Preface

This book deals with homotopy theory, which is one of the main branches of algebraic topology. The ideas and methods of homotopy theory have pervaded many parts of topology as well as many parts of mathematics. A general approach in these areas has been to reduce a geometric, analytic, or topological problem to a homotopy problem, and to then attempt to solve the homotopy problem, usually by algebraic methods. Thus, in addition to being interesting and important in its own right, homotopy theory has been successfully applied to geometry, analysis, and other parts of topology. There are several treatments of homotopy theory in general categories. However we confine ourselves to a study of classical homotopy theory, that is, homotopy theory of topological spaces and continuous functions. There are a number of books devoted to classical homotopy theory as well as extensive expositions of it in books on algebraic topology. This book differs from those in that the unifying theme by which the subject is developed is the Eckmann–Hilton duality theory.

The Eckmann–Hilton theory has been around for about fifty years but there appears to be no book-length exposition of it, apart from the early lecture notes of Hilton [40]. There are advantages, both expository and pedagogical, to presenting homotopy theory in this way. Dual concepts occur in pairs, such as H-space and co-H-space, fibration and cofibration, loop space and suspension, and so on, and so do many theorems. We often give complete details in describing one of these and only sketch its dual. This is done when the latter can essentially be derived by dualization. In this way we shorten the exposition by reducing the amount of repetitious material. This also allows the reader to test his or her understanding of the subject by supplying the missing details.

There is another advantage to studying Eckmann–Hilton duality theory. Frequently the dual of a result is known or trivial. But from time to time the dual result is neither of these and is in fact an interesting problem. This could give the reader material to work on.

A feature of this book is that it is designed primarily for students to learn the subject. The proofs in the text contain a great deal of detail. We also try to supplement the discussion of several of the concepts by explaining them intuitively. We provide many pictures and include a large number of exercises of varying degrees of difficulty at the end of each chapter. The exercises that have been used in the text are marked with a dagger (†) and the more difficult exercises are marked with an asterisk ($*$). It is generally regarded as important to do the exercises in order to learn the material. It has been said many times that mathematics is not a spectator sport.

This book has been written so that it can be used as a text for a university course in algebraic topology. We assume that the reader has been exposed to the basic ideas of the fundamental group, homology theory, and cohomology theory, material that is often covered in a first algebraic topology course. We state explicitly the results from these areas that we use and summarize the essential facts in the appendices. The text could also be used by mathematicians who wish to learn some homotopy theory. However, the book is not intended to introduce readers to current research in topology. There are many texts and survey articles that do this. Instead it is hoped that this book will provide a solid foundation for those who wish to work in topology or to learn more advanced homotopy theory.

We now summarize the text chapter by chapter. The first chapter contains a discussion of the notion of homotopy and its variations and related notions. We consider homotopy relative to a subset, homotopy of pairs, retracts, sections, homotopy equivalence, contractibility, and so on. Most of these should be familiar to the reader, but we present them for the sake of completeness. If X and Y are based spaces, we define the homotopy set $[X, Y]$ to be the set of homotopy classes of based maps $X \to Y$. Next CW complexes are introduced and some of their elementary properties established. These spaces play a major role in the rest of the book. Finally, there is a short section indicating some of the reasons for studying homotopy theory.

The next chapter deals with grouplike spaces and cogroups. The former is a group object in the category of based spaces and homotopy classes of maps. The latter is the categorical dual of a group object in this category. We consider loop spaces and suspensions, important examples of grouplike spaces and cogroups, respectively. This leads to a discussion of basic properties of the homotopy groups $[S^n, Y]$, where S^n is the n-sphere. We then define and construct spaces with a single nonvanishing homology group, called Moore spaces, and spaces with a single nonvanishing homotopy group, called Eilenberg–Mac Lane spaces. These give rise to homotopy groups with coefficients and to cohomology groups with coefficients. This gives a homotopical interpretation of cohomology groups. The chapter ends with a discussion of Eckmann–Hilton duality.

In Chapter 3 we discuss two dual classes of maps, fiber maps and cofiber maps. Fiber maps are defined by the covering homotopy property which is a well-known feature of covering spaces and fiber bundles. Cofiber maps appear often in topology because the inclusion map of a subcomplex of a CW complex into the complex is a cofiber map. A fiber map $E \to B$ determines a three term fiber sequence $F \to E \to B$, where $F \subseteq E$ is the fiber over the base point of B. A cofiber map $i : A \to X$ determines a cofiber sequence $A \to X \to Q$, where $Q = X/i(A)$ is the cofiber. We then study fiber bundles. We give examples of fiber bundles and these provide many examples of fiber sequences. We conclude the chapter by showing that any map can be factored as the composition of a homotopy equivalence and a fiber map or as the composition of a cofiber map and a homotopy equivalence.

The next chapter deals with exact sequences of homotopy sets. The main sequences are a long exact sequence associated to a fiber sequence and one associated to a cofiber sequence. By specializing these sequences we obtain the exact homotopy sequence of a fibration and the exact cohomology sequence of a cofibration. We next study the action of a grouplike space on a space and the coaction of a cogroup on a space. These give additional information on the exact sequences of homotopy sets. We then consider homotopy groups and define the relative homotopy groups of a pair of spaces. We discuss the exact homotopy sequence of a pair and the relative Hurewicz homomorphism. We conclude the chapter by introducing certain excision maps which are used in Chapter 6.

Chapter 5 is devoted to some applications of the exact sequences of the preceding chapter. We begin with two universal coefficient theorems. The first relates the cohomology groups with coefficients of a space to the integral cohomology groups of the space and the second relates the homotopy groups of a space with coefficients to the homotopy groups. Then we show how the operation of homotopy sets in Chapter 4 can be specialized to yield an operation of the fundamental group $\pi_1(Y)$ on the homotopy set $[X, Y]$. This operation is used to compare the based homotopy set $[X, Y]$ with the unbased homotopy classes of maps $X \to Y$. Finally we calculate some homotopy groups of several spaces including spheres, Moore spaces, and topological groups.

Chapter 6 contains the statement and proof of many of the important theorems of classical homotopy theory such as (1) the Serre theorem on the exact cohomology sequence of a fibration, (2) the Blakers–Massey theorem on the exact homotopy sequence of a cofibration, (3) the Hurewicz theorems which relates homology and homotopy groups, and (4) Whitehead's theorem regarding the induced homology homomorphism and the induced homotopy homomorphism. In the first part of the chapter we define homotopy pushouts and homotopy pullbacks and derive some of their properties. A major result that is used to prove both the Serre and Blakers–Massey theorems is that a certain homotopy-commutative square is a homotopy pushout square.

In Chapter 7 we discuss two basic and dual techniques for approximating a space by a sequence of simpler spaces. The obstruction theory developed in Chapter 9 is based on these approximations. The first technique, called the homotopy decomposition, assigns a sequence of spaces $X^{(n)}$ to a space X such that the ith homotopy group of $X^{(n)}$ is zero for $i > n$ and is isomorphic to the ith homotopy group of X for $i \leqslant n$. From the point of view of homotopy groups, the spaces $X^{(n)}$ approach X as n increases. The second technique, called the homology decomposition, is similar with homology groups in place of homotopy groups. We consider several properties and applications of these decompositions. In the last section of the chapter we generalize these decompositions from spaces to maps.

In Chapter 8 we derive some general results for the homotopy set $[X, Y]$. We give hypotheses in terms of cohomology and homotopy groups that imply that the set is countably infinite or finite. We consider some properties of the group $[X, Y]$ when X is a cogroup or Y is a grouplike space. We show that if Y is a grouplike space, then $[X, Y]$ is a nilpotent group whose nilpotency class is bounded above by the Lusternik–Schnirelmann category of X.

In the final chapter we consider two basic problems for mappings. In the first, called the extension problem, we seek to extend a map defined on a subspace to the whole space. In the second, called the lifting problem, we seek to lift a map into the base of a fibration to a map into the total space. These are two special cases of the extension-lifting problem. We develop an obstruction theory for this problem which gives a step-by-step procedure for obtaining the desired map. We present two approaches to the theory. For the first, we take a homotopy decomposition of the fiber map and assume that the desired map exists at the nth step. This determines an element in a cohomology group, whose vanishing is a necessary and sufficient condition for the map to exist at the $(n+1)$st step. In the final section we discuss a method for obtaining obstruction elements by taking homology decompositions. These elements are in homotopy groups with coefficients.

After Chapter 9 there are six appendices. These are of two types. One type consists of results whose proofs in the text would be a digression of the topics being treated. The proofs of these results appear in the appendix. The other type provides a summary and reference for those basic results about point-set topology, the fundamental group, homology theory, and category theory that are used in the text. Definitions are given, the results are stated, and in some cases the proof is either given or sketched.

In conclusion, I would like to acknowledge the many helpful suggestions of the following people: Robert Brown, Vladimir Chernov, Dae-Woong Lee, Gregory Lupton, John Oprea, Nicholas Scoville, Jeffrey Strom, and Dana Williams. I would like to express my appreciation to the following people at Springer: Katie Leach for editorial assistance, Rajiv Monsurate for advice on Tex, and Brian Treadway for drawing the figures. Finally, I am particularly indebted to Peter Hilton for having introduced me to this material and tutored me in it while I was a graduate student. To all these people, many thanks.

http://www.springer.com/978-1-4419-7328-3

Introduction to Homotopy Theory
Arkowitz, M.
2011, XIII, 344 p. 333 illus., Softcover
ISBN: 978-1-4419-7328-3