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## Preface

The reader of this book will need some background in real analysis, as in the first half of the first author's book *Real Analysis and Probability*. The book should be accessible to graduate students with that background as well as researchers. An interest in probability will also help with motivation.

Although on some topics we may have presented more or less final results, there are others that leave openings for research. The impetus for much of the work came from others' work in mathematical statistics, but we do not include any applications to statistics.

The book is mainly about some aspects of nonlinear analysis, some not much studied and some others previously studied but not in the same ways, and their applications to probability, as in the final Chapter 12. More specifically (to explain the book's title) we consider existence and smoothness questions for some concrete nonlinear operators acting on some concrete Banach spaces of functions. The book has relatively small overlaps, of the order of one or two chapters, with any previous book except for two lecture note volumes by the authors.

Here is a first example of what is done and distinctive in this book. If  $F$  and  $G$  are two functions such that  $F$  is defined on the range of  $G$ , one can form the composition  $H(x) \equiv F(G(x)) \equiv (F \circ G)(x)$ . When one mentions differentiability and composition, mathematicians tend to think of the chain rule, which is indeed an important fact, but we consider differentiability of the *two-function composition operator* we call  $TC$  which takes the pair of functions  $(F, G)$  into the function  $H$ . To take the derivative of this operator, we will assume  $F$  and  $G$  take values in Banach spaces. The domain of  $G$  need not have a linear or topological structure (it may be a measure space). What the differentiation will mean at some  $F, G$  is to take functions  $f$  and  $g$  approaching 0 in corresponding spaces, and to represent the increment  $(F + f) \circ (G + g) - F \circ G$  as  $A(f) + B(g)$  plus a remainder, where  $A$  and  $B$  are linear operators and the remainder becomes small in norm relative to  $f, g \rightarrow 0$ . The operator  $TC$  is linear in  $F$  for fixed  $G$ , but the remainder contains a term  $f \circ (G + g) - f \circ G$  which still depends on both  $f$  and  $g$ .

It seems to us that  $TC$  is a very natural operator deserving attention. It is treated in Chapter 8. Relatively more attention, e.g. Appell and Zabrejko [3], has been given to the operator  $G \mapsto F \circ G$  for fixed (nonlinear)  $F$ , and its extension to the case where  $F$  is a function of two variables, say  $x$  and  $y$ , and one forms a new function  $H(x) = F(x, G(x))$ . Such operators are treated in Chapters 6 and 7.

Among the most familiar of all Banach spaces are the  $L^p$  spaces, of equivalence classes of functions  $f$  such that  $|f|^p$  is integrable for a given measure  $\mu$ , where  $1 \leq p < \infty$ . A question is: for given  $p$  and  $s$  in  $[1, \infty)$ , for what space(s)  $\mathbb{F}$  of functions from  $\mathbb{R}$  into  $\mathbb{R}$  do we have Fréchet differentiability (defined in Chapter 5) of  $TC$  from  $\mathbb{F} \times L^s$  into  $L^p$ , say where the measure space is  $[0, 1]$  with Lebesgue measure? It turns out that for  $1 \leq s < p$ , we get degeneracy: even for fixed  $F$ , it must be constant (Corollary 7.36). For  $1 \leq p < s < \infty$ , consider the following. Let  $f$  be a real-valued function on  $\mathbb{R}$  and  $1 \leq p < \infty$ . Let  $v_p(f)$  be the supremum of  $\sum_{j=1}^n |f(x_j) - f(x_{j-1})|^p$  over all partitions  $x_0 = a < x_1 < \cdots < x_n = b$  and all  $n$ , called the  $p$ -variation of  $f$ . Finiteness of this is ordinary bounded variation when  $p = 1$ . Let  $\mathbb{F} = \mathcal{W}_p(\mathbb{R})$  be the space of all  $f$  such that  $v_p(f) < \infty$ , with the norm  $\|f\|_{[p]} = v_p(f)^{1/p} + \|f\|_{\sup}$  where  $\|f\|_{\sup} := \sup_x |f(x)|$ . Theorem 8.9 shows that  $TC$  is Fréchet differentiable from  $\mathcal{W}_p(\mathbb{R}) \times L^s$  into  $L^p$  of a finite measure space  $(\Omega, \mathcal{S}, \mu)$  at suitable  $F$  and  $G$ . Namely,  $F$  is differentiable, and its derivative  $F'$  satisfies a further condition. The image measure  $\mu \circ G^{-1}$  has a bounded density with respect to Lebesgue measure. The theorem gives a bound on the remainder in the differentiation of a given order in terms of  $s$  and  $p$ . The two paragraphs preceding Theorem 8.9 indicate how some of the conditions assumed are necessary or best possible and in particular, optimality of the  $\mathcal{W}_p$  norm on  $f$  for bounding the remainder term  $f \circ (G + g) - f \circ G$  (Proposition 7.28).

Countably additive signed measures are familiar objects in real analysis. For purposes of this book we need to consider Stieltjes-type integrals  $\int f dg$  where neither the integrator  $g$  nor the integrand  $f$  is of bounded variation. We found it useful to consider as integrators interval functions defined as follows. An *interval function*  $\mu$  is a function such that  $\mu(A)$  is defined for all intervals  $A$ , which may be restricted to subintervals of some given interval. Then  $\mu$  is called *additive* if  $\mu(A \cup B) = \mu(A) + \mu(B)$  for any two disjoint intervals  $A, B$  such that  $A \cup B$  is an interval, and  $\mu$  is called *upper continuous* if  $\mu(A_n) \rightarrow \mu(A)$  whenever intervals  $A_n$  decrease down to  $A$ . These properties would follow from, but do not imply, existence of an extension of  $\mu$  to a countably additive signed measure. For example, if  $F$  is a right-continuous function with left limits, not necessarily of bounded variation, one can define  $\mu_F((c, d]) := F(d) - F(c)$  and define  $\mu_F$  for other intervals by taking limits, giving an additive, upper continuous interval function  $\mu_F$ . For  $\mu$  upper continuous we define  $v_p(\mu)$ , for  $\mu$  which may be Banach-valued, as the supremum of  $\sum_i \|\mu(A_i)\|^p$  over all finite disjoint collections of intervals  $A_i$ . For not necessarily additive interval functions, the  $p$ -variation needs to be defined differently.

Beside composition, another natural operator is the product integral, which takes the coefficient functions (such as  $C(t)$  in (1.10) below) of a system of linear ordinary differential equations into a solution, under suitable conditions. Despite linearity of the system, the operator is nonlinear in the coefficients. Some entire books have been written on this operator. The product integral will take values in Banach algebra, namely, a Banach space in which a multiplication is defined satisfying usual algebraic conditions and  $\|xy\| \leq \|x\|\|y\|$ . A precise definition and some of the theory of Banach algebras are given in Chapter 4. An interval function  $\mu$  with values in a Banach algebra will be called *multiplicative* if  $\mu(A \cup B) = \mu(B)\mu(A)$  holds for any disjoint intervals  $A, B$  such that  $A \cup B$  is an interval, with  $s < t$  for each  $s \in A$  and  $t \in B$ . For  $1 \leq p < 2$ , a product integral operator  $\mu \mapsto \prod(\mathbb{I} + d\mu)$  is defined from additive to multiplicative interval functions and is an entire analytic function (as defined in Chapter 5) with respect to  $p$ -variation norms (Theorem 9.51) and serves to solve differential and integral equations. Whereas, for  $p > 2$ , finite  $p$ -variation of an additive, upper continuous  $\mu$  does not imply that the product integral even exists (Theorem 9.11).

Since  $p$ -variation gives sharp results about two natural operators, let's return to point functions  $f$  and consider the space  $\mathcal{W}_p(\mathbb{R})$  for  $1 \leq p < \infty$ . If  $G$  is any homeomorphism of  $\mathbb{R}$ , in other words, a continuous, strictly monotone (increasing or decreasing) function from  $\mathbb{R}$  onto itself, the map  $f \rightarrow f \circ G$  preserves  $\mathcal{W}_p$  and its norm. Invariance under this very large group holds for the spaces of all bounded continuous functions or all bounded functions with the norm  $\|f\|_{\text{sup}}$ . Other commonly considered spaces such as Sobolev spaces are of course highly useful, but they are invariant under much more restricted transformations. Whereas, the supremum norm gives no control of oscillations of a function and the  $\mathcal{W}_p$  norms do. We suppose the good properties of  $\|\cdot\|_p$  will give other uses than those we have found.

We also treat integrals, although with relatively little attention to the Lebesgue integral. Rather, let at first  $f$  and  $g$  be real functions of a real variable and consider Stieltjes-type integrals  $(f, g) \mapsto \int_a^b f dg$  where neither  $f$  nor  $g$  is necessarily of bounded variation. The given bilinear functional can be defined on various domains  $f \in \mathbb{F}$ ,  $g \in \mathbb{G}$  as will be seen. If  $f$  and  $g$  have suitable infinite-dimensional ranges, the integral can also be extended.

If  $1 \leq p < \infty$ ,  $1 \leq q < \infty$ ,  $v_p(f) < \infty$ ,  $v_q(g) < \infty$ , and  $p^{-1} + q^{-1} > 1$ , then a Stieltjes-type integral  $\int f dg$  can be defined, as had been shown by E. R. Love and L. C. Young in the late 1930's, with a corresponding inequality we call the *Love-Young inequality* (Corollary 3.91) and use often. Because of bilinearity, the differentiability is then immediate and simple.

Chapter 12 on probability and  $p$ -variation includes results from several research papers. Among others, Theorem 12.27 gives bounded  $p$ -variation of the sample paths of Markov processes (with values in metric spaces) for  $2 < p < \infty$  under a mild condition on expected lengths of increments, shown to be sharp. Corollary 12.43 extends the celebrated Komlós-Major-Tusnády theorem on convergence of the classical empirical process to a Brownian bridge

from the supremum norm to  $p$ -variation norms for  $2 < p < \infty$  with slower, but sharp, rates of convergence. Theorem 12.40 gives a sharp bound on the growth of the  $p$ -variation of the classical empirical process for  $1 \leq p < 2$ . These facts were published in three papers in *Annals of Probability* (one co-authored by one of us). Proposition 12.54 gives an example due to Terry Lyons, showing that for some processes  $X_t$  and  $Y_t$  which are each Brownian bridges, but which have an unusual joint distribution, an integral  $\int_0^1 X_t dY_t$  cannot be defined by any of the usual methods. The full work of Lyons and co-workers on “rough paths” is in progress and is beyond the scope of this book. In some cases where compact or “Hadamard” differentiability had been proved in the statistics literature, it might well be replaced by Fréchet differentiability with respect to  $p$ -variation norms, with a gain of remainder bounds. This and other problems we here leave as opportunities for readers.

Some parts of the book appeared, in different forms, in our earlier lecture note volumes Dudley and Norvaiša 1998 [55] and 1999 [54]. Improvements on or corrections to earlier results of ours or others are incorporated. The book also includes some new results published here for the first time as far as we know, as will be mentioned in the text or Notes for each.

*Guide to the reader:* Starred sections, specifically Sections \*2.7 and \*3.4, are not referred to in later chapters. Chapters of the book depend on earlier chapters as follows: Chapters 1 through 3 are basic in that Chapter 1 is a rather short introduction, and all later chapters refer to Chapter 3 many times each and directly or indirectly to Chapter 2. In the further sequence of chapters 4 through 8, each chapter has many references to the preceding one and directly or indirectly to intervening chapters. The Appendix relates only to Chapter 7. Chapter 12 on stochastic processes refers to, beyond Chapter 3, only two propositions in Chapter 9. Chapter 11, on Fourier series, refers only to the basic Chapters 1-3. Chapter 10, on nonlinear differential and integral equations, refers twice to Chapter 9, once to Chapter 7, once to Chapter 5, and many times to Chapter 3. Chapter 9, on multiplicative interval functions, the product integral, and linear differential and integral equations, refers to Chapters 1 through 6.

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