Preface

> On a dit, écrivais je (ou à peu près) dans une préface, que la géometrie est l'art de bien raisonner sur des figures mal faites.
>
> HENRI POINCARÉ [28]

In 1954, One hundred years after Henri Poincaré's birth, there was a special session during the International Congress of Mathematics, in the Netherlands, in honor of this great mathematician. The Russian mathematician, Pavel S. Aleksandrov, chosen to bring this about, started his speech by saying: ≪*To the question of what is Poincaré's relationship to topology, one can reply in a single sentence: he created it; but it is also possible to reply with a course of lectures in which Poincaré's fundamental topological results would be discussed in greater or lesser detail*≫ (see [3]). This is in part what we set out to do in this book.

Topology is a branch of mathematics that deals with the study of the qualitative properties of figures. Johann Benedikt Listing was among the first mathematicians who dedicated themselves to studying geometry in this sense and in 1847, he published a paper [23] in which he coined the term Topology. Bernhard Riemann's contribution to the birth of topology was also remarkable; after Riemann, Enrico Betti [5] studied manifolds through an invariant that generalizes Euler's for convex polyhedra. Betti's work provided Poincaré with the basis for his work in topology (called *Analysis Sitûs* by Poincaré).

Poincaré realized the importance of his new theory and wrote some twelve papers on Analysis Sitûs; indeed, here is what he stated in the paper he wrote at the request of the Swedish mathematician Gösta Mittag-Leffler [28]: ≪*Quant à moi, toutes les voies diverses où je m'étais engagé successivement me conduisaient à l'Analysis Sitûs. J'avais besoin des données de cette science pour poursuivre mes études sur les courbes définies par les équations différentielles et pour les étendre aux équations différentielles d'ordre supérieur et en particulier à celles du problème des trois corps. J'en avais besoin pour l'étude des périodes des intégrales multiples et pour l'application de cette étude au développement de la fonction pérturbatrice. Enfin j'entrevoyais dans l'Analysis Sitûs un moyen d'aborder un problème importante de la théorie des groupes, la recherche des groupes discrets ou des groupes finis contenus dans un groupe continu donné.*≫

In his first paper on Analysis Sitûs, published in 1895 [27], Poincaré defined manifolds in spaces with dimension greater than three and introduced the basic concept of *homeomorphism* defined as *the relation between two manifolds with the* *same qualitative properties* (see [28]). From a more up-to-date point of view, we may say that topology is the branch of mathematics concerning spaces with a certain structure (topological spaces), which are invariant under homeomorphisms; in other words, functions that are injective, surjective, and bicontinuous. The concept of homeomorphism allows us to group topological spaces into equivalence classes and consequently, to know in practical terms whether two of them are equal. This is the main purpose of topology.

This book consists of six chapters. In the first one, we present the basic concepts in Topology, Group Actions and Category Theory needed for developing the remainder of the book. The part concerning Category Theory is especially important since this book has been set in categorial terminology. This chapter could also serve as a basic text for a mini-course on General Topology.

In the second chapter, we study the category of simplicial complexes **Csim**, and two important covariant functors: the *geometric realization functor* from **Csim** to the category of topological spaces, and the *homology functor* from **Csim** to the category of graded Abelian groups. The geometric realization – $|K|$ – of a simplicial complex K is a polyhedron, assumed to be compact throughout this book. This is the chapter closest to Poincaré's initial paper. The Swiss mathematician Leonhard Euler was among the first ones to study one- and two-dimensional simplicial complexes; indeed, he used one-dimensional simplicial complexes and their geometric realization (graphs) in the famous problem about the seven bridges of Königsberg; later on, he also used two-dimensional simplicial complexes when he noticed that the relation

$$v - e + f = 2,$$

– where v is the number of vertices, e is the number of edges, and f is the number of faces – holds for every convex polyhedron in the elementary sense (namely, every edge is common to two faces and every face leaves the entire polyhedron to one of its sides).

By defining the so-called *Betti Numbers* for polyhedra of any dimension, Enrico Betti gave an initial generalization to the relation above; these are invariant under homeomorphims and, therefore, useful for classifying polyhedra. Based on these numbers, Poincaré developed a more complete characterization of polyhedra; in fact, in his 1895 [27] paper, Poincaré linked the Betti numbers to certain finitely generated Abelian groups associated with a polyhedron (integral homology groups of the polyhedron) and pointed out that the Betti numbers are ranks of homology groups. However, since homology groups are finitely generated Abelian groups, besides its free part (the one which gives its rank) they also have a torsion part; this too is an invariant by homeomorphisms. The combination of Betti numbers and torsion coefficients allows for a more complete analysis of polyhedra.

In Chap. II, homology groups are considered strictly from the simplicial point of view; in other words, the geometric structure of polyhedra is overlooked. In order to develop the theory (which is, at this point, of algebraic nature), one needs to define concepts in Homological Algebra (a branch of Algebra that sprang partly from Algebraic Topology): among other things, we prove the important Long Exact

Sequence Theorem in Sect. II.3, and in Sect. II.5 we define homology groups with coefficients in an arbitrary group. The Long Exact Sequence Theorem appears in simplicial homology in the case of a pair (K,L) of (oriented) simplicial complexes where L is a subcomplex of K; in fact, we may define an exact sequence of Abelian groups

$$\ldots \to H_n(L) \xrightarrow{H_n(i)} H_n(K) \xrightarrow{q_*(n)} H_n(K,L) \xrightarrow{\lambda_n} H_{n-1}(L) \to \ldots$$

where $H_n(K,L)$ are the relative homology groups.

The homology of polyhedra is defined in Chap. III which is, therefore, more geometric in nature than the previous one. The Simplicial Approximation Theorem (first proved by the American mathematician James Wadell Alexander [1]) provides the means by which one can pass from a simplicial approach to a geometric one. In practice, this theorem states that every map between two polyhedra may be "approached" by the geometric realization of a simplicial function between two triangulations of the polyhedra (reminding that any continuous curve on a plane can be approached by a polygonal line). The Long Exact Sequence Theorem appears also in this chapter, but here, the relative homology groups have a clearer meaning, since pairs of polyhedra $(|K|,|L|)$ have the Homotopy Extension Property, which allows us to prove that the group $H_n(|K|,|L|)$ is the homology group of the "quotient" polyhedron $|K|/|L|$. Samuel Eilenberg and Norman Steenrod [13] provided the formal method needed for constructing the long exact sequence of a pair of simplicial complexes or of polyhedra.

This chapter also contains an important application of the homology of polyhedra to the Theory of Fixed Points, namely, the Lefschetz Fixed Point Theorem, as well as several corollaries such as the Brouwer Fixed Point Theorem and the Fundamental Theorem of Algebra.

Computing homology groups of a polyhedron may pose serious difficulties, as in the case of the projective real spaces $\mathbb{R}P^n$. This is why we define *block homology*, based on ideas found in two classical books: Seifert–Threlfall [30] and Hilton–Wylie [17]. In closing this chapter, we dedicated Sect. III.6 to the proof of Eilenberg–MacLane's Acyclic Models Theorem [12], which allows us to compute the homology groups of the product of two polyhedra in terms of the homology groups of its factors. Somehow, this problem was already solved by H. Künneth [21] in 1924, when he established a relation between the Betti numbers and torsion coefficients of the product with the ones of each factor; the main difficulty resided precisely in strengthening Künneth's result by describing it in terms of homology groups. The paper [12] is one of the first written in terms of category theory, created in 1945 by Samuel Eilenberg and Saunders MacLane [11].

In Chap. IV we study cohomology: the homology groups

$$H_n(K;\mathbb{Q})$$

of an oriented complex K with rational coefficients have the structure of vector spaces on the rational field and may, therefore, be dualized. The possibility of dualizing such vector spaces led several mathematicians to consider "dualizing" homology groups, also when the coefficients are in an arbitrary Abelian group G. One of the first to study this possibility was James W. Alexander [2]; Solomon Lefschetz [22] wrote a detailed account of this paper. The new homology theory was soon called *cohomology* (it seems that it was Hassler Whitney [35] who coined this new term). As we might expect, the cohomology groups $H^n(|K|;\mathbb{Z})$ of a polyhedron $|K|$ are contravariant functors. The cohomology groups of a polyhedron are related to its homology groups by the Universal Coefficient Theorem; its proof (in terms of homologycal algebra) is given in this chapter. The cohomology of a polyhedron is an invariant stronger than the homology, since the cohomology with coefficients in a commutative ring with identity element (for instance, the ring of integers $\mathbb{Z}$) is also a ring. The product in such a ring is called *cup product*. In this way, we may obtain more precise information on the nature of the polyhedron. This chapter also introduces the *cap product* which is a bilinear relation of the type

$$\cap\colon\ H^p(|K|;\mathbb{Z}) \times H_{p+q}(|K|;\mathbb{Z}) \longrightarrow H_q(|K|;\mathbb{Z})\ ;$$

the cap product will be used in Chap. V for proving Poincaré's Duality Theorem.

Chapter V is divided into three sections: Manifolds, Closed Surfaces, and Poincaré Duality. In the first one, we introduce n-dimensional manifolds (without boundary) and triangulable n-manifolds. Then, we study closed surfaces, namely, path-connected, compact 2-manifolds: by a theorem due to Tibor Radó [29], these surfaces are triangulable. We prove that these manifolds can be classified into three types: the sphere S^2, the connected sums of two-dimensional tori (the torus T^2 and spheres with g handles), and the connected sums of real projective planes. Subsequently, by using block homology with coefficients in $\mathbb{Z}$, we prove that these three kinds of spaces are not homeomorphic. Finally, we prove Poincaré's Duality Theorem for connected, triangulable, and orientable n-manifolds V, that is to say, for triangulable n-manifods V such that $H_n(V;\mathbb{Z}) \cong \mathbb{Z}$. This very important theorem states that for every $0 \le p \le n$, $H^{n-p}(V;\mathbb{Z}) \cong H_p(V;\mathbb{Z})$ holds true.

In the last chapter (Chap. VI), we introduce another very important functor, from the category of polyhedra to that of groups (not necessarily Abelian), namely, *the fundamental group*, defined by Poincaré (see [27]). Next, we study a family of functors from the category of polyhedra to that of Abelian groups; we are talking about the *(higher) homotopy groups* $\pi_n(|K|,x_0)$ with $n \ge 2$. Only after Heinz Hopf wrote his 1931 paper [18], did mathematicians show interest in higher homotopy groups. In this paper, Hopf proved the existence of infinitely many different homotopy classes of maps from S^3 to S^2 (Satz 1); indeed, the isomorphism $\pi_3(S^2,\mathbf{e}_0) \cong \mathbb{Z}$ (see [26]) is deduced from the exact sequence of homotopy groups associated to the fibration $S^3 \to S^2$ with fiber S^1.

It is interesting to note that higher homotopy groups had already been introduced by Eduard Čech [6] during the Zürich International Mathematics Congress, in 1932; after that, Witold Hurewicz [19] studied these groups in depth. We approach homotopy groups by considering the set $[S^n,|K|]_*$ of all based homotopy classes of all maps $S^n \to |K|$ of a polyhedron $|K|$ and providing this set with a group operation, by means of a natural comultiplication of S^n (this is a map from S^n to the wedge product $S^n \vee S^n$). In this way, we bring about two endofunctors Σ and Ω of the category $\mathbf{Top}_*$ of based spaces, namely, the *suspension functor* and the *loop space functor*; this idea is based on the work of Beno Eckmann and Peter Hilton ([9], for instance). We close this chapter (and the book) with a small section on Obstruction Theory which combines cohomology and homotopy groups; Samuel Eilenberg [10] created this theory when researching the possibility of extending maps from a subpolyhedron of a polyhedron to the polyhedron itself.

Sections II.3, II.5, III.6, and IV.1 could be the basis for a course on Homological Algebra, a theory developed from the study of the homology of polyhedra; for further reading on this subject, we suggest either the book by Karl Gruenberg [15] or the book by Peter J. Hilton and Urs Stammbach [16].

In its first century of existence, Algebraic Topology has made remarkable progress, giving rise to new theories in mathematics, forging its way into various other mathematical branches, and solving seemingly unrelated yet important problems. The reader could, therefore, wish to go farther into this subject and so we give here some suggestions for further reading: we recommend the books by Peter Hilton and Shaun Wylie [17] (the reader may, at first, have some difficulty with its notation), Albrecht Dold [7] (a classic), and Edwin Spanier [32].

For the readers who wish to learn Homotopy Theory in more depth, we suggest the book by George Whitehead [33]. Finally, for reading on topological spaces and cellular structures in topology (CW-complexes), we recommend the book by Rudolf Fritsch and Renzo Piccinini [14].

The chapters of this book were developed from the material taught in the undergradute courses on higher geometry given by the authors at the University of Milano and the University of Milano–Bicocca. A first version was prepared by R. Piccinini some years ago, when he was a professor at the University of Milano–Bicocca. This volume is essentially the revision and completion of that material. Our many thanks to several colleagues and friends who have read the rough copy and made worthwhile suggestions: Keith Johnson, Sandro Levi, Augusto Minatta, Claudio Pacati, Robert Paré, Petar Pavešić, Nair Piccinini, Dorette Pronk, Alessandro Russo, Mauro Spreafico, and Richard Wood. Our colleague Delfina Roux must be thanked for having read with great attention the first draft of this volume, correcting the errors of language in the first Italian draft.

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Knowledge is of two kinds. We know a subject ourselves,
or we know where we can find information upon it.

SAMUEL JOHNSON
(Letter to Lord Chesterton, February 1755)

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