

Chapter 2

H-Spaces and Co-H-Spaces

2.1 Introduction

NOTATION AND STANDING ASSUMPTIONS

• From this chapter on, most of the spaces that we consider will be based and path-connected and have the based homotopy type of based CW complexes. Some notable exceptions to path-connectedness are the 0-sphere S^0 and the 0-skeleton of a CW complex. Unless otherwise stated, all functions under consideration will be continuous and based and all homotopies will preserve the base point. These restrictions are sometimes asserted explicitly for emphasis. We discuss unbased spaces, functions and homotopies from time to time. However, whenever doing so, we explicitly make note of the fact.

• We take all homology and cohomology to be reduced, so that a space has trivial zero-dimensional homology and cohomology.

• We adopt the following notation throughout: “ \simeq ” for homotopy of maps or same homotopy type of spaces, “ \cong ” for homeomorphism of spaces or isomorphism of groups and “ \sim ” for the relation of equivalence. Furthermore, if X is a set with an equivalence relation and $x \in X$, then $\langle x \rangle$ denotes the equivalence class containing x .

There are reasons for the restrictions on spaces listed above. First of all, nearly all of the spaces that are of interest to us are of this type. Second, these assumptions avoid having to add additional hypotheses to several theorems since CW complexes satisfy many of these hypotheses. But because of these assumptions, we must ensure that the constructions that we perform on spaces of the homotopy type of CW complexes yield spaces of the homotopy type of CW complexes. This is so, but the proofs in some instances are long and difficult. Presenting this material would take us far afield, and so we describe some proofs and give references for the others.

In this chapter we discuss the important notions of H-space and grouplike space and of co-H-space and cogroup. A grouplike space is the homotopy analogue of a group. It is a group object in the homotopy category. An H-

space is defined in the same way but without the assumption of associativity. Cogroups and co-H-spaces are the categorical duals of these in the homotopy category. We show that the set of homotopy classes of maps of any space into a grouplike space has an induced group structure as does the set of homotopy classes of maps of a cogroup into any space. We then consider the set of homotopy classes of maps from a cogroup into a grouplike space and show that the two group structures agree and are abelian. Loop spaces ΩY are examples of grouplike spaces and suspensions ΣX are examples of cogroups. We prove that there is a fundamental isomorphism $[\Sigma X, Y] \cong [X, \Omega Y]$. Since an n -sphere is a suspension, the set of homotopy classes of maps $[S^n, Y]$ is a group. These are the homotopy groups of Y , denoted $\pi_n(Y)$, and discussed in Section 2.4 and later in Section 4.5. Of particular interest in this section is a theorem which we call Whitehead's First Theorem which asserts that a map of CW complexes is a homotopy equivalence if and only if it induces an isomorphism of all homotopy groups. A natural generalization of spheres is Moore spaces which are spaces with a single nonvanishing homology group. Dually, Eilenberg–Mac Lane spaces are spaces with a single nonvanishing homotopy group. The existence and uniqueness up to homotopy type of these spaces are discussed. Homotopy groups with coefficients are then defined by using Moore spaces and (homotopical) cohomology groups with coefficients by using Eilenberg–Mac Lane spaces. The chapter ends with a discussion of Eckmann–Hilton duality.

2.2 H-Spaces and Co-H-Spaces

Before discussing H-spaces and co-H-spaces, we introduce some terminology that appears in the rest of the book. We assume that the reader is familiar with the concept of a commutative diagram of groups and homomorphisms and of spaces and maps. In commutative diagrams there is the initial point (a group or space), a terminal point (a group or space), and two compositions of homomorphisms or maps from the initial point to the terminal point. In the case of abelian groups, if one of the compositions is the negative of the other, then we say that the diagram *anticommutes* or is an *anticommutative diagram*. In the case of spaces, if the two compositions are homotopic, then we say that the diagram *homotopy-commutes*, *commutes up to homotopy*, or is a *homotopy-commutative diagram*.

Now we turn to the notions of a grouplike space and an H-space. Let Y be a space and recall that $j_1 : Y \rightarrow Y \times Y$ and $j_2 : Y \rightarrow Y \times Y$ are defined by $j_1(y) = (y, *)$ and $j_2(y) = (*, y)$ for all $y \in Y$.

Definition 2.2.1 A *grouplike space* consists of a space Y and two maps $m : Y \times Y \rightarrow Y$ and $i : Y \rightarrow Y$ such that

1. $m j_1 \simeq \text{id} \simeq m j_2 : Y \rightarrow Y$,

$$\begin{array}{ccc}
 Y & \xrightarrow{j_1} & Y \times Y \\
 & \searrow \text{id} & \downarrow m \\
 & & Y
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 Y & \xrightarrow{j_2} & Y \times Y \\
 & \searrow \text{id} & \downarrow m \\
 & & Y.
 \end{array}$$

$$2. \ m(m \times \text{id}) \simeq m(\text{id} \times m) : Y \times Y \times Y \rightarrow Y,$$

$$\begin{array}{ccc}
 Y \times Y \times Y & \xrightarrow{m \times \text{id}} & Y \times Y \\
 \downarrow \text{id} \times m & & \downarrow m \\
 Y \times Y & \xrightarrow{m} & Y.
 \end{array}$$

$$3. \ m(\text{id}, i) \simeq * \simeq m(i, \text{id}) : Y \rightarrow Y,$$

$$\begin{array}{ccc}
 Y & \xrightarrow{(\text{id}, i)} & Y \times Y \\
 & \searrow * & \downarrow m \\
 & & Y
 \end{array}
 \quad \text{and} \quad
 \begin{array}{ccc}
 Y & \xrightarrow{(i, \text{id})} & Y \times Y \\
 & \searrow * & \downarrow m \\
 & & Y,
 \end{array}$$

where $(\text{id}, i), (i, \text{id}) : Y \rightarrow Y \times Y$ are defined by $(\text{id}, i)(y) = (y, i(y))$ and $(i, \text{id})(y) = (i(y), y)$, for $y \in Y$.

A grouplike space is sometimes referred to as an *H-group*. The map m is called a *multiplication* and i is called a *homotopy inverse*. If only (1) holds, then Y (or more properly, the pair (Y, m)) is called an *H-space*. A space that is an H-space and a CW complex is called an *H-complex* and a grouplike space that is a CW complex is called a *grouplike complex*. We sometimes do not explicitly mention the multiplication or homotopy inverse and refer to a space Y as an H-space or grouplike space. Condition (2) is called *homotopy-associativity*. A homotopy-associative H-space is one in which (1) and (2) hold. In terms of the addition of maps defined below, condition (3) asserts that $\text{id} + i \simeq * \simeq i + \text{id}$. Therefore $[i]$ is the homotopy inverse of $[\text{id}]$ in the group $[Y, Y]$. From this we obtain the inverse of any $\alpha = [f] \in [X, Y]$ defined as $i_*(\alpha) = [if]$. We show in Proposition 8.4.4 that a homotopy-associative H-complex always has a homotopy inverse, and so is a grouplike complex. The H-space (Y, m) is *homotopy-commutative* if $mt \simeq m : Y \times Y \rightarrow Y$ where $t : Y \times Y \rightarrow Y \times Y$ is defined by $t(y, y') = (y', y)$, for $y, y' \in Y$.

Definition 2.2.2 Let (Y, m) and (Y', m') be H-spaces and $h : Y \rightarrow Y'$ a map. We call h an *H-map* if the following diagram is homotopy-commutative,

$$\begin{array}{ccc}
 Y \times Y & \xrightarrow{h \times h} & Y' \times Y' \\
 \downarrow m & & \downarrow m' \\
 Y & \xrightarrow{h} & Y'.
 \end{array}$$

This is written $h : (Y, m) \rightarrow (Y', m')$.

The space Y is a *topological group* if (Y, m, i) is a grouplike space such that equality holds instead of homotopy in all parts of Definition 2.2.1. In this case, it is customary to write $m(y, y')$ as yy' and $i(y)$ as y^{-1} . A grouplike space is thus the analogue of a group in homotopy theory. Similarly an H-map is the analogue of a homomorphism of groups. We give a class of examples in Section 2.3 of spaces that are grouplike, but not topological groups. For now we note that the spheres S^1 , S^3 , and S^7 are all H-spaces. The first two are in fact topological groups. Multiplication of complex numbers induces a multiplication on S^1 which makes it into a topological group and quaternionic multiplication does the same for S^3 . The sphere S^7 inherits its multiplication from the multiplication of octonions or Cayley numbers [49, pp. 448–449]. But the latter is not associative, and so S^7 is an H-space that is not a topological group. It has been proved [51] that this multiplication on S^7 is not homotopy-associative, so S^7 is not a grouplike space. The question of whether any other spheres have the structure of an H-space is a difficult one. A negative answer has been given by the work of several people with the major result due to Adams [1].

If (Y, m) is an H-space and X is any space, then the set $[X, Y]$ can be given an additively written binary operation which is defined as follows. Let $f, g : X \rightarrow Y$ and define $f + g = m(f \times g)\Delta = m(f, g)$

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f \times g} Y \times Y \xrightarrow{m} Y,$$

where Δ is the diagonal map. Then if $\alpha = [f]$ and $\beta = [g] \in [X, Y]$, we set $\alpha + \beta = [f + g]$. This is a well-defined binary operation on the set $[X, Y]$. We make some simple remarks about this operation.

- By (1), $f + * = m(f \times *)\Delta = m j_1 f \simeq f$. Therefore $\alpha + 0 = \alpha$, and similarly $0 + \alpha = \alpha$, where 0 is the homotopy class of the constant map. Thus for an H-space (Y, m) , the element $0 \in [X, Y]$ is a two-sided identity for the binary operation.
- If (3) holds, then, as mentioned earlier, $i_*(\alpha)$ is the inverse of α in $[X, Y]$.
- Clearly $m = p_1 + p_2 : X \times X \rightarrow X$, where $p_1, p_2 : X \times X \rightarrow X$ are the two projections, since $m(p_1 \times p_2)\Delta = m$.
- We obtain the *category of H-spaces* denoted \mathcal{H} consisting of H-spaces and homotopy classes of H-maps and the *category of grouplike spaces* denoted \mathcal{HG} consisting of grouplike spaces and homotopy classes of H-maps (see Appendix F).

We recall some categorical language and notation (Appendix F). Let $HoTop_*$ be the based homotopy category (consisting of spaces and homotopy classes of maps), let Gr be the category of groups, and let $Sets_*$ be the category of based sets. Furthermore, let \mathcal{B}_* be the category of based sets with a binary operation for which the basepoint is a two-sided identity and the morphisms are based functions preserving the binary operation (called

homomorphisms). In addition, \mathcal{AB}_* is the full subcategory of \mathcal{B}_* consisting of based sets for which the binary operation is commutative. Then there are forgetful functors $\mathcal{B}_* \rightarrow \mathbf{Sets}_*$ and $Gr \rightarrow \mathcal{B}_*$ (see Appendix F).

Now let Y be a fixed space and define a contravariant functor $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ by $\mathcal{F}_Y(X) = [X, Y]$ and $\mathcal{F}_Y(f) = f^* : [X', Y] \rightarrow [X, Y]$, where $f : X \rightarrow X'$. To say that $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through \mathcal{B}_* means that for every space X , the set $[X, Y]$ is a based set having a binary operation with the homotopy class of the constant map a two-sided identity and that $f^* : [X', Y] \rightarrow [X, Y]$ is a homomorphism for every map $f : X \rightarrow X'$. Similarly $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through Gr means that $[X, Y]$ is a group for every X with unit the homotopy class of the constant map and that $f^* : [X', Y] \rightarrow [X, Y]$ is a homomorphism.

Proposition 2.2.3

1. Y is an H-space if and only if $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through \mathcal{B}_* .
2. Y is a homotopy-commutative H-space if and only if $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through \mathcal{AB}_* .
3. Y is a grouplike space if and only if $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through Gr .

Proof. (1) Let (Y, m) be an H-space. We have already noted that $[X, Y]$ is a set with binary operation for which 0 is a two-sided identity. If $f : X \rightarrow X'$ is a map and $[a], [b] \in [X', Y]$, then

$$(a + b)f = m(a \times b)\Delta_{X'}f = m(af \times bf)\Delta_X = af + bf,$$

and so $f^*([a] + [b]) = f^*[a] + f^*[b]$. Thus $\mathcal{F}_Y : HoTop_* \rightarrow \mathbf{Sets}_*$ factors through \mathcal{B}_* . Conversely, suppose $[X, Y]$ is an object of \mathcal{B}_* for every X with the property that $f^* : [X', Y] \rightarrow [X, Y]$ is a homomorphism for every map $f : X \rightarrow X'$. Let the binary operation be denoted by $+$ and let $[*]$ be the two-sided identity, where $*$ is the constant map. Now define $m : Y \times Y \rightarrow Y$ by $[m] = [p_1] + [p_2] \in [Y \times Y, Y]$, where p_1 and p_2 are the two projections of $Y \times Y$ onto Y . Then, if $j_1, j_2 : Y \rightarrow Y \times Y$ are the two inclusions,

$$j_1^*[m] = [p_1j_1] + [p_2j_1] = [\text{id}_Y] + [*] = [\text{id}_Y],$$

and so $mj_1 \simeq \text{id}_Y$. Similarly, $mj_2 \simeq \text{id}_Y$. Therefore (Y, m) is an H-space.

- (2) If (Y, m) is homotopy-commutative and $[a], [b] \in [X, Y]$, then

$$a + b = m(a \times b)\Delta \simeq mt(a \times b)\Delta = m(b \times a)\Delta = b + a,$$

where $t : Y \times Y \rightarrow Y \times Y$ interchanges coordinates, and so $[X, Y]$ is commutative. Conversely, suppose $[X, Y]$ is commutative for all X . Let the multiplication m on Y be as defined in (1). Then

$$[mt] = t^*[m] = t^*([p_1] + [p_2]) = [p_1t] + [p_2t] = [p_2] + [p_1] = [p_1] + [p_2] = [m],$$

and so m is homotopy-commutative.

(3) We omit the proof which is like (1) and (2) but we record the following for later use. If \mathcal{F}_Y factors through Gr , then the multiplication m and the homotopy inverse i are defined by

$$[m] = [p_1] + [p_2] \quad \text{and} \quad [i] = -[\text{id}_Y]. \quad \square$$

We next introduce a definition and corollary of Proposition 2.2.3.

Definition 2.2.4 A *contravariant binary operation induced by Y* is a binary operation on $[X, Y]$ for every space X such that $0 \in [X, Y]$ is a two-sided identity and for every $f : X \rightarrow X'$, the function $f^* : [X', Y] \rightarrow [X, Y]$ is a homomorphism. A *contravariant group operation induced by Y* is similarly defined.

Then we have the following immediate consequence of Proposition 2.2.3.

Corollary 2.2.5 1. *There is a one-one correspondence between the set of homotopy classes of multiplications of Y and the set of contravariant binary operations induced by Y .*

2. *There is a one-one correspondence between the set of homotopy classes of grouplike multiplications of Y and the set of contravariant group operations induced by Y .*

The following result is frequently used.

Proposition 2.2.6 *If (Y, m) and (Y', m') are H -spaces and $h : (Y, m) \rightarrow (Y', m')$ an H -map, then $h_* : [X, Y] \rightarrow [X, Y']$ is a homomorphism of based sets with a binary operation. In particular, if Y and Y' are grouplike spaces, then $h_* : [X, Y] \rightarrow [X, Y']$ is a group homomorphism.*

Proof. Let $[a], [b] \in [X, Y]$; then

$$h(a + b) = hm(a \times b)\Delta \simeq m'(h \times h)(a \times b)\Delta = ha + hb.$$

Therefore h_* is a homomorphism. \square

To obtain the notion which is dual to that of a grouplike space, we reverse the arrows and replace the product with the wedge in Definition 2.2.1. As noted in Section 1.2, we regard $X \vee X \subseteq X \times X$ so that every element of $X \vee X$ is of the form $(x, *)$ or $(*, x')$, for $x, x' \in X$. Recall that $q_1 = p_1|_{X \vee X} : X \vee X \rightarrow X$ and $q_2 = p_2|_{X \vee X} : X \vee X \rightarrow X$, where $p_1, p_2 : X \times X \rightarrow X$ are the projections.

Definition 2.2.7 A *cogroup* consists of a space X and two maps $c : X \rightarrow X \vee X$ and $j : X \rightarrow X$ such that

1. $q_1 c \simeq \text{id} \simeq q_2 c : X \rightarrow X$.

2. $(c \vee \text{id})c \simeq (\text{id} \vee c)c : X \rightarrow X \vee X \vee X$

$$\begin{array}{ccc} X & \xrightarrow{c} & X \vee X \\ \downarrow c & & \downarrow c \vee \text{id} \\ X \vee X & \xrightarrow{\text{id} \vee c} & X \vee X \vee X. \end{array}$$

3. $\{\text{id}, j\}c \simeq * \simeq \{j, \text{id}\}c : X \rightarrow X$, where $\{\text{id}, j\} : X \vee X \rightarrow X$ is defined by $\{\text{id}, j\}(x, *) = x$ and $\{\text{id}, j\}(*, x) = j(x)$, for all $x \in X$, and $\{j, \text{id}\}$ is similarly defined.

A cogroup is also called a co-H-group, an H-cogroup, or a cogrouplike space. The map c is the *comultiplication* and j the *homotopy inverse*. If only (1) holds, then (X, c) or X is called a *co-H-space*. A co-H-space which is a CW complex is called a *co-H-complex*. Condition (2) is called *homotopy-associativity* (sometimes homotopy-coassociativity). We show in Proposition 8.4.4 that every simply connected, homotopy-associative co-H-complex has a homotopy inverse. The co-H-space X is called *homotopy-commutative* if $sc \simeq c : X \rightarrow X \vee X$, where $s : X \vee X \rightarrow X \vee X$ is defined by $s(x, *) = (*, x)$ and $s(*, x) = (x, *)$. We give examples of cogroups in Section 2.3 and show that all spheres and wedges of spheres of dimension ≥ 1 are cogroups. There are spaces that are co-H-spaces but not cogroups (see [9]). In addition, a co-H-space in the topological category (defined by equality of maps instead of homotopy of maps) is a one point space (see Exercise 2.4).

Definition 2.2.8 Let (X, c) and (X', c') be co-H-spaces and $g : X \rightarrow X'$ a map. We call g a *co-H-map* if there is a homotopy-commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ \downarrow c & & \downarrow c' \\ X \vee X & \xrightarrow{g \vee g} & X' \vee X'. \end{array}$$

This is written $g : (X, c) \rightarrow (X', c')$.

The set $[X, Y]$ has a binary operation when X is a co-H-space and Y is any space: let $f, g : X \rightarrow Y$ and let $\nabla : Y \vee Y \rightarrow Y$ be the folding map defined by $\nabla(y, *) = y$ and $\nabla(*, y) = y$, for $y \in Y$. We define $f + g = \nabla(f \vee g)c = \{f, g\}c$,

$$X \xrightarrow{c} X \vee X \xrightarrow{f \vee g} Y \vee Y \xrightarrow{\nabla} Y.$$

Then for $\alpha = [f]$ and $\beta = [g] \in [X, Y]$, we set $\alpha + \beta = [f + g]$.

As before $\alpha + 0 = \alpha = 0 + \alpha$ and $c = i_1 + i_2 : X \rightarrow X \vee X$, where $i_1, i_2 : X \rightarrow X \vee X$ are the two injections. In addition, if (3) holds, $j^*(\alpha) + \alpha = 0 = \alpha + j^*(\alpha)$, and so $j^*(\alpha)$ is the inverse of $\alpha \in [X, Y]$. We obtain the *category of co-H-spaces* \mathcal{CH} whose objects are co-H-spaces and whose morphisms are

homotopy classes of co-H-maps and a full (sub)category of cogroups \mathcal{CG} . Now let X be a fixed space and define a covariant functor $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ by $\mathcal{K}_X(Y) = [X, Y]$ and $\mathcal{K}_X(g) = g_* : [X, Y] \rightarrow [X, Y']$, where $g : Y \rightarrow Y'$. Then $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ factors through \mathcal{B}_* means that for every space Y , the set $[X, Y]$ is a based set with a binary operation with the homotopy class of the constant map a two-sided identity and that $g_* : [X, Y] \rightarrow [X, Y']$ is a homomorphism for every map $g : Y \rightarrow Y'$. Similarly $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ factors through Gr means that $[X, Y]$ is a group for every Y with unit the homotopy class of the constant map and that $g_* : [X, Y] \rightarrow [X, Y']$ is a homomorphism.

Proposition 2.2.9

1. X is a co-H-space if and only if $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ factors through \mathcal{B}_* .
2. X is a homotopy-commutative co-H-space if and only if $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ factors through \mathcal{AB}_* .
3. X is a cogroup if and only if $\mathcal{K}_X : HoTop_* \rightarrow Sets_*$ factors through Gr .
4. If (X, c) and (X', c') are co-H-spaces and $h : (X', c') \rightarrow (X, c)$ is a co-H-map, then $h^* : [X, Y] \rightarrow [X', Y]$ is a homomorphism of based sets with a binary operation. In particular, if X and X' are cogroups, then $h^* : [X, Y] \rightarrow [X', Y]$ is a group homomorphism.
5. If X is a co-H-space and $f, g : X \rightarrow Y$, then $(f + g)_* = f_* + g_* : H_n(X; G) \rightarrow H_n(Y; G)$ and $(f + g)^* = f^* + g^* : H^n(Y; G) \rightarrow H^n(X; G)$, for all $n \geq 0$ and abelian groups G .

Proof. The proofs of (1) – (3) are analogous to the proof of Proposition 2.2.3, therefore we omit them. We do note, however, that in (3), if \mathcal{K}_X factors through Gr , then the comultiplication c and homotopy inverse j are defined as follows,

$$c = i_1 + i_2 \quad \text{and} \quad j = -\text{id}_X,$$

where i_1 and i_2 are the two injections of $X \rightarrow X \vee X$. The proof of (4) is parallel to the proof of Proposition 2.2.6, and also omitted. We only prove (5) for homology. Let $\mu_X : H_n(X \vee X) \rightarrow H_n(X) \oplus H_n(X)$ be the isomorphism given by $\mu_X(z) = (q_{1*}(z), q_{2*}(z))$ for $z \in H_n(X \vee X)$. Consider the commutative diagram

$$\begin{array}{ccccccc}
 H_n(X) & \xrightarrow{c_*} & H_n(X \vee X) & \xrightarrow{(f \vee g)_*} & H_n(Y \vee Y) & \xrightarrow{\nabla_*} & H_n(Y) \\
 & \searrow \Delta & \downarrow \mu_X & & \downarrow \mu_Y & \nearrow \delta & \\
 & & H_n(X) \oplus H_n(X) & \xrightarrow{f_* \oplus g_*} & H_n(Y) \oplus H_n(Y), & &
 \end{array}$$

where Δ is the diagonal and $\delta(u, u') = u + u'$, for $u, u' \in H_n(Y)$. Then

$$(f + g)_* = \nabla_*(f \vee g)_*c_* = \delta(f_* \oplus g_*)\Delta = f_* + g_*,$$

and the result follows. \square

Analogous to Definition 2.2.4, we have the following for co-H-spaces.

Definition 2.2.10 A *covariant binary operation induced by X* is a binary operation on $[X, Y]$ for every space Y such that $0 \in [X, Y]$ is a two-sided identity and for every map $g : Y \rightarrow Y'$, the function $g_* : [X, Y] \rightarrow [X, Y']$ is a homomorphism. A covariant group operation induced by X is similarly defined.

We then have the following immediate consequence of Proposition 2.2.9.

Proposition 2.2.11 1. *There is a one-one correspondence between the set of homotopy classes of comultiplications of X and the set of covariant binary operations induced by X .*
 2. *There is a one-one correspondence between the set of homotopy classes of cogroup comultiplications of X and the set of covariant group operations induced by X .*

An interesting situation arises when (X, c) is a co-H-space and (Y, m) is an H-space. Then the comultiplication c and the multiplication m each induce a binary operation in $[X, Y]$.

Proposition 2.2.12 *If (X, c) is a co-H-space and (Y, m) is an H-space, then the binary operation $+_c$ in $[X, Y]$ obtained from c equals the binary operation $+_m$ in $[X, Y]$ obtained from m . In addition, this binary operation is abelian.*

Proof. For every $\alpha = [f], \beta = [g], \gamma = [h], \delta = [k] \in [X, Y]$, we prove

$$(\alpha +_m \beta) +_c (\gamma +_m \delta) = (\alpha +_c \gamma) +_m (\beta +_c \delta). \quad (2.1)$$

With $\Delta = \Delta_X$ and $\nabla = \nabla_Y$, the left-hand side of Equation 2.1 is represented by

$$\nabla(m(f \times g)\Delta \vee m(h \times k)\Delta)c = m\nabla_{Y \times Y}((f \times g) \vee (h \times k))(\Delta \vee \Delta)c$$

and the right-hand side of Equation 2.1 is represented by

$$m((\nabla(f \vee h)c) \times (\nabla(g \vee k)c))\Delta = m(\nabla \times \nabla)((f \vee h) \times (g \vee k))\Delta_{X \vee X}c.$$

But it is easily checked that

$$\nabla_{Y \times Y}((f \times g) \vee (h \times k))(\Delta \vee \Delta) = (\nabla \times \nabla)((f \vee h) \times (g \vee k))\Delta_{X \vee X},$$

and so Equation 2.1 is established. Now take $\beta = 0 = \gamma$ in Equation 2.1, getting

$$\alpha +_c \delta = \alpha +_m \delta.$$

This shows that the two binary operations agree. Next set $\alpha = 0 = \delta$ in Equation 2.1, getting

$$\beta +_c \gamma = \gamma +_m \beta.$$

This shows that the operation is abelian. \square

2.3 Loop Spaces and Suspensions

In this section we study loop spaces which are a class of grouplike spaces and suspensions which are a class of cogroups.

Definition 2.3.1 For a space B , the *loop space* ΩB is the subspace of B^I consisting of all paths l in B such that $l(0) = * = l(1)$. The loop space ΩB has the subspace topology of the space of paths B^I with the compact-open topology (see Appendix A). The elements of ΩB are called *loops* in B . If $g : B \rightarrow B'$ is a map, then $\Omega g : \Omega B \rightarrow \Omega B'$ is defined by $\Omega g(l) = gl$ (the composition of g and l).

Clearly if $g \simeq g' : B \rightarrow B'$, then $\Omega g \simeq \Omega g' : \Omega B \rightarrow \Omega B'$. We next define a map $m : \Omega B \times \Omega B \rightarrow \Omega B$ by

$$m(l, l')(t) = \begin{cases} l(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ l'(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

for $l, l' \in \Omega B$ and $t \in I$. We also define $i : \Omega B \rightarrow \Omega B$ by $i(l)(t) = l(1 - t)$, for $l \in \Omega B$ and $t \in I$.

The loop $m(l, l')$ consists of the loop l followed by the loop l' . That is, $m(l, l')$ is obtained by traversing the loop l at double speed followed by the loop l' also at double speed. The loop $i(l)$ is the loop l traversed in the opposite direction. We note that $m(l, l')$ is just the sum of paths $l + l'$ and $i(l)$ is $-l$, both of which were defined in Remark 1.4.7. We will see that the map m provides ΩB with grouplike structure.

If B has the homotopy type of a CW complex, then so does ΩB by a theorem of Milnor [70]. It also follows from Milnor's result that many of the path spaces such as B^I or EB also have the homotopy type of a CW complex whenever B does.

Proposition 2.3.2 *If B is a space, then ΩB is a grouplike space with multiplication m and homotopy inverse i . For any map $f : B \rightarrow B'$, the map $\Omega f : \Omega B \rightarrow \Omega B'$ is an H-map.*

Proof. We must first verify the three conditions in Definition 2.2.1.

(1) We show $\text{id} \simeq m j_1 : \Omega B \rightarrow \Omega B$ by defining a homotopy $F : \Omega B \times I \rightarrow \Omega B$. For $l \in \Omega B$ and $s, t \in I$, we set

$$F(l, s)(t) = \begin{cases} l(2t/(2-s)) & \text{if } 0 \leq t \leq \frac{2-s}{2} \\ * & \text{if } \frac{2-s}{2} \leq t \leq 1. \end{cases}$$

The other homotopy for (1) is similar.

(2) We show $m(m \times \text{id}) \simeq m(\text{id} \times m) : \Omega B \times \Omega B \times \Omega B \rightarrow \Omega B$ by defining a homotopy $G : \Omega B \times \Omega B \times \Omega B \times I \rightarrow \Omega B$. For $l, l', l'' \in \Omega B$ and $s, t \in I$, we set

$$G(l, l', l'', s)(t) = \begin{cases} l(4t/(1+s)) & \text{if } 0 \leq t \leq \frac{s+1}{4} \\ l'(4t-1-s) & \text{if } \frac{s+1}{4} \leq t \leq \frac{s+2}{4} \\ l''((4t-s-2)/(2-s)) & \text{if } \frac{s+2}{4} \leq t \leq 1. \end{cases}$$

(3) We show $*$ $\simeq m(\text{id}, i) : \Omega B \rightarrow \Omega B$ by defining a homotopy $H : \Omega B \times I \rightarrow \Omega B$. For $l \in \Omega B$ and $s \in I$, we set

$$H(l, s)(t) = \begin{cases} l(2st) & \text{if } 0 \leq t \leq \frac{1}{2} \\ l(2s(1-t)) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

The other homotopy for (3) is similar.

Finally, $m'(\Omega f \times \Omega f) = (\Omega f)m : \Omega B \times \Omega B \rightarrow \Omega B'$, where m' is the multiplication of $\Omega B'$. Therefore Ωf is an H-map. \square

In the proof of the previous proposition formal definitions of the required homotopies were given. However, it is helpful in understanding these homotopies to visualize them and say what they actually do.

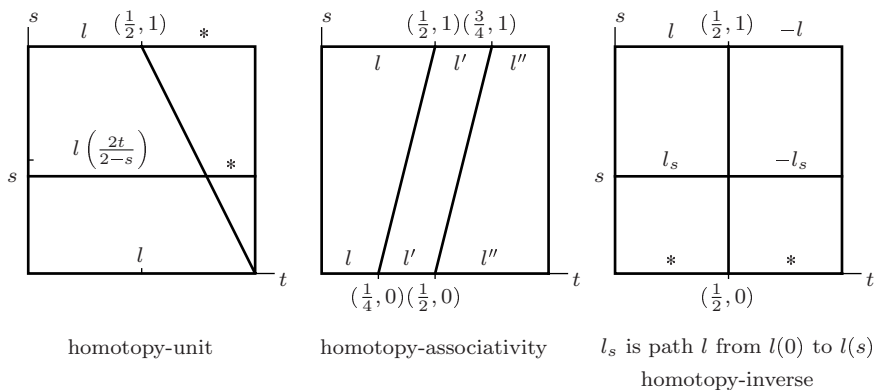


Figure 2.1

For example, in (3) we see that at time s the homotopy H applied to the path l is first the path l going from $l(0)$ to $l(s)$ and then is the path l in the opposite direction going from $l(s)$ to $l(0)$. Clearly this is the constant path $*$ when $s = 0$ and the path $m(l, il)$ when $s = 1$. A similar analysis can be made for the homotopies in (1) and (2).

Let $HoTop_*$ denote the homotopy category and let \mathcal{HG} denote the category of grouplike spaces. Then $\Omega : HoTop_* \rightarrow \mathcal{HG}$ defined by $\Omega(X) = \Omega X$ and

$\Omega[f] = [\Omega f]$ is a well-defined functor. Clearly $(\Omega B, m)$ is a grouplike space that is not in general a topological group. From Propositions 2.2.3 and 2.3.2 it follows that for any space B , ΩB induces natural group structure on $[X, \Omega B]$. In addition, if $f : B \rightarrow B'$ is a map, then $(\Omega f)_* : [X, \Omega B] \rightarrow [X, \Omega B']$ is a homomorphism.

Next we turn to suspensions.

Definition 2.3.3 For any space A , define the *suspension* ΣA (sometimes called the *reduced suspension*) to be the identification space

$$(A \times I)/(A \times \{0\} \cup \{*\} \times I \cup A \times \{1\}).$$

There is a map $c : \Sigma A \rightarrow \Sigma A \vee \Sigma A$ defined by

$$c\langle a, t \rangle = \begin{cases} (\langle a, 2t \rangle, *) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (*, \langle a, 2t - 1 \rangle) & \text{if } \frac{1}{2} \leq t \leq 1, \end{cases}$$

where $a \in A$, $t \in I$, and $*$ denotes the basepoint of ΣA . We also define $j : \Sigma A \rightarrow \Sigma A$ by $j\langle a, t \rangle = \langle a, 1 - t \rangle$. If $f : A \rightarrow A'$, then $\Sigma f : \Sigma A \rightarrow \Sigma A'$ is given by $\Sigma f\langle a, t \rangle = \langle f(a), t \rangle$.

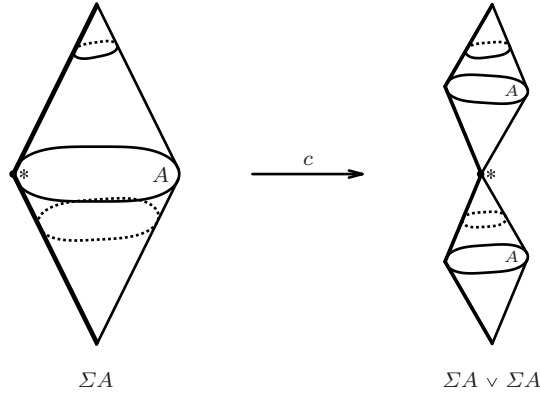


Figure 2.2

Clearly if $f \simeq f' : A \rightarrow A'$, then $\Sigma f \simeq \Sigma f' : \Sigma A \rightarrow \Sigma A'$.

There is another way to view the suspension. Let $C_0 X$ and $C_1 X$ be the two cones on X (see Section 1.4). Then $i_0 : X \rightarrow C_1 X$ is defined by $i_0(x) = \langle x, 0 \rangle$ and $i_1 : X \rightarrow C_0 X$ is defined by $i_1(x) = \langle x, 1 \rangle$. Then the suspension ΣX is homeomorphic to the identification space $C_0 X \vee C_1 X / \sim$, where $i_1(x) \sim i_0(x)$, for every $x \in X$.

Proposition 2.3.4 *For any space A , the space ΣA is a cogroup with co-multiplication c and homotopy inverse j . For any $f : A \rightarrow A'$, the map $\Sigma f : \Sigma A \rightarrow \Sigma A'$ is a co- H -map.*

The proof of this is completely analogous to that of Proposition 2.3.2 and is left as an exercise. However, after we give the proof of Proposition 2.3.5 we show how a proof can be derived from Proposition 2.3.2.

If \mathcal{CG} denotes the category of cogroups, it follows from Proposition 2.3.4, that $\Sigma : HoTop_* \rightarrow \mathcal{CG}$ is a functor defined by $\Sigma(A) = \Sigma A$ and $\Sigma(f) = \Sigma f$. By Proposition 2.2.9, for every space A , the set $[\Sigma A, Y]$ has group structure for every space Y such that a map $g : Y \rightarrow Y'$ induces a homomorphism $g_* : [\Sigma A, Y] \rightarrow [\Sigma A, Y']$. Moreover, a map $h : A' \rightarrow A$ induces a homomorphism $(\Sigma h)^* : [\Sigma A, Y] \rightarrow [\Sigma A', Y]$.

We have seen that if A and B are any two spaces, both $[\Sigma A, B]$ and $[A, \Omega B]$ are groups. If $f : \Sigma A \rightarrow B$ is a map, we define $\kappa(f) : A \rightarrow \Omega B$ by

$$\kappa(f)(a)(t) = f\langle a, t \rangle,$$

for $a \in A$ and $t \in I$.

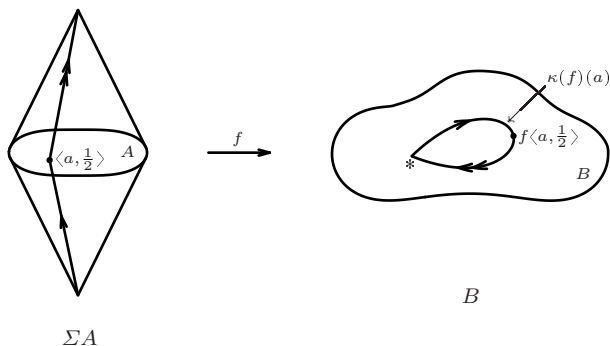


Figure 2.3

Clearly $\kappa(f)$ is well-defined and continuous (Appendix A). Furthermore, if f_t is a homotopy between $f : \Sigma A \rightarrow B$ and $f' : \Sigma A \rightarrow B$, then $\kappa(f_t)$ is a homotopy between $\kappa(f) : A \rightarrow \Omega B$ and $\kappa(f') : A \rightarrow \Omega B$. Thus κ induces $\kappa_* : [\Sigma A, B] \rightarrow [A, \Omega B]$. Similarly, if $g : A \rightarrow \Omega B$, we define $\bar{\kappa}(g) : \Sigma A \rightarrow B$ by $\bar{\kappa}(g)\langle a, t \rangle = g(a)(t)$, for $a \in A$ and $t \in I$. Then $\bar{\kappa}$ induces $\bar{\kappa}_* : [A, \Omega B] \rightarrow [\Sigma A, B]$. Now

$$\kappa(\bar{\kappa}(g))(a)(t) = (\bar{\kappa}(g))\langle a, t \rangle = g(a)(t),$$

and so $\kappa\bar{\kappa} = \text{id}$. In a like manner, $\bar{\kappa}\kappa = \text{id}$. Thus $\kappa_* : [\Sigma A, B] \rightarrow [A, \Omega B]$ is a bijection with inverse $\bar{\kappa}_* : [A, \Omega B] \rightarrow [\Sigma A, B]$. In addition, if $h : A' \rightarrow A$ and $k : B \rightarrow B'$ are maps, then

$$\kappa(f)h = \kappa(f\Sigma h) \quad \text{and} \quad (\Omega k)\kappa(f) = \kappa(kf),$$

for every $f : \Sigma A \rightarrow B$. Thus

$$h^*\kappa_* = \kappa_*(\Sigma h)^* \quad \text{and} \quad (\Omega k)_*\kappa_* = \kappa_*k_*.$$

Proposition 2.3.5 *For any spaces A and B , the bijection $\kappa_* : [\Sigma A, B] \rightarrow [A, \Omega B]$ is an isomorphism of groups.*

Proof. Let $f, g : \Sigma A \rightarrow B$ and consider $\kappa(f + g) = \kappa(\nabla(f \vee g)c) : A \rightarrow \Omega B$. Then for $a \in A$ and $t \in I$,

$$\begin{aligned} (\kappa(\nabla(f \vee g)c)(a))(t) &= \nabla(f \vee g)c\langle a, t \rangle \\ &= \begin{cases} \nabla(f \vee g)(\langle a, 2t \rangle, *) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \nabla(f \vee g)(*, \langle a, 2t - 1 \rangle) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \\ &= \begin{cases} f\langle a, 2t \rangle & \text{if } 0 \leq t \leq \frac{1}{2} \\ g\langle a, 2t - 1 \rangle & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

On the other hand, $\kappa(f) + \kappa(g) = m(\kappa(f) \times \kappa(g))\Delta : A \rightarrow \Omega B$. Then

$$\begin{aligned} (m(\kappa(f) \times \kappa(g))\Delta(a))(t) &= m(\kappa(f)(a), \kappa(g)(a))(t) \\ &= \begin{cases} (\kappa(f)(a))(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ (\kappa(g)(a))(2t - 1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases} \\ &= \begin{cases} f\langle a, 2t \rangle & \text{if } 0 \leq t \leq \frac{1}{2} \\ g\langle a, 2t - 1 \rangle & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

Thus $\kappa(f + g) = \kappa(f) + \kappa(g)$, and the result follows. \square

Definition 2.3.6 The isomorphism κ_* in Proposition 2.3.5 or its inverse $\bar{\kappa}_*$ is called the *adjoint isomorphism*. We say that f and $\kappa(f)$ and also α and $\kappa_*(\alpha)$ are *adjoint* to each other.

Using the fact that κ_* is a bijection and that $(\Omega B, m, i)$ is grouplike for all B , we now show that $(\Sigma A, c, j)$ is a cogroup for all A , where c and j are the maps defined in Definition 2.3.3. We have that $[A, \Omega B]$ is a group, with binary operation denoted $+$, and so $\kappa_* : [\Sigma A, B] \rightarrow [A, \Omega B]$ induces group structure with two-sided identity $[\ast]$ on $[\Sigma A, B]$, for all B . We denote this binary operation in $[\Sigma A, B]$ by $+$. Because any map $k : B \rightarrow B'$ induces a homomorphism $(\Omega k)_* : [A, \Omega B] \rightarrow [A, \Omega B']$, it follows from $(\Omega k)_*\kappa_* = \kappa_*k_*$ that $k_* : [\Sigma A, B] \rightarrow [\Sigma A, B']$ is a homomorphism. Therefore by Proposition 2.2.9(3), there exists a comultiplication \tilde{c} and a homotopy inverse \tilde{j} such that $(\Sigma A, \tilde{c}, \tilde{j})$ is a cogroup. We show that $c \simeq \tilde{c}$ and $j \simeq \tilde{j}$. By Definition 2.3.3, $\kappa(c) = \kappa(i_1) + \kappa(i_2)$, where $i_1, i_2 : \Sigma A \rightarrow \Sigma A \vee \Sigma A$ are the two injections. But $\tilde{c} = i_1 + i_2$ (see the proof of Proposition 2.2.9), and so $\kappa(\tilde{c}) = \kappa(i_1) + \kappa(i_2)$. Thus $\kappa(c) = \kappa(\tilde{c})$, and so $c \simeq \tilde{c}$. Finally $j = -\text{id}$ by

definition and so $\kappa(j) = -\kappa(\text{id})$. But $\kappa(\tilde{j}) = -\kappa(\text{id})$ (proof of Proposition 2.2.9). Therefore $j \simeq \tilde{j}$, and so $(\Sigma A, c, j)$ is a cogroup.

The suspension and loop space constructions can be iterated.

Definition 2.3.7 For spaces A and B and define $\Sigma^0 A = A$ and $\Omega^0 B = B$ and for integers $n \geq 1$,

$$\Sigma^n A = \Sigma(\Sigma^{n-1} A) \quad \text{and} \quad \Omega^n B = \Omega(\Omega^{n-1} B).$$

We next consider homotopy commutativity of iterated suspensions and loop spaces.

Proposition 2.3.8 *For spaces A and B , $\Sigma^n A$ is a homotopy-commutative cogroup and $\Omega^n B$ is a homotopy-commutative grouplike space, if $n \geq 2$.*

Proof. We just show that $\Sigma^n A$ is homotopy commutative. For any space Y , we have the following isomorphism of groups, $[\Sigma^n A, Y] \cong [\Sigma^{n-1} A, \Omega Y]$, for $n \geq 2$, by Proposition 2.3.5. The latter group is abelian by Proposition 2.2.12. By Proposition 2.2.9(2), $\Sigma^n A$ is homotopy-commutative. \square

Recall that the upper cap E_+^n of the unit n -sphere S^n is defined by $E_+^n = \{(x_1, x_2, \dots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$. The lower cap E_-^n of S^n is similarly defined by $x_{n+1} \leq 0$. Then $S^n = E_+^n \cup E_-^n$ and $S^{n-1} = E_+^n \cap E_-^n$.

Proposition 2.3.9 *For all $n \geq 1$, S^n is homeomorphic to ΣS^{n-1} .*

Proof. There are homeomorphisms $h_+ : E^n \rightarrow E_+^n$ and $h_- : E^n \rightarrow E_-^n$ defined by

$$\begin{aligned} h_+(x) &= \left(x, \sqrt{1 - |x|^2}\right) \quad \text{and} \\ h_-(x) &= \left(x, -\sqrt{1 - |x|^2}\right), \end{aligned}$$

for $x \in E^n$. Recall that $C_0 X = (X \times I)/(X \times \{0\} \cup \{*\} \times I)$ and $C_1 X = (X \times I)/(X \times \{1\} \cup \{*\} \times I)$. By Lemma 1.4.10, there is a homeomorphism $\tilde{K} : C_1(S^{n-1}) \rightarrow E^n$. Similarly by defining $L : S^{n-1} \times I \rightarrow E^n$ by $L(x, t) = (1-t)* + tx$, we obtain a homeomorphism $\tilde{L} : C_0(S^{n-1}) \rightarrow E^n$ as in Lemma 1.4.10. We compose \tilde{K} with h_+ to obtain a homeomorphism $\tau : C_1(S^{n-1}) \rightarrow E_+^n$ and we compose \tilde{L} with h_- to obtain a homeomorphism $\lambda : C_0(S^{n-1}) \rightarrow E_-^n$. Each of τ and λ restricted to S^{n-1} is the identity map of S^{n-1} . We regard ΣS^{n-1} as $C_1(S^{n-1}) \cup_{S^{n-1}} C_0(S^{n-1})$, the disjoint union of $C_1(S^{n-1})$ and $C_0(S^{n-1})$ with $S^{n-1} \subseteq C_1(S^{n-1})$ identified with $S^{n-1} \subseteq C_0(S^{n-1})$. Then the maps τ and λ yield a homeomorphism (see [Figure 2.4](#))

$$\Sigma S^{n-1} = C_1(S^{n-1}) \cup_{S^{n-1}} C_0(S^{n-1}) \cong E_+^n \cup E_-^n = S^n. \quad \square$$

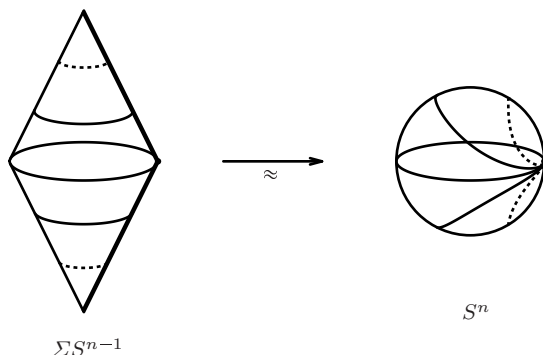


Figure 2.4

2.4 Homotopy Groups I

By Propositions 2.2.9, 2.3.4, and 2.3.9, the set $[S^n, Y]$ is a group for all spaces Y and all $n \geq 1$. These are the homotopy groups of Y .

Definition 2.4.1 For every space Y and $n \geq 0$, the set $[S^n, Y]$ is called the *n*th (ordinary) homotopy group of Y and is denoted $\pi_n(Y)$. For $n = 1$, it is called the *fundamental group* of Y .

We assume that the reader has had some exposure to the basic properties of fundamental groups. For review, we have presented the topics on the fundamental group that we use in Appendix B. If $n \geq 1$, then $\pi_n(Y)$ is a group for all Y and a map $f : Y \rightarrow Y'$ induces a homomorphism $f_* : \pi_n(Y) \rightarrow \pi_n(Y')$. In general, $\pi_0(Y)$ is a set with a distinguished element and $f_* : \pi_0(Y) \rightarrow \pi_0(Y')$ is a function that preserves the distinguished element. For another characterization of $\pi_0(Y)$, see Exercise 2.24.

We next give a few elementary properties of homotopy groups. We give more information on homotopy groups in Section 4.5 and compute some of these groups in Section 5.6.

- For $n \geq 2$, the groups $\pi_n(Y)$ are abelian. This follows from Proposition 2.3.8.
- The fundamental group $\pi_1(Y)$ is abelian if Y is an H-space by Proposition 2.2.12. In general, $\pi_1(Y)$ is not abelian (Appendix B). If Y is a grouplike space, then $\pi_0(Y)$ is a group (Exercise 2.24).
- If $n \geq 1$, then $\pi_n(Y) \cong \pi_{n-1}(\Omega Y)$ as groups by Proposition 2.3.5. In particular, ΩY is path-connected if and only if $\pi_1(Y) = 0$ by Exercise 2.24.
- If $f \simeq g : Y \rightarrow Y'$, then $f_* = g_* : \pi_n(Y) \rightarrow \pi_n(Y')$, for all $n \geq 0$.
- If $f : X \rightarrow Y$ is a homotopy equivalence, then $f_* : \pi_n(Y) \rightarrow \pi_n(Y')$ is an isomorphism, for all $n \geq 0$. For if $g : Y \rightarrow X$ is a homotopy inverse of

f , then $fg \simeq \text{id}$. Therefore $f_*g_* = (fg)_* = \text{id}_* = \text{id}$. Similarly $gf \simeq \text{id}$ implies that $g_*f_* = \text{id}$. Therefore f_* is an isomorphism.

- Let $i : X \rightarrow Y$ be an inclusion and let $r : Y \rightarrow X$ be a retraction. Then $i_* : \pi_n(X) \rightarrow \pi_n(Y)$ is a monomorphism and $r_* : \pi_n(Y) \rightarrow \pi_n(X)$ is an epimorphism, for all n , since $r_*i_* = \text{id}$. In fact, $\pi_n(Y) = i_*\pi_n(X) \oplus \text{Ker } r_*$. This clearly holds if r is a homotopy retraction. It also holds if r is an arbitrary map and i is a section or homotopy section of r .
- If Y is contractible, then $\pi_n(Y) = 0$, for all $n \geq 0$. This follows because $\text{id} \simeq * : Y \rightarrow Y$, and so $\text{id} = (\text{id})_* = *_* = 0 : \pi_n(Y) \rightarrow \pi_n(Y)$, for all $n \geq 0$.
- For spaces Y and Y' , we have $\pi_n(Y \times Y') \cong \pi_n(Y) \oplus \pi_n(Y')$, for all $n \geq 0$. For, by Corollary 1.3.7, the function $\theta : \pi_n(Y) \oplus \pi_n(Y') \rightarrow \pi_n(Y \times Y')$ defined by $\theta([f], [g]) = [(f, g)]$, for $[f] \in \pi_n(Y)$ and $[g] \in \pi_n(Y')$, is a bijection with inverse function λ given by $\lambda[h] = (p_{1*}[h], p_{2*}[h])$. Thus λ is an isomorphism, and so $\pi_n(Y \times Y') \cong \pi_n(Y) \oplus \pi_n(Y')$. Furthermore, we define $\mu : \pi_n(Y) \oplus \pi_n(Y') \rightarrow \pi_n(Y \times Y')$ by $\mu(\alpha, \beta) = j_{1*}(\alpha) + j_{2*}(\beta)$, where $j_1 : Y \rightarrow Y \times Y'$ and $j_2 : Y' \rightarrow Y \times Y'$ are the two inclusions. Then $\lambda\mu = \text{id}$, so μ is an isomorphism and equals θ . These results clearly extend to the product of finitely many spaces.
- If Y and Y' are spaces of the homotopy type of CW complexes, then the fundamental group of the wedge $Y \vee Y'$ is the free product $\pi_1(Y) * \pi_1(Y')$ of $\pi_1(Y)$ and $\pi_1(Y')$ (Appendix B).
- If Y is a nonpath-connected space and X is the path-connected component of Y containing the basepoint, then the inclusion $i : X \rightarrow Y$ induces an isomorphism $i_* : \pi_n(X) \rightarrow \pi_n(Y)$, for all $n \geq 1$. This is since for any map $f : S^n \rightarrow Y$, we have that $f(S^n) \subseteq X$ because $f(S^n)$ is a path-connected space containing $*$. Similarly, for any homotopy $F : S^n \times I \rightarrow Y$, we have that $F(S^n \times I) \subseteq X$.

The result that the fundamental group of an H-space is abelian is easy to prove. The result that the fundamental group of a co-H-space is free, which we prove next, is more difficult. It requires some facts about free groups and free products of groups (Appendix B).

Let G be a group that is not necessarily abelian. For notational convenience, we write \bar{g} for the inverse g^{-1} of $g \in G$. We denote the free product of G with itself by $G * G$. If $g \in G$, then g regarded as an element of the first factor of $G * G$ is written g' and as an element of the second factor of $G * G$ is written g'' . Thus an element $\xi \in G * G$ can be written

$$\xi = \prod_{i=1}^p g_i' \bar{\gamma}_i'', \quad \text{where } g_i, \gamma_i \in G.$$

Then there are projection homomorphisms $p_1, p_2 : G * G \rightarrow G$ given by $p_1(\xi) = \prod g_i$ and $p_2(\xi) = \prod \bar{\gamma}_i$. We introduce the following notation:

$$E_G = \{\xi \in G * G \mid p_1(\xi) = p_2(\xi)\}.$$

Thus $\xi = \prod_{i=1}^p g_i' \bar{\gamma}_i'' \in E_G$ if and only if $\gamma_p \cdots \gamma_1 g_1 \cdots g_p = 1$. Then $\pi : E_G \rightarrow G$ is defined by $\pi = p_1|_{E_G} = p_2|_{E_G}$, and so $\pi(\xi) = \prod g_i = \prod \bar{\gamma}_i$. Finally, let $\xi_u = u'u'' \in E_G$, where $u \in G$ and let $\Xi_G = \{\xi_u \mid u \neq 1\}$.

The following result, which appears in [7, Prop.3.1], is based on ideas attributed to M. Kneser.

Lemma 2.4.2 *The group E_G is free with basis Ξ_G .*

Proof. It is clear that the set Ξ_G is an independent set. In order to write any expression $\xi = \prod g_i' \bar{\gamma}_i''$ that satisfies $\gamma_p \cdots \gamma_1 g_1 \cdots g_p = 1$ as a product of the ξ_u and their inverses, we use the following simple algorithm. For $1 \leq i \leq 2p$, define δ_i by the formulas

$$\delta_{2k} = \gamma_k \cdots \gamma_1 g_1 \cdots g_k \quad \text{and} \quad \delta_{2k+1} = \delta_{2k} g_{k+1}.$$

Thus $\delta_1 = g_1$, $\delta_2 = \gamma_1 g_1$, $\delta_3 = \gamma_1 g_1 g_2$, and so on, and $\delta_{2p} = 1$. Now one verifies that ξ is the alternating product

$$\xi = \prod_{i=1}^{2p} \xi_{\delta_i}^{(i)},$$

where $(i) = (-1)^{i+1}$ (Exercise 2.20). □

Proposition 2.4.3 *If X is a co-H-complex, then $\pi_1(X)$ is a free group.*

Proof. If $G = \pi_1(X)$, then as noted earlier, $\pi_1(X \vee X)$ is isomorphic to $G * G$, the free product of G with itself. Let $c : X \rightarrow X \vee X$ be a comultiplication and let $q_1, q_2 : X \vee X \rightarrow X$ be the projections. Because $q_i c \simeq \text{id}$, we have that c induces a homomorphism $s = c_* : G \rightarrow G * G$ such that $p_1 s = p_2 s = \text{id} : G \rightarrow G$. Thus s determines a homomorphism $\sigma : G \rightarrow E_G$ such that $\pi \sigma = \text{id}$. Therefore σ maps G isomorphically onto $\sigma(G) \subseteq E_G$. By Lemma 2.4.2, E_G is free. Since a subgroup of a free group is free [39, p. 85], $\sigma(G)$ is free. Hence $G = \pi_1(X)$ is free. □

Next we present additional results on homotopy groups. We begin with a definition.

Definition 2.4.4 A path-connected space Y is said to be *n-connected*, if $\pi_i(Y) = 0$, for all $i \leq n$. A 1-connected space Y is also called *simply connected*. A map $f : X \rightarrow Y$ is called an *n-equivalence* (also called an *n-connected map*), if $f_* : \pi_i(X) \rightarrow \pi_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$. A map $f : X \rightarrow Y$ is a *weak (homotopy) equivalence* or an ∞ -equivalence if $f_* : \pi_n(X) \rightarrow \pi_n(Y)$ is an isomorphism for all n .

Lemma 2.4.5 *Let (X, A) be a based, relative CW complex with $\dim(X, A) \leq n$, let B and Y be spaces (not necessarily of the homotopy type of CW complexes), and let $e : B \rightarrow Y$ be an n -equivalence, $n \leq \infty$. Let $j : A \rightarrow X$ be the*

inclusion and assume that there are maps $f : X \rightarrow Y$ and $g : A \rightarrow B$ and a diagram

$$\begin{array}{ccc} A & \xrightarrow{g} & B \\ \downarrow j & \nearrow \tilde{g} & \downarrow e \\ X & \xrightarrow{f} & Y, \end{array}$$

such that $eg \simeq_L fj$, for some homotopy $L : A \times I \rightarrow Y$. Then there exists a map $\tilde{g} : X \rightarrow B$ such that $\tilde{g}j = g$ and a homotopy $F : X \times I \rightarrow Y$ such that $e\tilde{g} \simeq_F f$, where $F|_{A \times I} = L$.

This lemma, which is the major step in proving Whitehead's theorem 2.4.7, follows from the HELP lemma 4.5.7 which is proved in Section 4.5 after we have discussed the relative homotopy groups.

From Lemma 2.4.5 we can easily prove the following proposition.

Proposition 2.4.6 *Let X be a based CW complex, let B and Y be spaces (not necessarily of the homotopy type of CW complexes), and let $e : B \rightarrow Y$ be an n -equivalence, $n < \infty$. Then $e_* : [X, B] \rightarrow [X, Y]$ is an injection if $\dim X < n$ and a surjection if $\dim X \leq n$. If $n = \infty$, then $e_* : [X, B] \rightarrow [X, Y]$ is a bijection for any based CW complex X .*

Proof. We first show that e_* is onto if $\dim X \leq n$. Let $[f] \in [X, Y]$, set $A = \{*\}$, and define $g : A \rightarrow B$ to be the constant map. We then apply Lemma 2.4.5 to f and g and obtain a map $\tilde{g} \in [X, B]$ such that $e_*[\tilde{g}] = [f]$. Thus e_* is onto.

Now assume that $\dim X < n$ and $eg_0 \simeq_F eg_1$ for $g_0, g_1 : X \rightarrow B$. Let $X' = X \times I$ and so $\dim X' \leq n$. We set $A' = X \times \partial I \cup \{*\} \times I$ and define $G : A' \rightarrow B$ by

$$G(x, i) = g_i(x) \quad \text{and} \quad G(*, t) = *,$$

for $x \in X$, $t \in I$, and $i = 0, 1$. Since $\dim(X', A') \leq n$, we can apply Lemma 2.4.5 to F and G . We get a homotopy $H : X \times I \rightarrow B$ such that $H|_{A'} = G$. Then $g_0 \simeq_H g_1$, and so e_* is one-one. \square

There are two important theorems due to J. H. C. Whitehead which we shall arbitrarily call Whitehead's first theorem and Whitehead's second theorem. We now prove Whitehead's first theorem [92].

Theorem 2.4.7 *If $f : X \rightarrow Y$ is a map of CW complexes, then f is a weak equivalence if and only if f is a homotopy equivalence.*

Proof. We only prove that if f is a weak equivalence, it is a homotopy equivalence, since the other implication has been proved. Consider the function $f_* : [Y, X] \rightarrow [Y, Y]$. By Proposition 2.4.6, f_* is a bijection. Therefore there is a map $g : Y \rightarrow X$ such that $fg \simeq \text{id}_Y$. But $fgf \simeq f$ and so $f_*[gf] = f_*[\text{id}_X]$, where $f_* : [X, X] \rightarrow [X, Y]$. This latter f_* is a bijection, and so $gf \simeq \text{id}_X$. Thus f is a homotopy equivalence. \square

Remark 2.4.8 Whitehead's first theorem is useful to show that a map is a homotopy equivalence. For this we would prove that the map induces isomorphisms of all homotopy groups. Because our spaces have the homotopy type of CW complexes, it would follow that the map is a homotopy equivalence. We frequently use this remark without comment.

We observe that it is not sufficient that $\pi_n(X) \cong \pi_n(Y)$, for all n , for X and Y to have the same homotopy type. By Whitehead's first theorem, there should be a map $f : X \rightarrow Y$ that induces an isomorphism of all homotopy groups. An example of the nonsufficiency is given in 5.6.2.

Theorem 2.4.9 *Let X and Y be path-connected spaces (not necessarily of the homotopy type of CW complexes), let $f : X \rightarrow Y$ be a map, and let $n \geq 0$. Then there is a space K such that (K, X) is a relative CW complex having relative cells of dimensions $\geq n+1$ with the following property. There exists a map $\bar{f} : K \rightarrow Y$ such that $\bar{f}|_X = f$ and $f_* : \pi_i(K) \rightarrow \pi_i(Y)$ is an isomorphism for $i > n$ and a monomorphism for $i = n$.*

Proof. In the proof we write h_{*i} for the induced homotopy homomorphism $h_* : \pi_i(W) \rightarrow \pi_i(Z)$, for any map $h : W \rightarrow Z$. The idea of the proof is to attach $(n+1)$ -cells to X to kill $\text{Ker } f_{*n}$ and then attach additional $(n+1)$ -cells to map onto $\pi_{n+1}(Y)$. This process is then repeated. We begin by choosing generators $[g_\alpha]_{\alpha \in A}$ of $\text{Ker } f_{*n}$, where $g_\alpha : S_\alpha^n \rightarrow X$ and $S_\alpha^n = S^n$. Then the g_α determine $g : \bigvee_{\alpha \in A} S_\alpha^n \rightarrow X$ and we attach $(n+1)$ -cells to X by g to form the adjunction space $X' = X \cup_g \bigvee E_\alpha^{n+1}$. Since $fg \simeq *$, the map fg can be extended to $\bigvee E_\alpha^{n+1}$ by Lemma 1.4.10 and Proposition 1.4.9. This extension and f determine a map $f' : X' \rightarrow Y$ such that $f'|_X = f$. Then $f'_{*n+1} : \pi_{n+1}(X') \rightarrow \pi_{n+1}(Y)$, and we choose elements $[h_\beta] \in \pi_{n+1}(Y)$ for $\beta \in B$ that are a set of generators. Then the h_β determine $h : \bigvee_{\beta \in B} S_\beta^{n+1} \rightarrow Y$ and we form $X^{n+1} = X' \vee \bigvee_{\beta \in B} S_\beta^{n+1}$ and define $f^{n+1} : X^{n+1} \rightarrow Y$ by $f^{n+1} = \{f', h\}$. Note that (X^{n+1}, X) is a relative CW complex.

Let $k : X \rightarrow X'$ and $l : X' \rightarrow X^{n+1}$ be inclusion maps and let $j = lk : X \rightarrow X^{n+1}$. Then there is a commutative diagram

$$\begin{array}{ccccc}
 & \pi_i(X) & & & \\
 & \downarrow k_{*i} & \searrow f_{*i} & & \\
 j_{*i} \swarrow & \pi_i(X') & \xrightarrow{f'_{*i}} & \pi_i(Y). & \\
 & \downarrow l_{*i} & \nearrow f_{*i}^{n+1} & & \\
 & \pi_i(X^{n+1}) & & &
 \end{array}$$

We claim that f_{*n}^{n+1} is a monomorphism and f_{*n+1}^{n+1} is an epimorphism. If $\gamma \in \text{Ker } f_{*n}^{n+1}$, then $\gamma = j_{*n}(\delta)$ for some $\delta \in \text{Ker } f_{*n}$, since j_{*n} is an epimorphism by Proposition 1.5.24. Therefore

$$\delta = \sum_{\alpha \in A'} n_{\alpha} [g_{\alpha}],$$

where $A' \subseteq A$ is a finite subset and $n_{\alpha} \in \mathbb{Z}$. But

$$k_{*n}(\delta) = \sum_{\alpha \in A'} n_{\alpha} [kg_{\alpha}] = 0,$$

because $kg_{\alpha} \simeq *$. Therefore $\gamma = j_{*n}(\delta) = l_{*n}(k_{*n}(\delta)) = 0$. Hence f_{*n+1}^{n+1} is a monomorphism.

Next we show that f_{*n+1}^{n+1} is an epimorphism. Given $\epsilon \in \pi_{n+1}(Y)$, we have $\epsilon = \sum_{\beta \in B'} m_{\beta} [h_{\beta}]$, where $B' \subseteq B$ is a finite subset and $m_{\beta} \in \mathbb{Z}$. Since $X^{n+1} = X' \vee \bigvee_{\beta \in B} S_{\beta}^{n+1}$, we let $i_{\beta} : S_{\beta}^{n+1} \rightarrow X^{n+1}$ be the inclusion maps and set $\xi = \sum_{\beta \in B'} m_{\beta} [i_{\beta}]$ in $\pi_{n+1}(X^{n+1})$. Then

$$f_{*n+1}^{n+1}(\xi) = \sum_{\beta \in B'} m_{\beta} [h_{\beta}] = \epsilon,$$

and so f_{*n+1}^{n+1} is an epimorphism. This proves the claim.

We then apply this construction to f^{n+1} and obtain an extension $f^{n+2} : X^{n+2} \rightarrow Y$ such that f_{*n+1}^{n+2} is a monomorphism and f_{*n+2}^{n+2} is an epimorphism. Because f_{*n+1}^{n+1} is an epimorphism, it follows that f_{*n+1}^{n+2} is an isomorphism and f_{*n+2}^{n+2} is an epimorphism.

We continue this process and obtain maps $f^k : X^k \rightarrow Y$, for all $k > n$. We set $X^n = X$ and form the space $K = \cup_{k \geq n} X^k$ with the weak topology determined by the X^k . Then the f^k determine a map $\bar{f} : K \rightarrow Y$ which is an extension of f . Therefore \bar{f}_{*i} is an isomorphism for $i > n$ and a monomorphism for $i = n$ by Proposition 1.5.24. This completes the proof. \square

The following corollary is frequently used.

Corollary 2.4.10

1. Let X and Y be path-connected spaces and let $f : X \rightarrow Y$ be an n -equivalence, $n \geq 0$. Then there exists a space K obtained from X by attaching cells of dimensions $\geq n+1$ and there exists a map $\bar{f} : K \rightarrow Y$ such that $\bar{f}|_X = f$ and \bar{f} is a weak equivalence.
2. Let Y be a k -connected space (not necessarily of the homotopy type of a CW complex) with $k \geq 0$. Then there exists a CW complex K and a weak equivalence $f : K \rightarrow Y$ such that $K^k = \{*\}$. In particular, if Y is any path-connected space, there exists a CW complex K with $K^0 = \{*\}$ and a weak equivalence $f : K \rightarrow Y$.
3. If Y is a k -connected space of the homotopy type of a CW complex, $k \geq 0$, then there exists a CW complex K of the homotopy type of Y such that $K^k = \{*\}$. In particular, $H_i(Y) = 0$ for $i \leq k$.

Proof. (1) Let K be the space constructed in Theorem 2.4.9. By Proposition 1.5.24, the inclusion map $i : X \rightarrow K$ is an n -equivalence. This and the fact that f is an n -equivalence implies that \bar{f} is an n -equivalence. By Theorem 2.4.9, \bar{f} is a weak equivalence.

(2) By hypothesis, the map $\{*\} \rightarrow Y$ is a k -equivalence. We then apply Part (1) to obtain the desired result.

(3) We apply Whitehead's first theorem 2.4.7 to (2). \square

Definition 2.4.11 For any space Y (not necessarily of the homotopy type of a CW complex), a CW complex K together with a weak equivalence $K \rightarrow Y$ is called a *CW approximation* to Y .

The existence of a CW approximation for *any* space, gives some indication of the importance of CW complexes in homotopy theory. It has been shown in [69] how to construct a CW approximation functorially. We do not prove this. However, the following remark is a consequence.

Remark 2.4.12 If $a : K \rightarrow X$ and $b : L \rightarrow Y$ are two CW approximations and $f : X \rightarrow Y$ is a map, then there exists a map $h : K \rightarrow L$, unique up to homotopy, such that $fa \simeq bh$. It follows that the homotopy type of a CW approximation of a space is uniquely determined by the homotopy type of the space.

Proposition 2.4.6 gives conditions for an induced map of homotopy sets to be a bijection. The following similar result is very useful.

Proposition 2.4.13 *Let (X, A) be a relative CW complex such that all relative cells have dimension $\geq n + 2$, let $i : A \rightarrow X$ be the inclusion map, and let Y be a space. Then $i^* : [X, Y] \rightarrow [A, Y]$ is an injection if $\pi_j(Y) = 0$ for $j > n + 1$ and is a surjection if $\pi_j(Y) = 0$ for $j > n$.*

Proof. We first show that i^* is onto if $\pi_j(Y) = 0$ for $j > n$. Let $f : A \rightarrow Y$ be a map and consider the relative $(n + 2)$ -skeleton

$$(X, A)^{n+2} = A \cup \bigcup_{\gamma \in C} e_{\gamma}^{n+2},$$

for $\gamma \in C$. Let $\phi_{\gamma} : S_{\gamma}^{n+1} \rightarrow (X, A)^{n+1} = A$ be an attaching function. By Exercise 2.25, $f\phi_{\gamma} \simeq_{\text{free}} h_{\gamma} : S_{\gamma}^{n+1} \rightarrow Y$, for some based map h_{γ} . By hypothesis, $h_{\gamma} \simeq *$ and so $f\phi_{\gamma}$ is freely homotopic to a constant function. By Corollary 1.4.11, $f\phi_{\gamma}$ extends to a free map $\tilde{f}_{\gamma} : E_{\gamma}^{n+2} \rightarrow Y$. These functions together with f determine a map $f^{n+2} : (X, A)^{n+2} \rightarrow Y$ that extends f . Next we write

$$(X, A)^{n+3} = (X, A)^{n+2} \cup \bigcup_{\delta \in D} e_{\delta}^{n+3}$$

with attaching maps $\psi_{\delta} : S_{\delta}^{n+2} \rightarrow (X, A)^{n+2}$, where $\delta \in D$. Then as before $f^{n+2}\psi_{\delta}$ is freely homotopic to a constant function, and so f^{n+2} extends to

a map $f^{n+3} : (X, A)^{n+3} \rightarrow Y$. We continue in this way and obtain a map $g : X \rightarrow Y$ such that $gi = f$. Thus i^* is onto.

Next we show that i^* is one-one if $j > n + 1$. Suppose $f, g : X \rightarrow Y$ and $fi \simeq_F gi$. Then f, g and F determine a map $F' : X \times \partial I \cup A \times I \rightarrow Y$. We then apply the previous argument to the relative CW complex $(X \times I, X \times \partial I \cup A \times I)$ and the map F' to obtain an extension $G : X \times I \rightarrow Y$ of F' . Thus $f \simeq_G g$, and so i^* is one-one. \square

We next discuss a relation between the homotopy groups and the homology groups of a space. We begin by defining the *Hurewicz homomorphism* $h_n : \pi_n(Y) \rightarrow H_n(Y)$, for any space Y and integer $n \geq 1$. Let $\alpha = [f] \in \pi_n(Y)$. Then $f : S^n \rightarrow Y$ induces a homomorphism $f_* : H_n(S^n) \rightarrow H_n(Y)$. We fix a generator $\gamma_n \in H_n(S^n) \cong \mathbb{Z}$ for all $n \geq 1$ and set $h_n(\alpha) = f_*(\gamma_n) \in H_n(Y)$. Clearly h_n is well-defined. By Proposition 2.2.9, $(f + g)_*(\gamma_n) = f_*(\gamma_n) + g_*(\gamma_n)$, and thus h_n is a homomorphism. Also, it is easily seen that if $k : Y \rightarrow Y'$ is a map, then the following diagram is commutative

$$\begin{array}{ccc} \pi_n(Y) & \xrightarrow{k_*} & \pi_n(Y') \\ \downarrow h_n & & \downarrow h'_n \\ H_n(Y) & \xrightarrow{k_*} & H_n(Y'), \end{array}$$

where h_n and h'_n are Hurewicz homomorphisms.

We next wish to prove that the Hurewicz homomorphism is an isomorphism in a special case. For this, we first introduce the notion of the degree of a map.

Definition 2.4.14 Let $f : S^n \rightarrow S^n$ be a map, $n \geq 1$, and let $f_* : H_n(S^n) \rightarrow H_n(S^n)$ be the induced homology homomorphism. We define an integer, the *degree of f* , denoted $\deg f$, by $f_*(\gamma_n) = (\deg f)\gamma_n$, where $\gamma_n \in H_n(S^n) \cong \mathbb{Z}$ is a generator. The definition is clearly independent of the choice of generator.

Lemma 2.4.15 Let $f, g : S^n \rightarrow S^n$.

1. $f \simeq g \Rightarrow \deg f = \deg g$.
2. $\deg(fg) = (\deg f)(\deg g)$.
3. $\deg(f + g) = \deg f + \deg g$.

Proof. Only (3) requires proof and this follows from Proposition 2.2.9. \square

Thus the degree yields a homomorphism $\deg : \pi_n(S^n) \rightarrow \mathbb{Z}$.

Proposition 2.4.16 For $n \geq 1$, the homomorphism $\deg : \pi_n(S^n) \rightarrow \mathbb{Z}$ is an isomorphism and so $[\text{id}]$ a generator of $\pi_n(S^n) \cong \mathbb{Z}$.

Proof. Since $\deg(\text{id}) = 1$, it follows that \deg is onto. We show that \deg is one-one in Appendix D. \square

This is an important result that plays a crucial role in what follows.

We introduce some notation before returning to the Hurewicz homomorphism. If G_α is an abelian group for $\alpha \in A$, then $\bigoplus_\alpha G_\alpha$ denotes the direct sum of the G_α . If $f_\alpha : G_\alpha \rightarrow H$ is a homomorphism of abelian groups for every α , we denote by $\{f_\alpha\} : \bigoplus_\alpha G_\alpha \rightarrow H$ the homomorphism determined by the f_α . Similarly, if the G_α are groups (not necessarily abelian), we let $*_\alpha G_\alpha$ denote the free product of the G_α (Appendix B). Homomorphisms $f_\alpha : G_\alpha \rightarrow H$ of groups determine a homomorphism $\{f_\alpha\} : *_\alpha G_\alpha \rightarrow H$. Now consider the wedge of n -spheres $\bigvee_\alpha S_\alpha^n$ for $\alpha \in A$, where A is any index set and let $i_\alpha : S_\alpha^n \rightarrow \bigvee_\alpha S_\alpha^n$ be the inclusion. Then it is known [39, p. 126] that $\{i_{\alpha*}\} : \bigoplus_\alpha H_n(S_\alpha^n) \rightarrow H_n(\bigvee_\alpha S_\alpha^n)$ is an isomorphism.

Lemma 2.4.17 1. For $n \geq 2$, $\{i_{\alpha*}\} : \bigoplus_{\alpha \in A} \pi_n(S_\alpha^n) \rightarrow \pi_n(\bigvee_{\alpha \in A} S_\alpha^n)$ is an isomorphism.

2. $\{i_{\alpha*}\} : *_{\alpha \in A} \pi_1(S_\alpha^1) \rightarrow \pi_1(\bigvee_{\alpha \in A} S_\alpha^1)$ is an isomorphism.

Proof. We assume that each sphere S_α^n is a CW complex with two cells (Example 1.5.10(5)).

(1) The result is clear if A consists of one element. Now let $A = \{\alpha_1, \dots, \alpha_k\}$ be a finite set with $k \geq 2$. Let $W = \bigvee_{i=1}^k S_{\alpha_i}^n$ and $P = \prod_{i=1}^k S_{\alpha_i}^n$ and let $j : W \rightarrow P$ be the inclusion. Then P is a CW complex and W is a subcomplex such that the $n+1$ -skeleton $P^{n+1} = W$. By Proposition 1.5.24, $j_* : \pi_n(W) \rightarrow \pi_n(P)$ is an isomorphism. But if $j_{\alpha_i} : S_{\alpha_i}^n \rightarrow \prod_{i=1}^k S_{\alpha_i}^n$ is the inclusion, then $\{j_{\alpha_i*}\} : \bigoplus_{i=1}^k \pi_n(S_{\alpha_i}^n) \rightarrow \pi_n(P)$ is an isomorphism by the discussion at the beginning of this section. From this (1) follows when A is finite. Now let A be infinite and let $f : S^n \rightarrow \bigvee_\alpha S_\alpha^n$ be a map. Since $f(S^n)$ is compact, there is a finite set $\{\alpha_1, \dots, \alpha_k\}$ such that $f(S^n) \subseteq \bigvee_{i=1}^k S_{\alpha_i}^n$ by Lemma 1.5.6. Therefore $[f] \in \pi_n(\bigvee_\alpha S_\alpha^n)$ is in the image of $\pi_n(\bigvee_{i=1}^k S_{\alpha_i}^n) \rightarrow \pi_n(\bigvee_\alpha S_\alpha^n)$, for some set $\{\alpha_1, \dots, \alpha_k\}$. Consequently $\{i_{\alpha*}\} : \bigoplus_\alpha \pi_n(S_\alpha^n) \rightarrow \pi_n(\bigvee_\alpha S_\alpha^n)$ is onto. To show that $\{i_{\alpha*}\}$ is one-one, we observe that any homotopy $F : S^n \times I \rightarrow \bigvee_\alpha S_\alpha^n$ has compact image and so factors through a homotopy $F' : S^n \times I \rightarrow \bigvee_{i=1}^k S_{\alpha_i}^n$ for some finite set $\{\alpha_1, \dots, \alpha_k\}$. This completes the proof of (1).

(2) This is proved in Appendix B as Proposition B.3. \square

The following proposition contains a special case of the Hurewicz theorem for a wedge of spheres of the same dimension. The full Hurewicz theorem is proved in Section 6.4 as Theorem 6.4.8.

Proposition 2.4.18

1. If $n \geq 1$, then $\pi_i(S^n) = 0$ for $i < n$ and $h_n : \pi_n(S^n) \rightarrow H_n(S^n) \cong \mathbb{Z}$ is an isomorphism.

2. Let $S_\alpha^n = S^n$ for all α in some set A .

a. If $n \geq 2$, then the Hurewicz homomorphism $h_n : \pi_n(\bigvee_{\alpha \in A} S_\alpha^n) \rightarrow H_n(\bigvee_{\alpha \in A} S_\alpha^n)$ is an isomorphism.

b. The Hurewicz homomorphism $h_1 : \pi_1(\bigvee_{\alpha \in A} S^1_\alpha) \rightarrow H_1(\bigvee_{\alpha \in A} S^1_\alpha)$ is an epimorphism.

Proof. (1) Let $i < n$ and give each of S^i and S^n the CW decomposition with two cells. If $f : S^i \rightarrow S^n$ is any map, then by Corollary 1.5.23, f is homotopic to a cellular map $f' : S^i \rightarrow S^n$. Thus $f'(S^i) = *$ and so $f \simeq *$. Therefore $\pi_i(S^n) = 0$. To determine h_n , consider the commutative diagram

$$\begin{array}{ccc} & & \mathbb{Z} \\ & \nearrow \text{deg} & \downarrow \cong \\ \pi_n(S^n) & \xrightarrow{h_n} & H_n(S^n), \end{array}$$

where the vertical arrow is the isomorphism that assigns to the integer k the element $k\gamma_n \in H_n(S^n)$, for γ_n a generator of $H_n(S^n)$. The result follows from Proposition 2.4.16.

(2) For Part (a) consider the commutative diagram

$$\begin{array}{ccc} \bigoplus_\alpha \pi_n(S^n_\alpha) & \xrightarrow{\{i_{\alpha*}\}} & \pi_n(\bigvee_\alpha S^n_\alpha) \\ \downarrow \bigoplus_\alpha h_\alpha & & \downarrow h_n \\ \bigoplus_\alpha H_n(S^n_\alpha) & \xrightarrow{\{i_{\alpha*}\}} & H_n(\bigvee_\alpha S^n_\alpha), \end{array}$$

where $h_\alpha : \pi_n(S^n_\alpha) \rightarrow H_n(S^n_\alpha)$ and h_n are Hurewicz homomorphisms. Because the horizontal homomorphisms are isomorphisms and the h_α are isomorphisms, h_n is an isomorphism. Part (b) is a special case of Proposition B.5. \square

The last result of this section is part of Whitehead's second theorem 6.4.15.

Proposition 2.4.19 *Let X and Y be path-connected CW complexes, let $f : X \rightarrow Y$ be a map, and let $n \geq 1$ be an integer. If f is an n -equivalence, then $f_* : H_i(X) \rightarrow H_i(Y)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$.*

Proof. By Corollary 2.4.10(1) and Exercise 2.26, there is a CW complex K containing X such that K is obtained from X by adjoining cells of dimensions $\geq n+1$ and there is a weak homotopy equivalence $\bar{f} : K \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow j & \nearrow \bar{f} & \\ K & & \end{array}$$

where j is the inclusion map. By Whitehead's first theorem 2.4.7, \bar{f} is a homotopy equivalence, and so $\bar{f}_* : H_i(K) \rightarrow H_i(Y)$ is an isomorphism for

all $i < n$ and an epimorphism for $i = n$ if and only if the same holds for $j_* : H_i(X) \rightarrow H_i(K)$. All the cells of K of dimension $\leq n$ lie in X , thus the relative homology group $H_i(K, X) = 0$, for all $i \leq n$. From the exact homology sequence of the pair (K, X) , it follows that $j_* : H_i(X) \rightarrow H_i(K)$ is an isomorphism for all $i < n$ and an epimorphism for $i = n$. \square

The complete second theorem of Whitehead (Theorem 6.4.15) has another part in which the roles of homology and homotopy groups are interchanged. This theorem is proved in Section 6.4 as a consequence of the relative Hurewicz theorem.

2.5 Moore Spaces and Eilenberg–Mac Lane Spaces

The following two lemmas are useful in our discussion of Moore spaces and Eilenberg–Mac Lane spaces.

Lemma 2.5.1 *Let $n \geq 1$ and let X be a based CW complex with $(n-1)$ -skeleton $X^{n-1} = \{*\}$ and $\dim X \leq n+1$, that is, $X = X^n \cup \bigcup_{\beta \in B} e_{\beta}^{n+1}$, for $X^n = \bigvee_{\alpha \in A} S_{\alpha}^n$, where S_{α}^n are n -spheres and e_{β}^{n+1} are open $(n+1)$ -cells. Let Y be a space and let $\phi : \pi_n(X) \rightarrow \pi_n(Y)$ be a homomorphism. Then there exists a map $f : X \rightarrow Y$ such that $f_* = \phi : \pi_n(X) \rightarrow \pi_n(Y)$.*

Proof. Let $k : X^n \rightarrow X$ and $i_{\alpha} : S_{\alpha}^n \rightarrow X^n$ be the inclusions maps. Then there are homomorphisms

$$\pi_n(X^n) \xrightarrow{k_*} \pi_n(X) \xrightarrow{\phi} \pi_n(Y)$$

and we define $f_{\alpha} : S_{\alpha}^n \rightarrow Y$ by $\phi k_*[i_{\alpha}] = [f_{\alpha}]$. The f_{α} determine a map $f^n : X^n \rightarrow Y$ such that $f^n i_{\alpha} = f_{\alpha}$. Therefore

$$f_*^n[i_{\alpha}] = [f_{\alpha}] = \phi k_*[i_{\alpha}].$$

By Lemma 2.4.17, the $[i_{\alpha}]$ are generators of $\pi_n(X^n)$, and so $f_*^n = \phi k_*$. Let $h_{\beta} : S_{\beta}^n \rightarrow X^n$ be an attaching function for e_{β}^{n+1} . By Exercise 2.25 and Lemma 1.5.3 we may assume that h_{β} is a (based) map. Then $kh_{\beta} \simeq *$ since kh_{β} factors through the contractible space $E_{\beta}^{n+1} = E^{n+1}$. Hence $f_*^n[h_{\beta}] = \phi k_*[h_{\beta}] = 0$. Thus $f^n h_{\beta} \simeq *$ for every $\beta \in B$, and consequently f^n extends to a map $f : X \rightarrow Y$. But $\phi k_* = f_*^n = f_* k_*$ and $k_* : \pi_n(X^n) \rightarrow \pi_n(X)$ is onto by Proposition 1.5.24. Therefore $f_* = \phi$. \square

Lemma 2.5.2 *For every abelian group G and $n \geq 1$, there exists a based CW complex L with the following properties: the $(n-1)$ -skeleton $L^{n-1} = \{*\}$, $\dim L \leq n+1$, and for all $i \geq 0$,*

$$H_i(L) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Proof. We take a presentation $G = F/R$, where F is free-abelian and $R \subseteq F$. We choose bases $\{x_\alpha\}_{\alpha \in A}$ and $\{r_\beta\}_{\beta \in B}$ of F and R , respectively. Let S_α^n be the n -sphere indexed by $\alpha \in A$ and define L^n to be the wedge $\bigvee_{\alpha \in A} S_\alpha^n$. The Hurewicz homomorphism $h_n : \pi_n(L^n) \rightarrow H_n(L^n)$ is an isomorphism for $n \geq 2$ and an epimorphism for $n = 1$ by Proposition 2.4.18. If $\beta \in B$, then $r_\beta \in R \subseteq F \cong H_n(L^n)$. We choose a map $k_\beta : S_\beta^n \rightarrow L^n$ such that $h_n[k_\beta] = r_\beta$. Using k_β , we attach $(n+1)$ -cells e_β^{n+1} to L^n and form $L = L^n \cup \bigcup_{\beta \in B} e_\beta^{n+1}$. To complete the proof we use the CW homology of L (Appendix C). The i th chain group $C_i(L)$ is the free-abelian group generated by the i -cells, and so

$$C_i(L) \cong \begin{cases} F & \text{if } i = n \\ R & \text{if } i = n + 1 \\ 0 & \text{if } i \neq 0, n, n + 1. \end{cases}$$

Furthermore, it is not difficult to show that the boundary homomorphism $C_{n+1}(L) \rightarrow C_n(L)$ can be identified with the inclusion $R \subseteq F$ (see [64, Prop. 8.2.12]). Thus L has the desired homology. \square

We turn to Moore spaces.

Definition 2.5.3 Let G be an abelian group and n an integer ≥ 2 . A based CW complex X is called a *Moore space* of type (G, n) if X is 1-connected and

$$H_i(X) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

In Lemma 2.5.2, a Moore space L of type (G, n) has been constructed. We denote this Moore space (or any space homeomorphic to it) by $M(G, n)$.

We note a few properties of $M(G, n)$.

Lemma 2.5.4

1. The Hurewicz homomorphism $h_n : \pi_n(M(G, n)) \rightarrow H_n(M(G, n))$ is an isomorphism, and so $\pi_n(M(G, n)) = G$.
2. If $\phi : G \rightarrow H$ is a homomorphism of abelian groups, then there exists a map $f : M(G, n) \rightarrow M(H, n)$ such that $f_* = \phi : H_n(M(G, n)) \rightarrow H_n(M(H, n))$.

Proof. (1) From the construction of $M(G, n)$ in Lemma 2.5.2 as $(\bigvee_{\alpha \in A} S_\alpha^n) \cup \bigcup_{\beta \in B} e_\beta^{n+1}$, we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \pi_n(\bigvee_{\beta \in B} S_\beta^n) & \xrightarrow{k_*} & \pi_n(\bigvee_{\alpha \in A} S_\alpha^n) & \xrightarrow{i_*} & \pi_n(M(G, n)) \longrightarrow 0 \\ & & \downarrow h'_n & & \downarrow h''_n & & \downarrow h_n \\ 0 & \longrightarrow & H_n(\bigvee_{\beta \in B} S_\beta^n) & \xrightarrow{k_*} & H_n(\bigvee_{\alpha \in A} S_\alpha^n) & \xrightarrow{i_*} & H_n(M(G, n)) \longrightarrow 0, \end{array}$$

where h_n , h'_n , and h''_n are Hurewicz homomorphisms, $k : \bigvee_{\beta \in B} S^n_\beta \rightarrow \bigvee_{\alpha \in A} S^n_\alpha$ is determined by the k_β , and i is the inclusion. The bottom row is exact, h'_n and h''_n are isomorphisms (2.4.18), $ik \simeq *$, and the upper i_* is onto by Proposition 1.5.24. Thus the top row is exact and so h_n is an isomorphism.

(2) Let $\phi : G \rightarrow H$ be a homomorphism and let $L = M(G, n)$ and $M = M(H, n)$. Consider the Hurewicz homomorphisms $h' : \pi_n(L) \rightarrow H_n(L) = G$ and $h : \pi_n(M) \rightarrow H_n(M) = H$ which are isomorphisms by (1). By Lemma 2.5.1, there exists a map $f : L \rightarrow M$ such that $f_* = h^{-1}\phi h' : \pi_n(L) \rightarrow \pi_n(M)$. Hence the induced homology homomorphism $f_* = \phi : H_n(L) \rightarrow H_n(M)$. \square

Remark 2.5.5 We note that several choices have been made in the construction of $M(G, n)$ such as the presentation of G and the choice of generators of R and F . We show in Proposition 6.4.16 that the homotopy type of a Moore space of type (G, n) depends only on G and n . However, in spite of the notation, $M(G, n)$ is not functorial in G . For now, when we write $M(G, n)$ we assume that it has been constructed relative to a choice of presentation and of generators.

The spaces $M(G, n)$ can easily be described in special cases. We have $M(\mathbb{Z}, n) = S^n$ and if F is a free-abelian group with a basis whose cardinality is the same as some set A , then $M(F, n) = \bigvee_{\alpha \in A} S^n_\alpha$. Furthermore, $M(\mathbb{Z}_m, n)$ can be taken to be the space $S^n \cup_m e^{n+1}$ obtained by attaching an $(n+1)$ -cell to S^n by a map $m : S^n \rightarrow S^n$ of degree m . For $n = 1$, we define the Moore space $M(\mathbb{Z}, 1) = S^1$. We do not consider $M(G, 1)$ for other groups G (see [89]). For $n \geq 3$, $M(G, n) \cong \Sigma M(G, n-1)$ (see Exercise 3.1). In addition, if L is the space of Lemma 2.5.2 with $n = 1$, then $M(G, 2) \cong \Sigma L$. Thus all Moore spaces $M(G, n)$ are suspensions, in fact, $M(G, n)$ is a double suspension if $n \geq 3$ or if $n = 2$ and G is free-abelian. Hence $[M(G, n), X]$ is a group for all X . It is abelian if $n \geq 3$ or if $n = 2$ and G is free-abelian. At the end of this section we discuss the reason for making the definition of Moore space in terms of homology groups instead of cohomology groups. We compare Moore spaces with spaces with a single nonvanishing cohomology group.

The suspension structure of Moore spaces $M(G, n)$ enables us to define homotopy groups with coefficients.

Definition 2.5.6 Let G be an abelian group and let $n \geq 1$ (assuming $G = \mathbb{Z}$ if $n = 1$). Then for every space X , define the *n th homotopy group of X with coefficients in G* by

$$\pi_n(X; G) = [M(G, n), X].$$

Other notation (which we do not use) for this is $\pi_n(G; X)$ (see [40, p. 10]).

Note that $\pi_n(X; \mathbb{Z}) = \pi_n(X)$, the n th (ordinary) homotopy group of X . Furthermore, a map $h : X \rightarrow X'$ induces a homomorphism $h_* : \pi_n(X; G) \rightarrow$

$\pi_n(X'; G)$ defined as the induced homomorphism $h_* : [M(G, n), X] \rightarrow [M(G, n), X']$.

Homotopy groups with coefficients are discussed in the sequel. In particular, we present a universal coefficient theorem in Section 5.2 that expresses homotopy groups with coefficients in terms of ordinary homotopy groups.

We next turn to Eilenberg–Mac Lane spaces.

Definition 2.5.7 Let G be an abelian group and let n be an integer ≥ 1 . An *Eilenberg–Mac Lane space* of type (G, n) is a space X of the homotopy type of a based CW complex such that for every $i \geq 1$,

$$\pi_i(X) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

An Eilenberg–Mac Lane space of type (G, n) is denoted $K(G, n)$.

The following lemma is used to show that Eilenberg–Mac Lane spaces exist.

Lemma 2.5.8 *If X is a space, then for every $m \geq 1$, there exist spaces $W^{(m)}$ and inclusion maps $j_m : X \rightarrow W^{(m)}$ such that*

1. $\pi_i(W^{(m)}) = 0$ for $i > m$.
2. $j_{m*} : \pi_i(X) \rightarrow \pi_i(W^{(m)})$ is an isomorphism for $i \leq m$.
3. $W^{(m)}$ is obtained from X by attaching cells of dimension $\geq m + 2$.

Proof. Parts (1) and (3) follow immediately from Theorem 2.4.9 by taking $Y = \{*\}$ and $n = m + 1$. Part (2) is a consequence of Proposition 1.5.24. \square

Next we show that Eilenberg–Mac Lane spaces exist and obtain some of their properties.

Proposition 2.5.9 *Let G be an abelian group and n an integer ≥ 1 .*

1. *There exists an Eilenberg–Mac Lane space $K(G, n)$.*
2. *If $\phi : G \rightarrow H$ is a homomorphism, then there exists a map $h : K(G, n) \rightarrow K(H, n)$ such that $h_* = \phi : \pi_n(K(G, n)) \rightarrow \pi_n(K(H, n))$.*
3. *Any two Eilenberg–Mac Lane spaces of type (G, n) have the same homotopy type.*

Proof. (1) We first construct an Eilenberg–Mac Lane space of type (G, n) when $n \geq 2$. We apply Lemma 2.5.8 with X equal to the Moore space $M(G, n)$. Then $W^{(n)}$ is an Eilenberg–Mac Lane space of type (G, n) by Lemma 2.5.4. For $n = 1$, we set $K(G, 1) = \Omega K(G, 2)$.

(2) Let $\phi : G \rightarrow H$ be a homomorphism and let $K = K(G, n)$ and $L = K(H, n)$ be any Eilenberg–Mac Lane spaces. Let X be the Eilenberg–Mac Lane space of type (G, n) constructed in (1). We construct a map $f : X \rightarrow L$

such that $f_* = \phi : \pi_n(X) \rightarrow \pi_n(L)$. If $j : X^{n+1} \rightarrow X$ is the inclusion, then $j_* : \pi_n(X^{n+1}) \rightarrow \pi_n(X)$ is an isomorphism by Proposition 1.5.24. The $n+1$ -skeleton $X^{n+1} = M(G, n)$ and L satisfy the hypotheses of Lemma 2.5.1, and so there exists $f^{n+1} : X^{n+1} \rightarrow L$ such that $f_*^{n+1} = \phi j_* : \pi_n(X^{n+1}) \rightarrow \pi_n(L)$. Now apply Proposition 2.4.13 to the relative CW complex (X, X^{n+1}) and the map f^{n+1} . We conclude that there is a map $f : X \rightarrow L$ such that $fj \simeq f^{n+1}$, and so $f_* = \phi$. Similarly the identity map $\text{id} : G \rightarrow G$ yields a map $g : X \rightarrow K$ such that $g_* = \text{id}$. Then g is a homotopy equivalence by Whitehead's first theorem. Thus if $h = fg^{-1} : K(G, n) \rightarrow K(H, n)$, we have $h_* = \phi$.

(3) If K and L are both Eilenberg–Mac Lane spaces of type (G, n) , then the identity homomorphism $\text{id}_G : G \rightarrow G$ induces a map $h : K \rightarrow L$ by (2). Then h is a homotopy equivalence. \square

In general Eilenberg–Mac Lane spaces are infinite-dimensional complexes and are not easy to describe. There are a few that are familiar spaces and we mention these now: $K(\mathbb{Z}, 1) = S^1$, $K(\mathbb{Z}_2, 1) = \mathbb{RP}^\infty$, infinite-dimensional real projective space, and $K(\mathbb{Z}, 2) = \mathbb{CP}^\infty$, infinite-dimensional complex projective space (see Exercise 5.19).

It can be shown that for any group G (not necessarily abelian), Eilenberg–Mac Lane spaces of type $(G, 1)$ exist and are unique up to homotopy. However, we have no need to consider the spaces $K(G, 1)$ when G is non-abelian (except in Exercise 2.31).

Now let $K(G, n+1)$ be an Eilenberg–Mac Lane space with $n \geq 1$. We apply the loop space functor to this space and have by Proposition 2.3.5 that

$$\pi_i(\Omega K(G, n+1)) \cong \pi_{i+1}(K(G, n+1)) = \begin{cases} G & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Hence, as previously noted in the case $n = 1$, $\Omega K(G, n+1)$ is an Eilenberg–Mac Lane space $K(G, n)$. In fact, $\Omega^k K(G, n+k)$ is an Eilenberg–Mac Lane space of type (G, n) . Therefore, for any space X , the set $[X, K(G, n)]$ has abelian group structure.

We next indicate how Eilenberg–Mac Lane spaces give rise to cohomology groups.

Definition 2.5.10 For any space X , abelian group G and integer $n \geq 1$, we define the n th homotopical cohomology group of X with coefficients in G as

$$H^n(X; G) = [X, K(G, n)].$$

If $h : X' \rightarrow X$ is a map, then $h^* : H^n(X; G) \rightarrow H^n(X'; G)$ is just the induced homomorphism $h^* : [X, K(G, n)] \rightarrow [X', K(G, n)]$.

Remark 2.5.11 It can be shown that if X is a CW complex, then the homotopical cohomology groups $H^n(X; G)$ are isomorphic to the singular cohomology groups $H_{\text{sing}}^n(X; G)$. There are several proofs of this: one uses the

Brown representation theorem [39, p. 448] and another uses obstruction theory [91, p. 250]. In addition, in [46] Huber gives an isomorphism between the homotopical cohomology groups and the Čech cohomology groups. We give a simple proof of the isomorphism with singular cohomology in Section 5.3.

We note that we can define a function $\rho : H^n(X; G) \rightarrow H_{\text{sing}}^n(X; G)$ as follows. If $[f] \in H^n(X; G)$, then $f_{\text{sing}}^* : H_{\text{sing}}^n(K(G, n); G) \rightarrow H_{\text{sing}}^n(X; G)$ is the induced homomorphism of singular cohomology. Note that $H_n(K(G, n)) = G$ since $K(G, n)$ can be taken to be $M(G, n)$ with cells of dimension $\geq n + 2$ attached. Thus there is an element $b^n \in H_{\text{sing}}^n(K(G, n); G)$, called the n th basic class, which is defined by $\mu(b^n) = \text{id}$, where $\mu : H_{\text{sing}}^n(K(G, n); G) \rightarrow \text{Hom}(H_n(K(G, n)), G) \cong \text{Hom}(G, G)$ is the epimorphism in the universal coefficient theorem for cohomology (Appendix C). Then set $\rho[f] = f_{\text{sing}}^*(b^n)$. It is shown that ρ is an isomorphism in Theorem 5.3.2.

We need some properties of homotopical cohomology groups in the following chapters. These properties are known to hold for CW or singular cohomology groups. But since we do not prove the equivalence of the latter cohomology groups with the homotopical cohomology groups until Theorem 5.3.2, we next establish these properties.

Lemma 2.5.12 *Let X and Y be path-connected, based CW complexes, let $f : X \rightarrow Y$ be a map and let $n \geq 1$ be an integer. If f is an n -equivalence, then, for every group G , $f^* : H^i(Y; G) \rightarrow H^i(X; G)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$.*

Proof. (1) By Corollary 2.4.10(1), there is a CW complex K obtained from X by adjoining cells of dimensions $\geq n + 1$ and a homotopy equivalence $\bar{f} : K \rightarrow Y$ such that $\bar{f}j = f$, where $j : X \rightarrow K$ is the inclusion map. Therefore $f^* : H^i(Y; G) \rightarrow H^i(X; G)$ is an isomorphism for $i < n$ and a monomorphism for $i = n$ if and only if the same holds for $j^* : H^i(K; G) \rightarrow H^i(X; G)$. But the latter follows at once from Proposition 2.4.13. \square

Next we let X be a space, G an abelian group, and $n \geq 1$ an integer. We define a homomorphism $\eta_\pi : H^n(X; G) \rightarrow \text{Hom}(\pi_n(X), G)$ by $\eta_\pi[f] = f_* : \pi_n(X) \rightarrow \pi_n(K(G, n)) = G$.

Lemma 2.5.13 *If X is an $(n-1)$ -connected based CW complex, $n \geq 1$, then $\eta_\pi : H^n(X; G) \rightarrow \text{Hom}(\pi_n(X), G)$ is an isomorphism. (For $n = 1$ we assume that $\pi_1(X)$ is abelian.)*

Proof. By Corollary 2.4.10(3), we may assume $X^{n-1} = \{*\}$. We first prove that $\eta_\pi : H^n(X; G) \rightarrow \text{Hom}(\pi_n(X), G)$ is an isomorphism when $X = X^{n+1}$. By Lemma 2.5.1, η_π is onto. Now let $f : X \rightarrow K$, where $K = K(G, n)$, and assume that $\eta_\pi[f] = 0$. Since X^n is a wedge of n -spheres, $f|X^n \simeq * : X^n \rightarrow K$. But (X, X^n) has the homotopy extension property, and so there is a map $f' : X \rightarrow K$ such that $f \simeq f'$ and $f'|X^n = *$. Therefore f' induces a map $\tilde{f}' : X/X^n \rightarrow K$ such that $\tilde{f}'q = f'$, where $q : X \rightarrow X/X^n$ is the

projection. Since X/X^n is a wedge of $(n+1)$ -spheres, $\tilde{f}' \simeq *$. Hence $f' \simeq *$, and so $f \simeq *$. Therefore η_π is an isomorphism when $\dim X \leq n+1$. For an arbitrary $(n-1)$ -connected CW complex X , the inclusion map $X^{n+1} \rightarrow X$ is an $(n+1)$ -equivalence by the cellular approximation theorem. Thus $\pi_n(X) \cong \pi_n(X^{n+1})$ and, by Lemma 2.5.12, $H^n(X; G) \cong H^n(X^{n+1}; G)$. Therefore $\eta_\pi : H^n(X; G) \rightarrow \text{Hom}(\pi_n(X), G)$ is an isomorphism. \square

We next prove the Hopf classification theorem [91, p. 244].

Theorem 2.5.14 *If X is a CW complex of dimension $\leq n$, then there is a bijection between $[X, S^n]$ and $H^n(X)$.*

Proof. Let $K(\mathbb{Z}, n)$ be the Eilenberg–Mac Lane space constructed in the proof of Proposition 2.5.9 with $G = \mathbb{Z}$ and let $i : S^n \rightarrow K(\mathbb{Z}, n)$ be the inclusion. Then i induces $i_* : [X, S^n] \rightarrow [X, K(\mathbb{Z}, n)] = H^n(X)$. The $n+1$ -skeleton of $K(\mathbb{Z}, n)$ is S^n , therefore Proposition 1.5.24 shows that i_* is a bijection. \square

NOTE We denote the homotopical cohomology groups by $H^n(X; G)$ and refer to them as cohomology groups.

In the Eckmann–Hilton duality theory (discussed in the next section), cohomology groups with coefficients are dual to homotopy groups with coefficients. The former are defined as homotopy classes of maps with codomain an Eilenberg–Mac Lane space and the latter as homotopy classes of maps with domain a Moore space. Eilenberg–Mac Lane spaces are spaces with a single nonvanishing homotopy group, thus it would appear that the dual notion should be a co-Moore space, that is, a space with a single nonvanishing cohomology group. However, we have instead taken the dual to be a Moore space, that is, a space with a single nonvanishing homology group. The reason for this is that co-Moore spaces of type (G, n) do not exist for every group G (see [39, pp. 318–319]). Therefore to ensure the existence of homotopy groups with coefficients for any abelian group G , they have been defined in terms of Moore spaces.

We carry this discussion of co-Moore spaces a bit further. For a finitely generated abelian group G , write $G = F \oplus T$, where F is a free-abelian group and T is a finite abelian group. If $C(G, n)$ denotes a co-Moore space of type (G, n) , then a simple calculation of cohomology shows that $M(F, n) \vee M(T, n-1)$ is a $C(G, n)$. In particular, if $G = \mathbb{Z}_m$, then $M(\mathbb{Z}_m, n-1)$ is a $C(\mathbb{Z}_m, n)$. As noted above, we could have defined the homotopy groups of X with coefficients in \mathbb{Z}_m using co-Moore spaces by

$$\tilde{\pi}_n(X; \mathbb{Z}_m) = [C(\mathbb{Z}_m, n), X].$$

Then

$$\tilde{\pi}_n(X; \mathbb{Z}_m) = [M(\mathbb{Z}_m, n-1), X] = \pi_{n-1}(X; \mathbb{Z}_m).$$

However, we use Definition 2.5.6 for homotopy groups with coefficients.

2.6 Eckmann–Hilton Duality I

Our exposition is based on the duality theory of Eckmann and Hilton that we have referred to several times without explanation. We now discuss this topic. The first appearance of the duality in the literature was in the papers [27, 28, 29] of Eckmann and Hilton and the book [40] by Hilton. This duality principle differs from certain other duality principles which are formal and automatic. For example, in projective geometry there is a duality which asserts that every definition remains meaningful and every theorem true if we interchange the words *point* and *line* (and consequently other pairs of words such as *colinear* and *concurrent*, *side* and *vertex*, and so on)[22, Chap. 3]. Parts of the Eckmann–Hilton duality are formal and automatic (usually those parts that can be described in categorical terms). However, much of it is intuitive, informal, and heuristic.

We begin with the aspect of the Eckmann–Hilton duality which depends on duality in a category and we refer to some fundamental facts about categories and functors from Appendix F. With any category \mathcal{C} , we associate a *dual category* \mathcal{C}^{op} (also called the *opposite category*). The objects of \mathcal{C}^{op} are precisely those of \mathcal{C} , but the set of morphisms from an object X in \mathcal{C}^{op} to an object Y in \mathcal{C}^{op} , denoted $\mathcal{C}^{op}(X, Y)$, is defined to be $\mathcal{C}(Y, X)$. If composition of morphisms in \mathcal{C}^{op} is denoted by $*$ and composition in \mathcal{C} by juxtaposition, then $f * g = gf$.

Now suppose that Σ is a statement or concept that is meaningful in a category \mathcal{C} . Then we can apply it to the category \mathcal{C}^{op} and interpret it as a statement or concept in \mathcal{C} . This latter statement or concept in \mathcal{C} that is denoted Σ^* is the *dual* of Σ . If $\Sigma = \Sigma^*$, then we say that Σ is *self-dual*.

For example, recall from Appendix F the notion of categorical product. If X and Y are objects in a category \mathcal{C} , their categorical product is an object P in \mathcal{C} together with morphisms $p_1 : P \rightarrow X$ and $p_2 : P \rightarrow Y$ such that the following holds. If $f : A \rightarrow X$ and $g : A \rightarrow Y$ are any morphisms, then there exists a unique morphism $\theta : A \rightarrow P$ such that $p_1\theta = f$ and $p_2\theta = g$. If Σ denotes this concept in category \mathcal{C} , then the dual concept Σ^* in \mathcal{C} is the following. Suppose X and Y are objects in \mathcal{C} and there is an object C in \mathcal{C} together with morphisms $i_1 : X \rightarrow C$ and $i_2 : Y \rightarrow C$ such that if $f : X \rightarrow B$ and $g : Y \rightarrow B$ are any morphisms, then there exists a unique morphism $\theta : C \rightarrow B$ with $\theta i_1 = f$ and $\theta i_2 = g$. Then C is the coproduct in \mathcal{C} of X and Y , and so the coproduct is dual to the product. We can investigate various categories to see if the product or the coproduct exists and, if so, if it is given by a well-known construction. This can of course be done for any Σ and Σ^* .

The main categories that we consider are the topological category Top_* and the homotopy category $HoTop_*$ (Appendix F). In Top_* the product of X and Y is just their cartesian product $X \times Y$ with p_1 and p_2 the two projections. The coproduct is the wedge $X \vee Y$ with i_1 and i_2 the two injections. For the product and coproduct in the homotopy category $HoTop_*$ we take the cartesian product $X \times Y$ with homotopy classes of p_1 and p_2 for the former

and the wedge $X \vee Y$ with homotopy classes of i_1 and i_2 for the latter (see Lemma 1.3.6). Thus the product and the wedge as just defined are dual in $HoTop_*$. Other examples of dual concepts in $HoTop_*$ are homotopy retracts and homotopy sections, and H-spaces and co-H-spaces.

There is another aspect of Eckmann–Hilton duality that is more obscure and which often takes the form of a duality between functors or constructions. We illustrate this with an example. Suppose $f : X \rightarrow Y$ is a map. Then $f \simeq *$ if and only if there is a contractible space T such that f factors through T , that is, there are maps $i : X \rightarrow T$ and $\tilde{f} : T \rightarrow Y$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow i & \nearrow \tilde{f} & \\ T & & \end{array}$$

This happens if there is a functor C that assigns a contractible space $C(X)$ to every space X and a map $i_X : X \rightarrow C(X)$ with the above property. The cone on X does this, as we know by Proposition 1.4.9. If we dualize this by reversing the direction of the maps, then we seek a functor E such that $E(Y)$ is contractible for every space Y and a map $p_Y : E(Y) \rightarrow Y$ with the following property: $f \simeq * : X \rightarrow Y$ if and only if there is a map $\hat{f} : X \rightarrow E(Y)$ such that the following diagram commutes

$$\begin{array}{ccc} Y & \xleftarrow{f} & X \\ \uparrow p_Y & \nwarrow \hat{f} & \\ E(Y) & & \end{array}$$

We have seen that the path space EY has this property (Proposition 1.4.9). One might raise the question of why the dual of a contractible space is a contractible space. This could be argued as follows. The dual of an identity morphism in a category is an identity morphism because the defining property of id_X is $\text{id}_X f = f$ and $g \text{id}_X = g$, for all morphisms f and g . The constant morphisms have a similar defining property, and so the dual of a constant map is a constant map. Thus identity morphisms and constant morphisms are self-dual. But in Top_* a contractible space is one in which the identity map is homotopic to the constant map. Thus it is reasonable to regard the dual of a contractible space to be a contractible space.

We digress briefly to comment further on this example in order to indicate the origin of the Eckmann–Hilton duality. The preceding discussion can be transferred to the category of (left) R -modules in the following way. A homomorphism $\phi : A \rightarrow B$ of R -modules is called i -nullhomotopic if it can be extended to some injective R -module Q that contains A . This could be regarded as the analogue of extending a map of spaces to the cone of

the domain. We then say that two R -homomorphisms from A to B are i -homotopic if their difference is i -nullhomotopic. Alternatively, $\phi : A \rightarrow B$ is p -nullhomotopic if it can be factored through some projective R -module P that has B as a quotient. Then P could be regarded as the analogue in this category of the path space of the codomain. Two homomorphisms would then be p -homotopic if their difference is p -nullhomotopic. It can be shown that the notions of i -homotopy and p -homotopy in the category of R -modules do not agree. Furthermore, by taking Q/A we obtain an analogue of the suspension and by taking the kernel of $P \rightarrow B$ we obtain an analogue of the loop space. It was the realization that injective modules and their quotients play the role of cones and suspensions and that projective modules and their kernels play the role of path spaces and loop spaces in the category of left R -modules that was the beginning of the Eckmann–Hilton duality [40, Chap. 13].

We return to discussing the cone and path space functors in the category Top_* . We observe that they are adjoint functors. This implies that for a map $f : X \rightarrow Y$, there is a one–one correspondence between maps $F : CX \rightarrow Y$ such that $Fi_X = f$ and maps $\tilde{F} : X \rightarrow EY$ such that $p_Y \tilde{F} = f$. The correspondence is just the adjoint one given by

$$\tilde{F}(x)(t) = F\langle x, t \rangle,$$

for $x \in X$ and $t \in I$ (Proposition 1.3.4). Thus the two notions of nullhomotopy in the category of R -modules become the single notion of ordinary nullhomotopy for the category of spaces and maps. We say that the cone functor C and the path space functor E are dual functors (in addition to being adjoint). The suspension ΣX is a quotient of the map $i_X : X \rightarrow CX$ and the loop space ΩY is a “kernel” of the map $p_Y : EY \rightarrow Y$, therefore we view the functors Σ and Ω as dual to each other. We have already noted that these two functors are adjoint in the homotopy category $HoTop_*$, namely,

$$[\Sigma X, Y] \cong [X, \Omega Y]$$

by Proposition 2.3.5. Therefore we regard the functors Σ and Ω as dual and adjoint in $HoTop_*$. In a similar way the reduced cylinder $X \times I$ and the path space X^I can be regarded as dual and adjoint functors of X .

Furthermore, the homotopy groups appear to have properties dual to those of the cohomology groups. For example, the homotopy groups are covariant functors and the cohomology groups are contravariant functors. Moreover, there is a formula that expresses the homotopy groups of a product as a product of homotopy groups and an analogous formula for the cohomology groups of a wedge. In addition, the homotopy groups of the loop space of X are isomorphic to those of X (with a shift in degree) and a similar statement holds for the cohomology of a suspension. But there are also important differences. The fundamental group of a space is not necessarily abelian, but all cohomology groups are abelian. The homotopy groups for most common spaces such as CW complexes are not necessarily zero from some degree

on (see Section 5.6), whereas the cohomology groups are zero above the dimension of the space (although this difference may not indicate a failure of duality). In spite of this, we do view ordinary homotopy groups and integral cohomology as being informally dual to each other. Although the duality becomes more tenuous when we assert that specific spaces are dual to each other, we regard Eilenberg–Mac Lane spaces and Moore spaces as duals in a weak sense. Therefore we think of homotopy groups with coefficients as dual to cohomology groups with coefficients.

We return to discussing duality in Section 6.5 after we have presented more material. There we also discuss some of the interesting, unusual, and anomalous features of duality.

Exercises

Exercises marked with $(*)$ may be more difficult than the others. Exercises marked with (\dagger) are used in the text.

2.1. (\dagger) Let (Y, m) be an H-space and assume that $(Y \times Y, Y \vee Y)$ has the homotopy extension property. Prove that there a multiplication m' on Y that is homotopic to m and such that $m'(y, *) = y$ and $m'(*, y) = y$, for all $y \in Y$.

2.2. (\dagger) Let $f : X \rightarrow Y$ be a map which has a left homotopy inverse. Prove that if Y is an H-space, then X is an H-space. With this multiplication on X , is f an H-map? What condition will ensure that if Y is homotopy-associative, then X is homotopy-associative?

2.3. Let Y be a grouplike space with multiplication m and homotopy inverse i . Define a *commutator map* $\phi : Y \times Y \rightarrow Y$ by $\phi = (p_1 + p_2) + (ip_1 + ip_2)$. Prove

1. (Y, m) is homotopy-commutative if and only if $\phi \simeq *$.
2. If $\alpha = [f]$, $\beta = [g] \in [X, Y]$, then the group commutator $[\alpha, \beta] = [\phi(f, g)]$.
3. If k is a positive integer, set $k\phi = \phi + \cdots + \phi$ (k terms) : $Y \times Y \rightarrow Y$. Show that $m + k\phi$ is a multiplication on Y .
4. Dualize (1)–(3) to cogroups.

2.4. Prove that if “ \simeq ” is replaced by equality in the definition that (X, c) is a co-H-space, then $X = \{*\}$.

2.5. Is S^0 a (nonpath-connected) co-H-space?

2.6. Prove that a space X admits a comultiplication if and only if the diagonal map $\Delta : X \rightarrow X \times X$ can be factored up to homotopy through $X \vee X$. Prove that a space X admits a multiplication if and only if the folding map $\nabla : X \vee X \rightarrow X$ can be extended up to homotopy to $X \times X$.

2.7. Let X be an $(n-1)$ -connected space with $n \geq 2$.

1. Prove that if $\dim X \leq 2n-1$, then there is a comultiplication on X . If $\dim X \leq 2n-2$, prove that any two comultiplications on X are homotopic.
2. Prove that if $\pi_i(X) = 0$ for $i \geq 2n-1$, then there is a multiplication on X . Prove that any two multiplications on X are homotopic if $\pi_i(X) = 0$ for $i \geq 2n$.

In Exercises 2.8–2.14 also consider the dual of the given problem.

2.8. Let (Y, m) be an H-space, let $f, g : X \rightarrow Y$ be maps and let $f+g : X \rightarrow Y$ be their sum. Prove for any space A , that $(f+g)_* = f_* + g_* : [A, X] \rightarrow [A, Y]$.

2.9. Let (X, c) and (X', c') be co-H-spaces and let $g : X' \rightarrow X$ be a map. Prove the following generalization of Proposition 2.2.9: g is a co-H-map if and only if for every space Y and every $\alpha, \beta \in [X, Y]$, we have $(\alpha + \beta)[g] = \alpha[g] + \beta[g]$.

2.10. (*) Let $j_{\Sigma X} : \Sigma X \rightarrow \Sigma X$ be the homotopy inverse map defined by $j_{\Sigma X}\langle x, t \rangle = \langle x, 1-t \rangle$, for $x \in X$ and $t \in I$. Consider the double suspension $\Sigma^2 X$ and the map $\tau : \Sigma^2 X \rightarrow \Sigma^2 X$ defined by $\tau\langle x, s, t \rangle = \langle x, t, s \rangle$. Prove that $j_{\Sigma^2 X} \simeq \tau \simeq \Sigma j_{\Sigma X}$.

2.11. (†) Define a map $\theta : \Sigma(X_1 \vee X_2) \rightarrow (\Sigma X_1) \vee (\Sigma X_2)$ by $\theta\langle (x_1, *), t \rangle = (\langle x_1, t \rangle, *)$ and $\theta\langle (*, x_2), t \rangle = (*, \langle x_2, t \rangle)$, for $x_1 \in X_1$, $x_2 \in X_2$ and $t \in I$. If $i_j : X_j \rightarrow X_1 \vee X_2$ and $\iota_j : \Sigma X_j \rightarrow (\Sigma X_1) \vee (\Sigma X_2)$ are inclusions and $q_j : X_1 \vee X_2 \rightarrow X_j$ and $\chi_j : (\Sigma X_1) \vee (\Sigma X_2) \rightarrow \Sigma X_j$ are projections, $j = 1, 2$, then prove that (1) $\theta \Sigma i_j = \iota_j$ and $\chi_j \theta = \Sigma q_j$, (2) θ is a homeomorphism with inverse $\{\Sigma i_1, \Sigma i_2\}$ and (3) $\theta \simeq \iota_1 \Sigma q_1 + \iota_2 \Sigma q_2$.

2.12. (†) Let (X, c_X) be a co-H-space and $\theta : \Sigma(X \vee X) \rightarrow \Sigma X \vee \Sigma X$ the homeomorphism of Exercise 2.11. Show that $\theta \Sigma c_X$ is a comultiplication that is homotopic to $c_{\Sigma X}$, the suspension comultiplication on ΣX .

2.13. Show that if X and Y are co-H-spaces, then $X \vee Y$ is a co-H-space. If X and Y are both homotopy-associative or both homotopy-commutative, does the same hold for $X \vee Y$? Let $\theta : \Sigma(X \vee Y) \rightarrow \Sigma X \vee \Sigma Y$ be the homeomorphism of Exercise 2.11. Show that θ is a co-H-map.

2.14. Prove that if X is a co-H-space, then ΣX is homotopy-commutative.

2.15. (*) Find another homotopy in the proof of Proposition 2.3.2(3).

2.16. For any space A , prove that $(A \times I)/(A \times \partial I)$ is homeomorphic to ΣB , for some space B . What is B ?

2.17. Consider the cofiber sequence $X \rightarrow CX \rightarrow \Sigma X$, the fiber sequence $\Omega Y \rightarrow EY \rightarrow Y$ and a map $f : \Sigma X \rightarrow Y$. Find a map $\theta : CX \rightarrow EY$ such that the following diagram commutes

$$\begin{array}{ccccc} X & \longrightarrow & CX & \longrightarrow & \Sigma X \\ \downarrow \tilde{f} & & \downarrow \theta & & \downarrow f \\ \Omega Y & \longrightarrow & EY & \longrightarrow & Y, \end{array}$$

where \tilde{f} is the adjoint of f .

2.18. (*) (†) Prove: $\pi_1(X) = 0$ if and only if for paths $f, g : I \rightarrow X$ such that $f(0) = g(0) = *$ and $f(1) = g(1)$, we have $f \simeq g \text{ rel } \partial I$.

2.19. If G is a group we define a *comultiplication* on G to be a homomorphism $s : G \rightarrow G * G$ such that $p_1 s = \text{id} = p_2 s : G \rightarrow G$.

1. Prove that G admits a comultiplication if and only if $\pi : E_G \rightarrow G$ has a right inverse.
2. Prove that G admits a comultiplication if and only if G is a free group.

Note that Proposition 2.4.3 asserts that the functor π_1 carries a space with a comultiplication to a group with a comultiplication.

2.20. (†) In the proof of Lemma 2.4.2 verify that

$$\xi = \prod_{i=1}^{2p} \xi_{\delta_i}^{(i)}.$$

2.21. If X is an H-space and a co-H-space, prove that $\pi_1(X)$ is 0 or \mathbb{Z} . Give examples of these spaces.

2.22. Let X and Y be co-H-complexes that are not simply connected. Prove that $X \times Y$ is not a co-H-space.

2.23. (†) Prove that a path-connected CW complex X is contractible if and only if $\pi_q(X) = 0$ for all $q \geq 1$.

2.24. (†) Let X be a space that is not necessarily path-connected. Show that there is a bijection μ from $\pi_0(X)$ to the set of path-components of X . Show that if (X, m) is a grouplike space, then m induces group structure on $\pi_0(X)$ and on the set of path-components of X such that μ is an isomorphism.

2.25. (†) Let X and Y be spaces with X a CW complex and Y path-connected. If $f : X \rightarrow Y$ is a free map, prove that there is a (based) map $g : X \rightarrow Y$ such that $f \simeq_{\text{free}} g$.

2.26. (†) In Corollary 2.4.10(1) show that if X is a CW complex, then K can be taken to be a CW complex containing X as a subcomplex.

2.27. Let $i : A \rightarrow X$ be a cofiber map, where A and X are not necessarily of the homotopy type of CW complexes. Prove that there exists a relative CW complex (K, A) and a weak equivalence $f : K \rightarrow X$ such that $f|_A = i$.

2.28. Let X and Y be spaces (not necessarily of the homotopy type of CW complexes). We define $X \simeq Y$ if there exist spaces X_1, X_2, \dots, X_n such that $X_1 = X$, $X_n = Y$ and for $i = 1, 2, \dots, n-1$, there exists a weak equivalence $X_i \rightarrow X_{i+1}$ or a weak equivalence $X_{i+1} \rightarrow X_i$. Prove that $X \simeq Y \iff X$ and Y have CW approximations of the same homotopy type. For this problem you can assume the result stated in Remark 2.4.12.

2.29. How many homotopy classes of homotopy retractions are there of the inclusion $i_1 : S^n \rightarrow S^n \vee S^n$?

2.30. (*) Let A be a set and let $F(A)$ be the free group generated by A . For every $\alpha \in A$, let $p_\alpha : F(A) \rightarrow F\{\alpha\} \cong \mathbb{Z}$ denote the projection. Prove that

$$[F(A), F(A)] = \bigcap_{\alpha \in A} \text{Ker } p_\alpha,$$

where $[F(A), F(A)]$ is the commutator subgroup of $F(A)$. Note that $F(A) \cong \ast_{\alpha \in A} F\{\alpha\}$.

2.31. Let G be any group and let X be an Eilenberg–Mac Lane space of type $(G, 1)$. Prove that X is an H-space if and only if G is abelian. (You may assume existence and basic properties of $K(G, 1)$'s.)

2.32. (*) (†) Prove that $H_{n+1}(K(G, n)) = 0$, for $n > 1$.

2.33. Given a sequence of abelian groups G_1, G_2, \dots , show that there exists a path-connected CW complex X such that $H_i(X) = G_i$ for all i . Show a similar result for homotopy groups.

2.34. (†) If G is an abelian group and $n \geq 1$, prove that the Hurewicz homomorphism $h_n : \pi_n(K(G, n)) \rightarrow H_n(K(G, n))$ is an isomorphism.

2.35. (*) (†) If $f, g : X \rightarrow K(G, n)$ are maps and X is $(n-1)$ -connected, $n \geq 1$, then prove that $(f+g)_* = f_* + g_* : H_i(X) \rightarrow H_i(K(G, n))$ provided $i < 2n-1$.

2.36. In analogy to $\rho : H^n(X; G) \rightarrow H_{\text{sing}}^n(X; G)$ defined after Remark 2.5.11, define a homomorphism $\rho' : \pi_n(X; G) \rightarrow H_n(X; G)$ and show that ρ' is the Hurewicz homomorphism when $G = \mathbb{Z}$.

2.37. (Cf. Lemma 2.5.13) If X is a space, then $[X, S^1]$ is an abelian group since S^1 is a commutative topological group. Show that the group $[X, S^1]$ contains no non-zero elements of finite order. (It may be helpful to use the covering space $\mathbb{R} \rightarrow S^1$.)

2.38. Prove the following generalization of the Hopf classification theorem. If X is a CW complex of dimension $\leq n$, then there is a bijection between $[X, M(G, n)]$ and $H^n(X; G)$. Formulate and prove the dual result.

2.39. Consider the natural map $j : X \vee Y \rightarrow X \times Y$.

1. Use the characterization of $X \vee Y$ as a categorical coproduct in $HoTop_*$ and the characterization of $X \times Y$ as a categorical product in $HoTop_*$ to define $[j]$.
2. Show that $[j]$ is self-dual.

<http://www.springer.com/978-1-4419-7328-3>

Introduction to Homotopy Theory

Arkowitz, M.

2011, XIII, 344 p. 333 illus., Softcover

ISBN: 978-1-4419-7328-3