

Chapter 2

State Estimation Over an Unreliable Network

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Abstract In this chapter, we consider Kalman filtering over a packet-delaying network. Given the probability distribution of the delay, we can completely characterize the filter performance via a probabilistic approach. We assume the estimator maintains a buffer of length D so that at each time k , the estimator is able to retrieve all available data packets up to time $k - D + 1$. Both the cases of sensor with and without necessary computation capability for filter updates are considered. When the sensor has no computation capability, for a given D , we give lower and upper bounds on the probability for which the estimation error covariance is within a prescribed bound. When the sensor has computation capability, we show that the previously derived lower and upper bounds are equal to each other. An approach for determining the minimum buffer length for a required performance in probability is given and an evaluation on the number of expected filter updates is provided.

Keywords Kalman filter · Networked control systems · Packet-delaying networks · Estimation theory · Probabilistic performance

2.1 Introduction

The Kalman filter has played a central role in systems theory and has found wide applications in many fields, such as control, signal processing, and communications. In the standard Kalman filter, it is assumed that sensor data are transmitted along perfect communication channels and are available to the estimator either instantaneously or with some fixed delays, and no interaction between communication and control is considered. This abstraction has been adopted until recently when networks, especially wireless networks, are used in sensing and control systems for

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transmitting, data from sensor to controller and/or from controller to actuator. While having many advantages such as low cost and flexibility, networks also induce many new issues due to their limited capabilities and uncertainties, such as limited bandwidth, packet losses, and latency. On the other hand, in wireless sensor networks, sensor nodes also have limited computation capability in addition to their limitations in communications. These constraints undoubtedly affect system performance or even stability and cannot be neglected when designing estimation and control algorithms, which has inspired a lot of research in control with communication constraints; see the survey [3] and the references therein.

In recent years, networked control problems have gained much interest. In particular, the state estimation problem over a network has been widely studied. The problem of state estimation and stabilization of a linear time-invariant system over a digital communication channel, which has a finite bandwidth capacity was introduced by Wong and Brockett [19, 20] and further pursued by others (e.g., in [1, 7, 10, 17]). Sinopoli et al. [15] discussed how packet loss can affect state estimation. They showed there exists a certain threshold of the packet loss rate above which the state estimation error diverges in the expected sense, i.e., the expected value of the error covariance matrix becomes unbounded as time goes to infinity. They also provided lower and upper bounds of the threshold value. Following the spirit of [15], Liu and Goldsmith [5] extended the idea to the case, where there are multiple sensors and the packets arriving from different sensors are dropped independently. They provided similar bounds on the packet loss rate for a stable estimate, again in the expected sense. Huang and Dey [4], Xie and Xie [21] characterize packet losses as a Markov chain and give some sufficient and necessary stability conditions under the notion of peak covariance stability. The drawback of using mean covariance matrix as a stability measure is that it may conceal the fact that events with arbitrarily low probability may make the mean value diverge. Different from [4, 15, 21], Shi et al. [13] investigate the stability of the Kalman filter through a probabilistic approach.

The problem of state estimation and control with delayed measurements is not new and has been studied even before the emergence of networked control [11, 22]. Nilsson [9] analyzed delays that are either fixed or random according to a Markov chain. He solves the LQG optimal control problem for the different delay models. It has been well known that discrete-time systems with constant or known time-varying bounded measurement delays may be handled by state augmentation in conjunction with the standard Kalman filtering or by the reorganized innovation approach in [23, 24]. Although sensor data are usually time-stamped and thus transmission delays are known to the filter, the delays in networked systems are random in nature. For example, the ZigBee/IEEE 802.15.4 protocol is widely used in sensor network and wireless control applications [25]. When multiple sensor nodes simultaneously access the channel, a random waiting time is generated by the CSMA/CA algorithm for each node before they try to access the channel again. Thus, the experienced delay for data measurement is typically random.

For the problem of randomly delayed measurements, Ray et al. [11] present a modification of the conventional minimum variance state estimator to accommodate

the effects of the random arrival of measurements, whereas a suboptimal filter in the least mean square sense is given in [22]. In [6], a recursive minimum variance state estimator is presented for linear discrete-time partially observed systems, where the observations are transmitted by communication channels with randomly independent delays. Using covariance information, recursive least-squares linear estimators for signals with random delays are studied in [8]. Furthermore, the filtering problems with random delays and missing measurements have been investigated in [12, 16, 18] via the linear matrix inequality and the Riccati equation approaches, respectively. Note that most of the aforementioned work is concerned with the optimal or suboptimal average design, where the mean filtering error covariance is taken with respect to a random i.i.d. variable that characterizes the random delay in addition to the process and measurement noises and the initial state. Thus, the derived filter is in fact suboptimal when the delay is known online. There has been no systematic analysis on the performance of the Kalman filter, which offers the optimal filtering performance for systems with random measurement delay available online.

The goal of this chapter is to study the performance of Kalman filter under random measurement delay. We assume that the probability distribution of the delay is given and aim to give a complete characterization of filter performance. Due to the limited computation capability of the filtering center and also in consideration of the fact that a late arriving measurement related to the system state in the far past may not contribute much to the improvement of the accuracy of the current estimate, it is practically important to determine a proper buffer length for measurement data within which a measurement will be used to update the current state and beyond which the data will be discarded. The buffer provides a tradeoff between performance and computational load. In the chapter, for a given buffer length, we shall give lower and upper bounds for the probability at which the filtering error covariance is within a prescribed bound, i.e., $\Pr[P_k \leq M]$ for some given M . The upper and lower bounds can be easily evaluated by the probability distribution of the delay and the system dynamics. An approach for determining the minimum buffer length for a required performance in probability is given and an evaluation on the number of expected filter updates is provided. Both the cases of sensor with and without necessary computation capability for filter updates are considered.

2.2 Problem Setup

We consider the problem of state estimation over a packet-delaying network as seen from Fig. 2.1. The process dynamics and sensor measurement equation are given as follows:

$$x_{k+1} = Ax_k + w_k, \quad (2.2.1)$$

$$y_k = Cx_k + v_k. \quad (2.2.2)$$

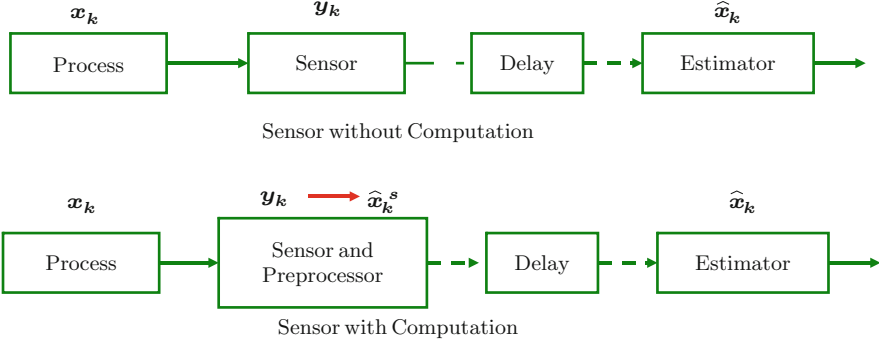


Fig. 2.1 System block diagram

In the above equations, $x_k \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the observation vector, $w_{k-1} \in \mathbb{R}^n$ and $v_k \in \mathbb{R}^m$ are zero mean white Gaussian random vectors with $\mathbb{E}[w_k w_j'] = \delta_{kj} Q$, $Q \geq 0$, $\mathbb{E}[v_k v_j'] = \delta_{kj} R$, $R > 0$, $\mathbb{E}[w_k v_j'] = 0 \forall j, k$, where $\delta_{kj} = 0$ if $k \neq j$ and $\delta_{kj} = 1$ otherwise. We assume that the pair (A, C) is observable, and $(A, \sqrt{Q})$ is controllable.

Depending on its computational capability, the sensor can either send y_k or preprocess y_k and send $\hat{x}_k^s$ to the remote estimator, where $\hat{x}_k^s$ is defined at the sensor as

$$\hat{x}_k^s \triangleq \mathbb{E}[x_k | y_1, \dots, y_k].$$

The two cases correspond to the two scenarios in Fig. 2.1, i.e., sensor without/with computation capability.

After taking a measurement at time k , the sensor sends y_k (or $\hat{x}_k^s$) to a remote estimator for generating the state estimate. We assume that the measurement data packets from the sensor are to be sent across a packet-delaying network, with negligible quantization effects, to the estimator. Each y_k (or $\hat{x}_k^s$) is delayed by d_k times, where d_k is a random variable described by a probability mass function f , i.e.,

$$f(j) = \Pr[d_k = j], j = 0, 1, \dots \quad (2.2.3)$$

For simplicity, we assume d_{k_1} and d_{k_2} are independent if $k_1 \neq k_2$. We further assume that d_k carries no information about the state, e.g., d_k is independent of w_k, v_k and the initial state x_0 . Similar to [14], we can use a Markov chain to model consecutive data packet delays, and the results extend straightforward to that case. Note that the i.i.d packet drop with drop rate $1 - \gamma$ considered in the literature can be treated as a special case here, i.e.,

$$f(0) = \gamma, f(\infty) = 1 - \gamma, f(j) = 0, 1 \leq j < \infty.$$

Thus, the theory developed here includes the packet drop analysis as well.

Define the following state estimate and other quantities at the remote estimator

$$\begin{aligned}\hat{x}_k^- &\triangleq \mathbb{E}[x_k | \text{all data packets up to } k-1], \\ \hat{x}_k &\triangleq \mathbb{E}[x_k | \text{all data packets up to } k], \\ P_k^- &\triangleq \mathbb{E}[(x_k - \hat{x}_k^-)(x_k - \hat{x}_k^-)' | \text{all data packets up to } k-1], \\ P_k &\triangleq \mathbb{E}[(x_k - \hat{x}_k)(x_k - \hat{x}_k)' | \text{all data packets up to } k].\end{aligned}$$

Assume the estimator discards any data y_k (or $\hat{x}_k^s$) that are delayed by D times or more. Given the system and the network delay models in (2.2.1)–(2.2.3), we are interested in the following problems.

1. How should $\hat{x}_k$ be computed?
2. What is the relationship between P_k and D ?
3. For a given $M \geq 0$ and $\epsilon \in [0, 1]$, what is the minimum D such that

$$\Pr[P_k \leq M] \geq 1 - \epsilon.$$

In the rest of the chapter, we provide solutions to the above three problems for each of the two scenarios in Fig. 2.1.

The following terms that are frequently used in subsequent sections are defined below. It is assumed that (A, C, Q, R) are the same as they appear in Sect. 2.2; $\lambda_i(A)$ is the i th eigenvalue of the matrix A ; $X \in \mathbb{S}_+^n$ where $\mathbb{S}_+^n$ is the set of n by n positive semidefinite matrices; $h, g : \mathbb{S}_+^n \rightarrow \mathbb{S}_+^n$ are functions defined below; Y_i is a random variable, where the underlying sample spaces will be clear from its context.

$$\begin{aligned}h(X) &\triangleq AXA' + Q \\ g(X) &\triangleq h(X) - AXC'[CXC' + R]^{-1}CXA' \\ \tilde{g}(X) &\triangleq X - XC'[CXC' + R]^{-1}CX \\ h \circ g(X) &\triangleq h(g(X)) \\ h^t(X) &\triangleq \underbrace{h \circ \dots \circ h}_{t \text{ times}}(X) \\ \Pr[Y_1|Y_2] &\triangleq \Pr[Y_1] \text{ given } Y_2\end{aligned}$$

2.3 Sensor Without Computation Capability

In this section, we consider the first scenario in Fig. 2.1, i.e., the sensor has no computation and sends y_k to the remote estimator. We assume C is full rank, and without loss of generality, we assume C^{-1} exists. The general C case will be considered in Sect. 2.4.1.

2.3.1 Modified Kalman Filtering

Let γ_t^k be the indicator function for y_t at time $k, t \leq k$, which is defined as follows.

$$\gamma_t^k = \begin{cases} 1, & y_t \text{ received at time } k, \\ 0, & \text{otherwise.} \end{cases}$$

Further define $\gamma_{k-i} \triangleq \sum_{j=0}^i \gamma_{k-i-j}^k$, i.e., γ_{k-i} indicates whether y_{k-i} is received by the estimator at or before k .

Assume $\hat{x}_{k-1}$ is optimal. Depending on whether y_k is received or not, i.e., $\gamma_k^k = 1$ or 0, $(\hat{x}_k, P_k)$ is known to be computed by a modified Kalman filter (**MKF**) [15]. We write $(\hat{x}_k, P_k)$ in compact form as follows.

$$(\hat{x}_k, P_k) = \mathbf{MKF}(\hat{x}_{k-1}, P_{k-1}, \gamma_k^k y_k),$$

which represents the following set of equations:

$$\begin{cases} \hat{x}_k^- = A\hat{x}_{k-1}, \\ P_k^- = AP_{k-1}A' + Q, \\ K_k = P_k^- C' [CP_k^- C' + R]^{-1}, \\ \hat{x}_k = A\hat{x}_{k-1} + \gamma_k^k K_k (y_k - CA\hat{x}_{k-1}), \\ P_k = (I - \gamma_k^k K_k C) P_k^-. \end{cases}$$

Assume $\gamma_k^k = 1$ for all k , then **MKF** reduces to the standard Kalman filter. In this case, P_k^- and P_k can be shown to satisfy

$$P_k^- = g(P_{k-1}^-), \quad P_k = \tilde{g} \circ h(P_{k-1}).$$

Let P^* be the unique positive semidefinite solution¹ to $g(X) = X$, i.e., $P^* = g(P^*)$. Define $\overline{P}$ as $\overline{P} \triangleq \tilde{g}(P^*)$. Then we have

$$\tilde{g} \circ h(\overline{P}) = \tilde{g} \circ h \circ \tilde{g}(P^*) = \tilde{g} \circ g(P^*) = \tilde{g}(P^*) = \overline{P},$$

where we use the fact that $h \circ \tilde{g} = g$. In other words,

$$P^* = \lim_{k \rightarrow \infty} P_k^-, \quad \overline{P} = \lim_{k \rightarrow \infty} P_k.$$

¹ Since (A, C) is assumed to be observable and $(A, \sqrt{Q})$ controllable, from standard Kalman filtering analysis, P^* exists.

2.3.2 Optimal Estimation with Delayed Measurements

As y_{k-i} may arrive at time k due to the delays introduced by the network, we can improve the estimation quality by recalculating $\hat{x}_{k-i}$ utilizing the new available measurement y_{k-i} . Once $\hat{x}_{k-i}$ is updated, we can update $\hat{x}_{k-i+1}$ in a similar fashion. This is the basic idea contained in the flow diagram in Fig. 2.2. Proposition 2.3.1 summarizes the main estimation result.

Proposition 2.3.1. *Let $y_{k-i}, i \in [0, D-1]$ be the oldest measurement received by the estimator at time k . Then $\hat{x}_k$ is computed by $i+1$ MKFs as*

$$\begin{aligned} (\hat{x}_{k-i}, P_{k-i}) &= \mathbf{MKF}(\hat{x}_{k-i-1}, P_{k-i-1}, y_{k-i}) \\ (\hat{x}_{k-i+1}, P_{k-i+1}) &= \mathbf{MKF}(\hat{x}_{k-i}, P_{k-i}, \gamma_{k-i+1} y_{k-i+1}) \\ &\vdots \\ (\hat{x}_{k-1}, P_{k-1}) &= \mathbf{MKF}(\hat{x}_{k-2}, P_{k-2}, \gamma_{k-1} y_{k-1}) \\ (\hat{x}_k, P_k) &= \mathbf{MKF}(\hat{x}_{k-1}, P_{k-1}, \gamma_k y_k). \end{aligned}$$

Furthermore, $\hat{D}$, the average number of MKF used at each time k is given by

$$\hat{D} = \prod_{i=1}^{D-1} (1 - f(i)) + \sum_{j=2}^D \prod_{i=j}^{D-1} (1 - f(i)) f(j-1) j, \quad (2.3.1)$$

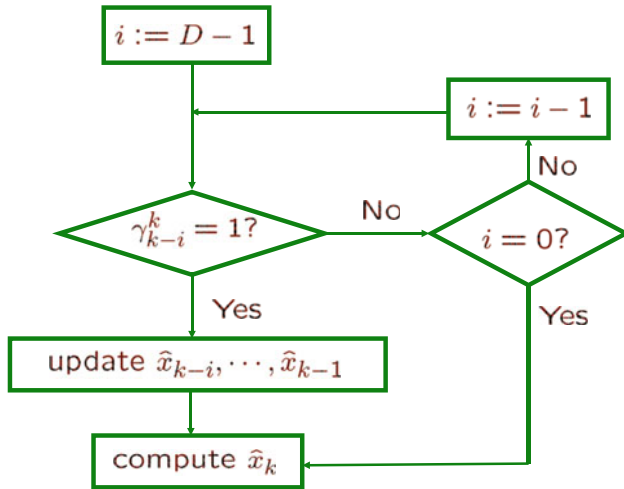


Fig. 2.2 Optimal estimation: sensor without computation capability

where

$$\prod_{i=D}^{D-1} (1 - f(i)) \triangleq 1.$$

Proof. We know that the estimate $\hat{x}_k$ is generated from the estimate of $\hat{x}_{k-1}$ together with $\gamma_k y_k$ at time k through an **MKF**. Similarly, the estimate $\hat{x}_{k-1}$ is generated from the estimate of $\hat{x}_{k-2}$ together with $\gamma_{k-1} y_{k-1}$ at time k through an **MKF**, etc. This recursion for $i + 1$ steps corresponds to the $i + 1$ **MKF**s stated in the theorem. Let $\hat{D}_k$ be the number of **MKF** used at each time k . Notice that $1 \leq \hat{D}_k \leq D$. Thus,

$$\hat{D} = \sum_{j=1}^D j \Pr[\hat{D}_k = j].$$

Consider $\Pr[\hat{D}_k = 1]$. Since $\hat{D}_k = 1$ iff $\gamma_{k-i}^k = 0$ for all $1 \leq i \leq D - 1$, we have

$$\Pr[\hat{D}_k = 1] = \Pr[\gamma_{k-i}^k = 0, 1 \leq i \leq D - 1].$$

As $\Pr[\gamma_{k-i}^k = 0] = 1 - f(i)$, we obtain

$$\Pr[\hat{D}_k = 1] = \prod_{i=1}^{D-1} (1 - f(i)).$$

Similarly, for $2 \leq j \leq D - 1$, we have

$$\Pr[\hat{D}_k = j] = \prod_{i=j}^{D-1} (1 - f(i)) f(j - 1)$$

and when $j = D$,

$$\Pr[\hat{D}_k = j] = f(D - 1).$$

Therefore, we obtain $\hat{D}$ in (2.3.1). $\square$

Remark 2.3.2. Similar results on estimation with delayed packets can be derived in other ways (e.g. [6]) but the result in Proposition 2.3.1 fits our intended uses later in the paper.

Remark 2.3.3. Although this section focuses on when C is invertible, proposition 2.3.1 does not need to assume C to be invertible. Depending on whether (A, C) is detectable or not, the error covariance matrix may diverge. However, the algorithm always produces the optimal estimate at each time.

Notice that in the first **MKF**, $\gamma_{k-i}^k = 1$ and hence $\gamma_{k-i} = 1$. As a result, we simply write $\gamma_{k-i} y_{k-i} = y_{k-i}$.

2.3.3 Lower and Upper Bounds of $\Pr[P_k \leq M]$

Since d_k is random and described by the probability mass function f , γ_{k-i}^k ($i = 0, \dots, D - 1$) is also random. As a consequence, P_k computed as in Proposition 2.3.1 is a random variable. Define $\hat{\gamma}_i(D)$ as

$$\hat{\gamma}_i(D) \triangleq \begin{cases} \sum_{j=0}^i f(j), & \text{if } 0 \leq i < D, \\ \sum_{j=0}^{D-1} f(j), & \text{if } i \geq D. \end{cases}$$

Recall that γ_{k-i} indicates whether y_{k-i} is received by the estimator at or before k , so it is easy to verify that

$$\Pr[\gamma_{k-i} = 1] = \hat{\gamma}_i(D). \quad (2.3.2)$$

Define $\overline{M} \triangleq C^{-1}RC^{-1'}$. Then we have the following result that shows the relationship between P_k and $\overline{M}$.

Lemma 2.3.4. *For any $k \geq 1$, if $\gamma_k = 1$, then $P_k \leq \overline{M}$.*

Proof. As $\gamma_k = 1$, we have $P_k = \tilde{g} \circ h(P_{k-1}) \leq \overline{M}$, where the inequality is from Lemma 2.A.2 in Appendix 2.A. $\square$

Remark 2.3.5. We can also interpret Lemma 2.3.4 as follows. One way to obtain an estimate $\tilde{x}_k$ when $\gamma_k = 1$ is simply by inverting the measurement, i.e., $\tilde{x}_k = C^{-1}y_k$. Therefore,

$$\tilde{e}_k = C^{-1}v_k \text{ and } \tilde{P}_k = \mathbb{E}[\tilde{e}_k \tilde{e}_k'] = C^{-1}RC^{-1'} = \overline{M}.$$

Since Kalman filter is optimal among the set of all linear filters, we must have $P_k \leq \tilde{P}_k = \overline{M}$.

Recall that $\overline{P}$ is defined in Sect. 2.3.1 as the steady-state error covariance for the Kalman filter. For $M \geq \overline{M}$, let us define $k_1(M)$ and $k_2(M)$ as follows:

$$k_1(M) \triangleq \min\{t \geq 1 : h^t(\overline{M}) \not\leq M\}, \quad (2.3.3)$$

$$k_2(M) \triangleq \min\{t \geq 1 : h^t(\overline{P}) \not\leq M\}. \quad (2.3.4)$$

We sometimes write $k_i(M)$ as k_i , $i = 1, 2$ for simplicity for the rest of the chapter. The following lemma shows the relationship between $\overline{P}$ and $\overline{M}$ as well as k_1 and k_2 .

Lemma 2.3.6. (1) $\overline{P} \leq \overline{M}$; (2) $k_1 \leq k_2$ whenever either k_i is finite, $i = 1, 2$.

Proof. 1. $\overline{P} = \tilde{g}(P^*) \leq \overline{M}$ where the inequality is from Lemma 2.A.2 in Appendix 2.A.

2. Without loss of generality, we assume k_2 is finite. If k_1 is finite, and $k_1 > k_2$, then according to their definitions, we must have

$$M \geq h^{k_1-1}(\overline{M}) \geq h^{k_1-1}(\bar{P}) \geq h^{k_2}(\bar{P}),$$

which violates the definition of k_2 . Notice that we use the property that h is nondecreasing as well as $h(\bar{P}) \geq \bar{P}$ from Lemma 2.A.1 and 2.A.3 in Sect. 2.A in the Appendix. Similarly, we can show that k_1 cannot be infinite. Therefore, we must have $k_1 \leq k_2$. $\square$

Lemma 2.3.7. Assume $P_0 \geq \bar{P}$. Then $P_k \geq \bar{P}$ for all $k \geq 0$.

Proof. Since **MKF** is used at each time k ,

$$P_k = \hat{f}_k^k \circ \hat{f}_{k-1}^k \cdots \hat{f}_1^k(P_0) \geq \bar{P},$$

where $\hat{f}_{k-i}^k = h$ or $\hat{f}_{k-i}^k = \tilde{g} \circ h$ depending on the packet arrival sequence². The inequality is from Lemma 2.A.1 in Appendix 2.A. $\square$

Define N_k as the number of consecutive packets not received by k , i.e.,

$$N_k \triangleq \min\{t \geq 0 : \gamma_{k-t} = 1\}. \quad (2.3.5)$$

Define

$$\theta(k_i, D) \triangleq \prod_{j=0}^{k_i-1} (1 - \hat{\gamma}_j(D)). \quad (2.3.6)$$

It is easy to see that

$$\theta(k_1, D) \geq \theta(k_2, D).$$

Lemma 2.3.8. Let k_1, k_2 , and N_k be defined according to (2.3.3)–(2.3.5). Then

$$\Pr[N_k \geq k_i] = \theta(k_i, D), i = 1, 2. \quad (2.3.7)$$

Proof.

$$\Pr[N_k \geq k_i] = \Pr[\gamma_{k-i} = 0, 0 \leq i \leq k_i - 1] = \theta(k_i, D).$$

$\square$

Theorem 2.3.9. Assume $\bar{P} \leq P_0 \leq \overline{M}$. For any $M \geq \overline{M}$, we have

$$1 - \theta(k_1, D) \leq \Pr[P_k \leq M] \leq 1 - \theta(k_2, D). \quad (2.3.8)$$

Proof. We divide the proof into two parts. For the rest of the proof, all probabilities are conditioned on the given D .

² Note that we use the superscript k in $\hat{f}_{k-i}^k$ to emphasize that it depends on the current time, k . For example, if $d_{k-i} = i + 1$, i.e., $\gamma_{k-i} = 0$ and $\gamma_{k-i}^{k+1} = 1$, then $\hat{f}_{k-i}^k = h$ and $\hat{f}_{k-i}^{k+1} = \tilde{g} \circ h$.

1. Let us first prove $1 - \theta(k_1, D) \leq \Pr[P_k \leq M]$, or in other words,

$$1 - \Pr[N_k \geq k_1] \leq \Pr[P_k \leq M].$$

As $\gamma_k = 1$ or 0 , there are in total 2^k possible realizations of γ_1 to γ_k as seen from Fig. 2.3. Let Σ_1 denote those packet arrival sequences of γ_1 to γ_k such that $N_k \geq k_1$. Similarly, let Σ_2 denote those packet arrival sequences such that $N_k < k_1$. Let $P_k(\sigma_i)$ be the error covariance at time k when the underlying packet arrival sequence is σ_i , where $\sigma_i \in \Sigma_i, i = 1, 2$. Consider a particular $\sigma_2 \in \Sigma_2$. As $\gamma_{k-k_1+1} = 1$, from Lemma 2.3.4, $P_{k-k_1+1} \leq \bar{M}$. Therefore, we have

$$P_k(\sigma_2) \leq h^{k_1-1}(P_{k-k_1+1}) \leq h^{k_1-1}(\bar{M}) \leq M,$$

where the first and second inequalities are from Lemma 2.A.1 in Appendix 2.A and the last inequality is from the definition of k_1 . In other words,

$$\Pr[P_k \leq M | \sigma_2] = 1.$$

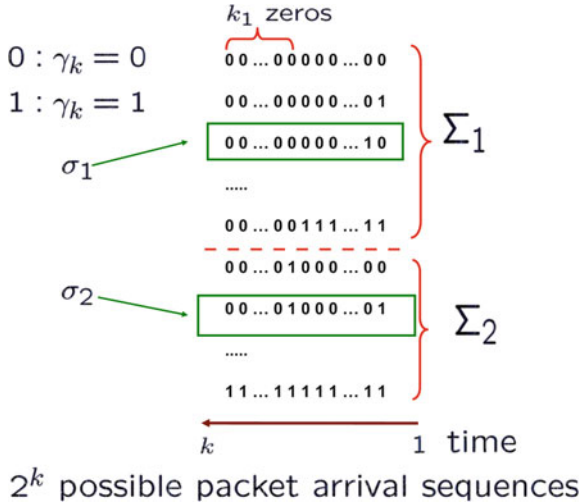


Fig. 2.3 $N_k \geq k_1$

Therefore,

$$\begin{aligned}
 \Pr[P_k \leq M] &= \sum_{\sigma \in \Sigma_1 \cup \Sigma_2} \Pr[P_k \leq M | \sigma] \Pr(\sigma) \\
 &= \sum_{\sigma_1 \in \Sigma_1} \Pr[P_k \leq M | \sigma_1] \Pr(\sigma_1) + \sum_{\sigma_2 \in \Sigma_2} \Pr[P_k \leq M | \sigma_2] \Pr(\sigma_2) \\
 &\geq \sum_{\sigma_2 \in \Sigma_2} \Pr[P_k \leq M | \sigma_2] \Pr(\sigma_2) \\
 &= \sum_{\sigma_2 \in \Sigma_2} \Pr(\sigma_2) = \Pr(\Sigma_2) = 1 - \Pr(\Sigma_1) = 1 - \Pr[N_k \geq k_1],
 \end{aligned}$$

where the first equality is from the Total Probability Theorem, the second equality holds as Σ_1 and Σ_2 are disjoint, and the third inequality holds as the first sum is nonnegative. The rest equalities are easy to see.

2. We now prove $\Pr[P_k \leq M] \leq 1 - \theta(k_2, D)$, or in other words

$$\Pr[P_k \leq M] \leq 1 - \Pr[N_k \geq k_2].$$

Let Σ'_1 denote those packet arrival sequences of γ_1 to γ_k such that $N_k \geq k_2$ and Σ'_2 denote those packet arrival sequences such that $N_k < k_2$ (Fig. 2.4). Consider $\sigma'_1 \in \Sigma'_1$. Let

$$s(\sigma'_1) = \min\{t \geq 1 : \gamma_{k-t} = 1 | \sigma'_1\}.$$

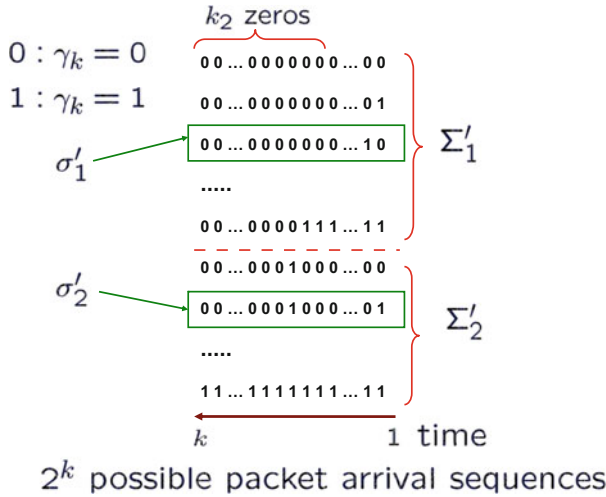


Fig. 2.4 $N_k \geq k_2$

As $\sigma'_1 \in \Sigma'_1$, we must have $s \geq k_2$. Consequently,

$$P_k(\sigma'_1) = h^{s(\sigma'_1)}(P_{k-s(\sigma'_1)}) \geq h^{s(\sigma'_1)}(\bar{P}),$$

where the inequality is from Lemma 2.3.7. Therefore we conclude that

$$P_k(\sigma'_1) \not\leq M.$$

Otherwise,

$$h^{s(\sigma'_1)}(\bar{P}) \leq P_k(\sigma'_1) \leq M,$$

which violates the definition of k_2 . In other words,

$$\Pr[P_k \leq M | \sigma'_1] = 0.$$

Therefore, we have

$$\begin{aligned} \Pr[P_k \leq M] &= \sum_{\sigma \in \Sigma_1 \cup \Sigma_2} \Pr[P_k \leq M | \sigma] \Pr(\sigma) \\ &= \sum_{\sigma'_1 \in \Sigma'_1} \Pr[P_k \leq M | \sigma'_1] \Pr(\sigma'_1) + \sum_{\sigma'_2 \in \Sigma'_2} \Pr[P_k \leq M | \sigma'_2] \Pr(\sigma'_2) \\ &= \sum_{\sigma'_2 \in \Sigma'_2} \Pr[P_k \leq M | \sigma'_2] \Pr(\sigma'_2) \\ &\leq \sum_{\sigma'_2 \in \Sigma'_2} \Pr(\sigma'_2) = \Pr(\Sigma'_2) = 1 - \Pr(\Sigma'_1) = 1 - \Pr[N_k \geq k_2], \end{aligned}$$

where the inequality is from the fact that

$$\Pr[P_k \leq M | \sigma'_2] \leq 1 \text{ for any } \sigma'_2 \in \Sigma'_2.$$

□

2.3.4 Computing the Minimum D

Assume we require that

$$\Pr[P_k \leq M] \geq 1 - \epsilon, \quad (2.3.9)$$

then according to (2.3.8), a sufficient condition is that

$$\theta(k_1, D) \leq \epsilon. \quad (2.3.10)$$

And a necessary condition is that

$$\theta(k_2, D) \leq \epsilon. \quad (2.3.11)$$

For a given M , define

$$\epsilon_1(M) \triangleq \theta(k_1, k_1 - 1), \quad (2.3.12)$$

$$\epsilon_2(M) \triangleq \theta(k_2, k_2 - 1). \quad (2.3.13)$$

2.3.4.1 Sufficient Minimum D

Notice that $\theta(k_1, D)$ is decreasing when $1 \leq D \leq k_1 - 1$ and remains constant when $D \geq k_1$. Hence if $\epsilon < \epsilon_1(M)$, no matter how large D is, there is *no guarantee* that $\Pr[P_k \leq M] \geq 1 - \epsilon$. If $\epsilon \geq \epsilon_1(M)$, then the minimum D_s such that *guarantees* $\Pr[P_k \leq M] \geq 1 - \epsilon$ is given by

$$D_s = \min\{D : \theta(k_1, D) \leq \epsilon, 1 \leq D \leq k_1 - 1\}. \quad (2.3.14)$$

2.3.4.2 Necessary Minimum D

Similarly, $\theta(k_2, D)$ is decreasing when $1 \leq D \leq k_2 - 1$ and remains constant when $D \geq k_2$. Hence if $\epsilon < \epsilon_2(M)$, no matter how large D is, it is guaranteed that $\Pr[P_k \leq M] > 1 - \epsilon$. If $\epsilon \geq \epsilon_1(M)$, then the minimum D_s such that it is *possible* that $\Pr[P_k \leq M] \geq 1 - \epsilon$ is given by

$$D_s = \min\{D : \theta(k_2, D) \leq \epsilon, 1 \leq D \leq k_2 - 1\}. \quad (2.3.15)$$

Example 2.3.10. Consider (2.2.1) and (2.2.2) with

$$A = 1.4, C = 1, Q = 0.2, R = 0.5.$$

We model the packet delay as a Poisson distribution with mean d , i.e., the probability density function $f(i)$ satisfies

$$f(i) = \frac{d^i e^{-d}}{i!}, i = 0, 1, \dots,$$

where $d = \mathbb{E}[d_k]$ denotes the mean value of the packet delay.

When $M = 50$ (Fig. 2.5), it is calculated that $k_1(M) = k_2(M) = 7$, hence $\theta(k_1, D) = \theta(k_2, D)$ and $\theta(7, 6) = 0.0313$. Thus, we can find the minimum D that guarantees $\Pr[P_k \leq 50] \geq 1 - \epsilon$ for any $\epsilon \geq 0.0313$. For any $\epsilon < 0.0313$, no matter how large D is, $\Pr[P_k \leq 50] < 1 - \epsilon$.

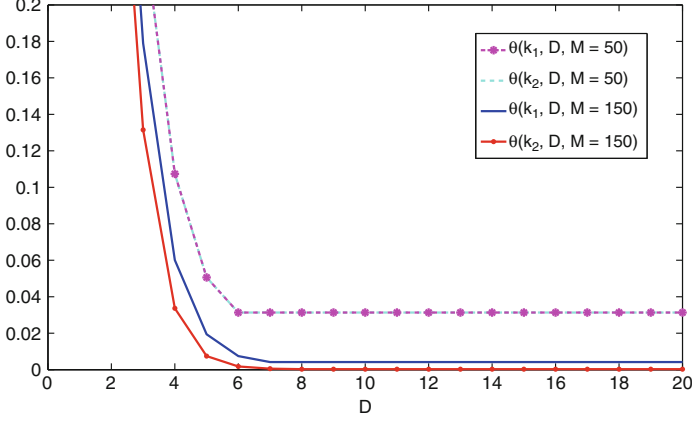


Fig. 2.5 $\theta(k_i, D)$ for different M

When $M = 150$, it is calculated that $k_1(M) = 8$ and $k_2(M) = 9$, hence $\theta(k_1, D) > \theta(k_2, D)$. We also find that $\theta(8, 7) = 0.0042$ and $\theta(9, 8) = 0.0003$. Therefore if $\epsilon > 0.0042$, we can find minimum D that guarantees $\Pr[P_k \leq 150] \geq 1 - \epsilon$; if $\epsilon < 0.0003$, no matter how large D is, $\Pr[P_k \leq 150] > 1 - \epsilon$.

Remark 2.3.11. We find the minimum D that gives the desired filter performance, i.e., $\Pr[P_k \leq M] \geq 1 - \epsilon$ for a given M and ϵ . We can also find the minimum D when requiring $\mathbb{E}[P_k]$ to be stable, i.e., $\lim_{k \rightarrow \infty} \mathbb{E}[P_k] < \infty$, and we provide the detailed analysis in Appendix 2.B using the theory developed in this section.

2.4 Sensor with Computation Capability

In this section, we consider the second scenario in Fig. 2.1, i.e., the sensor has necessary computation capability and sends $\hat{x}_k^s$ to the remote estimator. We assume all the variables in this section, e.g., γ_t^k, γ_k , etc. are the same as they are defined in Sect. 2.3 unless they are explicitly defined.

Consider the case when k is sufficiently large so that the Kalman filter enters steady state at the sensor side, i.e., $P_k^s = \bar{P}$. It is clear that the optimal estimation at the remote estimator is as follows. If $\gamma_k = 1$, then $\hat{x}_k = \hat{x}_k^s$ and $P_k = P_k^s = \bar{P}$. If $\gamma_k = 0$ and $\gamma_{k-1} = 1$, then $\hat{x}_k = A\hat{x}_{k-1}^s$ and $P_k = h(\bar{P})$. This is repeated until we examine γ_{k-D+1} . The full optimal estimation algorithm is presented in Fig. 2.6.

Note that in the first scenario (Fig. 2.2), i.e., sensor has no computation capability, we examine the sequence from γ_{k-D+1}^k to γ_k^k , while in the second scenario, (Fig. 2.6), we examine the sequence from γ_k to γ_{k-D+1} .

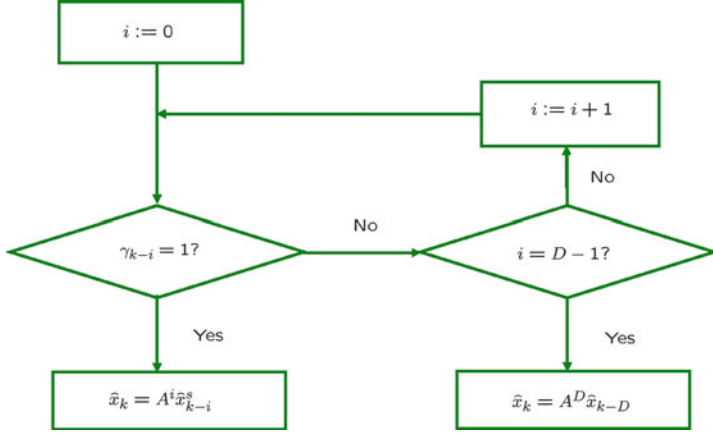


Fig. 2.6 Optimal estimation: sensor with computation capability

Theorem 2.4.1. Assume k is sufficiently large such that $P_k^s = \bar{P}$. For any $M \geq \bar{P}$, we have

$$\Pr[P_k \leq M] = 1 - \theta(k_2, D). \quad (2.4.1)$$

Proof. For the rest of the proof, all probabilities are conditioned on the given D . Let σ'_i and $\Sigma'_i, i = 1, 2$ be defined in the same way as in the proof of Theorem 2.3.9 (see Fig. 2.4). Clearly for any $\sigma'_2 \in \Sigma'_2$,

$$P_k(\sigma'_2) \leq h^{k_2-1}(\bar{P}) \leq M.$$

The first inequality is from the fact that $\gamma_{k-k_2+1} = 1$ and hence $P_{k-k_2+1} = \bar{P}$. The second inequality is from the definition of k_2 . In other words,

$$\Pr[P_k \leq M | \sigma'_2] = 1.$$

Similar to the proof of Theorem 2.3.9, for $\sigma'_1 \in \Sigma'_1$, let us define

$$s = s(\sigma'_1) \triangleq \min\{t \geq 1 : \gamma_{k-t} = 1 | \sigma'_1\}.$$

As $\sigma'_1 \in \Sigma'_1$, $s \geq k_2$. Therefore

$$P_k(\sigma'_1) = h^s(\bar{P}) \not\leq M.$$

In other words, $\Pr[P_k \leq M | \sigma'_1] = 0$. Therefore

$$\begin{aligned}
 \Pr[P_k \leq M] &= \sum_{\sigma' \in \Sigma'_1 \cup \Sigma'_2} \Pr[P_k \leq M | \sigma'] \Pr(\sigma') \\
 &= \sum_{\sigma'_1 \in \Sigma'_1} \Pr[P_k \leq M | \sigma'_1] \Pr(\sigma'_1) + \sum_{\sigma'_2 \in \Sigma'_2} \Pr[P_k \leq M | \sigma'_2] \Pr(\sigma'_2) \\
 &= \sum_{\sigma'_2 \in \Sigma'_2} \Pr[P_k \leq M | \sigma'_2] \Pr(\sigma'_2) \\
 &= \sum_{\sigma'_2 \in \Sigma'_2} \Pr(\sigma'_2) = \Pr(\Sigma'_2) = 1 - \Pr(\Sigma'_1) = 1 - \Pr[N_k \geq k_2].
 \end{aligned}$$

□

Computing $\Pr[N_k \geq k_2]$ follows exactly the same way as in Sect. 2.3.4. Since we have a strict equality in (2.4.1), in order that

$$\Pr[P_k \leq M] \geq 1 - \epsilon$$

a necessary and sufficient condition is that

$$\Pr[N_k \geq k_2] \leq \epsilon. \quad (2.4.2)$$

Therefore, the minimum D^* that guarantees (2.4.2) to hold is given by

$$D^* = \min\{D : \theta(k_2, D) \leq \epsilon, 1 \leq D \leq k_2 - 1\}. \quad (2.4.3)$$

Note that since $\theta(k_2, D) \geq \theta(k_2, k_2 - 1) = \epsilon_2(M)$, D^* from the above equation exists if and only if $\epsilon \geq \epsilon_2(M)$.

2.4.1 When C Is Not Full Rank

We use Theorem 2.4.1 to tackle the case when C is not full rank for the first scenario, i.e., sensor without computation capability. Since (A, C) is observable, there exists r ($2 \leq r \leq n$) such that

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix}$$

is full rank. In this section, we consider the special case when $r = 2$, and in particular, we assume $\begin{bmatrix} C \\ CA \end{bmatrix}^{-1}$ exists. The idea readily extends to the general case.

Unlike the case when C^{-1} exists, and y_k is sent across the network, here we assume that the previous measurement y_{k-1} is sent along with y_k . This only requires that the sensor has a buffer that stores y_{k-1} . Then if $\gamma_k = 1$, both y_k and y_{k-1} are received. Thus, we can use the following linear estimator to generate $\hat{x}_k$

$$\hat{x}_k = A \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix}.$$

The corresponding error covariance can be calculated as

$$P_k = AM_1A' + Q - A \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} CQ \\ 0 \end{bmatrix} - \begin{bmatrix} CQ \\ 0 \end{bmatrix}' \begin{bmatrix} CA \\ C \end{bmatrix}^{-1'} A',$$

where

$$M_1 = \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} CQC' + R & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} CA \\ C \end{bmatrix}^{-1'}.$$

Therefore once the packet for time k is received, i.e., $\gamma_k = 1$, we have

$$P_k = AM_1A' + Q - A \begin{bmatrix} CA \\ C \end{bmatrix}^{-1} \begin{bmatrix} CQ \\ 0 \end{bmatrix} - \begin{bmatrix} CQ \\ 0 \end{bmatrix}' \begin{bmatrix} CA \\ C \end{bmatrix}^{-1'} A' \triangleq \tilde{P}.$$

Now, if we treat $\tilde{P}$ as the steady-state error covariance at the sensor side, i.e., by letting $P_k^s = \tilde{P}$, and define

$$k_v \triangleq \min\{t \geq 1 : h^t(\tilde{P}) \not\leq M\},$$

we immediately obtain

$$\Pr[P_k \leq M] = 1 - \theta(k_v, D). \quad (2.4.4)$$

Remark 2.4.2. Although we give the exact expression of $\Pr[P_k \leq M]$ in (2.4.4), we have to point out that $\theta(k_v, D) \geq \theta(k_2, D)$, as $\tilde{P} \geq \bar{P}$ due to the optimality of Kalman filter. Thus, the case that sensor has computation capability leads to better filter performance, which is illustrated from the vector system example in the next section.

2.5 Examples

2.5.1 Scalar System

Consider the same parameters as in Example 2.3.10, i.e.,

$$A = 1.4, C = 1, Q = 0.2, R = 0.5$$

and

$$f(i) = \frac{d^i e^{-d}}{i!}, i = 0, 1, \dots$$

Figure 2.7 shows the values of $f(i)$ for $0 \leq i \leq 20$ for $d = 3$ and 5, respectively.

2.5.1.1 Sensor Without Computation Capability

We run a Monte Carlo simulation for different parameters. Figures 2.8–2.10 show the results when D and d take different values. From Fig. 2.11, we can see that both smaller d and larger D lead to larger $\Pr[P_k \leq M]$, which confirms the theory developed in this chapter. We also notice that when $d = 3$, the filter's performances using $D = 10$ and $D = 5$ only differ slightly (though the former one is better than the latter one), which confirms that using a large buffer may not improve the filter performance drastically.

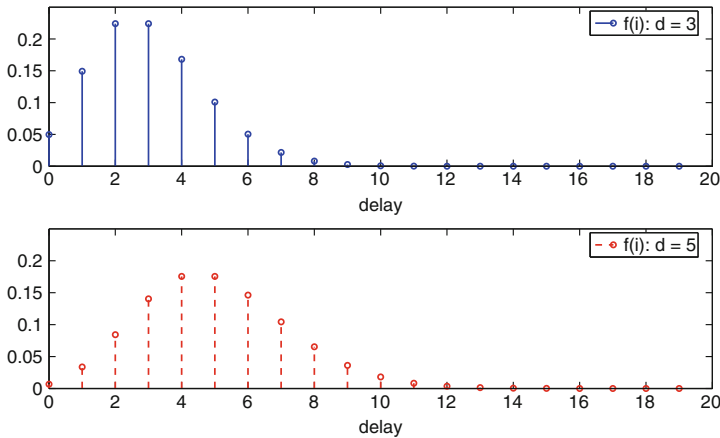


Fig. 2.7 Poisson distribution with $d = 3$ and $d = 5$

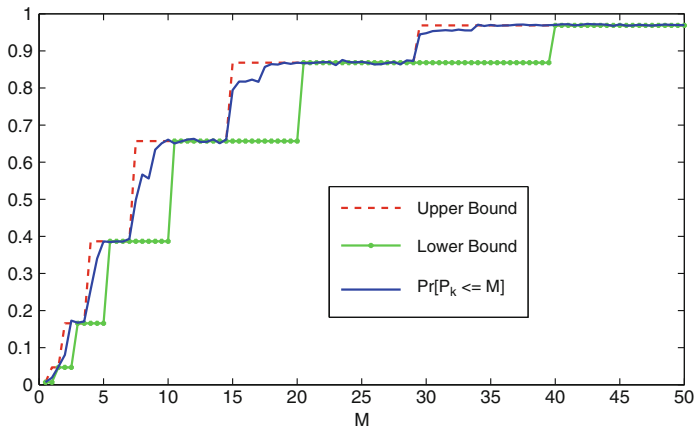


Fig. 2.8 $\Pr[P_k \leq M = 10], d = 5$

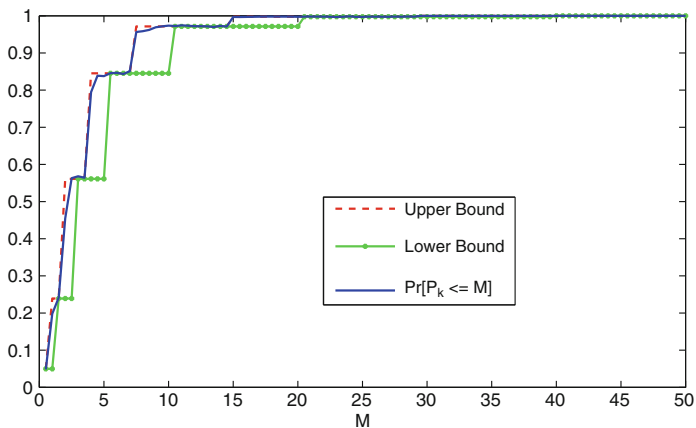


Fig. 2.9 $\Pr[P_k \leq M = 10], d = 3$

2.5.1.2 Sensor with Computation Capability

We run a Monte Carlo simulation for the case when the sensor has computation capability. Figure 2.12 shows the result when $D = 10$ and $d = 5$. As we can see, the predicted value of $\Pr[P_k \leq M]$ from (2.4.1) matches well with the actual value.

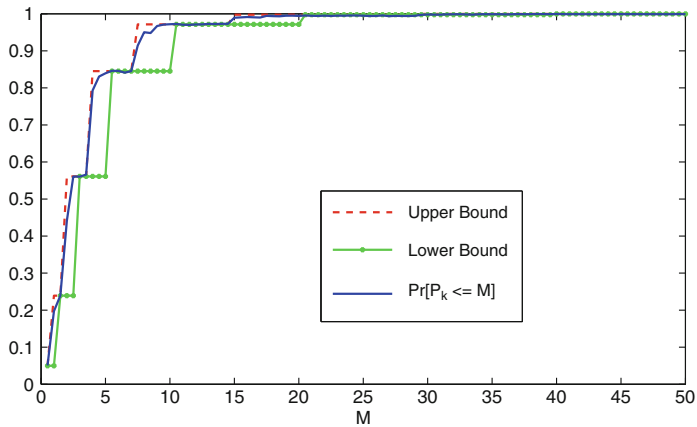


Fig. 2.10 $\Pr[P_k \leq M = 5], d = 3$

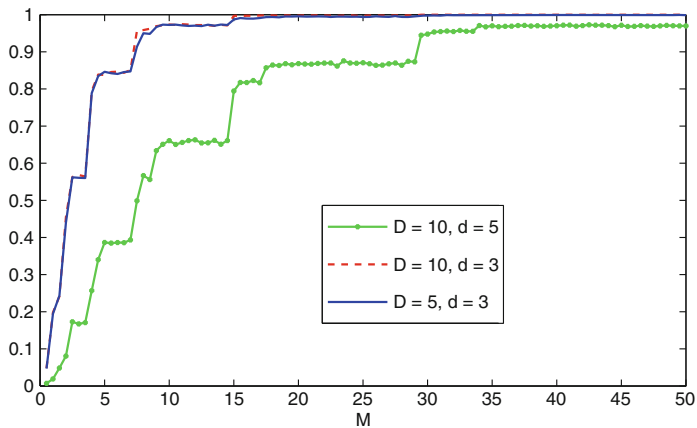


Fig. 2.11 Comparison of the three simulations

2.5.2 Vector System

Consider a vehicle moving in a two-dimensional space according to the standard constant acceleration model, which assumes that the vehicle has zero acceleration except for a small perturbation. The state of the vehicle consists of its x and y positions as well as velocities. Assume a sensor measures the positions of the vehicle

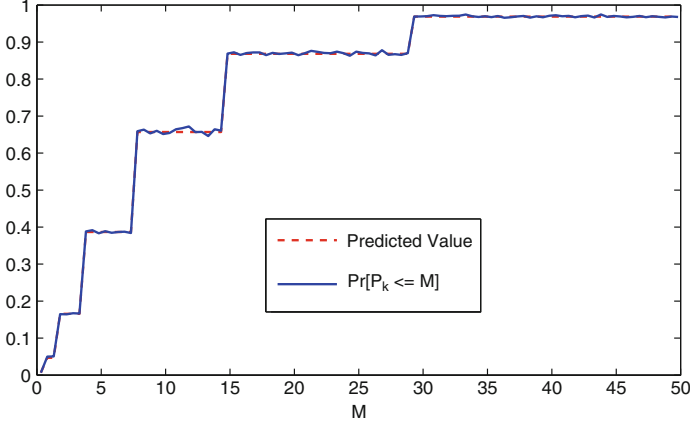


Fig. 2.12 $\Pr[P_k \leq M = 10]$, $d = 5$

and sends the measurements to a remote estimator over a packet-delaying network. The system parameters are given according to (2.2.1)–(2.2.2) as follows:

$$A = \begin{bmatrix} 1 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

The process and measurement noise covariances are $Q = \mathbf{diag}(0.01, 0.01, 0.01, 0.01)$ and $R = \mathbf{diag}(0.001, 0.001)$. We assume the same delay profile as in the scalar system example with $D = 5$ and $d = 3$.

We run a Monte Carlo simulation for both cases when the sensor has or has not computation capability. As we can see from Fig. 2.13, the predicted values of $\Pr[P_k \leq M]$ from (2.4.1) and (2.4.4) match well with the actual values. We also notice that when sensor has computation capability, the actual filter performance is better than when sensor has no computation capability, as stated in Remark 2.4.2. In Fig. 2.13, the M in the x-axis means $M \times I_4$, where I_4 is the identity matrix of dimension 4.

2.A Supporting Lemmas

Lemma 2.A.1. *For any $0 \leq X \leq Y$,*

$$\begin{aligned} h(X) &\leq h(Y), \quad g(X) \leq g(Y), \\ \tilde{g}(X) &\leq \tilde{g}(Y), \quad \tilde{g}(X) \leq X, \\ h \circ \tilde{g}(X) &= g(X), \quad g(X) \leq h(X). \end{aligned}$$

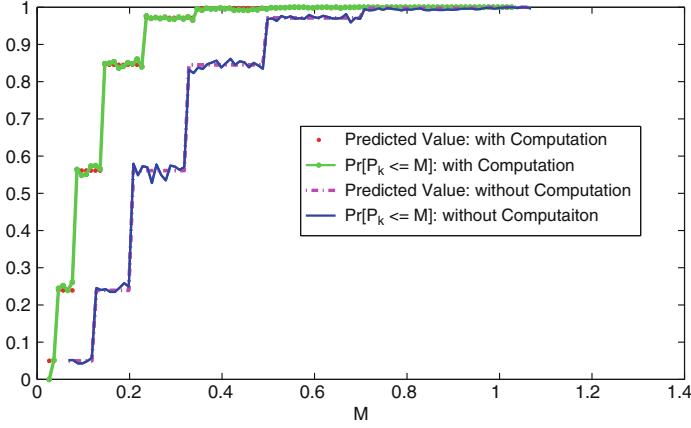


Fig. 2.13 $\Pr[P_k \leq M = 5], d = 3$

Proof. $h(X) \leq h(Y)$ holds as $h(X)$ is affine in X . Proof for $g(X) \leq g(Y)$ can be found in Lemma 1-c in [15]. As $\tilde{g}$ is a special form of g by setting $A = I$ and $Q = 0$, we immediately obtain $\tilde{g}(X) \leq \tilde{g}(Y)$. Next, we have

$$\tilde{g}(X) = X - XC'[CXC' + R]^{-1}CX \leq X$$

and

$$\begin{aligned} h \circ \tilde{g}(X) &= h(X - XC'[CXC' + R]^{-1}CX) \\ &= A(X - XC'[CXC' + R]^{-1}CX)A' + Q \\ &= g(X). \end{aligned}$$

Finally, we have

$$g(X) = h(X) - AXC'[CXC' + R]^{-1}CXA' \leq h(X).$$

□

Lemma 2.A.2. For any $X \geq 0$, $\tilde{g}(X) \leq \overline{M}$.

Proof. For any $t > 0$, we have

$$\tilde{g}(t\overline{M}) = \frac{t}{t+1}\overline{M} \leq \overline{M}.$$

For all $X \geq 0$, since $\overline{M} > 0$, it is clear that there exists $t_1 > 0$ such that $t_1\overline{M} > X$. Therefore,

$$\tilde{g}(X) \leq \tilde{g}(t_1\overline{M}) \leq \overline{M}.$$

□

Lemma 2.A.3. $\overline{P} \leq h(\overline{P})$.

Proof.

$$h(\overline{P}) = h \circ \tilde{g}(P^*) = g(P^*) = P^* \geq \tilde{g}(P^*) = \overline{P},$$

where the first and the last equality are from the definition of $\overline{P}$, the third equality is from the definition of P^* . The rest equality and inequality are from Lemma 2.A.1. $\square$

Lemma 2.A.4. *Let X be a continuous random variable defined on $[0, \infty)$ and let $F(x) = \Pr[X \leq x]$. Then*

$$\mathbb{E}[X] = \int_0^\infty [1 - F(x)]dx.$$

Proof. See Lemma (4) in [2], page 93. $\square$

2.B Evaluate $\mathbb{E}[P_k]$ and Its Stability

Consider (2.2.1) and (2.2.2) with

$$A = a > 1, Q = q > 0, C = c > 0, R = r > 0.$$

For scalar systems, $\Pr[P_k \leq M]$ is the cumulative distribution function of the random variable P_k (for a given D). Therefore, we can find $\mathbb{E}[P_k]$ from Theorem 2.3.9 or Theorem 2.4.1 by using Lemma 2.A.4 in Appendix 2.A, i.e., we write $\mathbb{E}[P_k]$ as

$$\begin{aligned} \mathbb{E}[P_k] &= \int_0^\infty (1 - \Pr[P_k \leq M])dM \\ &= \int_0^{\overline{M}} (1 - \Pr[P_k \leq M])dM + \int_{\overline{M}}^\infty (1 - \Pr[P_k \leq M])dM. \end{aligned}$$

Let us consider the case when sensor has no computation capability. The following results extends trivially to the case when sensor has computation capability. Using the fact

$$0 \leq \Pr[P_k \leq M] \leq 1,$$

we have

$$\mathbb{E}[P_k] \leq \overline{M} + \int_{\overline{M}}^\infty (1 - \Pr[P_k \leq M])dM \leq \overline{M} + \int_{\overline{M}}^\infty \theta(k_1, D)dM,$$

and

$$\mathbb{E}[P_k] \geq \int_{\overline{M}}^{\infty} (1 - \Pr[P_k \leq M]) dM \geq \int_{\overline{M}}^{\infty} \theta(k_2, D) dM.$$

Recall that $k_1(M) = \min\{t \geq 1 : h^t(\overline{M}) \not\leq M\}$ and

$$\begin{aligned} h^t(\overline{M}) &= a^{2t}\overline{M} + q(1 + a^2 + \dots + a^{2t-2}) = \left(\overline{M} + \frac{q}{a^2 - 1}\right)a^{2t} - \frac{q}{a^2 - 1} \\ &= c_1 a^{2t} - c_2 \end{aligned}$$

where

$$c_1 = \overline{M} + \frac{q}{a^2 - 1}, c_2 = \frac{q}{a^2 - 1},$$

therefore for any $t \geq 1$,

$$k_1(M) = t, \text{ if } c_1 a^{2t-2} - c_2 \leq M < c_1 a^{2t} - c_2.$$

Therefore,

$$\mathbb{E}[P_k] \leq \overline{M} + \int_{\overline{M}}^{\infty} \theta(k_1, D) dM = \overline{M} + \sum_{t=1}^{\infty} (c_1 a^{2t} - c_1 a^{2t-2}) \theta(t, D), \quad (2.B.1)$$

and

$$\mathbb{E}[P_k] \geq \int_{\overline{M}}^{\infty} \theta(k_2, D) dM = \sum_{t=1}^{\infty} (c'_1 a^{2t} - c'_1 a^{2t-2}) \theta(t, D), \quad (2.B.2)$$

where $c'_1 = \overline{P} + \frac{q}{a^2 - 1}$.

Lemma 2.B.1. *The minimum D that guarantees*

$$\lim_{k \rightarrow \infty} \mathbb{E}[P_k] < \infty$$

is given by

$$D_{\min} = \min \left\{ d : \sum_{i=0}^{d-1} f(i) > 1 - \frac{1}{a^2} \right\}. \quad (2.B.3)$$

Any other $D < D_{\min}$ leads to

$$\lim_{k \rightarrow \infty} \mathbb{E}[P_k] = \infty.$$

Proof. Recall that $\theta(k_i, D)$ is defined in (2.3.6) as

$$\theta(k_i, D) = \prod_{j=0}^{k_i-1} (1 - \hat{\gamma}_j(D)),$$

and $\hat{\gamma}_i(D) = \hat{\gamma}_D$ for any $i \geq D$. Thus from (2.B.1), in order that $\mathbb{E}[P_k] < \infty$, it is sufficient that

$$1 - \hat{\gamma}_D < \frac{1}{a^2},$$

or in other words,

$$\sum_{i=0}^{D-1} f(i) > 1 - \frac{1}{a^2}.$$

Similarly, if $D < D_{\min}$, i.e.,

$$1 - \hat{\gamma}_D \geq \frac{1}{a^2},$$

then from (2.B.2),

$$\lim_{k \rightarrow \infty} \mathbb{E}[P_k] = \infty.$$

□

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