cdot [h(x_i) - h(x_{i-1})]\|$ is arbitrarily small for any $y'_i, y''_i \in [x_{i-1}, x_i]$. Thus either f or h must be continuous at t .

The proof of (b) is similar except that one needs to restrict to partitions containing a point $t \in [a, b]$. \square

When f or h is continuous, the RS and RRS integrals coincide if either exists, as we will show in Theorem 2.42. Some classical sufficient conditions for existence of the two integrals are expressed using the total variation or the semivariation of the integrator. Let X, Y , and Z be Banach spaces related by means of a bounded bilinear operator B from $X \times Y$ into Z as in assumption

(1.14) and let $h: [a, b] \rightarrow Y$ with $-\infty < a < b < +\infty$. The *semivariation* of h on $[a, b]$ is defined by

$$w(h; [a, b]) := w_B(h; [a, b]) := \sup \left\{ \left\| \sum_{i=1}^n x_i \cdot [h(t_i) - h(t_{i-1})] \right\| : \{t_i\}_{i=0}^n \in \text{PP}[a, b], \{x_i\}_{i=1}^n \subset X, \max_i \|x_i\| \leq 1 \right\}, \quad (2.10)$$

where $\text{PP}[a, b]$ is the set of all point partitions of $[a, b]$. Then h is said to be of *bounded semivariation* on $[a, b]$ if $w_B(h; [a, b]) < +\infty$. As the notation $w_B(h; [a, b])$ indicates, the semivariation depends on the Banach spaces X , Y , and Z and the bilinear operator $B: (x, y) \mapsto x \cdot y$. Recall that the total variation of h on $[a, b]$ is the 1-variation of h on $[a, b]$, that is,

$$v(h; [a, b]) := \sup \{s_1(h; \kappa) : \kappa \in \text{PP}[a, b]\} \\ = \sup \left\{ \sum_{i=1}^n \|h(t_i) - h(t_{i-1})\| : \{t_i\}_{i=0}^n \in \text{PP}[a, b] \right\}.$$

The total variation $v(h; [a, b])$ does not depend on X , Z or B . It is clear that $w_B(h; [a, b]) \leq v(h; [a, b]) \leq +\infty$. Also, if $X = Y'$, $Z = \mathbb{R}$ and $B(x, y) = x(y)$, then $w_B(h; [a, b]) = v(h; [a, b])$. However in general, it is possible that $w_B(h; [a, b]) < \infty$ while $v(h; [a, b]) = +\infty$, as the following example shows:

Example 2.16. For each $t \in [0, 1]$, let $h(t) := 1_{[0, t]}$. Then h is a function from $[0, 1]$ into $Y = Z = \ell^\infty[0, 1]$, the Banach space of real-valued and bounded functions on $[0, 1]$ with the supremum norm. For a point partition $\{t_i\}_{i=0}^n$ of $[0, 1]$ and for any finite sequence $\{x_i\}_{i=1}^n \subset X = \mathbb{R}$ of real numbers in $[-1, 1]$, we have

$$\left\| \sum_{i=1}^n x_i [h(t_i) - h(t_{i-1})] \right\|_{\text{sup}} = \left\| \sum_{i=1}^n x_i 1_{(t_{i-1}, t_i]} \right\|_{\text{sup}} = \max_i |x_i| \leq 1.$$

Thus h has bounded semivariation on $[0, 1]$, with $w(h; [0, 1]) = 1$. Also, h has unbounded p -variation for any $p < \infty$. Indeed, if $0 < p < \infty$ and $\kappa = \{t_i\}_{i=0}^n$ is a point partition of $[0, 1]$, then

$$s_p(h; \kappa) = \sum_{i=1}^n \|1_{(t_{i-1}, t_i]}\|_{\text{sup}}^p = n.$$

Since n is arbitrary, $v_p(h; [0, 1]) = +\infty$. The function h is not regulated on $[0, 1]$ since it clearly does not satisfy condition (b) of Theorem 2.1. Moreover, for a real-valued function f on $[0, 1]$, the integral $(RS) \int_0^1 f dh$ exists if and only if f is continuous. Indeed, if f is continuous, then given $\epsilon > 0$, there is a $\delta > 0$ such that $|f(t) - f(s)| < \epsilon$ whenever $s, t \in [0, 1]$ and $|s - t| < \delta$. Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a tagged partition of $[0, 1]$ with mesh $|\tau| < \delta$.

For each $t \in [0, 1]$ there is an $i \in \{1, \dots, n\}$ such that either $t \in (t_{i-1}, t_i]$, or $t \in [0, t_1]$. In either case $|S_{RS}(f, dh; \tau)(t) - f(t)| = |f(s_i) - f(t)| < \epsilon$, and so $(RS) \int_0^1 f \, dh$ exists and equals f . Now, if f is not continuous then for some $t \in [0, 1]$, the difference $|S_{RS}(f, dh; \tau)(t) - f(t)|$ cannot be arbitrarily small for all tagged partitions with small enough mesh, proving the claim.

Now we are prepared to prove the following:

Theorem 2.17. *Assuming (1.14), if f or h is of bounded semivariation and the other is regulated, and they have no common one-sided discontinuities, then $(RRS) \int_a^b f \cdot dh$ exists.*

Proof. First suppose that h has bounded semivariation on $[a, b]$ and f is regulated. Given $\epsilon > 0$, by statement (b) of Theorem 2.1, there exists a partition $\lambda = \{z_l\}_{l=0}^k$ of $[a, b]$ such that

$$\text{Osc}(f; [z_{l-1}, z_l]) < \epsilon \quad \text{for } l = 1, \dots, k. \quad (2.11)$$

Since f and h have no common one-sided discontinuities, there exists a sequence $\mu = \{u_{l-1}, v_l : l = 1, \dots, k\} \subset (a, b)$ such that for each $l = 1, \dots, k$, $z_{l-1} < u_{l-1} < v_l < z_l$,

$$\min \{ \text{Osc}(f; [z_{l-1}, u_{l-1}]), \text{Osc}(h; [z_{l-1}, u_{l-1}]) \} < \epsilon/k \quad (2.12)$$

and

$$\min \{ \text{Osc}(f; [v_l, z_l]), \text{Osc}(h; [v_l, z_l]) \} < \epsilon/k. \quad (2.13)$$

Let $\tau = (\{t_j\}_{j=0}^m, \{s_j\}_{j=1}^m)$ be a tagged refinement of $\lambda \cup \mu$. For each $j = 1, \dots, m$, let $\tau_j = (\{t_{ji}\}_{i=0}^{n_j}, \{s_{ji}\}_{i=1}^{n_j})$ be any tagged partition of $[t_{j-1}, t_j]$. Then $\cup_j \tau_j$ is an arbitrary tagged refinement of τ . If $t_{j-1} \in \{z_0, \dots, z_{k-1}\}$ then

$$\begin{aligned} f(s_j) \cdot [h(t_j) - h(t_{j-1})] - S_{RS}(\tau_j) &= \\ [f(s_j) - f(t_j)] \cdot [h(t_j) - h(t_{j-1})] &+ \\ + \sum_{i=1}^{n_j} [f(t_j) - f(s_{ji})] \cdot [h(t_{ji}) - h(t_{j,i-1})]. \end{aligned}$$

The first product on the right side, say U_j , and the $i = 1$ term, say V_j , by (2.12), have the bound

$$\|U_j\| + \|V_j\| \leq 4(\epsilon/k) \max\{\|f\|_{\sup}, \|h\|_{\sup}\}. \quad (2.14)$$

At most one of t_{j-1} and t_j is a z_i . If $t_j \in \{z_1, \dots, z_k\}$ then

$$\begin{aligned} f(s_j) \cdot [h(t_j) - h(t_{j-1})] - S_{RS}(\tau_j) &= \\ [f(s_j) - f(t_{j-1})] \cdot [h(t_j) - h(t_{j-1})] &+ \\ + \sum_{i=1}^{n_j} [f(t_{j-1}) - f(s_{ji})] \cdot [h(t_{ji}) - h(t_{j,i-1})]. \end{aligned}$$

The first product on the right side and the $i = n_j$ term have the same bound as in (2.14) by (2.13). We apply the preceding $2k$ representations with the bounds (2.14). We bound the sum of the remaining terms for all j by $\max_{l=1,\dots,k} \text{Osc}(f; (z_{l-1}, z_l)) w_B(h; [a, b])$, and using (2.11) we get the bound

$$\|S_{RS}(\tau) - S_{RS}(\cup_j \tau_j)\| < \epsilon w_B(h; [a, b]) + 8\epsilon \max\{\|f\|_{\sup}, \|h\|_{\sup}\}.$$

Since $\epsilon > 0$ is arbitrary, the integral $(RRS) \int_a^b f \cdot dh$ exists.

Now suppose that f has bounded semivariation and h is regulated. Given $\epsilon > 0$, choose a partition $\lambda = \{z_l\}_{l=0}^k$ of $[a, b]$ such that (2.11) with h instead of f holds. Then choose a sequence $\mu = \{u_{l-1}, v_l : l = 1, \dots, k\}$ such that (2.12) and (2.13) hold. Again let $\tau = (\{t_j\}_{j=0}^m, \{s_j\}_{j=1}^m)$ be a tagged refinement of $\lambda \cup \mu$, and for each $j = 1, \dots, m$, let $\tau_j = (\{t_{ji}\}_{i=0}^{n_j}, \{s_{ji}\}_{i=1}^{n_j})$ be a tagged partition of $[t_{j-1}, t_j]$. If $t_j \notin \{z_1, \dots, z_k\}$, then summing by parts we have

$$\begin{aligned} f(s_j) \cdot [h(t_j) - h(t_{j-1})] - S_{RS}(\tau_j) &= \\ [f(s_j) - f(s_{j1})] \cdot [h(t_j) - h(t_{j-1})] &+ \\ + \sum_{i=1}^{n_j-1} [f(s_{j,i+1}) - f(s_{ji})] \cdot [h(t_{ji}) - h(t_j)]. \end{aligned}$$

If also $t_{j-1} \in \{z_0, \dots, z_{k-1}\}$, then for the first product on the right side, say T_j , by (2.12), we have the bound

$$\|T_j\| \leq 2(\epsilon/k) \max\{\|f\|_{\sup}, \|h\|_{\sup}\}. \quad (2.15)$$

The norm of the sum of the terms T_j for which neither t_{j-1} nor t_j is a z_i has the bound $\epsilon w_B(f; [a, b])$ by (2.11) with h instead of f . If $t_j \in \{z_1, \dots, z_k\}$, then summing by parts we have

$$\begin{aligned} f(s_j) \cdot [h(t_j) - h(t_{j-1})] - S_{RS}(\tau_j) &= \\ [f(s_j) - f(s_{j,n_j})] \cdot [h(t_j) - Ah(t_{j-1})] &+ \\ + \sum_{i=1}^{n_j-1} [f(s_{j,i+1}) - f(s_{ji})] \cdot [h(t_{ji}) - h(t_{j-1})]. \end{aligned}$$

The first product on the right side has the same bound as in (2.15) by (2.13). Applying the preceding representations for each $j = 1, \dots, m$, using the bounds (2.15) and (2.11) with h instead of f , we get the bound

$$\|S_{RS}(\tau) - S_{RS}(\cup_j \tau_j)\| \leq 4\epsilon \max\{\|f\|_{\sup}, \|h\|_{\sup}\} + 2\epsilon w_B(f; [a, b]).$$

As in the first part of the proof the integral $(RRS) \int_a^b f \cdot dh$ exists, proving the theorem. \square

2.3 The Refinement Young–Stieltjes and Kolmogorov Integrals

Let $f: [a, b] \rightarrow X$ and $h \in \mathcal{R}([a, b]; Y)$. Recall from Section 1.4 that for a point partition $\{t_i\}_{i=0}^n$ of $[a, b]$ with $a < b$, a tagged partition $(\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ is called a *Young tagged point partition* if $t_{i-1} < s_i < t_i$ for each $i = 1, \dots, n$. Given a Young tagged point partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$, define the *Young–Stieltjes sum* $S_{YS}(f, dh; \tau)$ based on τ by, recalling the definition (2.1),

$$\begin{aligned} S_{YS}(f, dh; \tau) &:= \sum_{i=0}^n (f \cdot \Delta_{[a,b]}^{\pm} h)(t_i) + \sum_{i=1}^n f(s_i) \cdot [h(t_i-) - h(t_{i-1}+)] \\ &= \sum_{i=1}^n \left\{ [f \cdot \Delta^+ h](t_{i-1}) + f(s_i) \cdot [h(t_i-) - h(t_{i-1}+)] + [f \cdot \Delta^- h](t_i) \right\}. \end{aligned} \quad (2.16)$$

The *refinement Young–Stieltjes* or *RYS* integral $(RYS) \int_a^b f \cdot dh$ is defined as 0 if $a = b$ or as

$$(RYS) \int_a^b f \cdot dh := \lim_{\tau} S_{YS}(f, dh; \tau)$$

if $a < b$, provided the limit exists in the refinement sense. The integral $(RYS) \int_a^b df \cdot h$ is defined, if it exists, via (1.15).

Proposition 2.18. *Let $f: [a, b] \rightarrow X$ and $h \in \mathcal{R}([a, b]; Y)$. If $a < b$, given a Young tagged point partition $\tau = (\kappa, \xi)$ of $[a, b]$, the Young–Stieltjes sum $S_{YS}(f, dh; \tau)$ can be approximated arbitrarily closely by Riemann–Stieltjes sums $S_{RS}(f, dh; \tilde{\tau})$ based on tagged refinements $\tilde{\tau}$ of κ such that all tags ξ of τ are tags of $\tilde{\tau}$. Thus, if $(RRS) \int_a^b f \cdot dh$ exists then so does $(RYS) \int_a^b f \cdot dh$, and the two are equal.*

Proof. The last conclusion holds if $a = b$ with both integrals defined as 0, so let $a < b$. For a Young tagged point partition $\tau = (\kappa, \xi) = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ of $[a, b]$, take a set $\mu = \{u_{i-1}, v_i\}_{i=1}^n \subset (a, b)$ such that $x_0 < u_0 < y_1 < v_1 < x_1 < \dots < x_{n-1} < u_{n-1} < y_n < v_n < x_n$. Let $\tilde{\tau} := (\kappa \cup \mu, \kappa \cup \xi)$, where each x_i , $i = 1, \dots, n$, is the tag for both $[v_i, x_i]$ and $[x_i, u_i]$. For $i = 1, \dots, n$, letting $u_{i-1} \downarrow x_{i-1}$ and $v_i \uparrow x_i$, it follows that the Riemann–Stieltjes sums $S_{RS}(f, dh; \tilde{\tau})$ converge to the Young–Stieltjes sum $S_{YS}(f, dh; \tau)$ and the conclusions follow. \square

Example 2.19. As in Example 2.14, for $a < c < b$, let ℓ_c be the indicator function of $[c, b]$ and let f be a real-valued function, both on $[a, b]$. For each Young tagged point partition $\tau = (\kappa, \xi)$ such that $c \in \kappa$, we have $S_{YS}(f, d\ell_c; \tau) = f(c)$. Therefore the integral $(RYS) \int_a^b f d\ell_c$ exists and has

value $f(c)$. In particular, $(RYS) \int_a^b \ell_c d\ell_c$ exists and has value 1, while the RRS integral does not exist, as shown in Example 2.14. More generally, for any real number r , let $\ell_c^r(x) := \ell_c(x)$ for $x \neq c$ and $\ell_c^r(c) := r$, so that $\ell_c^1 \equiv \ell_c$. Then $(RYS) \int_a^b f d\ell_c^r$ exists and has value $f(c)$ for any r .

In fact, the RYS integral $\int_a^b f dh$ does not depend on the values of the integrator h at its jump points in (a, b) . This is because the value of the Young–Stieltjes sum (2.16) does not depend on the values of h at jump points in the open interval (a, b) . The same is not true for the RRS integral. Indeed, by changing the value of the integrator h at a jump point t of both f and h one can make h discontinuous from the left or right at t and so destroy the necessary condition of Proposition 2.15. The RS integral, like the RYS integral, does not depend on values of the integrator at its discontinuity points. This holds because the RS integral is defined only under restrictive conditions. Specifically, by Proposition 2.15, if $(RS) \int_a^b f dh$ exists and h has a jump at $t \in (a, b)$ then f must be continuous at t . For $u \in \mathbb{R}$, let $h_u := h$ on $[a, t) \cup (t, b]$ and $h_u(t) := u$. Then for two Riemann–Stieltjes sums we have

$$S_{RS}(f, dh; (\kappa, \xi)) - S_{RS}(f, dh_u; (\kappa, \xi)) = \begin{cases} [f(s') - f(s'')] \cdot [h(t) - u] & \text{if } t \in \kappa, \\ 0 & \text{if } t \notin \kappa, \end{cases}$$

where $s' \in [t - \delta, t]$ and $s'' \in [t, t + \delta]$ for $\delta := |\kappa|$. Due to continuity of f at t the preceding difference can be made arbitrarily small if the mesh $|\kappa|$ is small enough.

Theorem 2.20. *Assuming (1.14), if f or h is of bounded semivariation and the other is regulated, then $(RYS) \int_a^b f \cdot dh$ exists.*

Proof. We can assume $a < b$. First suppose that h has bounded semivariation on $[a, b]$. Given $\epsilon > 0$, by statement (b) of Theorem 2.1, there exists a partition $\lambda = \{z_l\}_{l=0}^k$ of $[a, b]$ such that

$$\text{Osc}(f; (z_{l-1}, z_l)) < \epsilon \quad \text{for } l = 1, \dots, k. \quad (2.17)$$

Let $\tau = (\{t_j\}_{j=0}^m, \{s_j\}_{j=1}^m)$ be a Young tagged point partition which is a refinement of λ . For each $j = 1, \dots, m$, let $\tau_j = (\{t_{ji}\}_{i=0}^{n_j}, \{s_{ji}\}_{i=1}^{n_j})$ be a Young tagged point partition of $[t_{j-1}, t_j]$. Then $\cup_j \tau_j$ is a Young tagged refinement of τ and

$$\begin{aligned} S_{YS}(f, dh; \tau) - S_{YS}(f, dh; \cup_j \tau_j) \\ = \sum_{j=1}^m \left\{ \sum_{i=1}^{n_j} [f(s_j) - f(s_{ji})] \cdot [h(t_{ji}-) - h(t_{j,i-1}+)] \right. \\ \left. + \sum_{i=1}^{n_j-1} [f(s_j) - f(t_{ji})] \cdot [h(t_{ji}+) - h(t_{ji}-)] \right\}. \end{aligned}$$

Let $t_{j,i-1} < v_{j,2i-1} < v_{j,2i} < t_{ji}$ for $i = 1, \dots, n_j$ and $j = 1, \dots, m$. Consider sums

$$\sum_{r=2}^{2n_j} x_{j,r} \cdot [h(v_r) - h(v_{r-1})],$$

where $x_{j,2i} = f(s_j) - f(s_{ji})$ and $x_{j,2i+1} = f(s_j) - f(t_{ji})$ for $i = 1, \dots, n_j$ except that for $i = n_j$, $x_{j,2i+1} = 0$. Letting $v_{j,2i-1} \downarrow t_{j,i-1}$ and $v_{j,2i} \uparrow t_{ji}$, by (2.17), we get the bound

$$\|S_{YS}(f, dh; \tau) - S_{YS}(f, dh; \cup_j \tau_j)\| < 2\epsilon w_B(h; [a, b]).$$

Given any two Young tagged point partitions τ_1 and τ_2 of $[a, b]$, there exists a Young tagged refinement τ_3 of both and

$$\|S_{YS}(\tau_1) - S_{YS}(\tau_2)\| \leq \|S_{YS}(\tau_1) - S_{YS}(\tau_3)\| + \|S_{YS}(\tau_3) - S_{YS}(\tau_2)\|.$$

Thus by the Cauchy test under refinement, $(RYS) \int_a^b f \cdot dh$ exists.

Now suppose that f has bounded semivariation and h is regulated. With the same notation for Young tagged point partitions τ and τ_j , $j = 1, \dots, m$, we have

$$\begin{aligned} & S_{YS}(f, dh; \tau) - S_{YS}(f, dh; \cup_j \tau_j) \\ &= \sum_{j=1}^m [f(s_j) - f(s_{j1})] \cdot [h(t_j-) - h(t_{j-1}+)] + \sum_{j=1}^m d_j, \end{aligned}$$

where for each $j = 1, \dots, m$,

$$\begin{aligned} d_j &:= (f \cdot \Delta^+ h)(t_{j-1}) + f(s_{j1}) \cdot [h(t_j-) - h(t_{j-1}+)] + (f \cdot \Delta^- h)(t_j) \\ &\quad - S_{YS}(f, dh; \tau_j) \\ &= f(s_{j1}) \cdot [h(t_j-) - h(t_{j-1}+)] - f(s_{j,n_j}) \cdot h(t_j-) + f(s_{j1}) \cdot h(t_{j-1}+) \\ &\quad + \sum_{i=1}^{n_j-1} \left\{ [f(t_{ji}) - f(s_{ji})] \cdot h(t_{ji}-) + [f(s_{j,i+1}) - f(t_{ji})] \cdot h(t_{ji}+) \right\} \\ &= \sum_{i=1}^{n_j-1} \left\{ [f(t_{ji}) - f(s_{ji})] \cdot [h(t_{ji}-) - h(t_j-)] \right. \\ &\quad \left. + [f(s_{j,i+1}) - f(t_{ji})] \cdot [h(t_{ji}+) - h(t_j-)] \right\}. \end{aligned}$$

By Theorem 2.1(b), given $\epsilon > 0$ one can choose a partition $\lambda = \{z_l\}_{l=1}^k$ of $[a, b]$ such that (2.17) with h instead of f holds. If the Young tagged point partition τ is a refinement of λ , then by approximations as in the first half of the proof, we get the bound

$$\|S_{YS}(f, dh; \tau) - S_{YS}(f, dh; \cup_j \tau_j)\| < 3\epsilon w_B(f; [a, b]).$$

As in the first part of the proof the integral $(RYS) \int_a^b f \cdot dh$ exists by the Cauchy test under refinement, proving the theorem. \square

The Kolmogorov integral

Let f be an X -valued function on a nonempty interval J , which may be a singleton, and let μ be a Y -valued interval function on J . For a tagged interval partition $\mathcal{T} = (\{A_i\}_{i=1}^n, \{s_i\}_{i=1}^n)$ of J , define the *Kolmogorov sum* $S_K(J, \mathcal{T}) = S_K(f, d\mu; J, \mathcal{T})$ based on \mathcal{T} by

$$S_K(f, d\mu; J, \mathcal{T}) := \sum_{i=1}^n f(s_i) \cdot \mu(A_i). \quad (2.18)$$

For an additive interval function μ on J and an interval $A \subset J$, the *Kolmogorov integral* $\oint_A f \cdot d\mu$ is defined as 0 if $A = \emptyset$, or if A is nonempty, as the limit

$$\oint_A f \cdot d\mu := \lim_{\mathcal{T}} S_K(f, d\mu; A, \mathcal{T})$$

if it exists in the refinement sense, that is, $\oint_A f \cdot d\mu = z \in Z$ if for every $\epsilon > 0$ there is an interval partition \mathcal{B} of A such that for every refinement \mathcal{A} of \mathcal{B} and every tagged interval partition $\mathcal{T} = (\mathcal{A}, \xi)$, $\|S_K(f, d\mu; A, \mathcal{T}) - z\| < \epsilon$. Integrals of the form $\oint_A d\nu \cdot h$ are defined, if they exist, via (1.16).

If $A = \{a\} = [a, a]$ is a singleton then $(\{a\}, \{a\})$ is the only tagged interval partition of A , and so the Kolmogorov integral $\oint_A f \cdot d\mu$ always exists and equals $f(a) \cdot \mu(\{a\})$. Note that the Kolmogorov integral over a singleton need not be 0, whereas for the integrals with respect to point functions considered in this chapter, such integrals are 0.

Theorem 2.21. *Let f be an X -valued function on a nonempty interval J , and let μ be a Y -valued additive interval function on J . For $A, A_1, A_2 \in \mathfrak{I}(J)$ such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$, $\oint_A f \cdot d\mu$ exists if and only if both $\oint_{A_1} f \cdot d\mu$ and $\oint_{A_2} f \cdot d\mu$ exist, and then*

$$\oint_A f \cdot d\mu = \oint_{A_1} f \cdot d\mu + \oint_{A_2} f \cdot d\mu. \quad (2.19)$$

In particular, if the integral $\oint_J f \cdot d\mu$ is defined, then $\mathfrak{I}(J) \ni A \mapsto \oint_A f \cdot d\mu$ is a Z -valued additive interval function on J .

Proof. We can assume that A_1 and A_2 are nonempty. Let $\oint_A f \cdot d\mu$ be defined for a given $A \in \mathfrak{I}(J)$ and let B be a nonempty subinterval of A . To show that $\oint_B f \cdot d\mu$ is defined we use the Cauchy test. Given any two tagged interval partitions $\mathcal{T}_1, \mathcal{T}_2$ of B , let $\mathcal{T}'_1, \mathcal{T}'_2$ be extensions of $\mathcal{T}_1, \mathcal{T}_2$, respectively, to tagged interval partitions of A having the same subintervals of $A \setminus B$ and tags for them. Then

$$S_K(B, \mathcal{T}_1) - S_K(B, \mathcal{T}_2) = S_K(A, \mathcal{T}'_1) - S_K(A, \mathcal{T}'_2).$$

By assumption, the norm of the right side is small if \mathcal{T}'_1 and \mathcal{T}'_2 are both refinements of a suitable partition \mathcal{B} of A . Taking the trace \mathcal{B}_B of \mathcal{B} on B ,

it follows that the norm of the left side is small for tagged interval partitions $\mathcal{T}_1, \mathcal{T}_2$ which are refinements of \mathcal{B}_B . Thus the integral $\oint_B f \cdot d\mu$ exists by the Cauchy test.

Let $A, A_1, A_2 \in \mathfrak{I}(J)$ be such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \emptyset$. Let \mathcal{T}_1 and \mathcal{T}_2 be tagged interval partitions of A_1 and A_2 , respectively. Then $\mathcal{T} := \mathcal{T}_1 \cup \mathcal{T}_2$ is a tagged interval partition of A and

$$S_K(A, \mathcal{T}) = S_K(A_1, \mathcal{T}_1) + S_K(A_2, \mathcal{T}_2).$$

Thus if the two integrals on the right side of (2.19) exist then the integral on the left side exists, and the equality holds. The converse follows from the first part of the proof, proving the theorem. \square

The following shows that the interval function $\oint_A f \cdot d\mu$, $A \in \mathfrak{I}(J)$, is upper continuous if f is bounded and μ is upper continuous in addition to the assumptions of the preceding theorem.

Proposition 2.22. *Let f be a bounded X -valued function on a nonempty interval J , and let μ be a Y -valued upper continuous additive interval function on J . If the integral $\oint_J f \cdot d\mu$ is defined, then $\mathfrak{I}(J) \ni A \mapsto \oint_A f \cdot d\mu$ is a Z -valued upper continuous additive interval function on J .*

Proof. The interval function $\oint_A f \cdot d\mu$, $A \in \mathfrak{I}(J)$, is additive by Theorem 2.21. Let $A_1, A_2, \dots \in \mathfrak{I}(J)$ be such that $A_n \downarrow \emptyset$, and let $s_n \in A_n$ for each n . Since f is bounded and μ is upper continuous, it is enough to prove that

$$D_n := \oint_{A_n} f \cdot d\mu - f(s_n) \cdot \mu(A_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (2.20)$$

Let $J = \llbracket a, b \rrbracket$. For some $u \in J$ and all sufficiently large n , either A_n is left-open at $u \in \llbracket a, b \rrbracket$ or A_n is right-open at $u \in (a, b]$. By symmetry consider only the first case. Let $\epsilon > 0$. There exists a tagged interval partition \mathcal{T}_0 of $(u, b]$ such that any two Kolmogorov sums based on tagged refinements of \mathcal{T}_0 differ by at most ϵ . Let n_0 be such that A_{n_0} is a subset of the first interval in \mathcal{T}_0 , and let $n \geq n_0$. Choose a Kolmogorov sum based on a tagged interval partition \mathcal{T}'_n of A_n within ϵ of $\oint_{A_n} f \cdot d\mu$. Let \mathcal{T}_1 and \mathcal{T}_2 be two tagged refinements of \mathcal{T}_0 which coincide with \mathcal{T}'_n and $(A_n, \{s_n\})$, respectively, when restricted to A_n , and are equal outside of A_n . Then the norm of D_n does not exceed

$$\left\| \oint_{A_n} f \cdot d\mu - S_K(A_n, \mathcal{T}'_n) \right\| + \|S_K((u, b], \mathcal{T}_1) - S_K((u, b], \mathcal{T}_2)\| \leq 2\epsilon.$$

Since this is true for each $n \geq n_0$, (2.20) holds, proving the proposition. \square

The following is a consequence of Proposition 2.6 and the preceding proposition.

Corollary 2.23. *Let f be a bounded X -valued function on an interval $J := \llbracket a, b \rrbracket$ with $a < b$, and let μ be a Y -valued upper continuous additive interval function on J . If the integral $\oint_J f \cdot d\mu$ is defined, then for each $u, v \in J$ such that $u < v$,*

$$\begin{aligned} \lim_{t \uparrow v} \oint_{\llbracket a, t \rrbracket} f \cdot d\mu &= \oint_{\llbracket a, v \rrbracket} f \cdot d\mu, & \lim_{t \downarrow u} \oint_{\llbracket a, t \rrbracket} f \cdot d\mu &= \oint_{\llbracket a, u \rrbracket} f \cdot d\mu, \\ \lim_{t \downarrow u} \oint_{\llbracket t, b \rrbracket} f \cdot d\mu &= \oint_{\llbracket u, b \rrbracket} f \cdot d\mu, & \lim_{t \uparrow v} \oint_{\llbracket t, b \rrbracket} f \cdot d\mu &= \oint_{\llbracket v, b \rrbracket} f \cdot d\mu. \end{aligned}$$

Next we show that in defining the Kolmogorov integral $\oint_J f \cdot d\mu$ with respect to an *upper continuous* additive interval function μ it is enough to take partitions consisting of intervals in $\mathfrak{I}_{os}(J)$ (open intervals and singletons). We can assume that J is nondegenerate. First consider a closed interval $J = [a, b]$ with $a < b$. For a point partition $\{t_i\}_{i=0}^n$ of $[a, b]$, the collection

$$\left\{ \{t_0\}, (t_0, t_1), \{t_1\}, (t_1, t_2), \dots, (t_{n-1}, t_n), \{t_n\} \right\} \quad (2.21)$$

of subintervals of $[a, b]$ is called the *Young interval partition of $[a, b]$ associated to $\{t_i\}_{i=0}^n$* . A *tagged Young interval partition of $[a, b]$* is a set (2.21) together with the tags $t_0, s_1, t_1, s_2, \dots, s_n, t_n$, where $t_{i-1} < s_i < t_i$ for $i = 1, \dots, n$. For notational simplicity the tagged Young interval partition will be denoted by $(\{(t_{i-1}, t_i)\}_{i=1}^n, \{s_i\}_{i=1}^n)$.

Now a *Young interval partition* of any nonempty interval J is any interval partition consisting of singletons and open intervals. For a closed interval this coincides with the previous definition. A *tagged Young interval partition* of any interval J will be a Young interval partition together with a member (tag) of each interval in the partition. A tagged Young interval partition of any nondegenerate interval is given by specifying the open intervals in it and their tags, just as when the interval is closed. Let $\{t_i\}_{i=0}^n$ be a partition of $[u, v]$ with $u < v$, so that $u = t_0 < t_1 < \dots < t_n = v$. Then the *associated Young interval partition of $[u, v]$* is the set (2.21) if $\llbracket u, v \rrbracket = [u, v]$, or is the set (2.21) except for the left and/or right singletons $\{u\}$ or $\{v\}$ respectively if $\llbracket u, v \rrbracket$ is a left and/or right open interval. Given a Young interval partition (2.21) of $[a, b]$ and any subinterval $\llbracket u, v \rrbracket$ of $[a, b]$, the *trace Young interval partition of $\llbracket u, v \rrbracket$* is the one formed by nonempty intervals obtained by intersecting each subinterval in (2.21) with (u, v) and adjoining the endpoints $\{u\}$ and/or $\{v\}$ if $\llbracket u, v \rrbracket$ is a left and/or right closed interval.

Let $J = \llbracket a, b \rrbracket$ with $a < b$, let f be an X -valued function on J , and let μ be a Y -valued additive interval function on J . For a tagged Young interval partition $\mathcal{T} = (\{(t_{i-1}, t_i)\}_{i=1}^n, \{s_i\}_{i=1}^n)$ of J , let

$$S_{YS}(f, d\mu; J, \mathcal{T}) := S_K(f, d\mu; J, \mathcal{T}) \quad (2.22)$$

$$= \sum_{i=1}^n f(s_i) \cdot \mu((t_{i-1}, t_i)) + \sum_{i=0}^n f(t_i) \cdot \mu(\{t_i\} \cap J).$$

Assuming that μ is upper continuous, this sum can be approximated by Riemann–Stieltjes sums as follows. Let $h := R_{\mu,a}$ be the function defined by (2.3) and let $\{u_{i-1}, v_i\}_{i=1}^n \subset (a, b)$ be such that $t_0 < u_0 < s_1 < v_1 < t_1 < \dots < t_{n-1} < u_{n-1} < s_n < v_n < t_n$. Letting $\kappa := J \cap \{t_i\}_{i=0}^n$, $\tau := (\kappa \cup \{u_{i-1}, v_i\}_{i=1}^n, \kappa \cup \{s_i\}_{i=1}^n)$ is a tagged partition of $[c, d] \subset J$, where $c = a$ if $a \in J$, or $c = u_0$ otherwise, and $d = b$ if $b \in J$, or $d = v_n$ otherwise, and each t_i for $i = 1, \dots, n-1$ is the tag both for $[v_i, t_i]$ and for $[t_i, u_i]$. Next, as in the proof of Proposition 2.18, letting $u_{i-1} \downarrow t_{i-1}$ and $v_i \uparrow t_i$ for each $i = 1, \dots, n$, the Riemann–Stieltjes sum $S_{RS}(f, dh; \tau)$ converges to $S_{YS}(f, d\mu; J, \mathcal{T})$. Thus we have proved the following:

Lemma 2.24. *For f, μ, h, J and \mathcal{T} as above, the sum $S_{YS}(f, d\mu; J, \mathcal{T})$ can be approximated arbitrarily closely by Riemann–Stieltjes sums $S_{RS}(f, dh; \tau)$ based on tagged partitions τ of subintervals of J such that all tags of \mathcal{T} are tags of τ .*

Recall that the class of all Y -valued additive and upper continuous interval functions on a nonempty interval J is denoted by $\mathcal{AI}(J; Y)$. Now we can show that in defining the Kolmogorov integral with respect to an additive upper continuous interval function it is enough to take Young interval partitions.

Proposition 2.25. *Let J be a nondegenerate interval, let $\mu \in \mathcal{AI}(J; Y)$, and let $f: J \rightarrow X$. The integral $\int_J f \cdot d\mu$ is defined if and only if the limit $\lim_{\mathcal{T}} S_{YS}(f, d\mu; J, \mathcal{T})$ exists in the refinement sense, and then the integral equals the limit.*

Proof. It is enough to prove the “if” part. Let $J = [a, b]$ and let $\epsilon > 0$. Then there are a $z \in Z$ and a Young interval partition \mathcal{B} of $[a, b]$ such that for any Young interval partition \mathcal{Y} which is a refinement of \mathcal{B} and any tagged Young interval partition $\mathcal{T} = (\mathcal{Y}, \xi)$, $\|z - S_{YS}(f, d\mu; J, \mathcal{T})\| < \epsilon$. Let \mathcal{A} be a refinement of \mathcal{B} consisting of arbitrary subintervals of $[a, b]$. Let $\{u_i\}_{i=0}^n$ and $\{v_j\}_{j=0}^m$ be the sets of endpoints of intervals in \mathcal{A} and \mathcal{B} , respectively. For each $i \in \{0, \dots, n\}$ define a sequence $\{t_{ik}\}_{k \geq 1}$ as follows. If $\{u_i\}$ is a singleton in \mathcal{A} then let $t_{ik} := u_i$ for all k . If an interval $[\cdot, u_i]$ is in \mathcal{A} let $t_{ik} \downarrow u_i$, or if some $[u_i, \cdot] \in \mathcal{A}$ let $t_{ik} \uparrow u_i$, where in either case t_{ik} are not atoms of μ . This can be done by statement (c) of Proposition 2.6. For k large enough, $a = t_{0k} < t_{1k} < \dots < t_{nk} = b$. Then there is a unique Young interval partition \mathcal{Y}_k of $[a, b]$ associated to $\{t_{ik}\}_{i=0}^n$. Since each singleton $\{v_j\}$ equals $\{u_i\} \in \mathcal{A}$ for some i , \mathcal{Y}_k is a refinement of \mathcal{B} . The intervals in \mathcal{Y}_k converge to the intervals in \mathcal{A} as $k \rightarrow \infty$, except for singletons in \mathcal{Y}_k which are not atoms of μ , and hence do not contribute to sums. Let η be a set of tags for \mathcal{A} . Then each tag s of η is eventually in the interval of \mathcal{Y}_k corresponding to the interval of \mathcal{A} containing s . For such k , to form a set η_k of tags for \mathcal{Y}_k , take η and adjoin to it, for each i (if any exist) with $\{u_i\} \notin \mathcal{A}$, a t_{ik} with $\mu(\{t_{ik}\}) = 0$.

Thus there are Young–Stieltjes sums $S_{YS}(f, d\mu; J, (\mathcal{Y}_k, \eta_k))$ converging to $S_K(f, d\mu; J, (\mathcal{A}, \eta))$, proving the theorem in the case $J = [a, b]$. If $J = (a, b]$

and/or $J = \llbracket a, b \rrbracket$ then the proof is the same except that in the first case, $(u_0, c] \in \mathcal{A}$ for some c and we define $t_{0k} := u_0 = a$ for all k , while in the second case, $\llbracket d, u_n \rrbracket \in \mathcal{A}$ for some d and we define $t_{nk} := u_n = b$ for all k . The proof of Proposition 2.25 is complete. \square

For the next statement recall that $R_{\mu,a}$ is defined by (2.3).

Corollary 2.26. *Let $J = \llbracket a, b \rrbracket$ with $a < b$, let $\mu \in \mathcal{AI}(J; Y)$, let $h := R_{\mu,a}$, and let $f: J \rightarrow X$. Also for each $t \in [a, b]$, let*

$$\tilde{h}(t) := \begin{cases} h(t) & \text{if } t \in J, \\ h(a+) & \text{if } t = a \notin J, \\ h(b-) & \text{if } t = b \notin J, \end{cases} \quad \text{and} \quad \tilde{f}(t) := \begin{cases} f(t) & \text{if } t \in J, \\ 0 & \text{if } t = a \notin J, \\ 0 & \text{if } t = b \notin J. \end{cases} \quad (2.23)$$

Then $\oint_J f \cdot d\mu = (RYS) \int_a^b \tilde{f} \cdot d\tilde{h}$ if either side is defined.

Proof. By Proposition 2.6(f), h is regulated, and so \tilde{h} is well defined. By Theorem 2.8(b), $\mu = \mu_h$ defined by (2.2). Since there is a one-to-one correspondence between tagged Young partitions τ of $[a, b]$ and tagged Young interval partitions \mathcal{T} of J with $S_{YS}(f, d\tilde{h}; \tau) = S_{YS}(f, d\mu_h; J, \mathcal{T})$, the conclusion follows by Proposition 2.25. \square

Given a regulated function h on $[a, b]$, by Theorem 2.7 there is an additive, upper continuous interval function $\mu_h := \mu_{h, [a, b]}$ corresponding to h defined by (2.2) if $a < b$ or as 0 if $a = b$. Let

$$\oint_a^b f \cdot dh := \oint_{[a, b]} f \cdot d\mu_{h, [a, b]}, \quad (2.24)$$

provided the Kolmogorov integral is defined.

Proposition 2.27. *Let h be a Y -valued regulated function on $[a, b]$, and let $f: [a, b] \rightarrow X$. Then $\oint_a^b f \cdot dh = (RYS) \int_a^b f \cdot d\mu_h$ if either side is defined.*

Proof. We can assume that $a < b$. Since there is a one-to-one correspondence between tagged Young partitions τ and tagged Young interval partitions \mathcal{T} with $S_{YS}(f, dh; \tau) = S_{YS}(f, d\mu_h; [a, b], \mathcal{T})$, the conclusion again follows by Proposition 2.25. \square

In the special case that the interval function μ is a Borel measure on $[a, b]$, we will show that the Kolmogorov integral $\oint_{[a, b]} f \, d\mu$ agrees with the Lebesgue integral $\int_{[a, b]} f \, d\mu$ whenever both are defined.

Proposition 2.28. *Let μ be a finite positive measure on the Borel sets of $[a, b]$, and let f be a real-valued μ -measurable function on $[a, b]$. Then $\oint_{[a,b]} f d\mu = \int_{[a,b]} f d\mu$ whenever both integrals exist.*

Proof. We can assume that $a < b$. Suppose that the two integrals exist and let $\epsilon > 0$. Since μ is an additive and upper continuous interval function on $[a, b]$, by Proposition 2.25, there is a Young interval partition λ of $[a, b]$ such that if $(\{(t_{i-1}, t_i)\}_{i=1}^n, \{s_i\}_{i=1}^n)$ is a tagged Young partition which is a refinement of λ , then

$$\left| \sum_{i=0}^n f(t_i)\mu(T_i) + \sum_{i=1}^n f(s_i)\mu(T_{n+i}) - \oint_{[a,b]} f d\mu \right| < \epsilon, \quad (2.25)$$

where $T_i := \{t_i\}$ for $i = 0, \dots, n$ and $T_{n+i} := (t_{i-1}, t_i)$ for $i = 1, \dots, n$. Then $\{T_0, \dots, T_{2n}\}$ is a decomposition of $[a, b]$ into disjoint measurable sets. For $i = 1, \dots, n$, let $m_i := \inf\{f(s) : s \in T_{n+i}\}$ and $M_i := \sup\{f(s) : s \in T_{n+i}\}$. Letting $f(s_i) \downarrow m_i$ and $f(s_i) \uparrow M_i$ in (2.25) for each $i = 1, \dots, n$, we get that (2.25) holds with each $f(s_i)$ replaced by m_i , or each $f(s_i)$ replaced by M_i . Therefore, it follows that

$$\begin{aligned} -\epsilon &\leq \sum_{i=0}^n f(t_i)\mu(T_i) + \sum_{i=1}^n m_i\mu(T_{n+i}) - \oint_{[a,b]} f d\mu \\ &\leq \int_{[a,b]} f d\mu - \oint_{[a,b]} f d\mu \\ &\leq \sum_{i=0}^n f(t_i)\mu(T_i) + \sum_{i=1}^n M_i\mu(T_{n+i}) - \oint_{[a,b]} f d\mu \leq \epsilon. \end{aligned}$$

Since ϵ is arbitrary, $\int_a^b f d\mu = \oint_{[a,b]} f d\mu$. The proof of Proposition 2.28 is complete. \square

Corollary 2.29. *For a regulated real-valued function f on $[a, b]$ and a real-valued function h on $[a, b]$, right-continuous on $[a, b)$, if $(LS) \int_a^b f dh$ exists then so does $\oint_a^b f dh$ and the two integrals are equal.*

Proof. We can assume that $a < b$. Since $(LS) \int_a^b f dh$ exists, h is of bounded variation, and so $h = h^+ - h^-$, where h^+ and h^- are nondecreasing. By Theorem 2.20, $(RYS) \int_a^b f dh^+$ and $(RYS) \int_a^b f dh^-$ exist. Thus by Proposition 2.27, $\oint_a^b f dh^+$ and $\oint_a^b f dh^-$ exist. Since h is right-continuous at a , $\mu_{h,[a,b]}(\{a\}) = 0$. The conclusion then follows from (2.24), the obvious linearity of $h \mapsto \mu_h$ (Theorem 2.7), and Proposition 2.28. \square

The following is a change of variables theorem for Kolmogorov integrals.

Proposition 2.30. *For an interval J , let θ be a strictly increasing or strictly decreasing homeomorphism from an interval J onto $\theta(J)$. Let $\mu \in \mathcal{I}(\theta(J); Y)$ be additive and upper continuous, $\mu^\theta(A) := \mu(\theta(A))$ for $A \in \mathfrak{I}(J)$, and let $f: \theta(J) \rightarrow X$. Then*

$$\int_{\theta(J)} f \cdot d\mu = \int_J f \circ \theta \cdot d\mu^\theta$$

if either side is defined.

Proof. The statement follows from the one-to-one correspondence between tagged interval partitions of J and $\theta(J)$, and equality of corresponding Kolmogorov sums. \square

The Bochner integral

Next we define the Bochner integral of a Banach-space-valued function with respect to a measure and compare it to the Kolmogorov integral. Let S be a nonempty set, \mathcal{A} an algebra of subsets of S , and $(X, \|\cdot\|)$ a Banach space. An \mathcal{A} -simple function $f: S \rightarrow X$ will be a function $f := \sum_{i=1}^n x_i 1_{A_i}$ for some $x_i \in X$, $A_i \in \mathcal{A}$, and finite n . If μ is a subadditive function from \mathcal{A} into $[0, \infty]$, f will be called μ -simple if $\mu(A_i) < \infty$ for each i .

To define the Bochner integral, for a μ -simple function $f = \sum_{i=1}^n x_i 1_{A_i}$ and $A \in \mathcal{S}$, let $(Bo) \int_A f d\mu := \sum_{i=1}^n x_i \mu(A_i \cap A)$. It can be shown that $(Bo) \int f d\mu$ is well defined for μ -simple functions, just as for real-valued functions. If (X, d) is any metric space, then the function $(x, y) \mapsto d(x, y)$ is jointly continuous from $X \times X$ into $[0, \infty)$. If (X, d) is separable, then $d(\cdot, \cdot)$ is also jointly measurable for the Borel σ -algebra on each copy of X (e.g. [53, Propositions 2.1.4, 4.1.7]). If X is not separable, the joint measurability may fail (e.g. [53, §4.1, problem 11]). If (S, \mathcal{S}, μ) is a measure space and $(X, \|\cdot\|)$ is a Banach space, then a μ -measurable function f from S into X will be called μ -almost separably valued if there is a closed separable subspace Y of X such that $f^{-1}(X \setminus Y)$ is a μ -null set.

Definition 2.31. Let (S, \mathcal{S}, μ) be a measure space and let $X = (X, \|\cdot\|)$ be a Banach space. A function $f: S \rightarrow X$ will be called *Bochner μ -integrable* if there exist a sequence of μ -simple functions $\{f_k\}_{k \geq 1}$ and a sequence of μ -measurable real-valued functions $g_k \geq \|f - f_k\|$ such that $\lim_{k \rightarrow \infty} \int_S g_k d\mu = 0$.

Theorem 2.32. *Let $f: S \rightarrow X$ be Bochner μ -integrable. Then*

- (a) *f is μ -almost separably valued and measurable for the completion of μ ;*
- (b) *For each $A \in \mathcal{S}$ the limit*

$$\lim_{k \rightarrow \infty} \int_A f_k d\mu \tag{2.26}$$

exists and does not depend on the choice of $\{f_k\}$ satisfying the definition.

Proof. (a): Let $\{f_k\}_{k \geq 1}$ and $\{g_k\}_{k \geq 1}$ be as in Definition 2.31. Taking a subsequence, we can assume that $\int g_k d\mu < 4^{-k}$ for all k . Then there is a set $B \in \mathcal{S}$ with $\mu(B) = 0$ such that for $s \notin B$, $g_k(s) \rightarrow 0$, and so $f_k(s) \rightarrow f(s)$. The set of finite rational linear combinations of elements in the union of ranges of all f_k is countable, so its closure is a separable subspace of X , in which $f(s)$ takes values for $s \notin B$. Thus f is μ -almost separably valued and measurable for the completion of μ , since each f_k is μ -measurable, e.g. [53, Theorem 4.2.2]. Also,

$$\lim_{k \rightarrow \infty} \int_S \|f - f_k\| d\mu = 0, \quad (2.27)$$

where the Lebesgue integrals are well defined.

(b): For g and h μ -simple, and any measurable $A \subset S$, $\|(Bo) \int_A [g - h] d\mu\| \leq \int_S \|g - h\| d\mu$. It follows that that if the Bochner integral of f exists then the limit $(Bo) \int_A f_k d\mu$ exists and does not depend on the sequence of μ -simple functions $\{f_k\}$ satisfying (2.27). So the theorem is proved. \square

If f is Bochner μ -integrable, the limit

$$(Bo) \int_A f d\mu := \lim_{k \rightarrow \infty} \int_A f_k d\mu \quad (2.28)$$

is called the *Bochner integral* with respect to μ of f over the set $A \in \mathcal{S}$.

Proposition 2.33. *Let (S, \mathcal{S}, μ) be a measure space and $(X, \|\cdot\|)$ a separable Banach space. Let f be a μ -measurable function from S into X . Then f is Bochner μ -integrable if and only if $\int_S \|f\| d\mu < \infty$. Moreover, if f is Bochner μ -integrable, then*

$$\left\| (Bo) \int_S f d\mu \right\| \leq \int_S \|f\| d\mu. \quad (2.29)$$

Proof. Suppose that f is Bochner μ -integrable, and let $\{f_k\}$ be a sequence of μ -simple functions satisfying Definition 2.31. By (2.27), for some k large enough, we have $\int \|f - f_k\| d\mu < 1$, and then $\int \|f\| d\mu < 1 + \int \|f_k\| d\mu < \infty$. Moreover, for each k , we have

$$\left\| (Bo) \int_S f d\mu \right\| \leq \left\| (Bo) \int_S [f - f_k] d\mu \right\| + \int_S \|f_k\| d\mu.$$

Letting $k \rightarrow \infty$ on the right side, the first term tends to zero by (2.28) and the second term tends to the right side of (2.29) by (2.27), proving (2.29).

For the converse implication, we can assume that $\int \|f\| d\mu \leq 1$. Let $\{x_j\}_{j=1}^\infty$ be a dense sequence in X . For $k = 1, 2, \dots$, let $A_k := \{s \in S : \|f(s)\| > 1/k\}$. Then $\mu(A_k) \leq k < \infty$. If $s \notin A_k$, set $g_k(s) := 0$. If $s \in A_k$, let $g_k(s) := x_j$ for the least j such that $\|x_j - f(s)\| \leq 1/k$. Then g_k is measurable and $\|g_k(s)\| \leq 2\|f(s)\|$ for all $s \in S$, so $\int \|g_k\| d\mu \leq 2$. Also, $\|(g_k - f)(s)\| \leq 1/k$ for all s , so by dominated convergence, with

$$\|(g_k - f)(s)\| \leq (1/k)1_{A_k}(s) + 1_{A_k^c}(s)\|f(s)\| \leq \|f(s)\|$$

for all s , $\int \|g_k - f\| d\mu \rightarrow 0$ as $k \rightarrow \infty$. Let $h_{kr}(s) := g_k(s)$ if $g_k(s) = 0$ or x_j for some $j \leq r$; otherwise let $h_{kr}(s) := 0$. Then h_{kr} is a μ -simple function and $\|h_{kr}(s)\| \leq \|g_k(s)\|$ for all s , so by dominated convergence, $\int \|h_{kr} - g_k\| d\mu \rightarrow 0$ as $r \rightarrow \infty$. Let $f_k = h_{kr}$ for r large enough so that $\int \|h_{kr} - g_k\| d\mu < 1/k$. Then $\int \|f_k - f\| d\mu \rightarrow 0$ as $k \rightarrow \infty$, so f is Bochner μ -integrable. \square

We show next that for a regulated function the Bochner and Kolmogorov integrals both exist and are equal.

Proposition 2.34. *Let X be a Banach space, f an X -valued regulated function on $[a, b]$, and μ a finite positive measure on the Borel sets of $[a, b]$. Then $(Bo) \int_{[a,b]} f d\mu$ and $\oint_{[a,b]} f d\mu$ both exist and are equal.*

Proof. We can assume $a < b$. The range of f is separable, so we can assume that X is separable. Since f is bounded by Corollary 2.2, the Bochner integral $(Bo) \int_a^b f d\mu$ exists by Proposition 2.33. The Kolmogorov integral $\oint_{[a,b]} f d\mu$ exists by Theorem 2.20, Corollaries 2.11 and 2.26, and Proposition 2.27 with $Y = Z = \mathbb{R}$ and $B(x, y) \equiv yx$, $x \in X$, $y \in \mathbb{R}$. To show that the two integrals are equal, let $\epsilon_k \downarrow 0$. By Theorem 2.1 and the definition of \oint , there exists a nested sequence $\{\lambda_k\}_{k \geq 1}$ of Young interval partitions $\lambda_k = \{(t_{i-1}^k, t_i^k)\}_{i=1}^{n(k)}$ of $[a, b]$ such that for each $k \geq 1$, $\max_i \text{Osc}(f; (t_{i-1}^k, t_i^k)) < \epsilon_k$ and for any tagged Young partition $\tau_k = (\lambda_k, \{s_i\}_{i=1}^{n(k)})$,

$$\left\| \oint_{[a,b]} f d\mu - S_{YS}(f, d\mu; \tau_k) \right\| < \epsilon_k.$$

For each $k \geq 1$, fix a tagged partition τ_k and let

$$f_k := \sum_{i=1}^{n(k)} f(s_i^k) 1_{(t_{i-1}^k, t_i^k)} + \sum_{i=0}^{n(k)} f(t_i^k) 1_{\{t_i^k\}}. \quad (2.30)$$

Then for each $k \geq 1$, f_k is a μ -simple function, $\int_{[a,b]} f_k d\mu = S_{YS}(f, d\mu; \tau_k)$ and $\|f - f_k\|_{\sup} < \epsilon_k$. Thus the two integrals are equal, proving the proposition. \square

We will need the following extension of Lebesgue's differentiation theorem to the indefinite Bochner integral.

Theorem 2.35. *Let λ be Lebesgue measure on $[a, b]$ and let $f: [a, b] \rightarrow X$ be Bochner λ -integrable. Then for λ -almost all $t \in (a, b)$,*

$$\lim_{s \rightarrow t} \frac{1}{t-s} \left[(Bo) \int_{[a,t]} f d\lambda - (Bo) \int_{[a,s]} f d\lambda \right] = f(t).$$

Proof. By (2.29), it is enough to prove that for λ -almost all $t \in (a, b)$,

$$\lim_{u \downarrow 0} \frac{1}{u} \int_{A_t} \|f - f(t)\| \, d\lambda = 0, \quad (2.31)$$

where $A_t = [t - u, t]$ or $A_t = [t, t + u]$. By Theorem 2.32(a), f is λ -almost separably valued. Let $\{x_n\}$ be a countable dense set in $f([a, b] \setminus N)$ for some λ -null set $N \subset [a, b]$. By the Lebesgue differentiation theorem (e.g. Theorem 7.2.1 in Dudley [53]), we have

$$\lim_{u \downarrow 0} \frac{1}{u} \int_{A_t} \|f - x_n\| \, d\lambda = \|f(t) - x_n\|$$

for almost all $t \in (a, b)$ and for all n . For any $t \notin N$ such that this holds for all n , it follows that

$$\begin{aligned} \limsup_{u \downarrow 0} \frac{1}{u} \int_{A_t} \|f - f(t)\| \, d\lambda &\leq \limsup_{u \downarrow 0} \frac{1}{u} \int_{A_t} \|f - x_n\| \, d\lambda + \|x_n - f(t)\| \\ &= 2\|x_n - f(t)\| \end{aligned}$$

for all n . Given $\epsilon > 0$, one can choose n such that $\|x_n - f(t)\| < \epsilon/2$, proving (2.31) for λ -almost all $t \in (a, b)$. \square

*The Bartle integral

Next we define the Bartle integral of a vector-valued function with respect to an additive vector measure and compare it to the Kolmogorov integral. As in the basic assumption (1.14), let X , Y , and Z be three Banach spaces and $B(\cdot, \cdot): X \times Y \rightarrow Z$, $x \cdot y = B(x, y)$, a bounded bilinear operator with norm ≤ 1 . Let S be a nonempty set, let \mathcal{A} be an algebra of subsets of S , and let μ be a finitely additive function from \mathcal{A} into Y . The *semivariation* of μ over a set $A \in \mathcal{A}$ is defined by

$$w(\mu; A) := w_B(\mu; A) := \sup \left\{ \left\| \sum_i x_i \cdot \mu(A_i) \right\| \right\},$$

where the supremum is taken over all partitions $\{A_i\}$ of A into finitely many disjoint members of \mathcal{A} and all finite sequences $\{x_i\} \subset X$ satisfying $\|x_i\| \leq 1$. As the notation $w_B(\mu; A)$ indicates, the semivariation depends on the Banach spaces X and Z and the bilinear operator $B: (x, y) \mapsto x \cdot y$. We say that μ has *bounded semivariation* if $w_B(\mu) := w_B(\mu; S) < \infty$. The *variation* of μ over a set $A \in \mathcal{A}$ is defined by

$$v(\mu; A) := \sup \left\{ \sum_i \|\mu(A_i)\| \right\},$$

where the supremum is taken over all partitions $\{A_i\}$ of A into finitely many disjoint members of \mathcal{A} . (The variation $v(\mu; A)$ does not depend on X , Z , or B .) We say that μ has *bounded variation* if $v(\mu) := v(\mu; S) < \infty$. The semivariation $w_B(\mu; \cdot)$ is a monotone, subadditive function on \mathcal{A} , while the variation $v(\mu; \cdot)$ is a monotone, additive function on \mathcal{A} . It is clear that if $A \in \mathcal{A}$ then $0 \leq w_B(\mu; A) \leq v(\mu; A) \leq +\infty$. Also, if $X = Y'$, $Z = \mathbb{R}$, and $B(x, y) = x(y)$, then $w_B(\mu) = v(\mu)$. However, in general, it is possible that $w_B(\mu; \cdot)$ is not additive and that $w_B(\mu; A) < \infty$ while $v(\mu; A) = +\infty$, as the following examples show:

Example 2.36. (a) Let \mathcal{A} be the algebra generated by subintervals of $[0, 1]$, and let λ be Lebesgue measure. Let μ be the extension to \mathcal{A} of the additive interval function of Example 2.4 having values in $L^\infty([0, 1], \lambda)$, that is, for $A \in \mathcal{A}$, $\mu(A)$ is the equivalence class containing 1_A . Then μ is an additive function from \mathcal{A} to $Z = Y = L^\infty$. For a set $A \in \mathcal{A}$, a finite partition $\{A_i\}$ of A into finitely many disjoint sets in \mathcal{A} and for any finite sequence $\{x_i\} \subset X = \mathbb{R}$ of real numbers in $[-1, 1]$, we have

$$\left\| \sum_i x_i \mu(A_i) \right\|_\infty = \left\| \sum_i x_i 1_{A_i} \right\|_\infty = \max_i |x_i| \leq 1.$$

Thus μ has bounded semivariation, with $w(\mu; A) = 1$ for $\lambda(A) > 0$ and $w(\mu; A) = 0$ otherwise. Thus $w(\mu; \cdot)$ is not additive. Also, μ has unbounded variation. Indeed, if $A \in \mathcal{A}$ and $\lambda(A) > 0$, then one can take a countable partition $\{B_i\}$ of A into disjoint sets each with $\lambda(B_i) > 0$. Then for $n > 1$, the sequence $\{A_i\}_{i=1}^n$ defined by $A_1 := B_1, \dots, A_{n-1} := B_{n-1}$ and $A_n := \cup_{k \geq n} B_k$ is a finite partition of A and

$$\sum_{i=1}^n \|\mu(A_i)\|_\infty = \sum_{i=1}^{n-1} \|1_{B_i}\|_\infty + \|1_{\cup_{k \geq n} B_k}\|_\infty = n.$$

Since n is arbitrary, $v(\mu; A) = +\infty$. It is easy to see that the extension to \mathcal{A} of the interval function μ of Example 2.4 with values in $\ell^\infty[0, 1]$ has the same properties.

(b) Let H be a real Hilbert space, $X = \mathbb{R}$, and $Y = Z = L(H, H)$, the Banach space of bounded linear operators from H into itself, with $x \cdot y := xy$. Let S be a set and \mathcal{A} an algebra of subsets of S . Then a finitely additive *projection-valued measure* will be a function from \mathcal{A} into Y whose values are orthogonal projections onto closed linear subspaces (which may be $\{0\}$ or H) such that if A and B in \mathcal{A} are disjoint, then $\mu(A \cup B) = \mu(A) + \mu(B)$, where $\mu(A)$ and $\mu(B)$ are projections onto orthogonal subspaces. It is easily seen that $w(\mu; A) = 1$ if $\mu(A)$ is a projection onto a non-zero subspace, since otherwise $w(\mu; A) = 0$. If $\|\mu(A_i)\| = 1$ for infinitely many disjoint subsets A_i of a given set A , then clearly $v(\mu; A) = +\infty$, so μ has unbounded variation.

Let μ be an additive Y -valued function on an algebra \mathcal{A} of subsets of S . A function will be called μ -simple if it is $w(\mu)$ -simple. It is convenient to define an

“outer” form of $w(\mu; \cdot)$ on all subsets of S as follows. If $B \subset S$, then $w^*(\mu; B)$ is defined to be the infimum of $w(\mu; A)$ over all $A \in \mathcal{A}$ such that $B \subset A$. Then $w^*(\mu; \cdot)$ agrees with $w(\mu; \cdot)$ on \mathcal{A} and $w^*(\mu; \cdot)$ is a monotone and subadditive function on the class of all subsets of S . We say that a subset B of S is a μ -null set if $w^*(\mu; B) = 0$. Notice that this definition agrees with the one given in the case when μ is a measure on a σ -algebra. The definition of a μ -essentially bounded function (see Section 1.4) extends to the present case using the new meaning of μ -null sets. A sequence of functions $\{f_k\}_{k \geq 1}$ from S to X will be said to *converge in outer μ -semivariation* to a function $f: S \rightarrow X$ if for each $\epsilon > 0$, $w^*(\mu; B_k) \rightarrow 0$ as $k \rightarrow \infty$, where $B_k := \{x \in S: \|f_k(x) - f(x)\| > \epsilon\}$. For a μ -simple function $f = \sum_{i=1}^n x_i 1_{A_i}$ for some $n < \infty$, $x_i \in X$, and $A_i \in \mathcal{S}$, the *indefinite integral* $I(f, d\mu)$ is defined by

$$I(f, d\mu)(A) := \int_A f \cdot d\mu := \sum_{i=1}^n x_i \cdot \mu(A \cap A_i)$$

for each $A \in \mathcal{S}$. The indefinite integral of a μ -simple function is independent of the representation in its definition.

The following definition is due to Bartle [11, Definition 1].

Definition 2.37. Assuming (1.14), let \mathcal{A} be an algebra of subsets of S and let $\mu: \mathcal{A} \rightarrow Y$ be an additive function. A function $f: S \rightarrow X$ is called *Bartle μ -integrable* over S if there is a sequence $\{f_k\}_{k \geq 1}$ of μ -simple functions from S to X satisfying the following conditions:

- (a) the sequence $\{f_k\}$ converges in outer μ -semivariation to f ;
- (b) the sequence $\{I(f_k, d\mu)\}$ of indefinite integrals has the property that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $A \in \mathcal{A}$ and $w(\mu; A) < \delta$, then $\|I(f_k, d\mu)(A)\| < \epsilon$ for all $k \geq 1$;
- (c) given $\epsilon > 0$ there is a set $A_\epsilon \in \mathcal{A}$ with $w(\mu; A_\epsilon) < \infty$ and such that if $B \in \mathcal{A}$ and $B \subset S \setminus A_\epsilon$ then $\|I(f_k, d\mu)(B)\| < \epsilon$ for all $k \geq 1$.

We show next that uniformly for $A \in \mathcal{A}$, the sequence $\{I(f_k, d\mu)(A)\}_{k \geq 1}$ converges to a well-defined limit

$$(Ba) \int_A f \cdot d\mu := \lim_{k \rightarrow \infty} I(f_k, d\mu)(A),$$

which will be called the *Bartle integral* of f over the set A .

Proposition 2.38. *If f is Bartle μ -integrable then for each $A \in \mathcal{A}$ the sequence of indefinite integrals $\{I(f_k, d\mu)(A)\}_{k \geq 1}$ in condition (b) converges in the norm of Z uniformly for $A \in \mathcal{A}$ and the limit does not depend on the sequence of μ -simple functions $\{f_k\}_{k \geq 1}$ satisfying the conditions of Definition 2.37.*

Proof. If $w(\mu, \cdot) \equiv 0$, then $I(f_k, d\mu) \equiv 0$ and the conclusion holds. Assume that $w(\mu, \cdot) \not\equiv 0$. Let $\{f_k\}$ be a sequence of μ -simple functions satisfying the

conditions of Definition 2.37 and let $\epsilon > 0$. Take δ as in condition (b) and $A_\epsilon \in \mathcal{A}$ with $w(\mu; A_\epsilon) \neq 0$ as in (c). By condition (a) there is a positive integer K_ϵ such that $w(\mu; U_{n,m}) < \delta$ for each $n, m \geq K_\epsilon$, where $U_{n,m} \in \mathcal{A}$ and $U_{n,m} \supset \{x \in S: \|f_m(x) - f_n(x)\| > (\epsilon/w(\mu; A_\epsilon))\}$. Let $A \in \mathcal{A}$ and let $n, m \geq K_\epsilon$. Then we have

$$\begin{aligned} & \left\| \int_A f_n \cdot d\mu - \int_A f_m \cdot d\mu \right\| \\ &= \|I(f_n - f_m, d\mu)(A)\| \\ &\leq \|I(f_n - f_m, d\mu)(A \cap U_{n,m})\| + \|I(f_n - f_m, d\mu)(A \cap U_{n,m}^c \cap A_\epsilon^c)\| \\ &\quad + \|I(f_n - f_m, d\mu)(A \cap U_{n,m}^c \cap A_\epsilon)\| \\ &\leq 2\epsilon + 2\epsilon + (\epsilon/w(\mu; A_\epsilon))w(\mu; A \cap U_{n,m}^c \cap A_\epsilon) \leq 5\epsilon. \end{aligned}$$

This proves the existence and the uniformity of the limit.

To prove the second part of the conclusion, let $\{f_k\}, \{g_k\}$ be two sequences of μ -simple functions satisfying the conditions of Definition 2.37, and let $\epsilon > 0$. We can and do assume that for each $k \geq 1$, f_k and g_k have constant values on the same sets, that is for some partition $\{A_i^k\}$ of S , $f_k = \sum_i x_i^k 1_{A_i^k}$ and $g_k = \sum_i y_i^k 1_{A_i^k}$. For the two sequences $\{f_k\}, \{g_k\}$, take the minimum of two values of δ as in condition (b) and the union of two sets A_ϵ with $w(\mu; A_\epsilon) \neq 0$ as in (c). By condition (a), there is an integer $K_\epsilon > 0$ such that $w(\mu; U_k) < \delta$ for each $k \geq K_\epsilon$, where $U_k \in \mathcal{A}$ and $U_k \supset \{x \in S: \|f_k(x) - g_k(x)\| > \epsilon/w(\mu; A_\epsilon)\}$. Let $A \in \mathcal{A}$. Then for each $k \geq K_\epsilon$,

$$\begin{aligned} & \left\| \int_A f_k \cdot d\mu - \int_A g_k \cdot d\mu \right\| \\ &= \|I(f_k - g_k, d\mu)(A)\| \\ &\leq \|I(f_k - g_k, d\mu)(A \cap U_k)\| + \|I(f_k - g_k, d\mu)(A \cap U_k^c \cap A_\epsilon^c)\| \\ &\quad + \|I(f_k - g_k, d\mu)(A \cap U_k^c \cap A_\epsilon)\| \\ &\leq 2\epsilon + 2\epsilon + (\epsilon/w(\mu; A_\epsilon))w(\mu; A \cap U_k^c \cap A_\epsilon) \leq 5\epsilon. \end{aligned}$$

This proves the second part of the conclusion of Proposition 2.38. \square

For the vector measure μ of Example 2.36(a) with values in $L^\infty([0, 1], \lambda)$, a function $f: [0, 1] \rightarrow \mathbb{R}$ is Bartle μ -integrable if and only if there exists a sequence $\{f_k\}_{k \geq 1}$ of real-valued and μ -simple functions on $[0, 1]$ such that $\|f_k - f\|_\infty \rightarrow 0$ as $k \rightarrow \infty$. Indeed, if f_k are μ -simple functions satisfying condition (a) of Definition 2.37, then for $A \in \mathcal{A}$, $\int_A f_k d\mu = f_k 1_A$. Thus if f is Bartle μ -integrable then by the preceding proposition, $(Ba) \int_A f d\mu = f 1_A$ and $\|f_k - f\|_\infty \rightarrow 0$ as $k \rightarrow \infty$, since $w(\mu; A) = 0$ implies $\lambda(A) = 0$. The converse is clear by checking the conditions of Definition 2.37.

If μ is the vector measure $A \mapsto 1_A$ as in Example 2.36(a) but with values in $\ell^\infty[0, 1]$, then a function $f: [0, 1] \rightarrow \mathbb{R}$ is Bartle μ -integrable if and only if

f is regulated on $[0, 1]$ as follows. The argument of the preceding paragraph gives that $f: [0, 1] \rightarrow \mathbb{R}$ is Bartle μ -integrable if and only if there exists a sequence $\{f_k\}_{k \geq 1}$ of real-valued and μ -simple functions on $[0, 1]$ such that $\|f_k - f\|_{\sup} \rightarrow 0$ as $k \rightarrow \infty$, since in the present case $w(\mu; A) = 0$ implies that A is empty. Moreover, each μ -simple function is a step function, and so f is regulated on $[0, 1]$ by Theorem 2.1.

Proposition 2.39. *Let \mathcal{A} be an algebra of subsets of a set S , and let μ be a Y -valued additive function on \mathcal{A} with bounded semivariation. If $f: S \rightarrow X$ is μ -essentially bounded and there is a sequence of μ -simple functions converging to f in outer μ -measure, then f is Bartle μ -integrable and*

$$\left\| (Ba) \int_S f \cdot d\mu \right\| \leq \|f\|_{\infty} w(\mu; S). \quad (2.32)$$

Proof. Let $\{f_k\}_{k \geq 1}$ be a sequence of μ -simple functions which converge to f in outer μ -measure. Let $\epsilon > 0$, let $M := \|f\|_{\infty} + 2\epsilon$, and let N be a μ -null set such that $\{x \in S: \|f(x)\| > \|f\|_{\infty} + \epsilon\} \subset N$. Then for each $k \geq 1$,

$$\{x \in S: \|f_k(x)\| > M\} \subset \{x \in S: \|f_k(x) - f(x)\| > \epsilon\} \cup N. \quad (2.33)$$

For each $k \geq 1$, let $f_k^M := f_k 1_{A_k}$, where $A_k := \{x \in S: \|f_k(x)\| \leq M\}$. Then

$$\begin{aligned} & \{x \in S: \|f_k^M(x) - f(x)\| > \epsilon\} \\ & \subset \{x \in S: \|f_k(x) - f(x)\| > \epsilon\} \cup \{x \in S: \|f_k(x)\| > M\}. \end{aligned}$$

Since $w^*(\mu; \cdot)$ is monotone and subadditive, by (2.33) it then follows that condition (a) of Definition 2.37 holds for the sequence $\{f_k^M\}_{k \geq 1}$ of μ -simple functions. Since the norm of each value of each μ -simple function f_k^M is bounded by M , it follows that for each $k \geq 1$ and any $A \in \mathcal{A}$,

$$\|I(f_k, d\mu)(A)\| \leq Mw(\mu; A), \quad (2.34)$$

proving condition (b). Condition (c) holds trivially because $w(\mu; S) < \infty$. Therefore the function f is Bartle μ -integrable. The bound (2.32) follows from (2.34) and Proposition 2.38 because ϵ is arbitrary. The proof of the proposition is complete. \square

Each additive interval function $\mu: \mathcal{I}[a, b] \rightarrow Y$ extends to an additive function $\tilde{\mu}$ on the algebra $\mathcal{A}[a, b]$ generated by subintervals of $[a, b]$. We show next that if $\tilde{\mu}$ has bounded semivariation, then any regulated function on $[a, b]$ is Bartle μ -integrable and the Bartle integral agrees with the Kolmogorov integral in this case.

Proposition 2.40. *Assuming (1.14), let μ be a Y -valued additive interval function on $[a, b]$ such that $\tilde{\mu}$ is of bounded semivariation, and let f be an X -valued regulated function on $[a, b]$. Then $(Ba) \int_{[a, b]} f \cdot d\tilde{\mu}$ and $\oint_{[a, b]} f \cdot d\mu$ both exist and are equal.*

Proof. By Theorem 2.1, f is bounded and is a uniform limit of step functions, and so it is Bartle μ -integrable by Proposition 2.39. Let $\epsilon_k \downarrow 0$. By Theorem 2.1 once again, there exists a nested sequence $\{\lambda_k\}_{k \geq 1}$ of Young partitions $\lambda_k = \{A_{ki} := (t_{i-1}^k, t_i^k)\}_{i=1}^{n(k)}$ of $[a, b]$ such that for each $k \geq 1$, $\max_i \text{Osc}(f; A_{ki}) < \epsilon_k$. As in the proof of Proposition 2.34, for each $k \geq 1$, fix a tagged partition $\tau_k = (\{A_{ki}\}_{i=1}^{n(k)}, \{s_i^k\}_{i=1}^{n(k)})$ and let f_k be defined by (2.30). Then for each $k \geq 1$, f_k is a step function, $\int_{[a,b]} f_k d\tilde{\mu} = S_{YS}(f, d\mu; \tau_k)$, and $\|f - f_k\|_{\text{sup}} < \epsilon_k$. By Proposition 2.38, we have

$$(Ba) \int_{[a,b]} f \cdot d\tilde{\mu} = \lim_{k \rightarrow \infty} \int_{[a,b]} f_k \cdot d\tilde{\mu} = \lim_{k \rightarrow \infty} S_{YS}(f, d\mu; \tau_k). \quad (2.35)$$

For any $k \geq 1$ and any tagged refinement $\tau = (\{A_j\}, \{x_j\})$ of τ_k , we have

$$\begin{aligned} & \|S_{YS}(f, d\mu; \tau) - S_{YS}(f, d\mu; \tau_k)\| \\ &= \left\| \sum_{i=1}^{n(k)} \sum_{x_j \in A_{ki}} [f(x_j) - f(s_i^k)] \cdot \mu(A_j) \right\| \leq \epsilon_k w(\tilde{\mu}; [a, b]). \end{aligned}$$

Thus the Kolmogorov integral $\oint_{[a,b]} f \cdot d\mu$ exists and equals the Bartle integral (2.35), completing the proof. \square

2.4 Relations between *RS*, *RRS*, and *RYS* Integrals

By Propositions 2.13 and 2.18, we have

$$(RS) \int_a^b f \cdot dh \longrightarrow (RRS) \int_a^b f \cdot dh \xrightarrow{h \in \mathcal{R}} (RYS) \int_a^b f \cdot dh. \quad (2.36)$$

Here \longrightarrow means that existence of the integral to the left of it implies that of the integral to the right of it, with the same value, and $\xrightarrow{h \in \mathcal{R}}$ means that the implication holds for h regulated. In this section we consider the question under what conditions on f and h the two arrows in (2.36) can be inverted.

By Proposition 2.15, if the Riemann–Stieltjes integral exists then the integrand and integrator cannot have common discontinuities. This condition is also sufficient for the first arrow in (2.36) to be inverted (Theorem 2.42). A necessary condition for the existence of the refinement Riemann–Stieltjes integral is that the integrand and integrator cannot have common one-sided discontinuities. This condition is also sufficient if the integrand and integrator satisfy suitable p -variation conditions, as we will show in Corollary 3.91. Thus it is tempting to guess that the second arrow in (2.36) is invertible provided f and h have no common one-sided discontinuities. However, Example 2.44

below shows that this is not true in general. We end this section with a result giving a general sufficient condition for the second arrow in (2.36) to be invertible (Proposition 2.46).

The following definition of full Stieltjes integral defines it as the (RYS) integral provided condition (b) holds.

Definition 2.41. Let f and h be regulated functions on $[a, b]$ with values in X and Y , respectively. We say that the *full Stieltjes integral* $(S) \int_a^b f \cdot dh$ exists, or f is full Stieltjes integrable with respect to h over $[a, b]$, if (a) and (b) hold, where

- (a) $(RYS) \int_a^b f \cdot dh$ exists,
- (b) if f and h have no common one-sided discontinuities, then $(RRS) \int_a^b f \cdot dh$ exists.

Then let $(S) \int_a^b f \cdot dh := (RYS) \int_a^b f \cdot dh$.

By (2.36), whenever the two integrals in (a) and (b) exist, they have the same value. Next we prove a relation stated above between RS and RRS integrals.

Theorem 2.42. Let f and h be bounded functions from $[a, b]$ into X and Y respectively. The integral $(RS) \int_a^b f \cdot dh$ exists if and only if both $(RRS) \int_a^b f \cdot dh$ exists and f, h have no common discontinuities. Moreover, if the two integrals exist then they are equal.

Proof. Propositions 2.15 and 2.13 imply the “only if” part. For the converse implication suppose that the integral $(RRS) \int_a^b f \cdot dh$ exists and the two functions f, h have no common discontinuities. Given $\epsilon > 0$, there exists a partition $\lambda = \{z_j\}_{j=0}^m$ of $[a, b]$ such that

$$\left\| S_{RS}(f, dh; (\kappa, \xi)) - (RRS) \int_a^b f \cdot dh \right\| < \epsilon/2 \quad (2.37)$$

for each tagged partition (κ, ξ) such that κ is a refinement of λ . Then choose a $\delta > 0$ such that for each interval $\Delta_j := [z_j - \delta, z_j + \delta] \cap [a, b]$, $j = 1, \dots, m-1$,

$$\text{Osc}(f; \Delta_j) \text{Osc}(h; \Delta_j) < \epsilon/(2m).$$

Let $d := \min_{1 \leq j \leq m} (z_j - z_{j-1})$ and let $\tau = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a tagged partition with mesh $|\tau| < d \wedge \delta$. Then each interval $[x_{i-1}, x_i]$ contains at most one point of λ . For $j = 1, \dots, m-1$, let $i(j)$ be the index in $\{1, \dots, n-1\}$ such that $z_j \in (x_{i(j)-1}, x_{i(j)})$. Let (κ, ξ) be the tagged partition of $[a, b]$ obtained from τ by replacing each tagged interval $([x_{i(j)-1}, x_{i(j)}], y_{i(j)})$, $j = 1, \dots, m-1$, by the pair of tagged intervals $([x_{i(j)-1}, z_j], z_j)$, $([z_j, x_{i(j)}], z_j)$ if $z_j < x_{i(j)}$, or by the tagged interval $([x_{i(j)-1}, x_{i(j)}], z_j)$ otherwise. Then κ is a refinement of λ and

$$\begin{aligned} & \|S_{RS}(f, dh; \tau) - S_{RS}(f, dh; (\kappa, \xi))\| \\ & \leq \sum_{j=1}^{m-1} \|f(y_{i(j)}) - f(z_j)\| \|h(x_{i(j)}) - h(x_{i(j)-1})\| < \epsilon/2. \end{aligned}$$

This in conjunction with (2.37) implies that $(RS) \int_a^b f \cdot dh$ exists and equals $(RRS) \int_a^b f \cdot dh$. The proof of Theorem 2.42 is complete. \square

In view of the preceding theorem the notion of the full Stieltjes integral can be extended as follows:

Corollary 2.43. *Let f and h be regulated functions on $[a, b]$ with values in X and Y , respectively. The full Stieltjes integral $(S) \int_a^b f \cdot dh$ exists if and only if (a), (b) of Definition 2.41 and (c) hold, where*

(c) *if f and h have no common discontinuities, then $(RS) \int_a^b f \cdot dh$ exists.*

Moreover, whenever two or more of the integrals in (a), (b), and (c) exist, they have the same value.

The following shows that there are regulated functions $f, h: [0, 1] \rightarrow \mathbb{R}$ having no common discontinuities such that the integral $\int_0^1 f \, dh$ exists and equals 0 as a refinement Young–Stieltjes integral but it does not exist as a refinement Riemann–Stieltjes integral.

Example 2.44. The functions f and h will be defined on $[0, 1]$ and equal to 0 everywhere outside countable disjoint subsets A and B , respectively. Let B be the union of the sets $\{i/p: i = 1, \dots, p-1\}$ over the prime numbers $p \geq 5$, $p \in \{5, 7, 11, \dots\}$. Define the set A recursively along prime numbers $p \geq 5$ by choosing irrational points $a_{p,i}$ in each interval $((i-1)/p, i/p)$, $i = 1, \dots, p$, different from all such points previously defined. For each prime $p \geq 5$, let $f(a_{p,2j}) := h(2j/p) := 0$ and $f(a_{p,2j-1}) := h((2j-1)/p) := 1/\sqrt{p}$ for $j = 1, 2, \dots$ and $2j < p$. The functions f and h each converge to 0 along any 1–1 enumeration of the sets A , B , respectively. Thus f and h are regulated and have no common discontinuities. The integral $(RYS) \int_0^1 f \, dh$ exists and equals 0; in fact, all its approximating Young–Stieltjes sums equal 0. For any partition of $[0, 1]$ there are Riemann–Stieltjes sums which are 0. For the partition $\kappa_p := \{i/p\}_{i=0}^p$ of $[0, 1]$, for each prime $p \geq 5$, consider the Riemann–Stieltjes sum

$$\begin{aligned} S_p := \sum_{j=1}^{[p/2]-1} & \left\{ f(a_{p,2j-1}) \left[h\left(\frac{2j-1}{p}\right) - h\left(\frac{2j-2}{p}\right) \right] \right. \\ & \left. + f(a_{p,2j}) \left[h\left(\frac{2j}{p}\right) - h\left(\frac{2j-1}{p}\right) \right] \right\} = \frac{[p/2]-1}{p}. \end{aligned}$$

Since $S_p \rightarrow 1/2$ as $p \rightarrow \infty$, the Riemann–Stieltjes integral does not exist. The refinement Riemann–Stieltjes integral also does not exist by Theorem 2.42.

Lemma 2.45. *For regulated functions f and h on $[a, b]$ the following two statements hold:*

- (1) *If at each point of $[a, b]$ at least one of f and h is right-continuous, then the integral $(RRS) \int_a^b f \cdot dh$ exists whenever in its definition the Riemann-Stieltjes sums converge for tagged partitions $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ with tags $y_i \in (x_{i-1}, x_i]$, $i = 1, \dots, n$.*
- (2) *If at each point of $(a, b]$, at least one of f and h is left-continuous, then the integral $(RRS) \int_a^b f \cdot dh$ exists whenever in its definition the Riemann-Stieltjes sums converge for tagged partitions $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ with tags $y_i \in [x_{i-1}, x_i)$, $i = 1, \dots, n$.*

Proof. We prove only (1) because a proof of (2) is symmetric. Thus we have that there is an $A \in \mathbb{R}$ with the following property: given $\epsilon > 0$ there exists a partition $\lambda = \{z_j\}_{j=0}^m$ of $[a, b]$ such that

$$\|S_{RS}(f, dh; (\kappa, \xi')) - A\| < \epsilon \quad (2.38)$$

for each tagged partition $(\kappa, \xi') = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ such that κ is a refinement of λ and $y_i \in (x_{i-1}, x_i]$ for $i = 1, \dots, n$. Choose a set $\zeta = \{u_j\}_{j=1}^m \subset (a, b)$ such that $z_{j-1} < u_j < z_j$ for $j = 1, \dots, m$ and

$$\max_{1 \leq j \leq m} \text{Osc}(f; [z_{j-1}, u_j]) \text{Osc}(h; [z_{j-1}, u_j]) < \epsilon/m. \quad (2.39)$$

Suppose that $(\kappa, \xi) = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ is a tagged partition of $[a, b]$ such that κ is a refinement of $\lambda \cup \zeta$. Define $\xi_e := \{t_i\}_{i=1}^n$ by $t_i := y_i$ for i odd, $t_i := x_i$ for i even. Define $\xi_o := \{s_i\}_{i=1}^n$ by $s_i := y_i$ for i even, $s_i := x_i$ for i odd. Also, let $\xi_r := \{x_i\}_{i=1}^n$. Then letting $S(\xi) := S_{RS}(f, dh; (\kappa, \xi))$, we have $S(\xi) + S(\xi_r) = S(\xi_e) + S(\xi_o)$. Thus

$$\|S(\xi) - A\| \leq \|S(\xi_r) - A\| + \|S(\xi_e) - A\| + \|S(\xi_o) - A\|. \quad (2.40)$$

By (2.38) with $\xi' = \xi_r$, the first term on the right does not exceed ϵ . If each t_i in ξ_e is not x_{i-1} then the same bound holds for the second term on the right. Otherwise, let I_e be the set of indices $i \in \{1, \dots, n\}$ such that $x_{i-1} = y_i = t_i \in \xi_e$. Thus for each $i \in I_e$, i must be odd, so $t_{i-1} = x_{i-1} = t_i$. Let $J \subset \{0, \dots, n\}$ be the set of indices i such that $x_i \in \lambda$. For $i \in I_e$, if $i-1 \notin J$ then we can and do replace the two tagged intervals $([x_{i-2}, x_{i-1}], t_{i-1})$ and $([x_{i-1}, x_i], t_i)$ by the tagged interval $([x_{i-2}, x_i], x_{i-1})$ without changing the Riemann-Stieltjes sum and where now the tag is in the interior of the interval and the partition is still a refinement of κ . For $i \in I_e$, if $i-1 \in J$ then if we replace $t_i (= x_{i-1})$ by x_i , we change the sum by at most ϵ/m for each i by (2.39), and since there are at most m such values of i , we change the total sum by at most ϵ . Therefore it follows by (2.38) that $\|S(\xi_e) - A\| \leq 2\epsilon$. Similarly one can show that the last term in (2.40) does not exceed 2ϵ , so $\|S(\xi) - A\| < 5\epsilon$. Since ϵ is arbitrary, the integral $(RRS) \int_a^b f \cdot dh$ exists and equals A , proving the lemma. \square

Proposition 2.46. *Let regulated functions f, h on $[a, b]$ be such that f is left-continuous on $(a, b]$ and h is right-continuous on $[a, b)$, or vice versa. If $(RYS) \int_a^b f \cdot dh$ exists then so does $(RRS) \int_a^b f \cdot dh$, and the two integrals have the same value.*

Proof. We can assume that neither f nor h is identically 0, since if one is, the conclusion holds. We prove the proposition when f is left-continuous on $(a, b]$ and h is right-continuous on $[a, b)$. A proof of the other case is symmetric. In that case by (1) of Lemma 2.45 and since f is left-continuous on $(a, b]$, it is enough to prove the convergence of Riemann–Stieltjes sums based on Young tagged partitions of $[a, b]$. To this aim we show that differences between Riemann–Stieltjes and Young–Stieltjes sums based on the same Young tagged partition can be made arbitrarily small by refinement of partitions. Let $\epsilon > 0$. By definition of the refinement Young–Stieltjes integral there exists a partition $\lambda = \{z_j\}_{j=0}^m$ of $[a, b]$ such that

$$\left\| S_{YS}(f, dh; \tau) - (RYS) \int_a^b f \cdot dh \right\| < \epsilon \quad (2.41)$$

for any Young tagged partition $\tau = (\kappa, \xi)$ such that κ is a refinement of λ . Choose a set $\zeta = \{v_{j-1}, u_j : j = 1, \dots, m\} \subset (a, b)$ such that for each $j = 1, \dots, m$, $z_{j-1} < v_{j-1} < u_j < z_j$,

$$\max(\|h\|_{\sup} \text{Osc}(f; [u_j, z_j]), \|f\|_{\sup} \text{Osc}(h; [z_{j-1}, v_{j-1}])) < \epsilon/(2m). \quad (2.42)$$

Let $(\kappa, \xi) = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a Young tagged partition of $[a, b]$ such that κ is a refinement of $\lambda \cup \zeta$. Then $S_{RS}(f, dh; (\kappa, \xi)) - S_{YS}(f, dh; (\kappa, \xi)) = S_1 + S_2$, where

$$\begin{aligned} S_1 &:= \sum_{i=1}^n [f(y_i) - f(x_i)] \cdot [h(x_i) - h(x_{i-1})], \\ S_2 &:= \sum_{i=1}^n [f(x_i) - f(y_i)] \cdot [h(x_{i-1}) - h(x_{i-1})]. \end{aligned} \quad (2.43)$$

Let $\xi' = \{y'_i\}_{i=1}^n$ be another set of tags for κ such that $x_{i-1} < y'_i < x_i$ for $i = 1, \dots, n$. By (2.41) with $\tau = (\kappa, \xi)$ and $\tau = (\kappa, \xi')$, we get the bound

$$\left\| \sum_{i=1}^n [f(y'_i) - f(y_i)] \cdot [h(x_{i-1}) - h(x_{i-1})] \right\| < 2\epsilon.$$

Letting $y'_i \uparrow x_i$ for $i = 1, \dots, n$, it follows that in (2.43), $\|S_2\| \leq 2\epsilon$. Let $J \subset \{0, \dots, n\}$ be the set of indices i such that $x_i \in \lambda$, and let $I := \{0, \dots, n\} \setminus J$. To bound $\|S_1\|$ define a partition $\kappa' = \{x'_i\}_{i=0}^n$ of $[a, b]$ by $x'_i := x_i$ if $i \in J$ and take any $x'_i \in (x_i, y_{i+1})$ if $i \in I$. Then κ' is a refinement of λ and (κ', ξ) is a Young tagged partition. Let $\xi' = \{y'_i\}_{i=1}^n$ be another set of tags for κ'

defined by $y'_i := y_i$ if $i \in J$ and $y'_i := x_i$ if $i \in I$. By (2.41) with $\tau = (\kappa', \xi)$ and $\tau = (\kappa', \xi')$, we get the bound

$$\left\| \sum_{i \in I} [f(y_i) - f(x_i)] \cdot [h(x'_i -) - h(x'_{i-1})] \right\| < 2\epsilon.$$

Letting $x'_i \downarrow x_i$ for $i \in I$, it follows that the sum of all terms in S_1 with $i \in I$ does not exceed 2ϵ . By (2.42), the norm of each term of S_1 with $i \in J$ does not exceed ϵ/m , and the norm of the total sum of such terms does not exceed ϵ . Therefore in (2.43), $\|S_1\| \leq 3\epsilon$. The proof of Proposition 2.46 is complete. \square

2.5 The Central Young Integral

L. C. Young [244] defined, up to endpoint terms given by Young [247], an integral which we will call the central Young integral, or the *CY* integral. The idea of the *CY* integral is to use the *RRS* integral, avoiding its lack of definition when f and h have common one-sided discontinuities by taking a right-continuous version of f and left-continuous version of h , or vice versa, and adding sums of jump terms to restore the desired value of $\int f \cdot dh$. For regulated functions the *CY* integral extends the *RYS* integral, as will be shown in Theorem 2.51, and thus in turn the *RRS* integral by Proposition 2.46.

Define functions $f_+^{(b)}$ and $f_-^{(a)}$ on $[a, b]$ with $a < b$ by

$$f_+^{(b)}(x) := \begin{cases} f_+(x) := f(x+) := \lim_{z \downarrow x} f(z) & \text{if } a \leq x < b, \\ f(b) & \text{if } x = b, \end{cases}$$

and

$$f_-^{(a)}(x) := \begin{cases} f_-(x) := f(x-) := \lim_{z \uparrow x} f(z) & \text{if } a < x \leq b, \\ f(a) & \text{if } x = a. \end{cases}$$

Recalling the definitions of Δ^+ , Δ^- , and Δ^\pm in and before (2.1), the *CY* integral has two equivalent forms, defined next:

Definition 2.47. Assuming (1.14), let f and h be regulated functions on $[a, b]$ with values in the Banach spaces X, Y respectively. If $a < b$ define the Y_1 integral

$$\begin{aligned} (Y_1) \int_a^b f \cdot dh &:= (RRS) \int_a^b f_+^{(b)} \cdot dh_-^{(a)} - \sum_{[a, b]} \Delta_{[a, b]}^+ f \cdot \Delta_{[a, b]}^\pm h \\ &= (RRS) \int_a^b f_+^{(b)} \cdot dh_-^{(a)} - [\Delta^+ f \cdot \Delta^+ h](a) \\ &\quad + [f \cdot \Delta^- h](b) - \sum_{(a, b)} \Delta^+ f \cdot \Delta^\pm h \end{aligned} \tag{2.44}$$

if the RRS integral exists and the sum converges unconditionally in Z as defined in Section 1.4. If $a = b$ define the Y_1 integral as 0. Similarly, if $a < b$ define the Y_2 integral

$$\begin{aligned} (Y_2) \int_a^b f \cdot dh &:= (RRS) \int_a^b f_-^{(a)} \cdot dh_+^{(b)} + \sum_{[a,b]} \Delta_{[a,b]}^- f \cdot \Delta_{[a,b]}^\pm h \\ &= (RRS) \int_a^b f_-^{(a)} \cdot dh_+^{(b)} + [f \cdot \Delta^+ h](a) \\ &\quad + [\Delta^- f \cdot \Delta^- h](b) + \sum_{(a,b)} \Delta^- f \cdot \Delta^\pm h \end{aligned} \quad (2.45)$$

if the RRS integral exists and the sum converges unconditionally in Z . If $a = b$ define the Y_2 integral as 0.

Integrals $(Y_j) \int_a^b df \cdot h$, $j = 1, 2$, are defined symmetrically, if they exist, as in (1.15).

It will be shown that the Y_1 integral exists if and only if the Y_2 integral does, and if so, the two integrals have the same value. This will be done by giving an alternative representation of the two integrals in terms of the refinement Young–Stieltjes integral. To this aim, for $a < b$ we define the functions $f_-^{(a,b)}$ and $f_+^{(a,b)}$ on $[a, b]$ respectively by

$$f_-^{(a,b)}(x) := \begin{cases} f(a) & \text{if } x = a, \\ f(x-) & \text{if } x \in (a, b), \\ f(b) & \text{if } x = b, \end{cases} \quad \text{and} \quad f_+^{(a,b)}(x) := \begin{cases} f(a) & \text{if } x = a, \\ f(x+) & \text{if } x \in (a, b), \\ f(b) & \text{if } x = b. \end{cases} \quad (2.46)$$

Proposition 2.48. *Assuming (1.14) and $a < b$, for regulated functions f and h on $[a, b]$ with values in X and Y , respectively,*

$$(Y_1) \int_a^b f \cdot dh = (RYS) \int_a^b f_+^{(a,b)} \cdot dh - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h \quad (2.47)$$

and

$$(Y_2) \int_a^b f \cdot dh = (RYS) \int_a^b f_-^{(a,b)} \cdot dh + \sum_{(a,b)} \Delta^- f \cdot \Delta^\pm h \quad (2.48)$$

if either side is defined, where the right side is defined if the refinement Young–Stieltjes integral exists and the sum converges unconditionally in Z .

Proof. We prove only (2.47) because a proof of (2.48) is symmetric. By Propositions 2.18 and 2.46, the refinement Riemann–Stieltjes integral in (2.44) exists if and only if it exists as a refinement Young–Stieltjes integral, and then both have the same value. Let $\tau = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a Young tagged partition of $[a, b]$. Then

$$\begin{aligned}
 S_{YS}(f_+^{(b)}, dh_-^{(a)}; \tau) &= \sum_{i=1}^n f(y_i+) \cdot [h(x_i-) - h(x_{i-1}+)] + [f_+ \cdot \Delta^+ h](a) \\
 &\quad + \sum_{i=1}^{n-1} [f_+ \cdot \Delta^\pm h](x_i) \\
 &= S_{YS}(f_+^{(a,b)}, dh; \tau) + [\Delta^+ f \cdot \Delta^+ h](a) - [f \cdot \Delta^- h](b).
 \end{aligned}$$

Therefore $(RYS) \int_a^b f_+^{(b)} \cdot dh_-^{(a)}$ exists if and only if $(RYS) \int_a^b f_+^{(a,b)} \cdot dh$ does. Also, (2.47) holds, proving the proposition. \square

The following statement relates the two refinement Young–Stieltjes integrals in (2.47) and (2.48) with the integral $(RYS) \int_a^b f \cdot dh$.

Lemma 2.49. *For regulated functions f and h on $[a, b]$ with $a < b$, the following three statements are equivalent:*

- (a) $(RYS) \int_a^b f \cdot dh$ exists;
- (b) $(RYS) \int_a^b f_+^{(a,b)} \cdot dh$ and $(RYS) \int_a^b [f_+^{(a,b)} - f] \cdot dh$ both exist;
- (c) $(RYS) \int_a^b f_-^{(a,b)} \cdot dh$ and $(RYS) \int_a^b [f - f_-^{(a,b)}] \cdot dh$ both exist.

If any one of the three statements holds then

$$\begin{aligned}
 (RYS) \int_a^b f \cdot dh &= (RYS) \int_a^b f_+^{(a,b)} \cdot dh - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h \\
 &= (RYS) \int_a^b f_-^{(a,b)} \cdot dh + \sum_{(a,b)} \Delta^- f \cdot \Delta^\pm h,
 \end{aligned} \tag{2.49}$$

where the two sums converge unconditionally.

Proof. We prove only the implication $(a) \Rightarrow (b)$ and (2.49) because the converse implication follows by linearity of the RYS integral and a proof of the rest is symmetric. Thus suppose that (a) holds. It is enough to show that $(RYS) \int_a^b [f_+^{(a,b)} - f] \cdot dh$ exists and equals the sum in (2.49). First we show that the sum in (2.49) converges unconditionally. Let $\epsilon > 0$. By definition of the refinement Young–Stieltjes integral there exists a partition λ of $[a, b]$ such that

$$\left\| S_{YS}(f, dh; \tau) - (RYS) \int_a^b f \cdot dh \right\| < \epsilon \tag{2.50}$$

for any Young tagged partition $\tau = (\kappa, \xi)$ of $[a, b]$ such that κ is a refinement of λ . Let $\zeta = \{u_j\}_{j=1}^m \subset (a, b)$ be a set disjoint from λ . Choose a refinement $\kappa = \{t_i\}_{i=0}^n$ of λ such that each u_j is in $(t_{i(j)-1}, t_{i(j)})$, $j = 1, \dots, m$, for some $t_{i(j)-1}, t_{i(j)} \in \kappa \setminus \lambda$, with $i(j) < i(j+1) - 1$ for $j = 1, \dots, m-1$. Define

$\xi' = \{s'_i\}_{i=1}^n$ and $\xi'' = \{s''_i\}_{i=1}^n$ by $u_j = s'_{i(j)} < s''_{i(j)} < t_{i(j)}$ for $j = 1, \dots, m$ and take any $s'_i = s''_i \in (t_{i-1}, t_i)$ for the other indices i . By (2.50) with $\tau = (\kappa, \xi')$ and $\tau = (\kappa, \xi'')$, we get that

$$\begin{aligned} & \left\| \sum_{j=1}^m [f(u_j) - f(s''_{i(j)})] \cdot [h(t_{i(j)}-) - h(t_{i(j)-1}+)] \right\| \\ &= \|S_{YS}(f, dh; (\kappa, \xi')) - S_{YS}(f, dh; (\kappa, \xi''))\| < 2\epsilon. \end{aligned}$$

Letting $s''_{i(j)} \downarrow u_j$, $t_{i(j)} \downarrow u_j$ and $t_{i(j)-1} \uparrow u_j$ for $j = 1, \dots, m$, as is possible, it follows that $\|\sum_{\zeta} \Delta^+ f \cdot \Delta^\pm h\| \leq 2\epsilon$ for any finite set $\zeta \subset (a, b)$ disjoint from λ . Thus the sum in (2.49) converges unconditionally in Z .

Now to prove the existence of $(RYS) \int_a^b [f_+^{(a,b)} - f] \cdot dh$ notice that for any Young tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, b]$,

$$\begin{aligned} & \|S_{YS}(f_+^{(a,b)} - f, dh; \tau) - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h\| \\ & \leq \left\| \sum_{i=1}^n \Delta^+ f(s_i) \cdot [h(t_i-) - h(t_{i-1}+)] \right\| \\ & \quad + \left\| \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h - \sum_{i=1}^{n-1} [\Delta^+ f \cdot \Delta^\pm h](t_i) \right\|. \end{aligned}$$

Using (2.50) and the unconditional convergence just proved, one can make the right side arbitrarily small by way of refinements of partitions, proving the lemma. \square

Now we are ready to prove the equality of Y_1 and Y_2 integrals.

Theorem 2.50. *Let $f \in \mathcal{R}([a, b]; X)$ and $h \in \mathcal{R}([a, b]; Y)$. Then $(Y_1) \int_a^b f \cdot dh = (Y_2) \int_a^b f \cdot dh$ if either side is defined.*

Proof. We can assume that $a < b$. Suppose that the Y_2 integral exists. Hence by Proposition 2.48, the refinement Young–Stieltjes integral $(RYS) \int_a^b f_-^{(a,b)} \cdot dh$ exists and (2.48) holds. Then by Lemma 2.49 applied to $f = f_-^{(a,b)}$, the integral $(RYS) \int_a^b f_+^{(a,b)} \cdot dh$ exists and

$$(RYS) \int_a^b f_-^{(a,b)} \cdot dh = (RYS) \int_a^b f_+^{(a,b)} \cdot dh - \sum_{(a,b)} \Delta^\pm f \cdot \Delta^\pm h,$$

where the sum converges unconditionally. Also, by linearity of unconditional convergence, the sum

$$\sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h = \sum_{(a,b)} \Delta^\pm f \cdot \Delta^\pm h - \sum_{(a,b)} \Delta^- f \cdot \Delta^\pm h$$

converges unconditionally. Therefore the right side of (2.47) is defined, proving the existence of the Y_1 integral and the relation, by (2.48),

$$\begin{aligned} (Y_2) \int_a^b f \cdot dh &= (RYS) \int_a^b f_-^{(a,b)} \cdot dh + \sum_{(a,b)} \Delta^- f \cdot \Delta^\pm h \\ &= (RYS) \int_a^b f_+^{(a,b)} \cdot dh - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm h = (Y_1) \int_a^b f \cdot dh. \end{aligned}$$

A proof of the converse implication is symmetric and we omit it. \square

Since the Y_1 and Y_2 integrals have been shown to coincide for regulated functions f and h , if either is defined, we now define the *central Young integral*, or *CY* integral, by

$$(CY) \int_a^b f \cdot dh := (Y_1) \int_a^b f \cdot dh = (Y_2) \int_a^b f \cdot dh. \quad (2.51)$$

The *CY* integral extends the *RYS* integral, as the following shows.

Theorem 2.51. *Assuming (1.14), for $f \in \mathcal{R}([a, b]; X)$ and $h \in \mathcal{R}([a, b]; Y)$ the following hold:*

- (a) *if $(RYS) \int_a^b f \cdot dh$ exists then so does $(CY) \int_a^b f \cdot dh$, and the two are equal;*
- (b) *letting $\hat{f} := [f_-^{(a,b)} + f_+^{(a,b)}]/2$ (cf. (2.46)), if $a < b$ and $(CY) \int_a^b f \cdot dh$ exists then so does $(RYS) \int_a^b \hat{f} \cdot dh$, the sum $\sum_{(a,b)} [f - \hat{f}] \cdot \Delta^\pm h$ converges unconditionally in Z , and*

$$(CY) \int_a^b f \cdot dh = (RYS) \int_a^b \hat{f} \cdot dh + \sum_{(a,b)} [f - \hat{f}] \cdot \Delta^\pm h. \quad (2.52)$$

Proof. For (a), suppose that $(RYS) \int_a^b f \cdot dh$ exists. By the implication (a) \Rightarrow (b) and relation (2.49) of Lemma 2.49, the right side of (2.47) is defined. Hence the *CY* integral exists and equals the *RYS* integral, proving (a).

For (b), suppose that $(CY) \int_a^b f \cdot dh$ exists. By Proposition 2.48, the right sides of (2.47) and (2.48) are defined. By linearity, $(RYS) \int_a^b \hat{f} \cdot dh$ exists and the sum $\sum_{(a,b)} [f - \hat{f}] \cdot \Delta^\pm h$ converges unconditionally in Z . Then (2.52) follows by adding (2.47) and (2.48). \square

If a function f has values at its jump points in (a, b) which are normalized, that is, $f = \hat{f} = [f_-^{(a,b)} + f_+^{(a,b)}]/2$, then by the preceding theorem $(CY) \int_a^b f \cdot dh$ and $(RYS) \int_a^b f \cdot dh$ both exist or both do not. In general the equivalence is not true, as the following shows:

Proposition 2.52. *There exist functions f and h on $[0, 1]$, where $f \in c_0((0, 1))$ (cf. Definition 2.9) and h is continuous, for which $(Y_2) \int_0^1 f \, dh$ exists, while $(RYS) \int_0^1 [f - f_-^{(0,1)}] \, dh$ and hence $(RYS) \int_0^1 f \, dh$ do not exist.*

Proof. Let $f(t) := k^{-1/2}$ if $t = 1/(3k)$ for $k = 1, 2, \dots$, and $f(t) := 0$ otherwise. Let $h(1/(3k+1)) := 0$ and $h(1/(3k-1)) := k^{-1/2}$ for $k = 1, 2, \dots$. Let $h(0) = h(1) := 0$ and let h be “linear in between,” i.e., linear on each closed interval where h is so far defined only at the endpoints. Since h is continuous and $f_-^{(0)} \equiv 0$, $(Y_2) \int_0^1 f \, dh$ exists and is 0. Since $f_-^{(0,1)} \equiv 0$, it is enough to show the nonexistence of the integral $(RYS) \int_0^1 f \, dh$. Let $\lambda = \{u_j\}_{j=0}^m$ be any partition of $[0, 1]$. Take the smallest m such that $t_m := 1/(3m+1) \leq u_1$. If $\kappa_m := \lambda \cup \{t_m\}$ then the contribution to any Young–Stieltjes sum based on κ_m coming from $[0, t_m]$ is 0. For $n > m$, consider partitions

$$\kappa_n := \kappa_m \cup \{1/(3k+1), 1/(3k-1) : k = m+1, \dots, n\}.$$

We form a Young–Stieltjes sum based on κ_n by letting it be the same as one for κ_m on $[t_m, 1]$, and by evaluating f at $1/(3k)$ for $k = m+1, \dots, n$. Then the part of our Young–Stieltjes sum for κ_n coming from $[0, t_m]$ is

$$\begin{aligned} & \sum_{k=m+1}^n f\left(\frac{1}{3k}\right) \left[h\left(\frac{1}{3k-1}\right) - h\left(\frac{1}{3k+1}\right) \right] \\ & \quad + 0 \cdot \left[h\left(\frac{1}{3k-2}\right) - h\left(\frac{1}{3k+1}\right) \right] \\ & = \sum_{k=m+1}^n k^{-1/2} k^{-1/2} \rightarrow \infty \end{aligned}$$

as $n \rightarrow \infty$. Thus two Young–Stieltjes sums for $(RYS) \int_0^1 f \, dh$, both based on refinements of λ , differ by an arbitrarily large amount. So $(RYS) \int_0^1 f \, dh$ does not exist. \square

2.6 The Henstock–Kurzweil Integral

A *gauge function* on $[a, b]$ is any function defined on $[a, b]$ with strictly positive values. Given a gauge function $\delta(\cdot)$ on $[a, b]$ with $a < b$, a tagged partition $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ of $[a, b]$ is δ -fine if $y_i - \delta(y_i) \leq x_{i-1} \leq y_i \leq x_i \leq y_i + \delta(y_i)$ for $i = 1, \dots, n$. Let $TP(\delta, [a, b])$ be the set of all δ -fine tagged partitions of $[a, b]$.

Lemma 2.53. *Let $\delta(\cdot)$ be a gauge function defined on an interval $[a, b]$ with $a < b$. Then:*

(a) $TP(\delta, [a, b])$ is nonempty, i.e. there exists a tagged partition

$$\zeta := (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n) \in TP(\delta, [a, b]).$$

(b) For any finite set $F \subset [a, b]$, ζ can be chosen so that $F \subset \{x_i\}_{i=0}^n$.

(c) In (b), ζ can instead be chosen so that $F \subset \{y_i\}_{i=1}^n$.

Proof. (a): The system of open intervals $\{(y - \delta(y), y + \delta(y)) : y \in [a, b]\}$ is an open cover of $[a, b]$. Since $[a, b]$ is compact, there is a finite subcover $\{J_i := (y_i - \delta(y_i), y_i + \delta(y_i)) : i = 1, \dots, n\}$ of $[a, b]$ such that any $n - 1$ of the J_i do not cover $[a, b]$. Since y_i for $i = 1, \dots, n$ are distinct, we can assume that $y_1 < \dots < y_n$. Also, $J_i \cap J_{i+1} \neq \emptyset$ for $i = 1, \dots, n - 1$, $J_i \cap J_j = \emptyset$ for $|i - j| > 1$, $i, j = 1, \dots, n$, $a \in J_1$ and $b \in J_n$. Hence one can find numbers $a = x_0 < x_1 < \dots < x_n = b$ such that $x_i \in J_i \cap J_{i+1}$ for $i = 1, \dots, n - 1$ and $y_i \in [x_{i-1}, x_i]$ for $i = 1, \dots, n$. Thus $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n) \in TP(\delta, [a, b])$, proving (a).

(b): We can assume that $F \subset (a, b)$. Let $F = \{\xi_j\}_{j=1}^k$ where $a < \xi_1 < \dots < \xi_k < b$. Let $\xi_0 := a$ and $\xi_{k+1} := b$. By part (a), each interval $[\xi_j, \xi_{j+1}]$, $j = 0, 1, \dots, k$, has a δ -fine tagged partition. Combining these, we get one for $[a, b]$, proving (b).

(c): Given a finite set $F \subset [a, b]$, $[a, b] \setminus \bigcup_{y \in F} (y - \delta(y)/2, y + \delta(y)/2)$ is a finite union of closed subintervals $[a_j, b_j]$ of $[a, b]$. Each of these subintervals has a δ -fine tagged partition. Also, $[a, b] \setminus \bigcup_j [a_j, b_j]$ is a finite union of disjoint intervals (which may be open or closed at either end). Decomposing such intervals into smaller ones, we obtain a set of such intervals each containing exactly one $y \in F$ and included in $(y - \delta(y), y + \delta(y))$. Putting together the tagged partitions of each $[a_j, b_j]$ and the tagged intervals indexed by $y \in F$, we get a δ -fine tagged partition of $[a, b]$ in which each $y \in F$ is a tag, proving (c). \square

Definition 2.54. Assuming (1.14), let $f : [a, b] \rightarrow X$ and $h : [a, b] \rightarrow Y$. If $a < b$, the *Henstock–Kurzweil integral*, or *HK integral*, $(HK) \int_a^b f \cdot dh$ is defined as an $I \in Z$, if it exists, with the following property: given $\epsilon > 0$ there is a gauge function $\delta(\cdot)$ on $[a, b]$ such that $\|S_{RS}(f, dh; \tau) - I\| < \epsilon$ for each δ -fine tagged partition τ of $[a, b]$. If $a = b$, let $(HK) \int_a^b f \cdot dh := 0$. The integral $(HK) \int_a^b df \cdot h$ is defined, if it exists, via (1.15).

Proposition 2.55 (Cauchy test). *Let $a < b$, $f : [a, b] \rightarrow X$, and $h : [a, b] \rightarrow Y$. The Henstock–Kurzweil integral $(HK) \int_a^b f \cdot dh$ is defined if and only if given $\epsilon > 0$ there is a gauge function $\delta(\cdot)$ on $[a, b]$ such that*

$$\|S_{RS}(f, dh; \tau_1) - S_{RS}(f, dh; \tau_2)\| < \epsilon \quad (2.53)$$

for each $\tau_1, \tau_2 \in TP(\delta, [a, b])$.

Proof. To prove the “if” part, for each $n \geq 1$, choose a gauge function $\delta_n(\cdot)$ on $[a, b]$ such that (2.53) holds with $\epsilon = 1/n$ for each $\tau_1, \tau_2 \in TP(\delta_n, [a, b])$. Replacing δ_n by $\min\{\delta_1, \dots, \delta_n\}$, we can assume that $\delta_1 \geq \delta_2 \geq \dots$. For $n = 1, 2, \dots$, let $S_n := S_{RS}(f, dh; \tau_n)$ for some $\tau_n \in TP(\delta_n, [a, b])$. Then $\{S_n : n \geq 1\}$ is a Cauchy sequence in Z , and hence convergent to a limit I . Given $\epsilon > 0$ choose an n such that $n > 2/\epsilon$ and $\|I - S_n\| < \epsilon/2$. Let $\delta := \delta_n$. Then for each $\tau \in TP(\delta, [a, b])$,

$$\|I - S_{RS}(f, dh; \tau)\| \leq \|I - S_n\| + \|S_n - S_{RS}(f, dh; \tau)\| < \epsilon/2 + 1/n < \epsilon.$$

Thus $(HK) \int_a^b f \cdot dh$ is defined. Since the converse implication is clear, the proof is complete. \square

Proposition 2.56. *Let $f : [a, b] \rightarrow X$ and $h : [a, b] \rightarrow Y$. Let $a < c < b$. Then $(HK) \int_a^b f \cdot dh$ exists if and only if $(HK) \int_a^c f \cdot dh$ and $(HK) \int_c^b f \cdot dh$ both exist, and then we have*

$$(HK) \int_a^b f \cdot dh = (HK) \int_a^c f \cdot dh + (HK) \int_c^b f \cdot dh. \quad (2.54)$$

The proof of this property is easy once we observe the following:

Lemma 2.57. *For any gauge function $\delta(\cdot)$ on $[a, b]$ and $c \in [a, b]$, there is a gauge function $\delta'(\cdot)$ on $[a, b]$ such that $\delta' \leq \delta$ and each δ' -fine tagged partition has c as a tag, in the interior of its interval in the relative topology of $[a, b]$.*

Proof. Suppose that $a \leq c \leq b$ and $\delta(\cdot)$ is a gauge function on $[a, b]$. Let $\delta'(c) := \delta(c)$, and let $\delta'(x) := \min\{\delta(x), |c - x|/2\}$ for $x \in [a, b] \setminus \{c\}$. For a δ' -fine tagged partition $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ of $[a, b]$, $c = y_i \in (x_{i-1}, x_i)$ if $c \in [x_{i-1}, x_i] \cap (a, b)$, $c = y_1 < x_1$ if $c = a$, or $c = y_n > x_{n-1}$ if $c = b$. \square

Proof of Proposition 2.56. Suppose that $I_1 := (HK) \int_a^c f \cdot dh$ and $I_2 := (HK) \int_c^b f \cdot dh$ both exist. Given $\epsilon > 0$, take gauge functions $\delta_1(\cdot)$ on $[a, c]$ for I_1 and $\epsilon/2$, and $\delta_2(\cdot)$ on $[c, b]$ for I_2 and $\epsilon/2$. By the preceding lemma, there is a gauge function $\delta(\cdot)$ on $[a, b]$ such that $\delta \leq \delta_1$ on $[a, c]$, $\delta \leq \delta_2$ on $(c, b]$, $\delta(c) \leq \min\{\delta_1(c), \delta_2(c)\}$, and c is a tag for each δ -fine tagged partition of $[a, b]$. Let $\tau = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a δ -fine tagged partition of $[a, b]$ and let $c = y_i$ for some $i \in \{1, \dots, n\}$. Writing $f(y_i) \cdot [h(x_i) - h(x_{i-1})] = f(y_i) \cdot [h(y_i) - h(x_{i-1})] + f(y_i) \cdot [h(x_i) - h(y_i)]$, we have $S_{RS}(f, dh; \tau) = S_{RS}(f, dh; \tau_1) + S_{RS}(f, dh; \tau_2)$ for suitable $\tau_1 \in TP(\delta_1, [a, c])$ and $\tau_2 \in TP(\delta_2, [c, b])$. Thus $\|I_1 + I_2 - S_{RS}(f, dh; \tau)\| < \epsilon$, proving the existence of $(HK) \int_a^b f \cdot dh$ and relation (2.54).

Now suppose that $I := (HK) \int_a^b f \cdot dh$ exists. Let $\delta(\cdot)$ be a gauge function on $[a, b]$ for I and $\epsilon/2$. By the preceding lemma we can and do assume that

c is a tag for each δ -fine tagged partition of $[a, b]$. Let $\delta_1(\cdot)$ and $\delta_2(\cdot)$ be the restrictions of $\delta(\cdot)$ to $[a, c]$ and $[c, b]$, respectively. Let $\tau'_1, \tau''_1 \in TP(\delta_1, [a, c])$ and let $\tau_2 \in TP(\delta_2, [c, b])$. Then $\tau' := \tau'_1 \cup \tau_2$ and $\tau'' := \tau''_1 \cup \tau_2$ are two δ -fine tagged partitions of $[a, b]$, and

$$\|S_{RS}(f, dh; \tau'_1) - S_{RS}(f, dh; \tau''_1)\| = \|S_{RS}(f, dh; \tau') - S_{RS}(f, dh; \tau'')\| < \epsilon.$$

Thus $(HK) \int_a^c f \cdot dh$ exists by the Cauchy test. Similarly it follows that $(HK) \int_c^b f \cdot dh$ exists. Since $\tau_1 \cup \tau_2 \in TP(\delta, [a, b])$ for each $\tau_1 \in TP(\delta_1, [a, c])$ and $\tau_2 \in TP(\delta_2, [c, b])$, (2.54) holds. \square

The Henstock–Kurzweil integral shares some elementary properties with other Stieltjes-type integrals considered above, as will be shown in Theorems 2.72 and 2.73. Essentially the next statement says that the HK integral exists if and only if corresponding improper integrals exist. Proposition 2.76 will show that this property is not shared by the RRS , RYS , or CY integral.

Proposition 2.58. *Assuming (1.14), let $f: [a, b] \rightarrow X$ and $h: [a, b] \rightarrow Y$ with $a < b$. The integral $(HK) \int_a^b f \cdot dh$ is defined if and only if the integral $(HK) \int_x^b f \cdot dh$ is defined for each $x \in (a, b)$ and the limit*

$$\lim_{x \downarrow a} \left\{ (HK) \int_x^b f \cdot dh + f(a) \cdot [h(x) - h(a)] \right\} \quad (2.55)$$

exists. Also, the integral $(HK) \int_a^b f \cdot dh$ is defined if and only if $(HK) \int_a^x f \cdot dh$ is defined for each $x \in (a, b)$ and the limit

$$\lim_{x \uparrow b} \left\{ (HK) \int_a^x f \cdot dh + f(b) \cdot [h(b) - h(x)] \right\}$$

exists. In either case, the limit and the integral over $[a, b]$ are equal.

Proof. We prove only the first part of Proposition 2.58 because a proof of the second part is symmetric. Suppose that $I := (HK) \int_a^b f \cdot dh$ is defined. Then $I(x) := (HK) \int_x^b f \cdot dh$ is defined for each $x \in (a, b]$ by Proposition 2.56. Given $\epsilon > 0$, let $\delta(\cdot)$ be a gauge function on $[a, b]$ such that any two Riemann–Stieltjes sums based on δ -fine tagged partitions differ by at most ϵ . For $x \in (a, a + \delta(a))$, by Proposition 2.56 and Lemma 2.53(a), there is a δ -fine tagged partition τ_x of $[a, x]$ such that the Riemann–Stieltjes sum based on τ_x is within ϵ of $(HK) \int_a^x f \cdot dh$. Also, let τ^x be any δ -fine tagged partition of $[x, b]$ and $\sigma_x := (\{a, x\}, a)$. Let τ_1 and τ_2 be two tagged partitions of $[a, b]$ which coincide with τ_x and σ_x , respectively, when restricted to $[a, x]$, and which both equal τ^x when restricted to $[x, b]$. Then τ_1 and τ_2 are δ -fine and

$$\begin{aligned} \|I - I(x) - f(a) \cdot [h(x) - h(a)]\| &\leq \left\| (HK) \int_a^x f \cdot dh - S_{RS}(f, dh; \tau_x) \right\| \\ &\quad + \|S_{RS}(f, dh; \tau_1) - S_{RS}(f, dh; \tau_2)\| \\ &< 2\epsilon. \end{aligned}$$

Since $x \in (a, a + \delta(a)]$ and $\epsilon > 0$ are arbitrary, the limit (2.55) exists and equals I . Thus the “only if” part of the proposition holds.

To prove the converse implication we need the following auxiliary statement, sometimes called Henstock’s lemma.

Lemma 2.59. *Suppose $a < b$ and $I := (HK) \int_a^b f \cdot dh$ is defined. Given $\epsilon > 0$, let $\delta(\cdot)$ be a gauge function on $[a, b]$ for I and ϵ , and let $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a δ -fine tagged partition of $[a, b]$. Then for any subset $J \subset \{1, \dots, n\}$,*

$$\left\| \sum_{i \in J} \left\{ f(y_i) \cdot [h(x_i) - h(x_{i-1})] - (HK) \int_{x_{i-1}}^{x_i} f \cdot dh \right\} \right\| \leq \epsilon. \quad (2.56)$$

Proof. Let $k := \text{card } J'$ for $J' := \{1, \dots, n\} \setminus J$, and let ϵ' be an arbitrary positive number. By Proposition 2.56, for each $i \in J'$, there is a δ -fine tagged partition τ_i of $[x_{i-1}, x_i]$ such that

$$\left\| S_{RS}(f, dh; \tau_i) - (HK) \int_{x_{i-1}}^{x_i} f \cdot dh \right\| < \epsilon'/k.$$

Let $\tau := \bigcup_{i \in J'} \tau_i \cup \bigcup_{i \in J} (\{x_{i-1}, x_i\}, y_i)$. Then τ is a δ -fine tagged partition of $[a, b]$, and the left side of (2.56) is equal to

$$\left\| S_{RS}(f, dh; \tau) - (HK) \int_a^b f \cdot dh - \sum_{i \in J'} \left\{ S_{RS}(f, dh; \tau_i) - (HK) \int_{x_{i-1}}^{x_i} f \cdot dh \right\} \right\|,$$

which is less than $\epsilon + \epsilon'$. Since ϵ' is arbitrary, (2.56) holds. \square

Continuation of the proof of Proposition 2.58. Suppose that the limit (2.55) exists. Let I be its value. Then $I(x) := (HK) \int_x^b f \cdot dh$ is defined for each $x \in (a, b)$.

Given $\epsilon > 0$, we will construct a gauge function $\delta(\cdot)$ on $(a, b]$ such that for each $x \in (a, b)$, $a + \delta(x) < x$ and $\|I(x) - S_{RS}(f, dh; \tau^x)\| \leq \epsilon$ for any δ -fine tagged partition τ^x of $[x, b]$. Suppose we have such a gauge function $\delta(\cdot)$ on $(a, b]$. Define $\delta(a)$ so that $\|I - I(x) - f(a) \cdot [h(x) - h(a)]\| < \epsilon$ for any $x \in (a, a + \delta(a)]$. Let $\tau = (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ be a δ -fine tagged partition of $[a, b]$. Then by construction of $\delta(\cdot)$, we have that $y_1 = a$, $\tau^{x_1} := (\{x_i\}_{i=1}^n, \{y_i\}_{i=2}^n)$ is a δ -fine tagged partition of $[x_1, b]$, and

$$\begin{aligned} \|I - S_{RS}(f, dh; \tau)\| &\leq \|I(x_1) - S_{RS}(f, dh; \tau^{x_1})\| \\ &\quad + \|I - I(x_1) - f(a) \cdot [h(x_1) - h(a)]\| \\ &< 2\epsilon. \end{aligned}$$

Thus the integral $(HK) \int_a^b f \cdot dh$ is defined and equals I .

To construct the desired gauge function $\delta(\cdot)$ on $(a, b]$, let $\{u_n: n \geq 0\}$ be a decreasing sequence of numbers such that $u_0 = b$ and $\lim_{n \rightarrow \infty} u_n = a$. Given $\epsilon > 0$, let $\delta_1(\cdot)$ be a gauge function on $[u_1, u_0]$ for $(HK) \int_{u_1}^{u_0} f \cdot dh$ and $\epsilon/2$. For $n \geq 2$, let $\delta_n(\cdot)$ be a gauge function on $[u_n, u_{n-2}]$ for $(HK) \int_{u_n}^{u_{n-2}} f \cdot dh$ and $\epsilon/2^n$. For $z \in (u_1, u_0]$ define $\delta(z)$ so that $0 < \delta(z) \leq \delta_1(z)$ and $u_1 < z - \delta(z)$. For $z \in (u_n, u_{n-1}]$ with $n > 1$, define $\delta(z)$ so that $0 < \delta(z) \leq \delta_n(z)$ and $u_n < z - \delta(z) < z + \delta(z) < u_{n-2}$. Thus $\delta(\cdot)$ is defined on $(a, b]$. Let $x \in (a, b)$ and let $\tau^x = (\{x_i\}_{i=0}^m, \{y_i\}_{i=1}^m)$ be a δ -fine tagged partition of $[x, b]$. For each $n \geq 1$, let $I_n := \{i: y_i \in (u_n, u_{n-1}]\}$, and let $J_n := \cup_{i \in I_n} [x_{i-1}, x_i]$. By construction, τ^x restricted to J_n is a δ_n -fine tagged partition of J_n . Since $J_1 \subset (u_1, u_0]$ and $J_n \subset (u_n, u_{n-2})$ for each $n \geq 2$, by Lemma 2.59, for each $n \geq 1$,

$$\left\| \sum_{i \in I_n} \left\{ f(y_i) \cdot [h(x_i) - h(x_{i-1})] - (HK) \int_{x_{i-1}}^{x_i} f \cdot dh \right\} \right\| \leq \frac{\epsilon}{2^n}.$$

Thus

$$\left\| (HK) \int_x^b f \cdot dh - S_{RS}(f, dh; \tau^x) \right\| \leq \sum_{n=1}^{\infty} \epsilon/2^n = \epsilon.$$

As noted earlier, this completes the proof of Proposition 2.58. \square

Proposition 2.60. *Let h be a real-valued function on $[a, b]$. If $(HK) \int_a^b f \cdot dh$ exists whenever $f = 1_J$ for an interval J , then h is regulated on $[a, b]$.*

Proof. Suppose not. Then we can assume that for some $u_k \downarrow c \in [a, b)$, the $h(u_k)$ do not converge. If $h(u_k)$ are unbounded let $\epsilon = 1$. If they are bounded let $\epsilon := 3^{-1}[\limsup_{k \rightarrow \infty} h(u_k) - \liminf_{k \rightarrow \infty} h(u_k)]$. Let $f_c := 1_{(c, b]}$. Take a gauge function $\delta(\cdot)$ on $[a, b]$ for ϵ and $(HK) \int_a^b f_c \cdot dh$. By Lemma 2.57, there exist a gauge function $\delta' \leq \delta$ and a δ' -fine tagged partition τ of $[a, b]$ with c as the tag for an interval $[x_{i-1}, x_i]$ and $c < x_i$. For all k large enough we have $u_k < x_i$. Then take a δ' -fine tagged refinement τ_k of τ where c is the tag for $[x_{i-1}, u_k]$, by Lemma 2.53(a) for $[u_k, x_i]$. The corresponding Riemann–Stieltjes sums

$$S_{RS}(f_c, dh; \tau_k) = h(b) - h(u_k)$$

vary by more than 2ϵ as $k \rightarrow \infty$, a contradiction. \square

Suppose a continuous function $f: [a, b] \mapsto \mathbb{R}$ is differentiable everywhere on (a, b) . It may be that f' is not Lebesgue integrable, e.g. if $[a, b] = [0, 1]$, $f(x) = x^2 \sin(\pi/x^2)$, $0 < x < 1$, $f(x) = 0$ elsewhere. Between 1910 and 1920, Denjoy and Perron (see the Notes) defined integrals such that for every such f' , $\int_a^x f'(t) dt = f(x) - f(a)$, $a \leq x \leq b$. The next fact shows that the later-invented Henstock–Kurzweil integral has this property:

Theorem 2.61. *Let f be a continuous function from $[a, b]$ to \mathbb{R} having a derivative $f'(x)$ for $a < x < b$ except for at most countably many x . Then $(HK) \int_a^x f'(t) dt$ exists and equals $f(x) - f(a)$ for $a \leq x < b$.*

Proof. The statement follows from Theorems 7.2 and 11.1 of Gordon [84]. \square

For another example, $f(x) := x^2 \sin(e^{1/x})$, $x \neq 0$, $f(0) := 0$, satisfies the conditions of Theorem 2.61 although f' has very wild oscillations.

*2.7 Ward–Perron–Stieltjes and Henstock–Kurzweil Integrals

Before defining the Ward–Perron–Stieltjes integral we need to define what are called major and minor functions. Given two real-valued functions f, h on $[a, b]$, we call M a *major function* of f with respect to h if $M(a) = 0$, M has finite values on $[a, b]$, and for each $x \in [a, b]$ there exists $\delta(x) > 0$ such that

$$\begin{aligned} M(z) &\geq M(x) + f(x)[h(z) - h(x)] & \text{if } x \leq z \leq \min\{b, x + \delta(x)\}, \\ M(z) &\leq M(x) + f(x)[h(z) - h(x)] & \text{if } \max\{a, x - \delta(x)\} \leq z \leq x. \end{aligned} \quad (2.57)$$

Thus for a major function M there exists a positive function $\delta_M(\cdot) = \delta(\cdot)$ on $[a, b]$ which satisfies (2.57). Let $\mathcal{U}(f, h)$ be the class of all major functions of f with respect to h , and let

$$U(f, h) := \begin{cases} \inf\{M(b) : M \in \mathcal{U}(f, h)\} & \text{if } \mathcal{U}(f, h) \neq \emptyset, \\ +\infty & \text{if } \mathcal{U}(f, h) = \emptyset. \end{cases}$$

A function m is a *minor function* of f with respect to h if $-m \in \mathcal{U}(-f, h)$. Let $\mathcal{L}(f, h)$ be the class of all minor functions of f with respect to h . Thus $m \in \mathcal{L}(f, h)$ provided $m(a) = 0$, m has finite values on $[a, b]$, and there exists a positive function $\delta(\cdot) = \delta_m(\cdot)$ on $[a, b]$ such that

$$\begin{aligned} m(z) &\leq m(x) + f(x)[h(z) - h(x)] & \text{if } x \leq z \leq \min\{b, x + \delta(x)\}, \\ m(z) &\geq m(x) + f(x)[h(z) - h(x)] & \text{if } \max\{a, x - \delta(x)\} \leq z \leq x. \end{aligned} \quad (2.58)$$

Let

$$L(f, h) := \begin{cases} \sup\{m(b) : m \in \mathcal{L}(f, h)\} & \text{if } \mathcal{L}(f, h) \neq \emptyset, \\ -\infty & \text{if } \mathcal{L}(f, h) = \emptyset. \end{cases}$$

Then the following is true:

Lemma 2.62. $L(f, h) \leq U(f, h)$.

Proof. The statement is true if either of the sets $\mathcal{L}(f, h)$ or $\mathcal{U}(f, h)$ is empty. Suppose that $\mathcal{L}(f, h)$ and $\mathcal{U}(f, h)$ are nonempty. Let $m \in \mathcal{L}(f, h)$ and $M \in \mathcal{U}(f, h)$. Also, let $\delta := \delta_m \wedge \delta_M$ and $w(x) := M(x) - m(x)$ for $x \in [a, b]$. By (2.57) and (2.58), for each $x \in (a, b]$, it follows that

$$\begin{aligned} w(z) &\geq w(x) & \text{if } x \leq z \leq \min\{b, x + \delta(x)\}, \\ w(z) &\leq w(x) & \text{if } \max\{a, x - \delta(x)\} \leq z \leq x. \end{aligned}$$

Therefore $\inf \{M(x) : M \in \mathcal{U}(f, h)\} - \sup \{m(x) : m \in \mathcal{L}(f, h)\}$ is a non-decreasing function of x . Then the statement of the lemma holds because $m(a) = M(a) = 0$. \square

Now we are ready to define the Ward–Perron–Stieltjes integral.

Definition 2.63. If $U(f, h) = L(f, h)$ is finite then the common value will be denoted by $(WPS) \int_a^b f \, dh$ and called the *Ward–Perron–Stieltjes integral*, or the *WPS integral*.

Note that if $a = b$ then $U(f, h) = L(f, h) = 0$, and so the integral $(WPS) \int_a^a f \, dh$ equals 0. First we show that the Ward–Perron–Stieltjes integral extends the refinement Riemann–Stieltjes integral.

Theorem 2.64. *If $(RRS) \int_a^b f \, dh$ exists then so does $(WPS) \int_a^b f \, dh$, and the two are equal.*

Proof. We can assume that $a < b$. For $a \leq u < v \leq b$, let $M(u, v)$ be the supremum of all Riemann–Stieltjes sums $S_{RS}(f, dh; \tau)$ based on tagged partitions τ of $[u, v]$, with $M(u, u) \equiv 0$. Then

$$\begin{aligned} M(u, y) &\geq M(u, x) + f(x)[h(y) - h(x)] & \text{if } u \leq x \leq y, \\ M(u, y) &\leq M(u, x) + f(x)[h(y) - h(x)] & \text{if } u \leq y \leq x. \end{aligned} \quad (2.59)$$

Suppose $(RRS) \int_a^b f \, dh$ exists and $\epsilon > 0$. Then there exists $\lambda = \{z_j : j = 0, \dots, m\} \in \text{PP}[a, b]$ such that

$$(RRS) \int_a^b f \, dh - \epsilon < S_{RS}(f, dh; \tau) < (RRS) \int_a^b f \, dh + \epsilon \quad (2.60)$$

for each tagged refinement τ of λ . For each $x \in (a, b]$, let $j(x)$ be the largest integer such that $z_{j(x)} \leq x$. Let $M(a) := 0$, and for each $x \in (a, b]$ let

$$M(x) := \sum_{j=1}^{j(x)} M(z_{j-1}, z_j) + M(z_{j(x)}, x),$$

where the sum over the empty set is 0. Define a positive function $\delta(\cdot)$ on $[a, b]$ by $\delta(x) := \min\{x - z_{j(x)}, z_{j(x)+1} - x\}$ if $x \in (z_{j(x)}, z_{j(x)+1})$, $\delta(a) := z_1 - a$,

$\delta(b) := b - z_{m-1}$, and $\delta(z_j) := \min\{z_j - z_{j-1}, z_{j+1} - z_j\}$ for $j = 1, \dots, m-1$. By (2.59), the functions M and δ so defined satisfy (2.57). Hence M is a major function of f with respect to h . By (2.60), it follows that $M(b) \leq (RRS) \int_a^b f \, dh + \epsilon$. Similarly one can define a minor function m such that $m(b) \geq (RRS) \int_a^b f \, dh - \epsilon$. Since ϵ is arbitrarily small, this proves the theorem. \square

Next is the main result of this section, which says that the Henstock–Kurzweil and Ward–Perron–Stieltjes integrals coincide.

Theorem 2.65. *The integral $(WPS) \int_a^b f \, dh$ exists if and only if $(HK) \int_a^b f \, dh$ exists, and then their values are equal.*

Proof. We can assume that $a < b$. Suppose $(WPS) \int_a^b f \, dh$ is defined. Then given $\epsilon > 0$, there exist a major function $M \in \mathcal{U}(f, h)$ and a minor function $m \in \mathcal{L}(f, h)$ such that

$$M(b) - \epsilon < (WPS) \int_a^b f \, dh < m(b) + \epsilon. \quad (2.61)$$

Let $\delta := \delta_M \wedge \delta_m$ and let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be any δ -fine tagged partition of $[a, b]$, which exists by Lemma 2.53(a). By (2.57), we have

$$\begin{aligned} M(t_i) - M(s_i) &\geq f(s_i)[h(t_i) - h(s_i)], \\ M(s_i) - M(t_{i-1}) &\geq f(s_i)[h(s_i) - h(t_{i-1})] \end{aligned} \quad (2.62)$$

for each $i = 1, \dots, n$. Adding up the $2n$ inequalities (2.62), we get the upper bound $M(b) \geq S_\delta(\tau)$ for the Riemann–Stieltjes sum $S_\delta(\tau) := S_{RS}(f, dh; \tau)$. Similarly, using (2.58), we get the lower bound $S_\delta(\tau) \geq m(b)$. By (2.61), it then follows that

$$(WPS) \int_a^b f \, dh - \epsilon < S_\delta(\tau) < (WPS) \int_a^b f \, dh + \epsilon.$$

Thus $(HK) \int_a^b f \, dh$ exists and has the same value as the (WPS) integral.

Now suppose $(HK) \int_a^b f \, dh$ is defined. Given $\epsilon > 0$ there exists a gauge function $\delta(\cdot)$ such that for each δ -fine tagged partition τ , the Riemann–Stieltjes sum $S_\delta(\tau) := S_{RS}(f, dh; \tau)$ has bounds

$$(HK) \int_a^b f \, dh - \epsilon < S_\delta(\tau) < (HK) \int_a^b f \, dh + \epsilon. \quad (2.63)$$

For each $x \in (a, b]$, let $\delta_x(\cdot)$ be the gauge function $\delta(\cdot)$ restricted to the interval $[a, x]$, and $m(x) := \inf \{S_{\delta_x}(\tau) : \tau \in TP(\delta_x, [a, x])\}$, $M(x) := \sup \{S_{\delta_x}(\tau) : \tau \in TP(\delta_x, [a, x])\}$. Let $m(a) = M(a) := 0$. Then $m(x)$ and $M(x)$ have finite values for each $x \in (a, b]$. Let $x \in (a, b)$ and $z \in (x, x + \delta(x) \wedge$

$b]$. Then for each $\tau = (\{a = t_0, \dots, t_m = x\}, \{s_i\}_{i=0}^m) \in TP(\delta_x, [a, x])$, $(\{t_0, \dots, t_m, z\}, \{s_1, \dots, s_m, x\}) \in TP(\delta_z, [a, z])$. Thus

$$S_{\delta_x}(\tau) + f(x)[h(z) - h(x)] \leq M(z)$$

for any $\tau \in TP(\delta_x, [a, x])$. Thus the first inequality in (2.57) holds. Similarly one can show that the second one and (2.58) also hold. Therefore m and M are minor and major functions respectively. By Lemma 2.62, the definitions of $L(f, h)$ and $U(f, h)$, and (2.63), it then follows that

$$(HK) \int_a^b f \, dh - \epsilon \leq m(b) \leq L(f, h) \leq U(f, h) \leq M(b) \leq (HK) \int_a^b f \, dh + \epsilon.$$

Since $\epsilon > 0$ is arbitrary, $L(f, h) = U(f, h)$ and equals the HK integral. The proof of Theorem 2.65 is complete. \square

Next we show that for integrators in the class $c_0((a, b)) = c_0((a, b), [a, b]; \mathbb{R})$ with $a < b$ (cf. Definition 2.9), the Henstock–Kurzweil and Ward–Perron–Stieltjes integrals differ from the refinement Young–Stieltjes, central Young, and Kolmogorov integrals.

Proposition 2.66. *For $f \in \mathcal{R}[a, b]$ and any $h \in c_0((a, b))$ if $a < b$ or $h \in \mathcal{R}[a, a]$, i.e. h is any function $\{a\} \rightarrow \mathbb{R}$ if $a = b$,*

$$(RYS) \int_a^b f \, dh = (CY) \int_a^b f \, dh = \int_{[a, b]} f \, d\mu_{h, [a, b]} = 0.$$

Proof. This follows from the definitions of the integrals because $h_-^{(a)} \equiv h_+^{(b)} \equiv \Delta_{[a, b]}^\pm h \equiv \mu_{h, [a, b]} \equiv 0$ if $a < b$ and $\mu_{h, [a, a]} \equiv 0$. \square

We will show in Proposition 2.68 that the analogous statement for the Henstock–Kurzweil integral does not hold.

Lemma 2.67. *For each $h \in c_0((a, b))$ with $a < b$ there exists a right-continuous $f \in \mathcal{R}[a, b]$ such that $\Delta^- f \equiv h$ on (a, b) .*

Proof. Let $h = \sum_i c_i 1_{[\xi_i]}$. For each $i \geq 1$, let $f_i(x) := 0$ if $x \in [a, \xi_i) \cup [\xi_i + \delta_i, b]$ for some $\delta_i > 0$, let $f_i(\xi_i) := c_i$, and let f_i be linear on $[\xi_i, \xi_i + \delta_i]$. If $\delta_i \rightarrow 0$ fast enough as $i \rightarrow \infty$ so that each $(\xi_i, \xi_i + \delta_i)$ contains no ξ_j with $|c_j| > |c_i|/2$, then $f := \sum_i f_i$ converges uniformly on $[a, b]$ and f has the stated properties. \square

Proposition 2.68. *There exist $h \in c_0((0, 1))$ and $f \in \mathcal{R}[0, 1]$ such that $(HK) \int_0^1 f \, dh$ is undefined.*

Proof. For k odd, $k = 1, 3, \dots, 2^n - 1$, $n = 1, 2, \dots$, let $h(k/2^n) := 1/2^{n/3}$ and $h = 0$ elsewhere. Then $h \in c_0((0, 1))$. By Lemma 2.67, take a right-continuous $f \in \mathcal{R}[0, 1]$ such that $\Delta^- f(k/2^n) = 1/2^{n/3}$ for $k = 1, 3, \dots, 2^n - 1$ and $n = 1, 2, \dots$.

Let $\delta(\cdot)$ be any strictly positive function on $[0, 1]$. By the category theorem (e.g. Theorem 2.5.2 in Dudley [53]), there is an interval $J = [c, d] \subset [0, 1]$ of length ϵ for some $\epsilon > 0$ such that $\delta(y) > \epsilon$ for a dense set S of y in J , and the endpoints of J are not dyadic rationals. So, all tagged partitions of J with tags in S are δ -fine.

For $r = 1, 2, \dots$, let $M(r)$ be the set of all binary rationals $k/2^n$ in J for integers n , $1 \leq n \leq r$, and odd integers $k \geq 1$. As $n \rightarrow +\infty$, the number of such $k/2^n$ for a fixed n is asymptotic to $2^{n-1}\epsilon$. For a given r write $M(r) = \{x_{2j-1}\}_{j=1}^m$ where $x_1 < x_3 < \dots < x_{2m-1}$ for some $m = m(r)$. For each $j \in \{1, \dots, m-1\}$, choose $x_{2j} \in (x_{2j-1}, x_{2j+1})$ which is not a dyadic rational, and let $x_0 := c$, $x_{2m} := d$, recalling that $J = [c, d]$. For each $j \in \{1, \dots, m\}$, choose $y_{2j-1} \in S \cap (x_{2j-2}, x_{2j-1})$ and $y_{2j} \in S \cap (x_{2j-1}, x_{2j})$ close enough to x_{2j-1} so that $|[f(y_{2j}) - f(y_{2j-1}) - \Delta^- f(x_{2j-1})]h(x_{2j-1})| < 1/|M(r)|$, where $|M(r)| = \text{card}(M(r))$. Then $\zeta_r := (\{x_j\}_{j=0}^{2m}, \{y_j\}_{j=1}^{2m})$ is a δ -fine tagged partition of J and

$$\left| -S_{RS}(f, dh; \zeta_r) - \sum_{j=1}^m \Delta^- f(x_{2j-1})h(x_{2j-1}) \right| < 1.$$

Since the latter sums are unbounded as r increases, so are the former Riemann–Stieltjes sums. Then combining ζ_r with a δ -fine tagged partition of $[0, c]$ (if $0 < c$) and a δ -fine tagged partition of $[d, 1]$ (if $d < 1$), which exist by Lemma 2.53(a), we get unbounded Riemann–Stieltjes sums for δ -fine tagged partitions of $[0, 1]$. Since the gauge function $\delta(\cdot)$ on $[0, 1]$ is arbitrary, $(HK) \int_0^1 f dh$ does not exist. \square

Under some restrictions on the integrator, the Henstock–Kurzweil integral will be shown to extend the refinement Young–Stieltjes integral. Note that in the following, h may be a function $R_{\mu, a}$ as in Proposition 2.6(f), right-continuous on (a, b) but not necessarily at a , so the RYS integral can be replaced by the Kolmogorov integral according to Corollary 2.26.

Theorem 2.69. *Suppose f and h are regulated on $[a, b]$, and if $a < b$, h is right-continuous or left-continuous on (a, b) . If $(RYS) \int_a^b f dh$ exists then $(HK) \int_a^b f dh$ exists and has the same value.*

Proof. We can assume that $a < b$ and h is right-continuous on (a, b) by symmetry. Given $\epsilon > 0$, let $\lambda = \{z_j\}_{j=0}^m$ be a partition of $[a, b]$ such that for any Young tagged refinement τ of λ ,

$$\left| S_{YS}(f, dh; \tau) - (RYS) \int_a^b f dh \right| < \epsilon. \quad (2.64)$$

Recall $\Delta_{(a,b)}^+ f$ defined before (2.1) and $f_+^{(a,b)}$ defined in (2.46). Then $\Delta_{(a,b)}^+ f \equiv f_+^{(a,b)} - f$ on $[a, b]$. By Lemma 2.49 we can assume that (2.64) holds also for f replaced by $f_+^{(a,b)}$ or by $\Delta_{(a,b)}^+ f$, taking λ as a common refinement $\lambda = \lambda' \cup \lambda''$ of partitions λ' for $f_+^{(a,b)}$ and $\epsilon/2$, and λ'' for $\Delta_{(a,b)}^+ f$ and $\epsilon/2$. So it suffices to prove the theorem for $f_+^{(a,b)}$ and for $\Delta_{(a,b)}^+ f$.

Define a gauge function $\delta(\cdot)$ on $[a, b]$ as follows: if $t \notin \lambda$, let $\delta(t) := \min_j |t - z_j|/2$. Then any $\delta(\cdot)$ -fine tagged partition must contain each z_j as a tag. For $j = 0, \dots, m$, define $\delta(z_j)$ such that $\delta(z_j) \leq \min_{i \neq j} |z_i - z_j|/3$ and

$$\|f\|_\infty \sum_{j=1}^m \left\{ \text{Osc}(h; [z_j - \delta(z_j), z_j]) + \text{Osc}(h; (z_{j-1}, z_{j-1} + 2\delta(z_{j-1})]) \right\} < \epsilon. \quad (2.65)$$

Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be any $\delta(\cdot)$ -fine tagged partition of $[a, b]$ and let

$$S_\delta(\tau) := S_{RS}(f, dh; \tau) = \sum_{i=1}^n f(s_i) [h(t_i) - h(t_{i-1})] \quad (2.66)$$

be the Riemann–Stieltjes sum based on τ . We can assume that $s_i = t_i = s_{i+1}$ never occurs, since if it did, the i th and $(i+1)$ st term in $S_\delta(\tau)$ could be replaced by $f(s_i)[h(t_{i+1}) - h(t_{i-1})]$, and we would still have a Riemann–Stieltjes sum based on a $\delta(\cdot)$ -fine tagged partition equal to $S_\delta(\tau)$. Since the tags $\{s_i\}_{i=1}^n$ must contain the points of λ , for each index $j \in \{0, \dots, m\}$ there is an index $i(j) \in \{1, \dots, n\}$ such that $z_j = s_{i(j)}$. Let $\mu := \{i(j) : j = 0, \dots, m\} \subset \{1, \dots, n\}$. For $j = 1, \dots, m-1$, by definition of $\delta(\cdot)$, we must have

$$t_{i(j)-1} < s_{i(j)} < t_{i(j)}. \quad (2.67)$$

We also have $i(0) = 1$, $s_1 = z_0 = t_0 = a$ and $i(m) = n$, $s_n = z_m = t_n = b$.

Now consider the case that f is right-continuous on (a, b) , which holds when $f = f_+^{(a,b)}$. We will show that one can find a Young–Stieltjes sum, based on a refinement of λ , which is arbitrarily close to the Riemann–Stieltjes sum $S_\delta(\tau)$. To this aim we will replace the values $h(t_i)$ in $S_\delta(\tau)$, $i = 1, \dots, n-1$, by values $h(x_i)$ at continuity points x_i of h , close to t_i . Then the Young–Stieltjes sum terms $f(x_i)\Delta^\pm h(x_i)$ will be 0. Define a Riemann–Stieltjes sum T by, for each $i = 1, \dots, n-1$ such that $t_i < s_{i+1}$, replacing t_i in $S_\delta(\tau)$ by a slightly larger $x_i > t_i$ which is a continuity point of h with $x_i < s_{i+1}$. Then $|T - S_\delta(\tau)| < \epsilon$. Define another Riemann–Stieltjes sum U equal to T except that for each i such that $s_{i+1} = t_i$ and $1 \leq i \leq n-1$, we replace s_{i+1} by y_{i+1} and t_i by x_i with $t_i < x_i < y_{i+1} < t_{i+1}$, where x_i is a continuity point of h and $y_{i+1} - t_i$ is as small as desired. Since f is right-continuous on (a, b) , we can make $|U - T| < \epsilon$. In either case we can assume that $s_i \leq t_i < x_i \leq s_{i+1} + 2\delta(s_i)$ for $i = 1, \dots, n-1$. Let $x_0 := a$ and $x_n := b$. Let $y_i := s_i$ for each $i = 2, \dots, n-1$ for which y_i was not previously defined. Thus $y_{i(j)} = s_{i(j)}$ for

$j = 1, \dots, m-1$ by (2.67). Choose any $y_1 \in (a, t_1)$ and $y_n \in (x_{n-1}, b)$. Let $V := S_{YS}(f, dh; \zeta)$ be the Young–Stieltjes sum based on the Young tagged point partition $\zeta := (\xi, \eta) := (\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$. Then

$$\begin{aligned} |V - U| &= |f(a)\Delta^+ h(a) + f(y_1)[h(x_1) - h(a+)] - f(a)[h(x_1) - h(a)] \\ &\quad + f(b)\Delta^- h(b) + f(y_n)[h(b-) - h(x_{n-1})] - f(b)[h(b) - h(x_{n-1})]| \\ &= |[f(y_1) - f(a)][h(x_1) - h(a+)] + [f(y_n) - f(b)][h(b-) - h(x_{n-1})]| < 2\epsilon \end{aligned}$$

by (2.65). Now let $\kappa := \lambda \cup \xi = \{u_l\}_{l=0}^k$, a partition of $[a, b]$ with $k = m+n-1$. Each interval (x_{i-1}, x_i) for $i = 2, \dots, n-1$ contains z_j if $i = i(j) \in \mu$, and otherwise contains no z_j and becomes an interval (u_{l-1}, u_l) for some l . In the latter case let $v_l := y_i$. In the former case we have for some l , $u_{l-2} = x_{i-1} < z_j = u_{l-1} < x_i = u_l$. Choose any $v_{l-1} := w_j$ with $u_{l-2} < v_{l-1} < u_{l-1}$ and $v_l := r_j$ with $u_{l-1} < v_l < u_l$. Let $W := S_{YS}(f, dh; (\kappa, \sigma))$ be the Young–Stieltjes sum based on the Young tagged point partition (κ, σ) , where $\sigma := \{v_l\}_{l=1}^k$. Then

$$\begin{aligned} |W - V| &= \left| \sum_{j=1}^{m-1} \left\{ f(w_j)[h(z_j-) - h(x_{i(j)-1})] + f(z_j)\Delta^- h(z_j) \right. \right. \\ &\quad \left. \left. + f(r_j)[h(x_{i(j)-}) - h(z_j)] - f(z_j)[h(x_{i(j)-}) - h(x_{i(j)-1})] \right\} \right| \\ &\leq \sum_{j=1}^{m-1} |[f(w_j) - f(z_j)][h(z_j-) - h(x_{i(j)-1})]| \\ &\quad + \sum_{j=1}^{m-1} |[f(r_j) - f(z_j)][h(x_{i(j)-}) - h(z_j)]| < 2\epsilon \end{aligned}$$

by (2.65). Here W is a Young–Stieltjes sum based on the Young tagged refinement (κ, σ) of λ , so $|W - (RYS) \int_a^b f \, dh| < \epsilon$. Thus $|S_\delta(\tau) - (RYS) \int_a^b f \, dh| < 7\epsilon$. The conclusion follows when f is right-continuous on (a, b) .

Next consider the case that $f \in c_0((a, b))$ (cf. Definition 2.9), which holds when $f = \Delta_{(a,b)}^+ f$. There is a set of tags $\{w_j\}_{j=1}^m$ for the partition $\lambda = \{z_j\}_{j=0}^m$ such that $(\lambda, \{w_j\}_{j=1}^m)$ is a Young tagged point partition of $[a, b]$ and $f(w_j) = 0$ for all j . Thus by (2.64),

$$\left| \sum_{j=1}^{m-1} f(z_j)\Delta^- h(z_j) - (RYS) \int_a^b f \, dh \right| < \epsilon. \quad (2.68)$$

By (2.67), $t_i \in \lambda$ only for $i = 0$ and n . For any set $\nu \subset \{1, \dots, n-1\}$, consider partitions $\kappa = \{u_l\}_{l=0}^k$, $k = m+n-1$, of $[a, b]$ consisting of λ , t_i for $i \in \nu$, and for each $i \in \{1, \dots, n-1\} \setminus \nu$, a continuity point of h close to t_i . Let $l(1) < \dots < l(n-1)$ be such that $\{u_{l(i)}\}_{i=1}^{n-1}$ are the u_l not in λ . Let

$\sigma = \{v_l\}_{l=1}^k$ be a set of tags for κ such that (κ, σ) is a Young point partition of $[a, b]$ and $f(v_l) = 0$ for all l . Then from (2.68) and (2.64) we obtain

$$\left| \sum_{i \in \nu} f(t_i) \Delta^- h(t_i) \right| < 2\epsilon. \quad (2.69)$$

Consider also partitions (κ, σ) defined in the same way except that for $\xi \subset \{1, \dots, n-1\}$, if $i \in \xi \setminus \mu$ and i is even, $i \notin \mu$ implies $l(i-1) = l(i) - 1$. We take $u_l = u_{l(i)}$ to be a continuity point of h a little larger than t_i , while $u_{l(i)-1}$ is a continuity point of h a little smaller than t_{i-1} and $v_{l(i)} = t_{i-1}$. For each $i = 1, \dots, n-1$ such that $u_{l(i)}$ is not yet defined, let it be a continuity point of h near t_i , which is then true for all $i = 1, \dots, n-1$. Let $f(v_l) = 0$ for other l as before. For even $i \in \xi \setminus \mu$, letting $u_{l(i)} \downarrow t_i$ and $u_{l(i)-1} \uparrow t_{i-1}$, it follows that

$$\left| \sum \{f(t_{i-1})[h(t_i) - h(t_{i-1}-)]: i \in \xi \setminus \mu, i \text{ even}\} \right| \leq 2\epsilon.$$

The same holds likewise for i odd. Thus

$$\left| \sum_{i \in \xi \setminus \mu} f(t_{i-1})[h(t_i) - h(t_{i-1}-)] \right| \leq 4\epsilon.$$

With $\nu = \{i-1: i \in \xi \setminus \mu\}$ in (2.69) this gives

$$\left| \sum_{i \in \xi \setminus \mu} f(t_{i-1})[h(t_i) - h(t_{i-1})] \right| \leq 6\epsilon.$$

If $s_i = t_{i-1}$ then $i \notin \mu$ by (2.67). Thus

$$\left| \sum \{f(t_{i-1})[h(t_i) - h(t_{i-1})]: t_{i-1} = s_i, i = 2, \dots, n-1\} \right| \leq 6\epsilon.$$

Here $f(t_{i-1})$ could be replaced by $f(t_{i-1}) - f(y_i)$, where $t_{i-1} < y_i < t_i$ and $f(y_i) = 0$. So the Riemann–Stieltjes sum (2.66) differs by at most 6ϵ from the Riemann–Stieltjes sum

$$S' := \sum_{i=1}^n f(w_i)[h(t_i) - h(t_{i-1})],$$

where $w_i := y_i$ if $t_{i-1} = s_i$ and $i = 2, \dots, n-1$, $w_i := s_i$ otherwise. Then $t_{i-1} < w_i \leq t_i$ for $i = 2, \dots, n-1$. The rest of the proof follows as in the case when f is right-continuous and $t_i < s_{i+1}$ for all $i = 1, \dots, n-1$, so that $U = T$ and $y_i = s_i$ for all $i = 2, \dots, n-1$. The proof of Theorem 2.69 is complete. \square

The following gives a partial converse to the preceding theorem. We say that the Young–Stieltjes sums for f and h are *unbounded on any subinterval of $[a, b]$* with $a < b$ if for all $a \leq c < d \leq b$, $\sup\{|S_{YS}(f, dh; \tau)|\} = +\infty$, where the supremum is taken over all Young tagged partitions τ of $[c, d]$. Then the sums based on partitions which are refinements of any given partition of $[c, d]$ are also unbounded.

Proposition 2.70. *Let $a < b$. Given f left- or right-continuous on (a, b) and $h \in \mathcal{R}[a, b]$, suppose that the Young–Stieltjes sums for f and h are unbounded on any subinterval of $[a, b]$. Then $(HK) \int_a^b f dh$ does not exist.*

Proof. Suppose that $(HK) \int_a^b f dh$ exists. Let $\delta(\cdot)$ be a gauge function and I a number such that for any δ -fine tagged partition $(\{x_i\}_{i=0}^n, \{y_i\}_{i=1}^n)$ of $[a, b]$,

$$\left| \sum_{i=1}^n f(y_i)[h(x_i) - h(x_{i-1})] - I \right| < 1. \quad (2.70)$$

By the category theorem (e.g. Theorem 2.5.2 in Dudley [53]), there are an interval $[c, d]$ and a positive integer m such that $\delta(x) > 1/m$ for all x in a dense set S in $[c, d]$. Replacing $[c, d]$ by a subinterval of itself, we can assume that $0 < d - c < 1/m$ and that $c, d \in S$. Thus every tagged partition $(\{\xi_i\}_{i=0}^k, \{\eta_j\}_{j=1}^k)$ of $[c, d]$ with tags $\eta_j \in S$ for $j = 1, \dots, k$ is δ -fine. Take a δ -fine tagged partition as in (2.70), where we can take some $y_i = c < y_\ell = d$ by Lemma 2.53(c). We can also take $x_i = c$ and $x_\ell = d$, as follows. If initially $y_i = c < x_i$, then let $x'_r := x_r$ for $r = 0, 1, \dots, i-1$ and $y'_r := y_r$ for $r = 1, \dots, i-1$. Let $x'_i := y'_i := c$ and for $r = i, \dots, n$, let $y'_{r+1} := y_r$ and $x'_{r+1} := x_r$. Then $(\{x'_i\}_{i=0}^{n+1}, \{y'_i\}_{i=1}^{n+1})$ is a δ -fine tagged partition of $[a, b]$ giving the same sum as in (2.70). We can take d as some x_ℓ likewise. Let κ be the resulting δ -fine tagged partition of $[a, b]$. We show next that the part of κ corresponding to $[c, d]$ can be replaced by a tagged partition of $[c, d]$ such that the corresponding Riemann–Stieltjes sum is arbitrarily large and the resulting tagged partition of $[a, b]$ is δ -fine.

For any $M < \infty$, by assumption, there is a Young tagged partition $\tau = (\{t_j\}_{j=0}^k, \{s_j\}_{j=1}^k)$ of $[c, d]$ such that $|S_{YS}(f, dh; \tau)| > M$. Since h is regulated, there exist $v_j < t_j$, $j = 1, \dots, k$, and $u_j > t_j$, $j = 0, \dots, k-1$, such that $t_{j-1} < u_{j-1} < s_j < v_j < t_j$ for $j = 1, \dots, k$ and

$$\begin{aligned} & |f(c)[h(u_0) - h(c)] + \sum_{j=1}^{k-1} f(t_j)[h(u_j) - h(v_j)] \\ & + \sum_{j=1}^k f(s_j)[h(v_j) - h(u_{j-1})] + f(d)[h(d) - h(v_k)]| > M. \end{aligned}$$

Now since f is right- or left-continuous, and S is dense, we can replace s_j by some $s'_j \in S$ as close to s_j as desired, making a small change in the sum. Likewise we can replace t_j by some $t'_j \in S$, where $u_{j-1} < s'_j < v_j < t'_j < u_j$ for

each $j = 1, \dots, k-1$. Recall that $c, d \in S$; the endpoints t_0, t_k are not replaced. We thus obtain a Riemann–Stieltjes sum Σ for a δ -fine tagged partition ζ of $[c, d]$ with $|\Sigma| > M$. Then joining ζ with the δ -fine tagged partitions of $[a, c]$ (if $a < c$) and $[d, b]$ (if $d < b$) given by the fixed tagged partition κ , we get unbounded Riemann–Stieltjes sums for δ -fine tagged partitions of $[a, b]$, contradicting (2.70). \square

Ward [238] stated and Saks [201, Theorem VI.8.1] gave a proof of the fact that the integral $(WPS) \int_a^b f \, dh$ is defined provided the corresponding Lebesgue–Stieltjes integral $(LS) \int_a^b f \, dh$ is defined, and then they are equal. By Theorem 2.65, the Henstock–Kurzweil integral is in the same relation with the LS integral. The following gives conditions under which the converse holds.

Theorem 2.71. *Let f be nonnegative on $[a, b]$ and let h be nondecreasing on $[a, b]$ and right-continuous on $[a, b)$. Then $(HK) \int_a^b f \, dh$ exists if and only if $(LS) \int_a^b f \, dh$ does, and then the two are equal.*

Proof. If $a = b$ both integrals are 0, so we can assume $a < b$. The McShane integral is defined as the HK integral except that in a tagged partition $(\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ the tags s_i need not be in the corresponding intervals $[t_{i-1}, t_i]$. By Corollary 6.3.5 of Pfeffer [185, p. 113], under the present hypotheses $(HK) \int_a^b f \, dh$ exists if and only if the corresponding McShane integral exists, and then the two are equal. The equivalence between the McShane and Lebesgue–Stieltjes integrals when $f \geq 0$ and h is nondecreasing was proved by McShane [167, pp. 552, 553]. Also, it follows from Theorem 4.4.7, Proposition 3.6.14, and Theorem 2.3.4 of Pfeffer [185]. The proof of the theorem is complete. \square

2.8 Properties of Integrals

In this section, by “ $\int f \, dh$ integral” we will mean an integral $\int f \, dh$ as opposed to an integral $\int f \, d\mu$ for an interval function μ . Suppose that the basic assumption (1.14) holds. For functions $f: [a, b] \rightarrow X$ and $h: [a, b] \rightarrow Y$, consider the following properties of an integral $\int_a^b f \, dh$:

I. For $u_1, u_2 \in \mathbb{K}$ and $f_1, f_2: [a, b] \rightarrow X$,

$$\int_a^b (u_1 f_1 + u_2 f_2) \, dh = u_1 \int_a^b f_1 \, dh + u_2 \int_a^b f_2 \, dh,$$

where the left side exists provided the right side does.

II. For $v_1, v_2 \in \mathbb{K}$ and $h_1, h_2: [a, b] \rightarrow Y$,

$$\int_a^b f \cdot d(v_1 h_1 + v_2 h_2) = v_1 \int_a^b f \cdot dh_1 + v_2 \int_a^b f \cdot dh_2,$$

where again the left side exists provided the right side does.

III. For $a < c < b$, $\int_a^b f \cdot dh$ exists if and only if both $\int_a^c f \cdot dh$ and $\int_c^b f \cdot dh$ exist, and then

$$\int_a^b f \cdot dh = \int_a^c f \cdot dh + \int_c^b f \cdot dh. \quad (2.71)$$

IV. If $a < b$ and $\int_a^b f \cdot dh$ exists then for each $t \in [a, b]$, taking a limit for $s \in [a, b]$ if $t = a$ or b , $I(t) = I(f, dh)(t) := \int_a^t f \cdot dh$ exists and

$$\lim_{s \rightarrow t} \left\{ I(t) - I(s) - f(t) \cdot [h(t) - h(s)] \right\} = 0. \quad (2.72)$$

Properties I and II imply that the operator $V \times W \ni (f, h) \mapsto \int_a^b f \cdot dh \in Z$ is bilinear for any function spaces V, W on which it is defined. If property IV holds, the indefinite integral $I(f, dh)(t)$, $t \in [a, b]$, is a regulated function whenever the integrator h is regulated, and then

$$\Delta^- I(t) = f(t) \cdot \Delta^- h(t) \quad \text{and} \quad \Delta^+ I(s) = f(s) \cdot \Delta^+ h(s) \quad \text{for} \quad a \leq s < t \leq b.$$

Theorem 2.72. *The RS, RRS, RYS, S, CY, HK and \nexists integrals satisfy properties I and II.*

Proof. We prove only property I because the proof of property II is symmetric. We can assume that $a < b$. Let $u_1, u_2 \in \mathbb{K}$, $f_1, f_2: [a, b] \rightarrow X$, and $h: [a, b] \rightarrow Y$. For the RS, RRS, and HK integrals, property I follows from bilinearity of the mapping $(f, h) \mapsto f \cdot h$ from $X \times Y$ to Z , and from the equality

$$S_{RS}(u_1 f_1 + u_2 f_2, dh; \tau) = u_1 S_{RS}(f_1, dh; \tau) + u_2 S_{RS}(f_2, dh; \tau),$$

valid for any tagged partition τ of $[a, b]$. The same argument with Riemann–Stieltjes sums replaced by Young–Stieltjes sums (2.16) implies property I for the YS integral, and so it holds for the full Stieltjes S integral. Also, property I holds for the \nexists integral by Proposition 2.27.

For the CY integral given by the Y_1 integral (2.44), let f_1, f_2 be regulated functions. On the interval $[a, b]$, we have $(u_1 f_1 + u_2 f_2)_+ = u_1 (f_1)_+ + u_2 (f_2)_+$ and $\Delta^+(u_1 f_1 + u_2 f_2) = u_1 \Delta^+ f_1 + u_2 \Delta^+ f_2$. Property I for the CY integral follows, since it holds for the RRS integral and by the linearity of unconditionally convergent sums. The proof of Theorem 2.72 is complete. \square

Theorem 2.73. *The RRS, RYS, S, CY, HK, and \nexists integrals satisfy property III.*

Proof. Let $a < c < b$ and let $f: [a, b] \rightarrow X$ and $h: [a, b] \rightarrow Y$. Suppose that $(RRS) \int_a^c f \cdot dh$ and $(RRS) \int_c^b f \cdot dh$ exist. For a tagged partition $\tau_1 = (\{x_i\}_{i=0}^n, \{t_i\}_{i=1}^n)$ of $[a, c]$ and a tagged partition $\tau_2 = (\{y_j\}_{j=0}^m, \{s_j\}_{j=1}^m)$ of $[c, b]$, let $\tau := (\{x_i\}_{i=0}^n \cup \{y_j\}_{j=1}^m, \{t_i\}_{i=1}^n \cup \{s_j\}_{j=1}^m)$, a tagged partition of $[a, b]$. Then the equality

$$S_{RS}(f, dh; \tau) = S_{RS}(f, dh; \tau_1) + S_{RS}(f, dh; \tau_2) \quad (2.73)$$

yields that $(RRS) \int_a^b f \cdot dh$ exists and (2.71) holds for the RRS integral. For the converse suppose that $(RRS) \int_a^b f \cdot dh$ exists. Applying the Cauchy test for pairs of tagged partitions τ' and τ'' of $[a, b]$ which are refinements of $\{a, c, b\}$ and induce the same tagged partition of $[c, b]$, it follows that $(RRS) \int_a^c f \cdot dh$ exists. Likewise $(RRS) \int_c^b f \cdot dh$ exists. The proof of property III for the RRS integral is complete.

The proof of property III for the RYS integral is the same except that the equality

$$S_{YS}(f, dh; \tau) = S_{YS}(f, dh; \tau_1) + S_{YS}(f, dh; \tau_2)$$

for Young tagged partitions as in (2.16) is used instead of (2.73). Thus property III also holds for the \nexists integral by Proposition 2.27. Since property III holds for the RRS and RYS integrals it also holds for the full Stieltjes integral $(S) \int_a^b f \cdot dh$ by definition 2.41 of the (S) integral and since f and h have a common one-sided discontinuity on $[a, b]$ if and only if they also have one on at least one of $[a, c]$ or $[c, b]$.

Since property III for the HK integral is proved by Proposition 2.56, it is left to prove property III for the CY integral. To this aim suppose that functions f, h are regulated and that the Y_1 integral $(Y_1) \int_a^b f \cdot dh$ exists. By the same property for the RRS integral already proved, it follows that the following three integrals exist and

$$(RRS) \int_a^b f_+^{(b)} \cdot dh_-^{(a)} = (RRS) \int_a^c f_+ \cdot dh_-^{(a)} + (RRS) \int_c^b f_+^{(b)} \cdot dh_-.$$

On the right side the integrand f_+ over $[a, c]$ depends on $f(c+)$ and the integrator h_- over $[c, b]$ depends on $h(c-)$. For the two integrals $(RRS) \int_a^c f_+ \cdot dh_-^{(a)}$ and $(RRS) \int_a^c f_+^{(c)} \cdot dh_-^{(a)}$, the difference between Riemann–Stieltjes sums for each based on the same tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, c]$ is

$$\begin{aligned} & f(s_n+) \cdot [h(t_n-) - h(t_{n-1}-)] - f_+^{(c)}(s_n) \cdot [h(t_n-) - h(t_{n-1}-)] \\ &= \Delta^+ f(t_n) \cdot [h(t_n-) - h(t_{n-1}-)] \end{aligned}$$

if $s_n = t_n$ or is 0 if $s_n \in [t_{n-1}, t_n)$. Taking a limit under refinement of tagged partitions of $[a, c]$, we can get $t_{n-1} \uparrow t_n = c$, and so $h(t_n-) - h(t_{n-1}-) \rightarrow 0$, while f is bounded. Therefore the following two integrals both exist and are equal:

$$(RRS) \int_a^c f_+ \cdot dh_-^{(a)} = (RRS) \int_a^c f_+^{(c)} \cdot dh_-^{(a)}.$$

Next, for the two integrals $(RRS) \int_c^b f_+^{(b)} \cdot dh_-$ and $(RRS) \int_c^b f_+^{(b)} \cdot dh_-^{(c)}$, the difference between Riemann–Stieltjes sums for each based on the same tagged partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[c, b]$ is

$$\begin{aligned} & f(s_1+) \cdot [h(t_1-) - h(c-)] - f(s_1+) \cdot [h(t_1-) - h(c)] \\ &= f(s_1+) \cdot \Delta^- h(c) \rightarrow f(c+) \Delta^- h(c) \end{aligned}$$

as $t_1 \downarrow t_0 = c$, which one can obtain by taking a limit under refinements of tagged partitions of $[c, b]$. Therefore the following two integrals both exist and the equation

$$(RRS) \int_c^b f_+^{(b)} \cdot dh_- = (RRS) \int_c^b f_+^{(b)} \cdot dh_-^{(c)} + f(c+) \Delta^- h(c)$$

holds. By the definition of the Y_1 integral (2.44), it then follows that the Y_1 integral exists over the subintervals $[a, c]$ and $[c, b]$, and we have

$$\begin{aligned} (Y_1) \int_a^b f \cdot dh &= (RRS) \int_a^c f_+ \cdot dh_-^{(a)} + (RRS) \int_c^b f_+^{(b)} \cdot dh_- \\ &\quad - [\Delta^+ f \cdot \Delta^+ h](a) - \sum_{(a,c)} \Delta^+ f \cdot \Delta^\pm h - [\Delta^+ f \cdot \Delta^\pm h](c) \\ &\quad - \sum_{(c,b)} \Delta^+ f \cdot \Delta^\pm h + [f \cdot \Delta^- h](b) \\ &= (RRS) \int_a^c f_+^{(c)} \cdot dh_-^{(a)} - [\Delta^+ f \cdot \Delta^+ h](a) - \sum_{(a,c)} \Delta^+ f \cdot \Delta^\pm h + [f \Delta^- h](c) \\ &\quad + (RRS) \int_c^b f_+^{(b)} \cdot dh_-^{(c)} - [\Delta^+ f \cdot \Delta^+ h](c) - \sum_{(c,b)} \Delta^+ f \cdot \Delta^\pm h + [f \Delta^- h](b) \\ &= (Y_1) \int_a^c f \cdot dh + (Y_1) \int_c^b f \cdot dh. \end{aligned}$$

The converse implication follows by applying the same arguments, and so property III holds for the CY integral. The proof of Theorem 2.73 is complete. \square

The Riemann–Stieltjes integral does not satisfy property III. Indeed, for the indicator functions $f := 1_{[1,2]}$ and $h := 1_{(1,2]}$ defined on $[0, 2]$, the RS integral exists over the intervals $[0, 1]$ and $[1, 2]$, but not over the interval $[0, 2]$. Notice that f and h for this example have a common discontinuity at $1 \in [0, 2]$. This feature of the Riemann–Stieltjes integral was observed by Pollard [187], who also showed that the refinement Riemann–Stieltjes integral does satisfy property III. A weaker form of this property can be stated as follows:

III'. For $a < c < b$, if $\int_a^b f \cdot dh$ exists then both $\int_a^c f \cdot dh$ and $\int_c^b f \cdot dh$ exist, and (2.71) holds.

Then we have the following result:

Proposition 2.74. *The RS integral satisfies property III'.*

Proof. Suppose that $(RS) \int_a^b f \cdot dh$ exists and let $a < c < b$. Adjoining equal tagged subintervals of $[c, b]$ to any two given tagged partitions τ'_1, τ'_2 of $[a, c]$, one can form two tagged partitions τ_1, τ_2 of $[a, b]$, without increasing $|\tau'_1| \vee |\tau'_2|$, and such that $S_{RS}(\tau'_1) - S_{RS}(\tau'_2) = S_{RS}(\tau_1) - S_{RS}(\tau_2)$. Thus one can apply the Cauchy test to prove the existence of $(RS) \int_a^c f \cdot dh$. Similarly one can show that $(RS) \int_c^b f \cdot dh$ also exists. The additivity relation (2.71) then follows from the equality $S_{RS}((\kappa, \xi)) = S_{RS}((\kappa', \xi')) + S_{RS}((\kappa'', \xi''))$, valid for any tagged partitions such that $\kappa = \kappa' \cup \kappa'', \xi = \xi' \cup \xi''$, and κ', κ'' are partitions of $[a, c]$ and $[c, b]$ respectively. \square

Theorem 2.75. *The RS, RRS, RYS, and HK integrals satisfy property IV.*

Proof. Suppose that $(RS) \int_a^b f \cdot dh$ exists. The indefinite integral $I_{RS}(u) := (RS) \int_a^u f \cdot dh$ is defined for $u \in (a, b]$ by Proposition 2.74 and is 0 when $u = a$ by the definition of the RS integral. Also by Proposition 2.74, we have $I_{RS}(u) - I_{RS}(v) = \text{sgn}(u - v)(RS) \int_{u \wedge v}^{u \vee v} f \cdot dh$. In light of symmetry of proofs, we prove only that

$$\lim_{v \uparrow u} \left\{ (RS) \int_v^u f \cdot dh - S_{RS}(f, dh; \sigma_v) \right\} = 0 \quad (2.74)$$

for any $a < u \leq b$, where for $a < v < u$, σ_v is the tagged partition $(\{v, u\}, \{u\})$ of $[v, u]$. Let $u \in (a, b]$. Given $\epsilon > 0$ there is a $\delta \in (0, u - a]$ such that any two Riemann–Stieltjes sums based on tagged partitions of $[a, u]$ with mesh less than δ differ by at most ϵ . For $u - \delta \leq v < u$, choose a Riemann–Stieltjes sum $S_{RS}(\tau_v) = S_{RS}(f, dh; \tau_v)$, based on a tagged partition τ_v of $[v, u]$, within ϵ of $(RS) \int_v^u f \cdot dh$. Let τ' and τ'' be two tagged partitions of $[a, u]$ with mesh less than δ which are equal when restricted to $[a, v]$ and which coincide with τ_v and σ_v , respectively, when restricted to $[v, u]$. Then

$$\begin{aligned} & \left\| (RS) \int_v^u f \cdot dh - S_{RS}(f, dh; \sigma_v) \right\| \\ & \leq \left\| (RS) \int_v^u f \cdot dh - S_{RS}(\tau_v) \right\| + \|S_{RS}(\tau') - S_{RS}(\tau'')\| \\ & < 2\epsilon \end{aligned} \quad (2.75)$$

for each v with $u - \delta \leq v < u$. Since $\epsilon > 0$ is arbitrary, (2.74) holds, proving property IV for the RS integral.

If $(RRS) \int_a^b f \cdot dh$ exists, we argue similarly. To prove (2.74) with RRS instead of RS , let $\epsilon > 0$. Then there is a partition $\lambda = \{t_j\}_{j=0}^m$ of $[a, u]$ such that any two Riemann–Stieltjes sums based on tagged refinements of λ differ by at most ϵ . For $v \in [t_{m-1}, u)$, choose a Riemann–Stieltjes sum, based on a tagged partition τ_v of $[v, u]$, within ϵ of $(RRS) \int_v^u f \cdot dh$. Let τ' and τ'' be two tagged refinements of λ which coincide with τ_v and σ_v , respectively, when restricted to $[v, u]$. Then (2.75) with RRS instead of RS holds for each $v \in [t_{m-1}, u)$, proving property IV for the RRS integral.

Suppose now that $(RYS) \int_a^b f \cdot dh$ exists. Let $u \in (a, b]$, and for $a < v < u$, let σ_v be a Young tagged partition of the form $(\{v, u\}, \{s\})$ of $[v, u]$. Since h is regulated by definition of the RYS integral, we have

$$\begin{aligned} \lim_{v \uparrow u} \left\{ S_{YS}(f, dh; \sigma_v) - f(u) \cdot [h(u) - h(v)] \right\} \\ = \lim_{v \uparrow u} \left\{ [f(s) - f(u)] \cdot [h(u-) - h(v+)] + [f(v) - f(u)] \cdot \Delta^+ h(v) \right\} = 0. \end{aligned}$$

Therefore and by property III again, we have to show that

$$\lim_{v \uparrow u} \left\{ (RYS) \int_v^u f \cdot dh - S_{YS}(f, dh; \sigma_v) \right\} = 0.$$

The proof is the same as for the RRS integral except that Riemann–Stieltjes sums are replaced by Young–Stieltjes sums. Since property IV for the HK integral follows from Proposition 2.58, the proof of Theorem 2.75 is complete. \square

Property IV for the HK integral agrees with the “only if” part of Proposition 2.58, which says that existence of the HK integral implies existence of improper versions. The following shows that the “if” part of Proposition 2.58 does not hold for the RRS integral, for the RYS integral by Propositions 2.18 and 2.46, nor for the CY integral by its definition.

Proposition 2.76. *There are continuous real-valued functions f, h on $[0, 1]$ such that $\lim_{t \uparrow 1} (RRS) \int_0^t f \, dh$ exists, but $(RRS) \int_0^1 f \, dh$ does not.*

Proof. Let $s_m := 1 - 1/m$ for $m = 1, 2, \dots$. For each m , let $f(s_{4m-2}) = f(s_{4m}) := 0$, $f(s_{4m-1}) := m^{-1/2}$, $f(0) = f(1) := 0$, and let f be linear and continuous on intervals between adjacent points where it has been defined. Let $h(s_{4m-3}) := 0$, so $h(0) = 0$, $h(s_{4m-2}) = h(s_{4m}) := m^{-1/2}$, $h(1) := 0$, and let h also be linear between adjacent points where it has been defined. Then f and h are both continuous. Since h is constant on each interval where f is non-zero, we have for $0 < t < 1$ that $(RRS) \int_0^t f \, dh = 0$, taking a partition containing t and all points $s_m < t$.

But for any partition $\kappa = \{t_i\}_{i=0}^n$ of $[0, 1]$, there exist arbitrarily large Riemann–Stieltjes sums for f, h based on refinements of κ , as follows. Let m_0

be the smallest m such that $s_{4m-3} > t_{n-1}$. Form a partition κ_N by adjoining to κ all points s_{4m-3}, s_{4m} for $m = m_0, m_0+1, \dots, m_0+N$, and form Riemann–Stieltjes sums containing terms $f(s_{4m-1})[h(s_{4m}) - h(s_{4m-3})] = 1/m$ for all such m , and 0 terms $f(s_{4m})[h(s_{4m+1}) - h(s_{4m})]$. As $N \rightarrow \infty$ these sums become arbitrarily large, as claimed. \square

The following fact is a change of variables formula for the full Stieltjes integral.

Proposition 2.77. *Let $\phi: [a, b] \rightarrow \mathbb{R}$ be a strictly monotone continuous function with range $[c, d] := \text{ran}(\phi)$. Assuming (1.14), let $f: [c, d] \rightarrow X$ and $h: [c, d] \rightarrow Y$. Then $(S) \int_a^b (f \circ \phi) \cdot d(h \circ \phi)$ exists if and only if $(S) \int_c^d f \cdot dh$ does, and*

$$(S) \int_a^b (f \circ \phi) \cdot d(h \circ \phi) = \begin{cases} (S) \int_c^d f \cdot dh & \text{if } \phi \text{ is increasing,} \\ -(S) \int_c^d f \cdot dh & \text{if } \phi \text{ is decreasing.} \end{cases}$$

Proof. We can assume that $a < b$. First suppose that ϕ is increasing. The inverse function ϕ^{-1} of ϕ is also continuous and increasing from $[c, d]$ onto $[a, b]$. Therefore for the two integrals the limits under refinement of partitions both exist or not simultaneously, and if they do exist they are equal, proving the first part of the proposition.

Now suppose that ϕ is decreasing on $[a, b]$. Then letting $\theta(x) := -x$ for $x \in [-b, -a]$, it follows that $\tilde{\phi} := \phi \circ \theta$ is an increasing continuous function on $[-b, -a]$. Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a tagged partition of $[a, b]$. Then $\tilde{\tau} := (\{-t_{n-i}\}_{i=0}^n, \{-s_{n-i}\}_{i=1}^n)$ is a tagged partition of $[-b, -a]$ and

$$S_{RS}(f \circ \phi, dh \circ \phi; \tau) = -S_{RS}(f \circ \tilde{\phi}, dh \circ \tilde{\phi}; \tilde{\tau}).$$

Also, if σ is a tagged refinement of τ then $\tilde{\sigma}$ is a tagged refinement of $\tilde{\tau}$. Thus

$$(RRS) \int_a^b f \circ \phi \cdot dh \circ \phi = -(RRS) \int_{-b}^{-a} f \circ \tilde{\phi} \cdot dh \circ \tilde{\phi},$$

where the two integrals exist or not simultaneously. Now applying the first part of the proposition to the integral on the right side, it follows that

$$(RRS) \int_{-b}^{-a} f \circ \tilde{\phi} \cdot dh \circ \tilde{\phi} = (RRS) \int_{\phi(b)}^{\phi(a)} f \cdot dh,$$

provided either integral is defined. The same is true for refinement Young–Stieltjes integrals, proving the proposition. \square

Next is a fact which allows interchange of integration with a linear map for a scalar-valued integrator.

Proposition 2.78. *Let \mathbb{K} be either the field \mathbb{R} or the field \mathbb{C} , and let L be a bounded linear mapping from a Banach space X to another Banach space Z . For $f: [a, b] \rightarrow X$ and $h: [a, b] \rightarrow \mathbb{K}$, if $(RS) \int_a^b f \cdot dh$ exists then so does $(RS) \int_a^b (L \circ f) \cdot dh$ and*

$$L\left((RS) \int_a^b f \cdot dh\right) = (RS) \int_a^b (L \circ f) \cdot dh. \quad (2.76)$$

Proof. We can assume that $a < b$. For any tagged partition τ of $[a, b]$, we have

$$L(S_{RS}(f, dh; \tau)) = S_{RS}(L \circ f, dh; \tau).$$

Since L is continuous, if $(RS) \int_a^b f \cdot dh$ exists then the right side has a limit as the mesh of τ tends to zero and (2.76) holds, proving the proposition. \square

Next it will be seen that the previous proposition does not extend to cases where both the integrand and integrator have multidimensional values, for any integral we consider.

Example 2.79. Let $X = Y = \mathbb{R}^2$, let $Z = \mathbb{R}$, and let the bilinear form from $X \times Y$ into Z be the usual inner product. Let g and h be any real-valued functions on $[0, 1]$ and ν any real-valued interval function on $[0, 1]$. Let $f(t) := (g(t), 0) \in \mathbb{R}^2$ and $H(t) := (0, h(t)) \in \mathbb{R}^2$ for any $t \in [0, 1]$. For any interval $J \subset [0, 1]$ let $\mu(J) := (0, \nu(J)) \in \mathbb{R}^2$. Let $L(x, y) := (y, x)$ for each $(x, y) \in \mathbb{R}^2$. Then for each form of integration we have defined, the integrals $\int f \cdot dH$ and $\int f \cdot d\mu$ exist and are 0, where $f = f_0^1$ or $f_{[0,1]}$, but the integrals $\int (L \circ f) \cdot dH$ and $\int (L \circ f) \cdot d\mu$ may not exist, or may exist with any value.

For all the integrals $\int_a^b f \cdot dg$ defined so far and properties proved for them, we have the symmetrical properties of integrals $\int_a^b df \cdot g$ by (1.15).

Integration by parts

We start with the classical integration by parts formula for the Riemann–Stieltjes and refinement Riemann–Stieltjes integrals.

Theorem 2.80. *Assuming (1.14), let $f: [a, b] \rightarrow X$ and $g: [a, b] \rightarrow Y$. For $\# = RS$ or $\# = RRS$, if $(\#) \int_a^b f \cdot dg$ exists then so does $(\#) \int_a^b df \cdot g$, and*

$$(\#) \int_a^b f \cdot dg + (\#) \int_a^b df \cdot g = f(b) \cdot g(b) - f(a) \cdot g(a) =: f \cdot g|_a^b. \quad (2.77)$$

Proof. We can assume that $a < b$. Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a tagged partition of $[a, b]$, and let $s_0 := t_0 = a$, $s_{n+1} := t_n = b$. Summing by parts, we have

$$\sum_{i=1}^n [f(t_i) - f(t_{i-1})] \cdot g(s_i) = f \cdot g \Big|_{t_0}^{t_n} - \sum_{i=0}^n f(t_i) \cdot [g(s_{i+1}) - g(s_i)]. \quad (2.78)$$

For $\# = RS$, notice that if $\max_i(t_i - t_{i-1}) < \delta$ for some $\delta > 0$, then $\max_i(s_{i+1} - s_i) < 2\delta$. Therefore if $(RS) \int_a^b f \cdot dg$ exists then by (2.78) and (1.15), $(RS) \int_a^b df \cdot g$ exists and (2.77) holds for $\# = RS$.

For $\# = RRS$, notice that the sum on the right side of (2.78) can be written as the sum

$$\sum_{i=0}^n \left\{ f(t_i) \cdot [g(s_{i+1}) - g(t_i)] + f(t_i) \cdot [g(t_i) - g(s_i)] \right\} =: S.$$

Thus if $\{t_i\}_{i=0}^n$ is a refinement of a partition λ of $[a, b]$ then S is a Riemann–Stieltjes sum based on a tagged refinement of λ . Hence if $(RRS) \int_a^b f \cdot dg$ exists then by (2.78) and (1.15), $(RRS) \int_a^b df \cdot g$ exists and (2.77) holds for $\# = RRS$. The proof is complete. \square

Next is an integration by parts formula for the central Young integral as defined in Section 2.5.

Theorem 2.81. *Assuming (1.14), let $f \in \mathcal{R}([a, b]; X)$ and $g \in \mathcal{R}([a, b]; Y)$. If $a < b$, suppose that at least one of the conditions (a) or (b) holds:*

- (a) *the sums $\sum_{(a,b)} \Delta^+ f \cdot \Delta^+ g$ and $\sum_{(a,b)} \Delta^- f \cdot \Delta^- g$ converge unconditionally in Z ;*
- (b) *on (a, b) , $f = [f_- + f_+]/2$ and $g = [g_- + g_+]/2$.*

If $(CY) \int_a^b f \cdot dg$ exists then so does $(CY) \int_a^b df \cdot g$, and

$$(CY) \int_a^b f \cdot dg + (CY) \int_a^b df \cdot g = f \cdot g \Big|_a^b + A, \quad (2.79)$$

where $A = 0$ if $a = b$ and otherwise $A = -[\Delta^+ f \cdot \Delta^+ g](a) + [\Delta^- f \cdot \Delta^- g](b) + B$ with

$$B := \begin{cases} -\sum_{(a,b)} \Delta^+ f \cdot \Delta^+ g + \sum_{(a,b)} \Delta^- f \cdot \Delta^- g & \text{if (a) holds} \\ 0 & \text{if (b) holds.} \end{cases}$$

Proof. We can assume that $a < b$. By the definitions of the CY integral (2.51) and the Y_1 integral (2.44),

$$\begin{aligned} (CY) \int_a^b f \cdot dg &= (RRS) \int_a^b f_+^{(b)} \cdot dg_-^{(a)} \\ &\quad - [\Delta^+ f \cdot \Delta^+ g](a) + [f \cdot \Delta^- g](b) - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm g, \end{aligned}$$

where the RRS integral exists and the sum converges unconditionally in Z . If (a) holds, by linearity of unconditional convergence, the sum $\sum_{(a,b)} \Delta^\pm f \cdot \Delta^- g$ converges unconditionally in Z , and

$$\sum_{(a,b)} \Delta^\pm f \cdot \Delta^- g - \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm g = \sum_{(a,b)} \Delta^- f \cdot \Delta^- g - \sum_{(a,b)} \Delta^+ f \cdot \Delta^+ g. \quad (2.80)$$

If (b) holds, since “ \cdot ” is bilinear,

$$\Delta^+ f \cdot \Delta^\pm g = \Delta^+ f \cdot (2\Delta^- g) = (2\Delta^+ f) \cdot \Delta^- g = \Delta^\pm f \cdot \Delta^- g.$$

Thus again the sum $\sum_{(a,b)} \Delta^\pm f \cdot \Delta^- g$ converges unconditionally in Z , and

$$\sum_{(a,b)} \Delta^\pm f \cdot \Delta^- g = \sum_{(a,b)} \Delta^+ f \cdot \Delta^\pm g. \quad (2.81)$$

Also, in case (a) or (b) the integration by parts theorem for the RRS integral (Theorem 2.80 with $\# = RRS$) yields that $(RRS) \int_a^b df_+^{(b)} \cdot g_-^{(a)}$ exists and

$$\begin{aligned} (RRS) \int_a^b f_+^{(b)} \cdot dg_-^{(a)} + (RRS) \int_a^b df_+^{(b)} \cdot g_-^{(a)} &= f_+^{(b)} \cdot g_-^{(a)} \Big|_a^b \\ &= f(b) \cdot g(b-) - f(a+) \cdot g(a). \end{aligned} \quad (2.82)$$

Hence by definition of the Y_2 integral (2.45), $(CY) \int_a^b df \cdot g$ exists and

$$\begin{aligned} (CY) \int_a^b df \cdot g &= (RRS) \int_a^b df_+^{(b)} \cdot g_-^{(a)} \\ &\quad + [\Delta^+ f \cdot g](a) + [\Delta^- f \cdot \Delta^- g](b) + \sum_{(a,b)} \Delta^\pm f \cdot \Delta^- g. \end{aligned}$$

Then by (2.80) in case (a) or (2.81) in case (b), it follows that

$$\begin{aligned} (CY) \int_a^b f \cdot dg + (CY) \int_a^b df \cdot g &= (RRS) \int_a^b f_+^{(b)} \cdot dg_-^{(a)} + (RRS) \int_a^b df_+^{(b)} \cdot g_-^{(a)} \\ &\quad + f \cdot g \Big|_a^b - f(b) \cdot g(b-) + f(a+) \cdot g(a) \\ &\quad - [\Delta^+ f \cdot \Delta^+ g](a) + [\Delta^- f \cdot \Delta^- g](b) + A. \end{aligned}$$

This together with (2.82) implies formula (2.79), proving the theorem. \square

For the refinement Young–Stieltjes integral, an integration by parts theorem analogous to the preceding one does not hold. Recall that by Proposition 2.52 there are real-valued functions f and h on $[0, 1]$, where f is in $c_0((0, 1))$ and h is continuous, for which $(CY) \int_0^1 f dh$ exists, while $(RYS) \int_0^1 f dh$ does not exist. For this pair of functions, $(RYS) \int_0^1 h df$ exists and equals 0 because $f_-^{(0)} \equiv f_+^{(1)} \equiv 0$, and $\sum \Delta^+ f \Delta^+ h = \sum \Delta^- f \Delta^- h = 0$ since h is continuous. However, by Theorem 2.51(a) and Theorem 2.81, we have the following corollary:

Corollary 2.82. *Under the conditions of Theorem 2.81, if $(RYS) \int_a^b f \cdot dg$ and $(RYS) \int_a^b df \cdot g$ both exist then (2.79) with CY replaced by RYS holds.*

For a regulated function f on $[a, b]$ with $a < b$ and $\hat{f} := [f_-^{(a,b)} + f_+^{(a,b)}]/2$, we have

$$\sum_{(a,b)} [f - \hat{f}] \cdot \Delta^\pm g + \sum_{(a,b)} \Delta^\pm f \cdot [g - \hat{g}] = - \sum_{(a,b)} \Delta^+ f \cdot \Delta^+ g + \sum_{(a,b)} \Delta^- f \cdot \Delta^- g,$$

provided that the sums converge unconditionally. Therefore by Theorem 2.51(b) and Theorem 2.81, the following form of integration by parts for the refinement Young–Stieltjes integral holds:

Corollary 2.83. *Assuming (1.14) and $a < b$, let $f \in \mathcal{R}([a, b]; X)$ and $g \in \mathcal{R}([a, b]; Y)$ be such that at least one of the two conditions (a) and (b) of Theorem 2.81 holds. If either $(CY) \int_a^b f \cdot dg$ or $(CY) \int_a^b df \cdot g$ exists then $(RYS) \int_a^b \hat{f} \cdot dg$ and $(RYS) \int_a^b df \cdot \hat{g}$ both exist, and*

$$(RYS) \int_a^b \hat{f} \cdot dg + (RYS) \int_a^b df \cdot \hat{g} = f \cdot g|_a^b - [\Delta^+ f \cdot \Delta^+ g](a) + [\Delta^- f \cdot \Delta^- g](b), \quad (2.83)$$

where $\hat{f} = f$ and $\hat{g} = g$ if the condition (b) of Theorem 2.81 holds.

Bounds for integrals

Here we give bounds for integrals assuming that either the integrand or integrator has bounded variation.

Theorem 2.84. *Assuming (1.14), $f: [a, b] \rightarrow X$, and $h: [a, b] \rightarrow Y$, we have*

- (a) *if f is regulated and h is of bounded variation then the full Stieltjes integral $(S) \int_a^b f \cdot dh$ is defined and for each $t \in [a, b]$,*

$$\left\| (S) \int_a^b f \cdot dh - f(t) \cdot [h(b) - h(a)] \right\| \leq \text{Osc}(f; [a, b]) v_1(h; [a, b]); \quad (2.84)$$

- (b) *if h is regulated and f is of bounded variation then the full Stieltjes integral $(S) \int_a^b f \cdot dh$ is defined and for each $t \in [a, b]$,*

$$\left\| (S) \int_a^b f \cdot dh - f(t) \cdot [h(b) - h(a)] \right\| \leq \text{Osc}(h; [a, b]) v_1(f; [a, b]). \quad (2.85)$$

Proof. We can assume that $a < b$. In both cases if f and h have no common discontinuities then the integral $\int_a^b f \cdot dh$ exists in the Riemann–Stieltjes sense by Theorems 2.42 and 2.17. If f and h have no common one-sided discontinuities then the integral $\int_a^b f \cdot dh$ exists in the refinement Riemann–Stieltjes sense by Theorem 2.17. Under the assumptions of the present theorem the integral

$\int_a^b f \cdot dh$ always exists in the refinement Young–Stieltjes sense by Theorem 2.20, and so the full Stieltjes integral $(S) \int_a^b f \cdot dh$ is defined.

To prove (a) it then suffices to prove it for the refinement Young–Stieltjes integral. Letting $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a Young tagged partition of $[a, b]$ and $t \in [a, b]$, we have the bound

$$\begin{aligned} & \|S_{YS}(f, dh; \tau) - f(t) \cdot [h(b) - h(a)]\| \\ & \leq \left\| \sum_{i=0}^n [f(t_i) - f(t)] \cdot (\Delta_{[a,b]}^\pm h)(t_i) \right\| + \left\| \sum_{i=1}^n [f(s_i) - f(t)] \cdot [h(t_i-) - h(t_{i-1}+)] \right\|. \end{aligned}$$

For each $i = 1, \dots, n$, letting $u_{i-1} \downarrow t_{i-1}$ and $v_i \uparrow t_i$, one can approximate $\Delta^+ h(a)$ by $h(u_0) - h(a)$, $\Delta^- h(b)$ by $h(b) - h(v_n)$, $\Delta^\pm h(t_i)$ by $h(u_i) - h(v_i)$ for $i = 1, \dots, n-1$ and $h(t_i-) - h(t_{i-1}+)$ by $h(v_i) - h(u_{i-1})$ for $i = 1, \dots, n$. For an arbitrary $\epsilon > 0$, this gives the further bound

$$\begin{aligned} & \leq \epsilon + \max_{0 \leq i \leq n} \|f(t_i) - f(t)\| \left(\|h(u_0) - h(a)\| + \|h(b) - h(v_n)\| + \sum_{i=1}^{n-1} \|h(u_i) - h(v_i)\| \right) \\ & \quad + \max_{1 \leq i \leq n} \|f(s_i) - f(t)\| \sum_{i=1}^n \|h(v_i) - h(u_{i-1})\| \leq \epsilon + \text{Osc}(f; [a, b]) v_1(h; [a, b]). \end{aligned}$$

Since $\epsilon > 0$ is arbitrary, the limit under refinements of Young tagged partitions gives the bound (2.84) for the refinement Young–Stieltjes integral, proving (a).

As in (a), it will suffice to prove (b) only for the refinement Young–Stieltjes integral. Let $\tau = (\{t_i\}_{i=0}^{2n}, \{s_i\}_{i=1}^{2n})$ be a Young tagged partition of $[a, b]$. For $j = 1, \dots, 2n$, let $\Delta_j f := f(u_j) - f(u_{j-1})$ with $\{u_j\}_{j=0}^{2n} = \{t_0, s_1, t_1, \dots, s_n, t_n\}$ and let $\{v_j\}_{j=1}^{2n} := \{t_0+, t_1-, t_1+, \dots, t_n-\}$. Summation by parts gives

$$S_{YS}(f, dh; \tau) = f(b) \cdot h(b) - f(a) \cdot h(a) - \sum_{j=1}^{2n} \Delta_j f \cdot h(v_{j-1}).$$

Adding to and subtracting from the right side first $f(b) \cdot h(a)$ and then $f(a) \cdot h(b)$, we get two representations

$$\begin{aligned} S_{YS}(f, dh; \tau) &= f(b) \cdot [h(b) - h(a)] - \sum_{j=1}^{2n} \Delta_j f \cdot [h(v_{j-1}) - h(a)] \\ &= f(a) \cdot [h(b) - h(a)] - \sum_{j=1}^{2n} \Delta_j f \cdot [h(v_{j-1}) - h(b)]. \end{aligned} \tag{2.86}$$

Thus for $t \in \{a, b\}$, we have the bound

$$\|S_{YS}(f, dh; \tau) - f(t) \cdot [h(b) - h(a)]\| \leq \text{Osc}(h; [a, b]) v_1(f; [a, b]). \tag{2.87}$$

Since τ is an arbitrary Young tagged partition of $[a, b]$, (2.85) holds if $t \in \{a, b\}$. Suppose that $t \in (a, b)$ and let $t = t_\nu$ for some $\nu \in \{1, \dots, n-1\}$. Then $\tau_1 := (\{t_i\}_{i=0}^\nu, \{s_i\}_{i=1}^\nu)$ and $\tau_2 := (\{t_i\}_\nu^n, \{s_i\}_{\nu+1}^n)$ are Young tagged partitions of $[a, t]$ and $[t, b]$ respectively. Applying the first representation in (2.86) to τ_1 and the second representation to τ_2 , it follows that

$$\begin{aligned} S_{YS}(f, dh; \tau) &= S_{YS}(f, dh; \tau_1) + S_{YS}(f, dh; \tau_2) \\ &= f(t) \cdot [h(b) - h(a)] - \sum_{i=1}^{\nu} \Delta_j f \cdot [h(v_{j-1}) - h(a)] \\ &\quad - \sum_{i=\nu+1}^n \Delta_j f \cdot [h(v_{j-1}) - h(b)], \end{aligned}$$

and so (2.87) holds for any $t \in (a, b)$ and any τ containing t as a partition point. Thus (2.85) holds for any $t \in [a, b]$. The proof of Theorem 2.84 is complete. \square

Corollary 2.85. *Under the assumptions of the preceding theorem, the indefinite full Stieltjes integral $I_S(f, dh)(t) := (S) \int_a^t f \cdot dh$, $t \in [a, b]$, is defined and has bounded variation.*

Proof. We can assume $a < b$. The indefinite Riemann–Stieltjes integral exists by the preceding theorem and property III' (see Proposition 2.74). The indefinite refinement Riemann–Stieltjes and Young–Stieltjes integrals exist by the preceding theorem and property III (see Theorem 2.73). Let $\kappa = \{t_i\}_{i=0}^n$ be a partition of $[a, b]$. Then using additivity of the three integrals proved as the already mentioned properties III' and III, we get the bound

$$\begin{aligned} s_1(I_S(f, dh); \kappa) &\leq \sum_{i=1}^n \left\| (S) \int_{t_{i-1}}^{t_i} f \cdot dh - f(t_{i-1}) \cdot [h(t_i) - h(t_{i-1})] \right\| \\ &\quad + \sum_{i=1}^n \|f(t_{i-1})\| \|h(t_i) - h(t_{i-1})\| \\ &\leq \left(\text{Osc}(f; [a, b]) + \|f\|_{\sup} \right) v_1(h; [a, b]), \end{aligned}$$

completing the proof of the corollary. \square

A substitution rule

Here is a substitution rule for the Riemann–Stieltjes integrals with integrators having bounded variation.

Proposition 2.86. *For a Banach space X , let $h \in \mathcal{W}_1[a, b]$, $g \in \mathcal{R}([a, b]; X)$, and $f \in \mathcal{R}[a, b]$ be such that the two pairs (h, g) and (h, f) have no common discontinuities, and $gf: [a, b] \rightarrow X$ is the function defined by pointwise multiplication. Then the following Riemann–Stieltjes integrals (including those in integrands) are defined and*

$$(RS) \int_a^b dI_{RS}(g, dh) \cdot f = (RS) \int_a^b gf \cdot dh = (RS) \int_a^b g \cdot dI_{RS}(dh, f), \quad (2.88)$$

where $I_{RS}(g, dh)(t) := (RS) \int_a^t g \cdot dh \in X$ and $I_{RS}(dh, f)(t) := (RS) \int_a^t f \, dh$ for $t \in [a, b]$, and \cdot denotes the natural bilinear mapping $X \times \mathbb{R} \rightarrow X$.

Proof. We can assume that $a < b$. Since the pairs (g, h) and (f, h) have no common discontinuities, the indefinite integrals $I_{RS}(g, dh)$ and $I_{RS}(dh, f)$ exist and are in $\mathcal{W}_1([a, b]; X)$ and $\mathcal{W}_1[a, b]$, respectively, by Corollary 2.85. Since the discontinuities of $I_{RS}(g, dh)$ and $I_{RS}(dh, f)$ are subsets of those of h , and so the pairs $(I_{RS}(g, dh), f)$ and $(g, I_{RS}(dh, f))$ have no common discontinuities, the leftmost and the rightmost integrals in (2.88) exist by the same corollary. Since gf is a regulated function on $[a, b]$ and the pair (gf, h) have no common discontinuities, the middle integral in (2.88) exists by the same corollary. We prove only the first equality in (2.88), since the proof of the second one is symmetric. Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a tagged partition of $[a, b]$. Then

$$\|S_{RS}(dI_{RS}(g, dh), f; \tau) - S_{RS}(gf, dh; \tau)\| \leq \|f\|_{\sup} R(\tau),$$

where

$$R(\tau) := \sum_{i=1}^n \left\| (RS) \int_{t_{i-1}}^{t_i} g \cdot dh - g(s_i) \cdot [h(t_i) - h(t_{i-1})] \right\|.$$

It is enough to prove that

$$\lim_{|\tau| \downarrow 0} R(\tau) = 0. \quad (2.89)$$

Let $\epsilon > 0$. By Theorem 2.1, there exists a point partition $\{z_j\}_{j=0}^m$ of $[a, b]$ such that

$$\text{Osc}(g; (z_{j-1}, z_j)) < \epsilon \quad \text{for } j \in \{1, \dots, m\}. \quad (2.90)$$

Let $a \leq u \leq z_j \leq v \leq b$ for some j . Splitting the interval $[u, v]$ into the parts $[u, z_j]$ and $[z_j, v]$ if z_j is not an endpoint of $[u, v]$, and then applying properties III' and IV for the Riemann–Stieltjes integral (see respectively Proposition 2.74 and Theorem 2.75), we have that

$$\left\| (RS) \int_u^v g \cdot dh - g(z_j) \cdot [h(v) - h(u)] \right\|$$

is small provided $v - u$ is small enough. Therefore and since g and h have no common discontinuities, there exists a $\delta > 0$ such that if $z_j \in [t_{i-1}, t_i]$ for some j and mesh $|\tau| < \delta$, then

$$\begin{aligned}
& \left\| (RS) \int_{t_{i-1}}^{t_i} g \cdot dh - g(s_i) \cdot [h(t_i) - h(t_{i-1})] \right\| \\
& \leq \left\| (RS) \int_{t_{i-1}}^{t_i} g \cdot dh - g(z_j) \cdot [h(t_i) - h(t_{i-1})] \right\| \\
& \quad + \left\| [g(z_j) - g(s_i)] \cdot [h(t_i) - h(t_{i-1})] \right\| < \epsilon / (m + 1).
\end{aligned}$$

This together with (2.90) and the bound of Theorem 2.84 yields

$$R(\tau) \leq \epsilon v_1(h; [a, b]) + (m + 1)\epsilon / (m + 1) = \epsilon [v_1(h; [a, b]) + 1]$$

if the mesh of the tagged partition τ is less than δ . The proof of (2.89), and hence of Theorem 2.86, is complete. \square

A chain rule formula

Here we give a representation of the full Stieltjes integral as defined in Definition 2.41, with respect to a composition $F \circ f$, where f has bounded variation and F is smooth.

Theorem 2.87. *Let $f: [a, b] \rightarrow \mathbb{R}^d$ be of bounded variation, let F be a real-valued C^1 function on \mathbb{R}^d and let $h \in \mathcal{R}[a, b]$. Then the composition $F \circ f$ is of bounded variation and*

$$\begin{aligned}
(S) \int_a^b h \, d(F \circ f) &= (S) \int_a^b h (\nabla F \circ f) \cdot df + \sum_{(a, b]} h \left\{ \Delta^-(F \circ f) - (\nabla F \circ f) \cdot \Delta^- f \right\} \\
&\quad + \sum_{[a, b)} h \left\{ \Delta^+(F \circ f) - (\nabla F \circ f) \cdot \Delta^+ f \right\}, \tag{2.91}
\end{aligned}$$

where the two sums converge absolutely if $a < b$ and equal 0 if $a = b$.

Proof. We can assume that $a < b$. Since F is a Lipschitz function on any bounded set (the range of f), $F \circ f$ is of bounded variation. Thus the two full Stieltjes integrals in (2.91) exist by Theorems 2.20, 2.17, and 2.42. Again since F is Lipschitz and ∇F is bounded on the range of f , the two sums in (2.91) converge absolutely. By Propositions 2.13 and 2.18, it is enough to prove the equality (2.91) for the RYS integrals. Let $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ be a Young tagged partition of $[a, b]$. Also let

$$S_-(\tau) := \sum_{\{t_1, \dots, t_n\}} h \left\{ \Delta^-(F \circ f) - (\nabla F \circ f) \cdot \Delta^- f \right\}$$

and

$$S_+(\tau) := \sum_{\{t_0, \dots, t_{n-1}\}} h \left\{ \Delta^+(F \circ f) - (\nabla F \circ f) \cdot \Delta^+ f \right\}.$$

Then we have the identity

$$S_{YS}(h, dF \circ f; \tau) = S_{YS}(h(\nabla F \circ f), df; \tau) + S_-(\tau) + S_+(\tau) + R(\tau), \quad (2.92)$$

where $R(\tau)$ is the sum

$$\sum_{i=1}^n h(s_i) \left\{ [F(f(t_i-)) - F(f(t_{i-1}+))] - (\nabla F(f(s_i))) \cdot [f(t_i-) - f(t_{i-1}+)] \right\}.$$

Let $\epsilon > 0$ and let B be a ball in \mathbb{R}^d containing the range of f . Since ∇F is uniformly continuous on B , there is a $\delta > 0$ such that $\|\nabla F(u) - \nabla F(v)\| < \epsilon$ whenever $\|u - v\| < \delta$ and $u, v \in B$. Moreover, since f is regulated, by Theorem 2.1(b), there exists a partition $\lambda = \{z_j\}_{j=0}^m$ of $[a, b]$ such that $\text{Osc}(f; (z_{j-1}, z_j)) < \delta$ for $j = 1, \dots, m$. Let τ be a Young tagged refinement of λ . Then for each $i = 1, \dots, n$, by the mean value theorem for $\lambda \mapsto F(\lambda f(t_i-) + (1 - \lambda)f(t_{i-1}+))$, $0 \leq \lambda \leq 1$, and by the chain rule of differentiation there is a $\lambda_i \in [0, 1]$ such that

$$F(f(t_i-)) - F(f(t_{i-1}+)) = \nabla F(\theta_i) \cdot [f(t_i-) - f(t_{i-1}+)],$$

where $\theta_i := \lambda_i f(t_i-) + (1 - \lambda_i)f(t_{i-1}+)$. Thus

$$\begin{aligned} |R(\tau)| &\leq \sum_{i=1}^n |h(s_i)| \|\nabla F(\theta_i) - \nabla F(f(s_i))\| \|f(t_i-) - f(t_{i-1}+)\| \\ &\leq \epsilon \|h\|_{\sup} \sum_{i=1}^n \|f(t_i-) - f(t_{i-1}+)\| \leq \epsilon \|h\|_{\sup} v_1(f; [a, b]). \end{aligned}$$

Since each of the four other sums in (2.92) converges under refinements of partitions to the respective four terms in (2.91), the stated equality (2.91) holds, proving the theorem. \square

Taking the C^1 function F defined by $F(u, v) := uv$, $u, v \in \mathbb{R}$, we obtain from the preceding theorem:

Corollary 2.88. *Let f, g be two real-valued functions on $[a, b]$ having bounded variation, and let $h \in \mathcal{R}[a, b]$. Then the product function fg has bounded variation, and*

$$\begin{aligned} (S) \int_a^b h \, d(fg) &= (S) \int_a^b h f \, dg + (S) \int_a^b h g \, df \\ &\quad - \sum_{(a,b]} h \Delta^- f \Delta^- g + \sum_{[a,b)} h \Delta^+ f \Delta^+ g, \end{aligned} \quad (2.93)$$

where the two sums converge absolutely if $a < b$ and equal 0 if $a = b$.

Proof. We can assume that $a < b$. The two full Stieltjes integrals on the right side of (2.93) exist by Theorems 2.20, 2.17, and 2.42. By linearity, their sum gives the integral $(S) \int_a^b h(\nabla F \circ (g, f)) \cdot d(g, f)$ with $F(u, v) := uv$ for $u, v \in \mathbb{R}$. Thus (2.93) follows from (2.91), proving the corollary. \square

Taking $h \equiv 1$ in the preceding corollary, we recover (for f, g of bounded variation) the integration by parts formula for the RYS integral obtained in Corollary 2.82.

2.9 Relations between Integrals

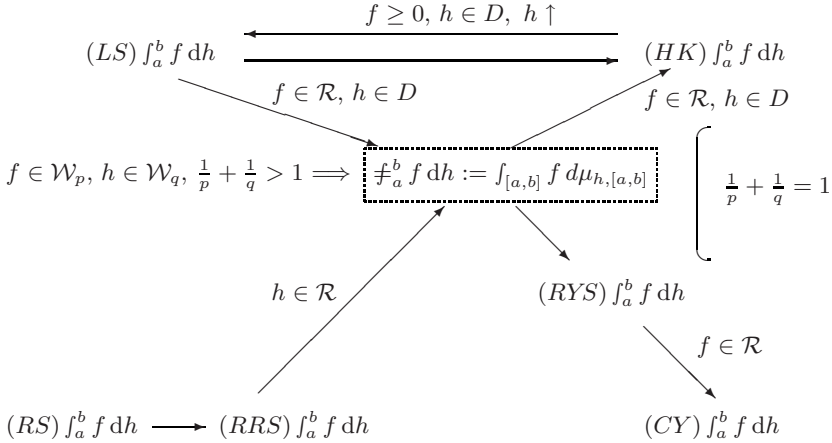


Fig. 2.1. Implications for integrals

In this section we explore whether, under some hypotheses, if one integral exists, then so does another, with the same value. Figure 2.1 summarizes some implications and Table 2.1 gives references for proofs. In Figure 2.1, \longrightarrow means that existence of the integral to the left of it implies that of the integral to the right of it, with the same value. For “ \longleftarrow ” left and right are interchanged. The marking “ $f \in \mathcal{R}$ ” or “ $h \in \mathcal{R}$ ” means that the implication holds for regulated f or h , respectively. The marking “ $h \in D$ ” means that the implication holds for a regulated and right-continuous h . Finally, “ \implies ” means that the condition to the left of it implies existence of the integral to the right of it. The condition $\frac{1}{p} + \frac{1}{q} = 1$ on the right has no arrows from (or to) it,

Table 2.1. References to proofs of the implications shown in Figure 2.1:

$(RS) \int_a^b f \, dh \longrightarrow (RRS) \int_a^b f \, dh$	Proposition 2.13
$f \in \mathcal{W}_q, h \in \mathcal{W}_p, \frac{1}{p} + \frac{1}{q} > 1 \implies \nexists_a^b f \, dh$	Corollary 3.95
$(RRS) \int_a^b f \, dh \longrightarrow \nexists_a^b f \, dh, \quad h \in \mathcal{R}$	Propositions 2.18 and 2.27
$(LS) \int_a^b f \, dh \longrightarrow \nexists_a^b f \, dh, \quad f \in \mathcal{R}, h \in D$	Corollary 2.29
$(LS) \int_a^b f \, dh \longleftrightarrow (HK) \int_a^b f \, dh, \quad f \geq 0, h \in D, h \uparrow$	Theorem 2.71
$\nexists_a^b f \, dh \longrightarrow (HK) \int_a^b f \, dh, \quad f \in \mathcal{R}, h \in D$	Theorem 2.69
$\nexists_a^b f \, dh \longrightarrow (RYS) \int_a^b f \, dh$	Proposition 2.27
$(RYS) \int_a^b f \, dh \longrightarrow (CY) \int_a^b f \, dh$	Theorem 2.51(a)
$f \in \mathcal{W}_p, g \in \mathcal{W}_q, \frac{1}{p} + \frac{1}{q} = 1, 1 < p < \infty$ does not imply that any of the above integrals exists	Proposition 3.104

signifying that there exist $f \in \mathcal{W}_p$ and $h \in \mathcal{W}_q$ with $\frac{1}{p} + \frac{1}{q} = 1, 1 < p, q < \infty$, such that $\int_a^b f \, dh$ does not exist for any of the definitions given. The figure is for real-valued functions and for $a < b$, although the implications not about LS or HK integrals hold for f and h X - and Y -valued respectively.

Proposition 2.89. *Let μ be an upper continuous additive X -valued interval function on $[a, b]$, and let ν be a Y -valued interval function on $[a, b]$ such that $\nu(\emptyset) = 0$. Let h be the point function on $[a, b]$ defined by $h(t) := \mu([a, t])$ for $a \leq t \leq b$, and let g be a Y -valued regulated function on $[a, b]$ such that $\nu([a, t)) = g(t-)$ for $a < t \leq b$. The Kolmogorov integral $\nexists_{[a, b]} \nu([a, \cdot)) \cdot d\mu$ exists if and only if $(RRS) \int_a^b g_-^{(a)} \cdot dh$ does, and if they exist then for each $a \leq t \leq b$,*

$$\nexists_{[a, t]} \nu([a, \cdot)) \cdot d\mu = (RRS) \int_a^t g_-^{(a)} \cdot dh.$$

Also for $a \leq c < b$, $\nexists_{(c, b]} \nu([a, \cdot)) \cdot d\mu$ exists if and only if $(RRS) \int_c^b g_-^{(c)} \cdot dh$ does, and if they exist then for each $t \in (c, b]$,

$$\nexists_{(c, t]} \nu([a, \cdot)) \cdot d\mu = (RRS) \int_c^t g_-^{(c)} \cdot dh.$$

Proof. We can assume that $a < b$. By Proposition 2.6(f), since μ is upper continuous and additive, h is regulated and right-continuous on $[a, b]$. By Propositions 2.18 and 2.46, since for any $a \leq c \leq t \leq b$, $g_-^{(c)}$ is left-continuous on $(c, t]$, the integral $(RRS) \int_c^t g_-^{(c)} \cdot dh$ exists if and only if $(RYS) \int_c^t g_-^{(c)} \cdot dh$ does, and the two integrals have the same value if they exist. Since $\nu(\emptyset) = 0$ and $\Delta^+ h(a) = 0$, we have that

$$S_{YS}(\nu([a, \cdot)), d\mu; [a, t], \tau) = S_{YS}(g_-^{(a)}, dh; [a, t], \tau)$$

for any tagged Young interval partition τ of $[a, t]$. Therefore by Proposition 2.25 and the definition of the (RYS) integral, the Kolmogorov integral

$\oint_{[a,t]} \nu([a, \cdot]) \cdot d\mu$ exists if and only if $(RRS) \int_a^t g_-^{(a)} \cdot dh$ does, which follows from existence of either integral for $t = b$, and the two integrals have the same value for each t if they exist, proving the first part of the lemma. To prove the second part, again let $a \leq c < t \leq b$. Since $\Delta^+ h(c) = 0$, we have that

$$S_{YS}(\nu([a, \cdot]), d\mu; (c, t], \tau) = S_{YS}(g_-^{(c)}, dh; [c, t], \tau)$$

for any tagged Young interval partition τ of $\llbracket c, t \rrbracket$. Therefore by Proposition 2.25 again, the Kolmogorov integral $\oint_{(c,t]} \nu([a, \cdot]) \cdot d\mu$ exists if and only if $(RYS) \int_c^t g_-^{(c)} \cdot dh$ does, which follows from existence of either integral for $t = b$, and the two integrals have the same values for each t . The proof of the proposition is complete. \square

The following relation between integrals of particular forms is used in Chapter 9. It follows from the preceding proposition and from definitions (1.15), (1.16) of the integrals with the reversed order.

Proposition 2.90. *Let μ be an upper continuous additive X -valued interval function on $[a, b]$, and let ν be a Y -valued interval function on $[a, b]$ such that $\nu(\emptyset) = 0$. Let h be the point function on $[a, b]$ defined by $h(t) := \mu([a, t])$ for $a \leq t \leq b$, and let g be a Y -valued regulated function on $[a, b]$ such that $\nu([a, t)) = g(t-)$ for $a < t \leq b$. The Kolmogorov integral $\oint_{[a,b]} d\mu \cdot \nu([a, \cdot])$ exists if and only if $(RRS) \int_a^b dh \cdot g_-^{(a)}$ does, and if they exist then for each $a \leq t \leq b$,*

$$\oint_{[a,t]} d\mu \cdot \nu([a, \cdot]) = (RRS) \int_a^t dh \cdot g_-^{(a)}.$$

Also for $a \leq c < b$, $\oint_{(c,b]} d\mu \cdot \nu([a, \cdot])$ exists if and only if $(RRS) \int_c^b dh \cdot g_-^{(c)}$ does, and if they exist then for each $t \in (c, b]$,

$$\oint_{(c,t]} d\mu \cdot \nu([a, \cdot]) = (RRS) \int_c^t dh \cdot g_-^{(c)}.$$

The additivity property of the Kolmogorov integral in conjunction with the preceding proposition implies the following additivity property.

Corollary 2.91. *Let $h \in \mathcal{R}([a, b]; X)$ be right-continuous and $g \in \mathcal{R}([a, b]; Y)$. For $a < c < b$, the integral $(RRS) \int_a^b dh \cdot g_-^{(a)}$ exists if and only if both $(RRS) \int_a^c dh \cdot g_-^{(a)}$ and $(RRS) \int_c^b dh \cdot g_-^{(c)}$ exist, and then*

$$(RRS) \int_a^b dh \cdot g_-^{(a)} = (RRS) \int_a^c dh \cdot g_-^{(a)} + (RRS) \int_c^b dh \cdot g_-^{(c)}.$$

Proof. Let $\mu := \mu_{h,[a,b]}$ and $\nu := \mu_{g,[a,b]}$ be the interval functions defined by (2.2), and let $\tilde{h}(t) := \mu([a, t])$ for $a \leq t \leq b$. Using Proposition 2.6 and the definition (2.3) of $R_{\mu,a}$, it then follows that $\tilde{h} = R_{\mu,a} = h - h(a)$, μ is an upper continuous additive interval function, $\nu(\emptyset) = 0$, and $g(t-) = R_{\nu,a}(t-) = \nu([a, t])$ for $a < t \leq b$. Therefore by additivity of the Kolmogorov integral (Theorem 2.21 extended to the integral (1.16) with the reversed order) and by Proposition 2.90, it follows that $(RRS) \int_a^b dh \cdot g_-^{(a)}$ exists if and only if both $(RRS) \int_a^c dh \cdot g_-^{(a)}$ and $(RRS) \int_c^b dh \cdot g_-^{(c)}$ exist, and then

$$\begin{aligned} (RRS) \int_a^b dh \cdot g_-^{(a)} &= (RRS) \int_a^b d\tilde{h} \cdot g_-^{(a)} = \int_{[a,b]} d\mu \cdot \nu([a, \cdot)) \\ &= \int_{[a,c]} d\mu \cdot \nu([a, \cdot)) + \int_{(c,b]} d\mu \cdot \nu([a, \cdot)) \\ &= (RRS) \int_a^c dh \cdot g_-^{(a)} + (RRS) \int_c^b dh \cdot g_-^{(c)}. \end{aligned}$$

The proof is complete. \square

2.10 Banach-Valued Contour Integrals and Cauchy Formulas

Throughout this section we assume that $a < b$. For suitable curves $\zeta(\cdot)$ in the complex plane \mathbb{C} and functions f , which may be Banach-valued, integrals $\int_{\zeta(\cdot)} f(\zeta) d\zeta$ will be defined and treated. The curves will be just as in the classical theory of holomorphic functions into \mathbb{C} , as follows.

Definition 2.92. A *curve* is a continuous function $\zeta(\cdot)$ from a nondegenerate interval $[a, b]$ into \mathbb{C} . A C^1 *arc* is a curve $\zeta(\cdot)$ such that for $\zeta \equiv \xi + i\eta$ with ξ and η real-valued, the derivative $\zeta'(t) = \xi'(t) + i\eta'(t)$ exists and is continuous and non-zero for $a < t < b$, and has limits $\zeta'(a+) = \lim_{t \downarrow a} \zeta'(t) \neq 0 \neq \zeta'(b-) = \lim_{t \uparrow b} \zeta'(t)$. A *piecewise C^1 curve* is a curve ζ on $[a, b]$ such that for some partition $a = t_0 < t_1 < \dots < t_n = b$, $\zeta(\cdot)$ is a C^1 arc on each $[t_{i-1}, t_i]$, and such that for each $i = 1, \dots, n-1$, the ratio $\zeta'(t_i+)/\zeta'(t_i-)$ is not real and negative, nor is $\zeta'(a+)/\zeta'(b-)$. A *closed curve* is a curve $\zeta(\cdot)$ from $[a, b]$ into \mathbb{C} such that $\zeta(a) = \zeta(b)$. The closed curve $\zeta(\cdot)$ is *simple* if $\zeta(s) \neq \zeta(t)$ for $a \leq s < t < b$.

If z_0 is a point not in the range of a closed curve $\zeta(\cdot)$, that is, $z_0 \notin \text{ran}(\zeta)$, then we can write $\zeta(t) - z_0 = r(t)e^{i\theta(t)}$ for some real $r(t) > 0$ and $\theta(t)$ which are continuous functions of t , with $\theta(b) - \theta(a) = 2n\pi$ for some integer n , called the *winding number* $w(\zeta(\cdot), z_0)$ of ζ around z_0 .

For example, if $[a, b] = [0, 1]$ and $\zeta(t) = e^{2\pi it}$, then $w(\zeta(\cdot), z) = 1$ if $|z| < 1$ and 0 if $|z| > 1$.

Proposition 2.93. *For a piecewise C^1 curve $\zeta(\cdot)$ on $[a, b]$ (not necessarily simple) and a point $z \notin \text{ran}(\zeta)$, we have*

$$w(\zeta(\cdot), z) = \frac{1}{2\pi i} (RS) \int_a^b \frac{d\zeta(t)}{\zeta(t) - z}.$$

Proof. In the representation $\zeta(t) = z + r(t)e^{i\theta(t)}$, $r(\cdot) > 0$ and $\theta(\cdot)$ are piecewise C^1 functions on $[a, b]$. Using elementary calculations, we get

$$\begin{aligned} \frac{1}{2\pi i} \int_a^b \frac{d\zeta(t)}{\zeta(t) - z} &= \frac{1}{2\pi i} \left[\int_a^b \frac{dr(t)}{r(t)} + \int_a^b \frac{de^{i\theta(t)}}{e^{i\theta(t)}} \right] \\ &= \frac{1}{2\pi i} \left[\log \frac{r(b)}{r(a)} + i(\theta(b) - \theta(a)) \right] = w(\zeta(\cdot), z) \end{aligned}$$

since $r(b) = r(a)$, proving the proposition. \square

A topological space (S, \mathcal{T}) is *connected* if it is not the union of two disjoint nonempty open sets. A subset $C \subset S$ is *connected* iff it is connected in its relative topology. Holomorphic functions with values in a Banach space are defined as follows:

Definition 2.94. Let X be a complex Banach space and let U be an open connected set in \mathbb{C} . A function $f: U \rightarrow X$ has a *Taylor expansion* around a point $u \in U$ if there are an $r > 0$ and a sequence $\{h_k\}_{k \geq 0} \subset X$ such that for all z with $|z - u| < r$, we have $z \in U$ and

$$f(z) = \sum_{k=0}^{\infty} (z - u)^k h_k, \quad (2.94)$$

where the series converges in X and is called the *Taylor series* of f around u . A function $f: U \rightarrow X$ is *holomorphic* on U if it has a Taylor expansion around each point of U .

We assume that holomorphic functions are defined on connected open sets, even if this assumption is not used until we deal with analytic continuation, as in Theorems 5.33 and 5.34, for the following reasons.

Let U and V be disjoint nonempty open subsets of \mathbb{C} . Let g and h be any two holomorphic functions from U and V respectively into \mathbb{C} . Let $f(z) := g(z)$ for $z \in U$ and $f(z) := h(z)$ for $z \in V$. Then f would be “holomorphic” on $U \cup V$. But suppose g and h can each be extended to be entire functions, holomorphic on all of \mathbb{C} , and $g \neq h$ on \mathbb{C} . When a function g has an entire extension, such an extension is unique and it seems unnatural to define a holomorphic function in a way that could contradict an entire extension.

More specifically, consider sums $f(z) = \sum_{k \geq 1} a_k / (z - z_k)$ with $\sum_k |a_k| < \infty$, called Borel series. Suppose that $|z_k| \geq 1$ for all k and that the set of all

limits of subsequences z_{k_j} with $k_j \rightarrow \infty$ as $j \rightarrow \infty$ is the unit circle $T^1 := \{z : |z| = 1\}$. Then the series converges on $U \cup V$ where $U := \{z : |z| < 1\}$ and $V := \{z : |z| > 1\} \setminus \{z_j\}_{j \geq 1}$. The sum $f(z)$ is holomorphic on U and on V . Each of U and V is open and connected, and they are disjoint. It can happen that $f \equiv 0$ on U but $f \not\equiv 0$ on V , e.g. [196, Theorem 4.2.5]. So again, analytic continuation from U would conflict with the values on V .

Other holomorphic functions on open sets $U \subset \mathbb{C}$ have no entire extension and non-unique maximal holomorphic extensions in \mathbb{C} . Let $\mathbb{C} = \{z = x + iy : x, y \in \mathbb{R}\}$ and $U := \{z \in \mathbb{C} : x > 0\}$. Consider the function $g(z) = \sqrt{z}$ for $z \in U$ (which has a branch point at 0). It has a holomorphic extension to the complement $\{z \in \mathbb{C} : y = 0, x \leq 0\}^c$ and another to $\{z \in \mathbb{C} : x = 0, y \leq 0\}^c$. Neither extension can be extended holomorphically to any further point. To obtain natural domains for holomorphic functions with branch points one takes Riemann surfaces (cf. [1, 2nd ed., 1966 §3.4.3]).

We show next that a Taylor expansion if it exists is unique and the series converges absolutely and uniformly.

Lemma 2.95. *Let X be a Banach space over \mathbb{K} and let $\{h_k\}_{k \geq 0} \subset X$. If the power series $\sum_{k \geq 0} t^k h_k$ converges absolutely and its sum is equal to zero for each $t \in \mathbb{K}$ such that $|t| \leq \delta$ for some $\delta > 0$, then $h_k \equiv 0$ for $k = 0, 1, \dots$*

Proof. Taking $t = 0$ we get $h_0 \equiv 0$. Suppose there is a least integer $k_0 \geq 1$ such that $h_{k_0} \neq 0$. For each t with $0 < |t| < \delta$, we have $\sum_{k \geq k_0} t^{k-k_0} h_k = 0$, and so

$$\|h_{k_0}\| \leq \sum_{k > k_0} \|h_k\| |t|^{k-k_0} = \frac{1}{\delta^{k_0}} \sum_{k > k_0} \|h_k\| \delta^k \left(\frac{|t|}{\delta}\right)^{k-k_0} \leq \frac{C}{\delta^{k_0}} \frac{|t|}{(\delta - |t|)},$$

where $C := \sup\{\|h_k\| \delta^k : k \geq 0\} < \infty$. Letting $|t| \rightarrow 0$ it follows that $h_{k_0} = 0$, a contradiction, proving that $h_k = 0$ for all $k = 1, 2, \dots$

Proposition 2.96. *Let X be a complex Banach space, and let U be an open connected set in \mathbb{C} . If $f : U \rightarrow X$ has a Taylor expansion (2.94) around $u \in U$ and the Taylor series converges on $B(u, r)$ for some $r > 0$, then it converges absolutely on $B(u, r)$ and uniformly on $\bar{B}(u, s)$ for any $s < r$. In particular, f is continuous on $B(u, r)$ and its Taylor expansion around u is unique.*

Proof. If $0 < s < r$ then $s^k \|h_k\| \rightarrow 0$ as $k \rightarrow \infty$, and so $\limsup_{k \rightarrow \infty} \|h_k\|^{1/k} \leq 1/s$. Since $s \in (0, r)$ is arbitrary, $\limsup_{k \rightarrow \infty} \|h_k\|^{1/k} \leq 1/r$. Thus the Taylor series (2.94) converges absolutely by the root test for series of positive numbers. By Lemma 2.95, the Taylor expansion (2.94) is unique. It also follows that the Taylor series converges uniformly on $\bar{B}(u, s)$ for any $s < r$, and so f is continuous on $B(u, r)$, proving the proposition. \square

Let $\zeta(\cdot)$ be a piecewise C^1 curve from $[a, b]$ into \mathbb{C} . Let X be a Banach space over \mathbb{C} and let f be a continuous function into X from a set in \mathbb{C} including the range of $\zeta(\cdot)$. Then the contour integral over $\zeta(\cdot)$ is defined by

$$\oint_{\zeta(\cdot)} f(\zeta) d\zeta := (RS) \int_a^b f(\zeta(t)) \cdot d\zeta(t),$$

where \cdot denotes the natural bilinear mapping $X \times \mathbb{C} \rightarrow X$.

The Cauchy integral formula will first be extended to Banach-valued functions for curves which are circles.

Proposition 2.97. *Let f be a holomorphic function from an open disk $U := \{z: |z - w| < r\} \subset \mathbb{C}$ into a complex Banach space X . Let $0 < t < r$ and $\zeta(\theta) := w + te^{i\theta}$ for $0 \leq \theta \leq 2\pi$. Then for any $z \in \mathbb{C}$ with $|z - w| < t$,*

$$f(z) = \frac{1}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{\zeta - z}. \quad (2.95)$$

Proof. Let L be any continuous \mathbb{C} -valued linear functional on X ($L \in X'$), so that for some $K < \infty$, $|L(x)| \leq K\|x\|$ for all $x \in X$. For each convergent series $f(v) = \sum_{k=0}^{\infty} (v - w)^k h_k$, we have a convergent series $L(f(v)) = \sum_{k=0}^{\infty} L(h_k)(v - w)^k$, and so $L \circ f$ is holomorphic from U into \mathbb{C} . Thus (2.95) holds for $L \circ f$ in place of f by the classical Cauchy integral formula (e.g. Ahlfors [1, 2nd ed., 1966 §4.2.2, Theorem 6]). Now L can be interchanged with the integral by Proposition 2.78. If $x, y \in X$ and $L(x) = L(y)$ for all $L \in X'$, then $x = y$ by the Hahn–Banach theorem. The conclusion follows. \square

For a function f from an open set $U \subset \mathbb{C}$ into a complex Banach space X and $z \in U$, the *derivative* $f'(z)$ is defined, if it exists, as

$$\lim_{w \rightarrow z} \frac{f(w) - f(z)}{w - z} \in X.$$

Let $f^{(1)} := f'$ and $n \geq 1$. If the n th derivative $f^{(n)}$ is defined in a neighborhood of z , then $f^{(n+1)}(z)$ is defined as $(f^{(n)})'(z)$, if it exists.

The classical Cauchy integral formula for derivatives extends directly to Banach spaces:

Theorem 2.98. *Under the hypotheses of Proposition 2.97, $f^{(n)}(z)$ exists for all $n = 1, 2, \dots$ and*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}.$$

The conclusion follows from Proposition 2.97 and the next lemma since a holomorphic function is bounded on C by Proposition 2.96.

Lemma 2.99. *Suppose that g is continuous on the range C of a piecewise C^1 curve $\zeta(\cdot)$. Then for each $n \geq 1$, the function*

$$I_n(z) = I_n(g; z) := \oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z)^n}$$

has a derivative on the complement C^c of C , and $I'_n(z) = nI_{n+1}(z)$.

Proof. We prove first that I_1 is continuous on C^c . Let $z_0 \in C^c$ and let $\delta > 0$ be such that the open ball $|z - z_0| < \delta$ does not intersect C . By restricting z to the smaller ball $|z - z_0| < \delta/2$, we have that $|\zeta(t) - z| > \delta/2$ for each $t \in [a, b]$. By linearity of the RS integral, we have

$$I_1(z) - I_1(z_0) = (z - z_0) \oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)}.$$

Using the bounds (2.84), it follows that

$$|I_1(z) - I_1(z_0)| \leq |z - z_0|(12/\delta^2)\|g \circ \zeta\|_{[a, b], \sup} v_1(\zeta; [a, b]),$$

and so $I_1 = I_1(g)$ is continuous at z_0 , and this is true for all continuous g on C .

Applying the first part of the proof to the function $h(\zeta) := g(\zeta)/(\zeta - z_0)$, we conclude that

$$\frac{I_1(z) - I_1(z_0)}{z - z_0} = \oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z)(\zeta - z_0)} = I_1(h; z) \rightarrow I_1(h; z_0) = I_2(g; z_0)$$

as $z \rightarrow z_0$. This proves the conclusion in the case $n = 1$.

The conclusion for an arbitrary n is proved by induction. Suppose that for some $n \geq 2$, $I'_{n-1}(g; z) = (n-1)I_n(g; z)$ for $z \in C^c$, for all continuous g on C . Again let $z_0 \in C^c$. Using the identity

$$\begin{aligned} I_n(z) - I_n(z_0) &= \left[\oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z)^{n-1}(\zeta - z_0)} - \oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z_0)^n} \right] \\ &\quad + (z - z_0) \oint_{\zeta(\cdot)} \frac{g(\zeta) d\zeta}{(\zeta - z)^n(\zeta - z_0)}, \end{aligned}$$

one can conclude that I_n is continuous at z_0 . Indeed, defining $h(\zeta) := g(\zeta)/(\zeta - z_0)$, the first term is equal to $I_{n-1}(h; z) - I_{n-1}(h; z_0)$ and tends to zero as $z \rightarrow z_0$ by the induction hypothesis, while in the second term the integral is bounded for z in a neighborhood of z_0 as shown in the first part of the proof. Now dividing the identity by $z - z_0$ and letting $z \rightarrow z_0$, the quotient in the first term tends to a derivative $I'_{n-1}(h; z_0)$, which by the induction hypothesis equals $(n-1)I_n(h; z_0) = (n-1)I_{n+1}(g; z_0)$. The remaining factor $I_n(h; z)$ is continuous as before, and so has the limit $I_n(h; z_0) = I_{n+1}(g; z_0)$. Thus $I'_n(z_0)$ exists and equals $nI_{n+1}(z_0)$. The proof of the lemma is complete. \square

Proposition 2.100. *Let X be a complex Banach space, and let f be a holomorphic function on a disk $B(u, r) \subset \mathbb{C}$ with values in X for some $u \in \mathbb{C}$ and $r > 0$. Then for each $z \in B(u, r)$,*

$$f(z) = f(u) + \sum_{k \geq 1} \frac{(z-u)^k}{k!} f^{(k)}(u)$$

is the Taylor expansion of f around u .

Proof. Let $0 < s < t < r$, let $\zeta(\theta) := u + te^{i\theta}$ for $0 \leq \theta \leq 2\pi$, and let $z \in B(u, t)$. Then by the Cauchy integral formula (Proposition 2.97),

$$f(z) = \frac{1}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{\zeta - z}.$$

For each $\zeta \in \text{ran}(\zeta)$, the series

$$\frac{1}{\zeta - z} = (\zeta - u)^{-1} \sum_{k=0}^{\infty} \left(\frac{z-u}{\zeta-u} \right)^k$$

converges absolutely, and uniformly if $|z-u| \leq s$. Since f is continuous on the range $\text{ran}(\zeta)$ by Proposition 2.96, it is bounded. Then term-by-term integration yields

$$f(z) = \sum_{k=0}^{\infty} \frac{(z-u)^k}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{(\zeta-u)^{k+1}}.$$

Using the bound (2.84) for the RS integral and the root test, it follows that the series converges absolutely. Now the conclusion follows by the Cauchy integral formula for derivatives (Theorem 2.98) and by the uniqueness of a Taylor expansion (Proposition 2.96). \square

Proposition 2.101. *If f is holomorphic from an open connected set $U \subset \mathbb{C}$ into a Banach space X and $w \in U$, then for*

$$g(z) := \begin{cases} \frac{f(z)-f(w)}{z-w}, & \text{if } z \neq w, \\ f'(w), & \text{if } z = w, \end{cases}$$

g is also holomorphic on U .

Proof. It is easily seen that g is holomorphic on $U \setminus \{w\}$, since $z \mapsto 1/(z-w)$ is, so it suffices to find a power series expansion for g around w . Let $f(z) = \sum_{k=0}^{\infty} (z-w)^k x_k$ for $|z-w| < r$, where $r > 0$ and $x_k := f^{(k)}(w)/k! \in X$ by Proposition 2.100. Then $g(z) = \sum_{k=0}^{\infty} (z-w)^k x_{k+1}$, also for $|z-w| < r$, with $g(w) = x_1 = f'(w)$, finishing the proof. \square

A topological space (S, \mathcal{T}) is *locally connected* iff \mathcal{T} has a base consisting of connected open sets. Clearly, any normed vector space with its usual topology is connected and locally connected. In particular, \mathbb{R} and \mathbb{C} are locally connected.

Theorem 2.102. *For any topological space (S, \mathcal{T}) and any nonempty connected set $A \subset S$ there is a unique connected set $B \supset A$ such that B is a maximal connected set for inclusion. Such a set B is always closed.*

Proof. The collection of connected sets including A is partially ordered by inclusion. The union of any inclusion-chain of connected sets is easily seen to be connected. Thus by Zorn's lemma there is a connected set $B_1 \supset A$ which is a maximal connected set for inclusion. Suppose B_2 is another such set. The union of two non-disjoint connected sets is easily seen to be connected. This gives a contradiction unless $B_1 = B_2$, so B is unique. Moreover, the closure of any connected set is easily shown to be closed. So by maximality, B is closed and the theorem is proved. \square

A maximal connected subset B of S is called a *component*. If A is a singleton $\{x\}$, it is always connected, and the B from the preceding theorem is called the *component of x* .

Proposition 2.103. *Let (S, \mathcal{T}) be a locally connected topological space. Then any component F of S is open as well as closed. Thus S has a unique decomposition as a union of disjoint open and closed components.*

Proof. Let F be a component of S and $x \in F$. Let V be a connected open neighborhood of x by local connectedness. Then $F \cup V$ is connected since F and V are and $F \cap V \supset \{x\} \neq \emptyset$. So $V \subset F$ and F is open. The rest follows. \square

Clearly, an open set in a locally connected space is locally connected, so it is in a unique way a union of disjoint nonempty connected open sets.

An open set $U \subset \mathbb{C}$ will here, following Ahlfors [1, 2nd ed., §4.4.2], be called *simply connected* if and only if U^c has no bounded component. A more general definition is based on the following.

Definition 2.104. A closed curve $\zeta(\cdot)$ from $[a, b]$ into a set U is *null-homotopic in U* if there exists a jointly continuous function H from $[a, b] \times [0, 1]$ into U such that $H(\cdot, 0) \equiv \zeta(\cdot)$, for some $w \in U$, $H(\cdot, 1) \equiv w$, and for $0 \leq u \leq 1$, $H(a, u) = H(b, u)$, so that each $H(\cdot, u)$ is a closed curve.

Theorem 2.105. *Let $U \subset \mathbb{C}$ be a connected nonempty open set. Then the following are equivalent:*

- (a) *any closed curve $\zeta(\cdot)$ with range in U is null-homotopic in U ;*

- (b) for every closed curve $\zeta(\cdot)$ into U and $z \in U^c$, $w(\zeta(\cdot), z) = 0$;
 (c) U^c has no bounded component;
 (d) U is homeomorphic to the open unit disk $\{z \in \mathbb{C}: |z| < 1\}$.

Proof. The implications (d) \rightarrow (a) \rightarrow (b) and (c) \rightarrow (b) are easy. The other implications are not as easy and are given in Newman [178, §§6.6 and 7.9]. Complex analysis texts also give proofs, of the stronger fact that in (d), unless $U = \mathbb{C}$, the homeomorphisms can be taken to be holomorphic with holomorphic inverses (Riemann mapping theorem, see e.g. Beardon [12, Theorem 11.1.1 and §11.2]). \square

In Theorem 2.105, (a) is the definition of simply connected for general topological spaces. It is not equivalent to (c) in \mathbb{R}^3 , for example.

Here is a Cauchy integral theorem for Banach-valued functions.

Theorem 2.106. *Let f be holomorphic from an open connected set $U \subset \mathbb{C}$ into a complex Banach space X and let $\zeta(\cdot)$ be a piecewise C^1 closed curve whose range is included in U . If (i) the winding number $w(\zeta(\cdot); z) = 0$ for each $z \notin U$, e.g. if (ii) U is simply connected, then*

$$\oint_{\zeta(\cdot)} f(\zeta) d\zeta = 0. \quad (2.96)$$

Proof. Condition (ii) implies (i) by Theorem 2.105, (c) \Rightarrow (b), or (a) \Rightarrow (b) for the general definition of simply connected. So we can assume (i). By Proposition 2.96, f is continuous on U , and so the integral (2.96) is defined. For any continuous linear functional $L \in X'$, $z \mapsto L(f(z))$ is holomorphic from U into \mathbb{C} . Thus by Proposition 2.78, we have

$$L\left(\oint_{\zeta(\cdot)} f(\zeta) d\zeta\right) = \oint_{\zeta(\cdot)} L(f(\zeta)) d\zeta.$$

The integral on the right is zero by the Cauchy theorem for multiply connected sets (e.g., Ahlfors [1, 2nd ed., §4.4.4 Theorem 18]). Thus by the Hahn–Banach theorem (e.g. [53, 6.1.5 Corollary]), it follows that (2.96) holds, proving the theorem. \square

Here are Cauchy integral formulas with winding numbers.

Theorem 2.107. *Let f be holomorphic from a connected and simply connected open set $U \subset \mathbb{C}$ into a complex Banach space X . Let $\zeta(\cdot)$ be a piecewise C^1 closed curve whose range C is included in U and let $z \in U \setminus C$. Then*

$$w(\zeta(\cdot), z)f(z) = \frac{1}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{\zeta - z}, \quad (2.97)$$

and for $n = 1, 2, 3, \dots$,

$$w(\zeta(\cdot), z)f^{(n)}(z) = \frac{n!}{2\pi i} \oint_{\zeta(\cdot)} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}. \quad (2.98)$$

Proof. On the right side of (2.97) write $f(\zeta) = [f(\zeta) - f(z)] + f(z)$. For the $f(\zeta) - f(z)$ term, the integral is 0 by Proposition 2.101 and Theorem 2.106. For the $f(z)$ term, we get the left side of (2.97) by Proposition 2.93, proving (2.97). We thus have (2.98) for $n = 0$. For a given $\zeta(\cdot)$ and $z \notin C$, $w(\zeta(\cdot), u)$ is constant for u in a neighborhood of z , so the multiplication by $w(\zeta(\cdot), u)$ can be interchanged with $d^n/du^n|_{u=z}$. On the right, we can use induction on n and Lemma 2.99, so (2.98) is proved. \square

If $w(\zeta(\cdot), z) = 1$, we get the familiar form of the formula, now for rather general closed curves $\zeta(\cdot)$.

2.11 Notes

Notes on Section 2.1. More properties of a real-valued function equivalent to being regulated are given by Theorem 2.1 in [54, Part III].

The correspondence between right-continuous regulated point functions and additive upper continuous interval functions given by Corollary 2.11 is similar to the well-known correspondence between right-continuous point functions of bounded variation and σ -additive set functions on the Borel σ -algebra.

Notes on Section 2.2. In T. J. Stieltjes's treatment of his integral $\int_a^b f dh$ in [227], the integrand f is continuous, while the integrator h is a monotone increasing function. The Riemann–Stieltjes integral $(RS) \int_a^b f dh$, for any f, h such that it exists, appeared in G. König [122], who apparently had been using the RS integral in his lectures for some time, as stated in [228, p. 247] and in [168, p. 314]. Szénácssy [228] has a chapter on König, where on p. 247 of the English translation, König's general definition of the RS integral is quoted. Interest in Stieltjes-type integrals flourished after F. Riesz [193] showed that the Stieltjes integral provides a representation of an arbitrary bounded linear functional on the space $C[0, 1]$ of all continuous functions on $[0, 1]$. Pollard [187] introduced the refinement Riemann–Stieltjes integral, showed that it extends the Riemann–Stieltjes integral, and proved that for h nondecreasing and f bounded, the integral $(RRS) \int_a^b f dh$ exists if and only if the Darboux–Stieltjes integral does. The latter integral is defined as $\sup_{\kappa} L(f, h; \kappa) = \inf_{\kappa} U(f, h; \kappa)$ if the equality holds, where for a partition $\kappa = \{t_i\}_{i=0}^n$ of $[a, b]$, $m(f; A) := \inf\{f(t) : t \in A\}$ and $M(f; A) := \sup\{f(t) : t \in A\}$,

$$\begin{cases} L(f, h; \kappa) := \sum_{i=1}^n m(f; [t_{i-1}, t_i])[h(t_i) - h(t_{i-1})], \\ U(f, h; \kappa) := \sum_{i=1}^n M(f; [t_{i-1}, t_i])[h(t_i) - h(t_{i-1})]. \end{cases} \quad (2.99)$$

F. A. Medvedev [168] gives an extensive account of the early history of Stieltjes-type integrals, and in particular gives some details about König's work on the RS integral.

S. Pollard [187] gave a detailed treatment of the refinement Riemann–Stieltjes integral. Earlier, a theory of limits based on refinements was treated by E. H. Moore [175]. Thus the RRS integral sometimes is called the Pollard–Moore–Stieltjes integral (see e.g. [94, p. 269]). The books of Gochman [82] and Hildebrandt [97, §§9–18] also contain detailed expositions of the RRS integral.

For real-valued functions, Propositions 2.13 and 2.15 were proved in Pollard [187, p. 90] and Smith [223], respectively.

Notes on Section 2.3. The refinement Young–Stieltjes integral developed in stages over a long period. This integral was rediscovered by several authors, and hence it is known by different names. According to T. H. Hildebrandt [94], the RYS integral originated in the work of W. H. Young [254]. In this work W. H. Young extended his approach to defining the Lebesgue integral, and the RYS integral appeared there as an auxiliary tool. At that time there was considerable interest in enlarging the class of functions integrable with respect to a monotone function h . Lebesgue [134], for example, showed that such a class could be the class of all Lebesgue summable functions with respect to dh when a Stieltjes-type integral is defined by means of the Lebesgue integral using a change of variables formula. Lebesgue suggested that it would be difficult to extend the Stieltjes integral to such general integrands by any other means. Recall that Radon's work which led to the Lebesgue–Stieltjes integral appeared a little later [190]. However, W. H. Young [254] showed that his method of monotonic sequences used in connection with the Lebesgue integral extends to an integration with respect to any function h of bounded variation almost without changes. The main change concerned the definition of the integral $\int f \, dh$ for a step function f . In this case, W. H. Young set

$$\int_a^b f \, dh := \sum_i \{ [f \Delta^+ h](x_{i-1}) + c_i [h(x_i-) - h(x_{i-1}+)] + [f \Delta^- h](x_i) \}, \quad (2.100)$$

provided $f(x) = c_i$ for $x \in (x_{i-1}, x_i)$. The method of monotonic sequences of W. H. Young was later extended by Daniell [37] to integrals of functions defined on abstract sets. A concise theory based on an integral of Stieltjes type was provided by W. H. Young's son L. C. Young in the form of a textbook [243], first published in 1927 (see also the presentation in Hildebrandt [97]). The integral, defined as a limit of Young–Stieltjes sums if it exists when the mesh of tagged partitions tends to zero, was mentioned by R. C. Young [251]. Full use of the refinement Young–Stieltjes integral, in the context of Fourier series, is due to L. C. Young [247].

Motivated by the weakness of the Riemann–Stieltjes integral, Ross [195] suggested an extension of the Darboux–Stieltjes integral, replacing (2.99) by sums reminiscent of Young–Stieltjes sums. Namely, let f be a real-valued func-

tion on $[a, b]$, and let h be a nondecreasing function on $[a, b]$. For a partition $\kappa = \{t_i\}_{i=0}^n$ of $[a, b]$, let

$$\begin{cases} L(f, dh; \kappa) := J(f, dh; \kappa) + \sum_{i=1}^n m(f; (t_{i-1}, t_i)) [h(t_i-) - h(t_{i-1}+)] \\ U(f, dh; \kappa) := J(f, dh; \kappa) + \sum_{i=1}^n M(f; (t_{i-1}, t_i)) [h(t_i-) - h(t_{i-1}+)], \end{cases}$$

where $J(f, dh; \kappa) := \sum_{i=1}^n f(t_i) \Delta_{[a, b]}^{\pm} h(t_i)$, and $m(f, A)$, $M(f, A)$ are defined as in (2.99). We say that f is Ross–Darboux–Stieltjes, or *RDS*, integrable on $[a, b]$ with respect to h , if $U(f, dh) := \inf_{\kappa} U(f, dh; \kappa) = \sup_{\kappa} L(f, dh; \kappa) =: L(f, dh)$, and then let $(RDS) \int_a^b f dh := U(f, dh) = L(f, dh)$. By Theorem 35.25 in [195], a bounded function f on $[a, b]$ is *RDS* integrable with respect to h if and only if there is $C \in \mathbb{R}$ such that given $\epsilon > 0$ there exists $\delta > 0$ such that $|S_{YS}(f, dh; \tau) - C| < \epsilon$ for each tagged Young interval partition $\tau = (\{t_i\}_{i=0}^n, \{s_i\}_{i=1}^n)$ of $[a, b]$ such that $\max\{h(t_i-) - h(t_{i-1}+)\} : i = 1, \dots, n < \delta$. Then using Theorem 2.1(b), it is easy to see that if $(RDS) \int_a^b f dh$ exists then so does $(RYS) \int_a^b f dh$ and the two are equal.

The Bartle integral given by Definition 2.37 is a general bilinear integral of a vector function with respect to an additive vector measure. According to Diestel and Uhl [41, p. 58], “Bartle [11] launched a theory of integration that includes most of the known integration procedures that have any claim to quality. His integral specializes to include the Bochner integral but does not include the general Pettis integral. Possibly workers in the theory of vector measures would be better off if they attempted to use the Bartle integral rather than inventing their own.” A survey of the history of vector integration can be found in Hildebrandt [95] and Bartle [11].

There are several different constructions of an integral of a Banach-space-valued function with respect to a nonnegative finite scalar-valued measure μ . The Bochner integral given by Definition 2.31 is one of the best known among such integrals. S. Bochner [21] defined his integral. The extension of Lebesgue’s differentiation theorem to Banach-valued functions, Theorem 2.35, is given e.g. by Diestel and Uhl [41, Theorem II.2.9, p. 49].

A function f taking values in a Banach space X is integrable with respect to μ in the sense of Dunford [56] if there exists a sequence of μ -simple functions $\{f_k\}_{k \geq 1}$ such that it converges to f μ -almost everywhere, and the sequence $\{\int_A f_k d\mu\}_{k \geq 1}$ converges in X for each $A \in \mathcal{S}$. The integral $(D) \int_A f d\mu$ is then defined to be the limit $\lim_{k \rightarrow \infty} \int_A f_k d\mu$, and is called the *second Dunford integral*. The integral does not depend on $\{f_k\}$ since it is a special case of the Bartle integral by Theorem 9 in [11], which is well defined by Proposition 2.38. The second Dunford integral clearly extends the Bochner integral. Hildebrandt [95, p. 123] showed that for μ -measurable functions f , the second Dunford integral coincides with several other integrals. Example 7 of Birkhoff [16, p. 377] gives a μ -measurable function f such that $\|f(\cdot)\|$ is not summable but is integrable in the sense of Dunford. For a scalar-valued measurable function f and a vector-valued measure μ , an integral analogous to the second Dunford integral is treated in Dunford and Schwartz [57, Definition IV.10.7].

Notes on Section 2.4. The “if” part of Theorem 2.42 was proved in Smith [223, p. 495].

Notes on Section 2.5. L. C. Young [244, p. 263] defined an extension of the Riemann–Stieltjes integral for regulated complex-valued functions f and h by

$$(Y_0) \int_a^b f \, dh := (RRS) \int_a^b f_+ \, dh_- + \sum_{a \leq t \leq b} [f(t) - f(t+)] [h(t+) - h(t-)]$$

if the RRS integral exists and the sum converges absolutely. However, the value of the Y_0 integral depends on $h(a-)$, $f(b+)$, and $h(b+)$, which need not be defined. If in the Y_0 integral we replace $f(t+)$ by $f_+^{(b)}(t)$, $h(t+)$ by $h_+^{(b)}(t)$, and $h(t-)$ by $h_-^{(a)}(t)$, then we get the Y_1 integral defined by (2.44). L. C. Young in 1938 gave such a convention for values at the endpoints in footnote 6 on p. 583 of [247], so one may suppose that he had these definitions in mind already in his 1936 paper.

The Y_1 and Y_2 integrals have been defined and proved equal when both exist in Dudley [49]. An extension of the Y_1 and Y_2 integrals to integrals of the form $\int f \cdot dh \cdot g$ with two integrands f and g was given in [54, Part II]. In Section 9.8 below such an extension is given for the RYS and Kolmogorov integrals.

Theorem 2.50 for functions with values in a Banach algebra is a special case of Theorem 3.7 in [54, Part II].

*Notes on Sections 2.6 and *2.7.* Saks [201] defines integrals due to Perron and Denjoy and gives references to their work, published beginning in 1912–1915. The Perron integral and one form of the Denjoy integral turned out to be equivalent to the Henstock–Kurzweil integral for integrals $\int_a^b g(x) \, dx$: Gordon [84, Chapter 11].

Ward [238] in 1936 defined an integral called a Perron–Stieltjes integral which includes both the Lebesgue–Stieltjes and refinement Riemann–Stieltjes integrals. Theorem 2.64 is Theorem 5 of Ward [238]. Kurzweil [129, Section 1.2] in 1957 defined the Henstock–Kurzweil integral and proved Theorem 2.65 on the equivalence of the WPS and HK integrals [129, Theorem 1.2.1]. Kurzweil ([129], [130], [131]), Henstock ([90], [91], [92]), and McShane [166] consider extended integrals for functions $U(\cdot, \cdot)$ of two variables, the second and third authors also on general spaces X , where $(HK) \int_a^b f \, dh$ is the special case $U(x, y) \equiv f(y)h(x)$ and $X = [a, b]$.

A discussion of Proposition 2.58 and its proof when $h(x) = x$ can be found in McLeod [165, Sections 1.5, 2.8, and 7.3].

While there has been relatively little literature about the refinement Young–Stieltjes and central Young integrals, there has been much more about Henstock–Kurzweil (or gauge) integrals, e.g. Lee Peng-Yee [184]. A 1991 book by Henstock [92] has a reference list of more than 1200 papers and books, mainly on the theme of “non-absolute integration,” if not necessarily about

the HK integral itself. In fact, Henstock [92] treats integration over general spaces (“division spaces”).

Schwabik [211, Theorem 3.1] proved Theorem 2.69 when h is of bounded variation and f is not necessarily regulated but either (a) f is bounded, or (b) f is arbitrary and for any t if $h(t-) = h(t+)$ then $h(t) = h(t-)$.

Notes on Section 2.8. According to Hildebrandt [94, p. 276], the integration by parts formula (2.79) for the refinement Young–Stieltjes integral was proved by de Finetti and Jacob [66] assuming that f and g are of bounded variation. Hewitt [93] proved an integration by parts formula similar to (2.83) for the Lebesgue–Stieltjes integral. Corollary 2.83 contains the result of Love [145], who calls the refinement Young–Stieltjes integral the refinement Ross–Riemann–Stieltjes integral, or the R^3S -integral.

Notes on Section 2.10. Ahlfors [1, 2nd ed., §4.4.4] wrote: “It was E. Artin who discovered that the characterization of homology by vanishing winding numbers ties in precisely with what is needed for Cauchy’s theorem. This idea has led to a remarkable simplification of earlier proofs.”

Proposition 2.93 is given e.g. in Ahlfors [1, 2nd ed., §4.2.1].



<http://www.springer.com/978-1-4419-6949-1>

Concrete Functional Calculus

Dudley, R.M.; Norvaiša, R.

2011, XII, 671 p., Hardcover

ISBN: 978-1-4419-6949-1