Chapter 2
Hahn–Banach and Banach Open Mapping Theorems

The Hahn-Banach theorem, in the geometrical form, states that a closed and convex set can be separated from any external point by means of a hyperplane. This intuitively appealing principle underlines the role of convexity in the theory. It is the first, and most important, of the fundamental principles of functional analysis. The rich duality theory of Banach spaces is one of its direct consequences. The second fundamental principle, the Banach open mapping theorem, is studied in the rest of the chapter.

A real-valued function p on a vector space X is called a *subadditive* if $p(x+y) \leq p(x) + p(y)$ for all $x, y \in X$. It is called *positively homogeneous* if for all $x \in X$ and $\alpha \geq 0$ it satisfies $p(\alpha x) = \alpha p(x)$. If p is subadditive and, moreover, $p(\alpha x) = |\alpha| p(x)$ for all $x \in X$ and all scalars α, then p is called a *seminorm* on X. Note that every norm is a seminorm. Note, too, that every positively homogeneous subadditive function is a convex function.

By a *linear functional* on a vector space X, we mean a linear mapping from X into $\mathbb{K}$.

Theorem 2.1 (Hahn, Banach) *Let Y be subspace of a real linear space X, and let p be a positively homogeneous subadditive functional on X. If f is a linear functional on Y such that $f(x) \leq p(x)$ for every $x \in Y$, then there is a linear functional F on X such that $F = f$ on Y and $F(x) \leq p(x)$ for every $x \in X$.*

Proof: Let $\mathcal{P}$ be the collection of all ordered pairs (M', f'), where M' is a subspace of X containing Y and f' is a linear functional on M' that coincides with f on Y and satisfies $f' \leq p$ on M'. $\mathcal{P}$ is nonempty as it contains the pair (Y, f). We partially order $\mathcal{P}$ by $(M', f') \prec (M'', f'')$ if $M' \subset M''$ and $f''\big|_{M'} = f'$. If $\{M_\alpha, f_\alpha\}$ is a chain, then $M' := \bigcup M_\alpha$ and a linear functional f' on M' defined by $f'(x) = f_\alpha(x)$ for $x \in M_\alpha$ satisfy $(M_\alpha, f_\alpha) \prec (M', f')$ for all α. By Zorn's lemma, $\mathcal{P}$ has a maximal element (M, F). We need to show that $M = X$.

Assume $M \neq X$, pick $x_1 \in X \backslash M$ and put $M_1 = \text{span}\{M, x_1\}$. We will find $(M_1, F_1) \in \mathcal{P}$ such that $(M, F) \prec (M_1, F_1)$, a contradiction. For a fixed $\alpha \in \mathbb{R}$ we define $F_1(x + tx_1) = F(x) + t\alpha$ for $x \in M$, $t \in \mathbb{R}$. Then F is linear. It remains to show that we can choose α so that $F_1 \leq p$.

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Due to the positive homogeneity of p and F, it is enough to choose α such that

$$\begin{matrix} F_1(x+x_1) \leq p(x+x_1) \\ F_1(x-x_1) \leq p(x-x_1) \end{matrix} \quad \text{for every } x \in M. \tag{2.1}$$

Indeed, for $t > 0$ we then have

$$F_1(x+tx_1) = tF_1\left(\tfrac{x}{t}+x_1\right) \leq tp\left(\tfrac{x}{t}+x_1\right) = p(x+tx_1)$$

and for $t = -\eta < 0$ we have

$$\begin{aligned} F_1(x+tx_1) &= F_1(x-\eta x_1) = \eta F_1\left(\tfrac{x}{\eta}-x_1\right) \\ &\leq \eta p\left(\tfrac{x}{\eta}-x_1\right) = p(x-\eta x_1) = p(x+tx_1). \end{aligned}$$

But (2.1) is equivalent to $(\alpha :=)\ F_1(x_1) \leq p(x+x_1) - F(x)$ and $(-\alpha =)$ $-F_1(x_1) \leq p(x-x_1) - F(x)$ for every $x \in M$. This in turn is equivalent to

$$F(y) - p(y-x_1) \leq \alpha \leq p(x+x_1) - F(x)$$

for every $x, y \in M$. Thus to find a suitable $\alpha \in \mathbb{R}$ we need to show that $\sup\{F(y) - p(y-x_1) :\ y \in M\} \leq \inf\{p(x+x_1) - F(x) :\ x \in M\}$. This is in turn equivalent to the statement that for every $x, y \in M$ we have

$$F(y) - p(y-x_1) \leq p(x+x_1) - F(x).$$

The latter reads $F(x+y) \leq p(x+x_1) + p(y-x_1)$, which is true as

$$F(x+y) \leq p(x+y) = p(x+x_1+y-x_1) \leq p(x+x_1) + p(y-x_1).$$

This completes the proof of Theorem 2.1. $\square$

2.1 Hahn–Banach Extension and Separation Theorems

Before we pass to normed space versions of the Hahn–Banach theorem, we need to establish the relationship between the real and the complex normed spaces.

Let X be a complex normed space. The space X is also a real normed space. We will denote this real version of X by $X_{\mathbb{R}}$.

On the other hand, if X is a real normed space, then $X \times X$ becomes a complex normed space $X_{\mathbb{C}}$ when its linear structure and norm are defined for $x, y, u, v \in X$ and $a, b \in \mathbb{R}$ by

$$\begin{gathered} (x, y) + (u, v) := (x+u, y+v) \\ (a+ib)(x, y) := (ax - by, bx + ay) \\ \|(x, y)\|_{\mathbb{C}} := \sup\{\|\cos(\theta)x + \sin(\theta)y\| :\ 0 \leq \theta \leq 2\pi\}. \end{gathered}$$

The set $X \times \{0\} := \{(x, 0) : x \in X\}$ is a closed $\mathbb{R}$-linear subspace of $X_{\mathbb{C}}$ which is—as a real space—isometric to X under the mapping $(x, 0) \mapsto x$. Conversely, $X_{\mathbb{C}} = \{h + ik : h, k \in X \times \{0\}\}$.

We will verify that $\|\cdot\|_{\mathbb{C}}$ is actually a norm on $X_{\mathbb{C}}$. It is clear that $\|\cdot\|_{\mathbb{C}}$ is non-negative, satisfies the triangle inequality, and factors real constants to their absolute value. If α is real and $z := (x, y) \in X_{\mathbb{C}}$, then

$$\begin{aligned}
\|e^{-i\alpha}z\|_{\mathbb{C}} &= \|\big(\cos(\alpha)x + \sin(\alpha)y, -\sin(\alpha)x + \cos(\alpha)y\big)\|_{\mathbb{C}} \\
&= \sup\{\|\cos(\theta)[\cos(\alpha)x + \sin(\alpha)y] + \sin(\theta)[-\sin(\alpha)x + \cos(\alpha)y]\| : 0 \le \theta \le 2\pi\} \\
&= \sup\{\|\cos(\theta + \alpha)x + \sin(\theta + \alpha)y\| : 0 \le \theta \le 2\pi\} \\
&= \sup\{\|\cos(\eta)x + \sin(\eta)y\| : 0 \le \eta \le 2\pi\} = \|z\|_{\mathbb{C}}.
\end{aligned}$$

Therefore $\|\cdot\|_{\mathbb{C}}$ is a norm on $X_{\mathbb{C}}$. Since $\max\{\|x\|, \|y\|\} \le \|(x, y)\|_{\mathbb{C}} \le \|x\| + \|y\|$, we have that the topology induced on $X_{\mathbb{C}} = X \times X$ by $\|\cdot\|_{\mathbb{C}}$ is equivalent to the product topology induced on $X \times X$ by $\|\cdot\|$.

We will now relate duals of X and $X_{\mathbb{R}}$. Consider the mapping $R\colon X^* \to {X_{\mathbb{R}}}^*$ defined by $R(f)(x) = \mathrm{Re}\big(f(x)\big)$ for $x \in X$, where $\mathrm{Re}(f(x))$ is the real part of $f(x)$. We claim that it is a norm-preserving mapping from X^* onto ${X_{\mathbb{R}}}^*$ and is linear as a mapping $(X^*)_{\mathbb{R}} \to {X_{\mathbb{R}}}^*$.

To see this claim, note that if X is a complex Banach space and $f \in X^*$, then $\sup_{z \in B_X} |f(z)| = \sup_{z \in B_X} |\mathrm{Re}\big(f(z)\big)|$. Indeed, for all z we have $|f(z)| \ge |\mathrm{Re}\big(f(z)\big)|$, so one inequality is clear. On the other hand, for $z \in B_X$ we write $f(z) = e^{i\alpha}|f(z)|$ and have $f(e^{-i\alpha}z) = e^{-i\alpha}f(z) = |f(z)|$. Thus $|\mathrm{Re}\big(f(e^{-i\alpha}z)\big)| = |f(z)|$ and $\|e^{-i\alpha}z\| = \|z\|$.

Now we show that R is onto ${X_{\mathbb{R}}}^*$. To $g \in {X_{\mathbb{R}}}^*$ we assign the functional defined on X by $G(x) = g(x) - ig(ix)$. Then G is linear over $\mathbb{R}$, but also

$$G(ix) = g(ix) - ig(-x) = g(ix) + ig(x) = i\big(g(x) - ig(ix)\big) = iG(x).$$

Therefore G is linear over $\mathbb{C}$ and hence $G \in X^*$. Moreover, $R(G) = g$.

Theorem 2.2 (Hahn, Banach) *Let Y be a subspace of a normed space X. If $f \in Y^*$ then there exists $F \in X^*$ such that $F\big|_Y = f$ and $\|F\|_{X^*} = \|f\|_{Y^*}$.*

Proof: First assume that X is a real normed space. Define a new norm $|||\cdot|||$ on X by $|||x||| = \|f\|_{Y^*}\|x\|$, where $\|\cdot\|$ is the original norm of X. We have $|f(y)| \le |||y|||$ for all $y \in Y$, so by Theorem 2.1 there is a linear functional F on X that extends f and $|F(x)| \le |||x|||$ $(= \|f\|_{Y^*}\|x\|)$ for every $x \in X$. Therefore $\|F\|_{X^*} := \sup\{|F(x)| : \|x\| \le 1\} \le \|f\|_{Y^*}$. Since F extends f, we obviously have $\|F\|_{X^*} \ge \|f\|_{Y^*}$ as well. Consequently $\|F\|_{X^*} = \|f\|_{Y^*}$.

Now assume that X is a complex normed space. Consider the linear functional $R(f)$ on $Y_{\mathbb{R}}$, where R is the isometry defined above. By the first part of this proof, we extend $R(f)$ to a linear functional $g \in {X_{\mathbb{R}}}^*$ that satisfies $\|g\|_{{X_{\mathbb{R}}}^*} = \|R(f)\|_{{Y_{\mathbb{R}}}^*} = \|f\|_{Y^*}$. Then the norm of the linear functional $F(x) := g(x) - ig(ix) \in X^*$ is equal to $\|g\|_{{X_{\mathbb{R}}}^*}$ $(= \|f\|_{Y^*})$.

The real part of F is g and thus $\operatorname{Re}\big(F\big|_Y\big) = \operatorname{Re}(f)$, that is, $R(F\big|_Y) = R(f)$. Since R is a bijection of Y^* onto $Y_{\mathbb{R}}{}^*$, we get $F\big|_Y = f$. □

Corollary 2.3 (Hahn, Banach) *Let X be a normed space. For every $x \in X$ there is $f \in S_{X^*}$ such that $f(x) = \|x\|$. In particular, $\|x\| = \max\{|f(x)| : f \in B_{X^*}\}$ for every $x \in X$.*

As a consequence, if $X \neq \{0\}$ then $X^* \neq \{0\}$ as well (see Corollary 3.33).

Proof: Put $Y = \operatorname{span}\{x\}$ and define $f \in Y^*$ by $f(tx) = t\|x\|$. Clearly $\|f\|_{Y^*} = 1$ and $f(x) = \|x\|$. Using Theorem 2.2 we extend f to a linear functional from X^* with the same norm. From $|f(x)| \leq \|f\|\,\|x\|$ we have $\sup_{f \in B_{X^*}} |f(x)| \leq \|x\|$. On the other hand, the linear functional constructed above shows that the supremum is attained and equal to $\|x\|$. □

Corollary 2.4 *Let $\{x_i\}_{i=1}^n$ be a linearly independent set of vectors in a normed space X and $\{\alpha_i\}_{i=1}^n$ be a set of real numbers. Then there is $f \in X^*$ such that $f(x_i) = \alpha_i$ for $i = 1, \dots, n$.*

Proof: Define a linear functional f on $\operatorname{span}\{x_i\}$ by $f(x_i) = \alpha_i$ for $i = 1, \dots, n$. Proposition 1.39 shows that f is continuous. The result follows from Theorem 2.2. □

Definition 2.5 *Let C be a convex subset of a normed space X and let $x \in C$. A non-zero linear functional $f \in X^*$ is called a* supporting functional *of C at x if $f(x) = \sup\{f(y) : y \in C\}$. The point x is said to be a* support point *of C (*supported by *f).*

By Corollary 2.3, for every $x \in S_X$ there is a supporting functional of B_X at x, and so x is a support point of B_X.

There exists a closed convex and bounded set C in a Banach space, having empty interior, and a point in C that is not a support point (see Exercise 2.17). However, every closed convex and bounded subset of a Banach space must have support points. This follows from Theorem 7.41.

Consider a Banach space X. If Y is a subset of X, we define its *annihilator* by $Y^\perp = \{f \in X^* : f(y) = 0$ for all $y \in Y\}$. Note that $Y^\perp$ is a closed subspace of X^*. Similarly, for a subset Y of X^* we define $Y_\perp = \{x \in X : f(x) = 0$ for every $f \in Y\}$, which is a closed subspace of X.

Note that if F is a subset of a Hilbert space H, then the orthogonal complement $F^\perp$ when considered a subspace of the dual H^* under the canonical duality (see Theorem 2.22) coincides with the annihilator $F^\perp$.

Proposition 2.6 *Let Y be a closed subspace of a Banach space X. Then $(X/Y)^*$ is isometric to $Y^\perp$ and Y^* is isometric to $X^*/Y^\perp$.*

Proof: Consider the mapping $\delta\colon Y^\perp \to (X/Y)^*$ defined by $\delta(x^*)\colon \hat{x} \mapsto x^*(x)$, where $x \in \hat{x}$. This definition is correct, since $x^*(x_1) = x^*(x_2)$ whenever $x_1, x_2 \in \hat{x}$

as $x^* \in Y^\perp$. To see that δ maps $Y^\perp$ onto $(X/Y)^*$, given $f \in (X/Y)^*$, define $x^* \in X^*$ by $x^*(x) = f(\hat{x})$, where $x \in \hat{x}$. Then $x^* \in Y^\perp$ and $\delta(x^*)(\hat{x}) = x^*(x) = f(\hat{x})$. To check that δ is an isometry, write

$$\|\delta(x^*)\| = \sup_{\|\hat{x}\|<1} |\delta(x^*)(\hat{x})| = \sup_{\|x\|<1} |x^*(x)| = \|x^*\|.$$

The middle equality follows since given $\|\hat{x}\| < 1$, there is $x \in \hat{x}$ such that $\|x\| < 1$. On the other hand, given $\|x\| < 1$, we have $\|\hat{x}\| < 1$.

To prove the second part of this proposition, define a mapping σ from Y^* into $X^*/Y^\perp$ by $\sigma(y^*) = \{\text{all extensions of } y^* \text{ on } X\}$. It is easy to see that $\sigma(y^*)$ is a coset in $X^*/Y^\perp$ and the Hahn–Banach theorem gives that $\|\sigma(y^*)\| = \|y^*\|_{Y^*}$. It follows that σ is a linear and onto mapping. $\square$

We will now establish several separation results.

Proposition 2.7 *Let Y be a closed subspace of a normed space X. If $x \notin Y$ then there is $f \in S_{X^*}$ such that $f(y) = 0$ for all $y \in Y$ and $f(x) = \operatorname{dist}(x, Y)$.*

Proof: Let $(0 <)\ d = \operatorname{dist}(x, Y)$. Put $Z = \operatorname{span}\{Y, x\}$ and define a linear functional f on Z by $f(y + tx) = td$ for $y \in Y$ and $t \in \mathbb{K}$. Clearly $f\big|_Y = 0$ and $f(x) = d$. For $u := y + tx$, where $y \in Y$ and t is a scalar such that $u \neq 0$, we have

$$\begin{aligned}|f(u)| = |t|\,d = \frac{|t|.\|u\|}{\|u\|}d = \frac{|t|.\|u\|}{\|y+tx\|}d = \frac{\|u\|}{\|(y/t)+x\|}d \\ = \frac{\|u\|d}{\|x - \big(-(y/t)\big)\|} \leq \frac{\|u\|d}{\operatorname{dist}(x, Y)} = \|u\|.\end{aligned}$$

Therefore $\|f\| \leq 1$.

On the other hand, there is a sequence $y_n \in Y$ such that $\|y_n - x\| \to d$. We have $d = |f(y_n) - f(x)| \leq \|f\| \cdot \|y_n - x\|$, so by passing to the limit when $n \to \infty$ we obtain $d \leq \|f\|d$.

Thus $\|f\| = 1$, and $f(x) = \operatorname{dist}(x, Y)$. Extending f on X with the same norm we obtain the desired functional. $\square$

Proposition 2.8 *Let X be a normed space. If X^* is separable, then X is separable.*

Proof: Choose a dense subset $\{f_n\}$ of S_{X^*}. For every $n \in \mathbb{N}$, pick $x_n \in S_X$ such that $f_n(x_n) > \frac{1}{2}$. Let $Y = \overline{\operatorname{span}}\{x_n\}$. As Y is separable (finite rational combinations of $\{x_n\}$ are dense in Y), it is enough to show that $X = Y$. If $Y \neq X$, then there is $f \in X^*$, $\|f\| = 1$ such that $f(x) = 0$ for every $x \in Y$. Let n be such that $\|f_n - f\| < \frac{1}{4}$. Then

$$\begin{aligned}|f(x_n)| = |f_n(x_n) - (f_n(x_n) - f(x_n))| \geq |f_n(x_n)| - |f_n(x_n) - f(x_n)| \\ \geq |f_n(x_n)| - \|f - f_n\| \cdot \|x_n\| > \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4},\end{aligned}$$

a contradiction. $\square$

To prove separation results for sets we need a new notion.

Definition 2.9 *Let C be a set in a normed space X. We define the* Minkowski functional *of C, $\mu_C : X \to [0, +\infty]$, by*

$$\mu_C(x) = \begin{cases} \inf\{\lambda > 0 : \ x \in \lambda C\}, & \text{if } \{\lambda > 0 : \ x \in \lambda C\} \neq \emptyset, \\ +\infty, & \text{if } \{\lambda > 0 : \ x \in \lambda C\} = \emptyset. \end{cases}$$

Lemma 2.10 *Let μ be a subadditive real function on a real normed space X. Then* (i) *μ is continuous if and only if it is continuous at* 0.
(ii) *If μ is continuous, every linear functional $f : X \to \mathbb{R}$ such that $f \leq \mu$ is also continuous.*

Proof: From the subadditivity of μ it follows easily that, for $x, y \in X$, $-\mu(y-x) \leq \mu(x) - \mu(y) \leq \mu(x - y)$, so μ is continuous if (and only if) it is continuous at 0. This proves (i). In order to prove (ii), use Exercise 2.1. $\square$

Lemma 2.11 *Let C be a convex neighborhood of* 0 *in a normed space X. Then its Minkowski functional μ_C is a finite non-negative positively homogeneous subadditive continuous functional. Moreover,* $\{x : \ \mu_C(x) < 1\} = \mathrm{Int}(C) \subset C \subset \overline{C} = \{x : \mu_C(x) \leq 1\}$.

Proof: Let $B_\delta = \{x : \ \|x\| \leq \delta\} \subset C$ for some $\delta > 0$. Since $0 \in C$, the point 0 is in λC for every $\lambda > 0$ and thus $\mu_C(0) = 0$. Given $x \in X \backslash \{0\}$, we get $\delta \frac{x}{\|x\|} \in B_\delta \subset C$, so $x \in \frac{\|x\|}{\delta} C$. Thus

$$(0 \leq) \ \mu_C(x) \leq \frac{\|x\|}{\delta} < \infty. \tag{2.2}$$

Given $\alpha, \lambda > 0$, clearly $x \in \lambda C$ if and only if $\alpha x \in \lambda \alpha C$. Therefore $\mu_C(\alpha x) = \alpha \mu_C(x)$ and thus μ_C is positively homogeneous. We claim that $\mu_C(x) < \lambda$ implies that $x \in \lambda C$. Indeed, there exists λ_0 such that $\mu_C(x) \leq \lambda_0 < \lambda$ and $x \in \lambda_0 C$. Then we can find $c \in C$ such that $x = \lambda_0 c$. Therefore

$$(x =) \ \lambda_0 c = \lambda \left(\frac{\lambda_0}{\lambda} c + \frac{(1 - \lambda_0)}{\lambda} 0 \right). \tag{2.3}$$

Since C is convex and $0 \in C$ we get $x \in \lambda C$ as claimed.

To prove subadditivity, let $x, y \in X$ and s, t such that $\mu_C(x) < s$, $\mu_C(y) < t$. By the former claim, $x \in sC$ and $y \in tC$. Then $x + y \in sC + tC$, and thus by the convexity

$$x + y \in (t + s)\Big(\frac{s}{t+s} C + \frac{t}{t+s} C \Big) \subset (t + s)C.$$

Therefore $\mu_C(x + y) \leq t + s$, so by the choice of s and t we have $\mu_C(x + y) \leq \mu_C(x) + \mu_C(y)$.

The continuity of μ_C at 0 follows from (2.2), so μ_C is continuous by Lemma 2.10. By the continuity of μ_C, the set $\{x \in X : \mu_C(x) < 1\}$ is open, and, by the claim, a subset of C, hence a subset of $\mathrm{Int}(C)$. It follows that, if $\mu_C(x) = 1$ and $0 < s < 1 < t$, then $sx \in \mathrm{Int}(C)$ and $tx \notin C$. Therefore, if $\mu_C(x) = 1$ then x belongs to the boundary of C and if $\mu_C(x) > 1$ then, again by the continuity of μ_C, $x \in \mathrm{Int}(X \setminus C)$. This proves the statement. □

Theorem 2.12 (Hahn, Banach) *Let C be a closed convex set in a normed space X. If $x_0 \notin C$ then there is $f \in X^*$ such that* $\mathrm{Re}\big(f(x_0)\big) > \sup\{\mathrm{Re}\big(f(x)\big) : x \in C\}$.

Proof: First, let X be a real space. We may assume without loss of generality that $0 \in C$, otherwise we consider $(C - x)$ and $x_0 - x$ for some $x \in C$. Let $\delta = \mathrm{dist}(x_0, C)$. Then δ is positive as C is closed. Set $D = \{x \in X : \mathrm{dist}(x, C) \le \delta/2\}$. Since $0 \in C$, we have $\frac{\delta}{4} B_X \subset D$ and so D contains 0 as an interior point. D is also closed, convex, and $x_0 \notin D$. Let μ_D be the Minkowski functional of D. Since D is closed and $x_0 \notin D$, we have $\mu_D(x_0) > 1$ (Exercise 2.21).

Define a linear functional on $\mathrm{span}\{x_0\}$ by $f(\lambda x_0) = \lambda \mu_D(x_0)$. Then on $\mathrm{span}\{x_0\}$ we have $f(\lambda x_0) \le \mu_D(\lambda x_0)$. For $\lambda \ge 0$ it is clear from the definition of f, for $\lambda < 0$ we have $f(\lambda x_0) = \lambda \mu_D(x_0) < 0$ while $\mu_D(\lambda x_0) \ge 0$. Extend f onto X by Theorem 2.1 and denote this extension by f again. Then $f(x) \le \mu_D(x)$ for every $x \in X$. The continuity of f follows from Lemma 2.10.

Since $\mu_D(x_0) > 1$ and $f(x_0) = \mu_D(x_0)$, we get $f(x_0) > 1$, so $f(x_0) > \sup\{f(x) : x \in C\}$.

If X is a complex space, we construct g from $X^*_{\mathbb{R}}$ as in the real case and then define $f(x) = g(x) - ig(ix)$. □

For simplicity we will state the following result only for the real case.

Proposition 2.13 *Let X be a real normed space.*
(i) *Let C be an open convex set in X. If $x_0 \notin C$ then there is $f \in X^*$ such that $f(x) < f(x_0)$ for all $x \in C$.*
(ii) *Let A, B be disjoint convex sets in X. If A is open then there is $f \in X^*$ such that $f(a) < \inf_{f(b):\, b \in B}$ for all $a \in A$.*

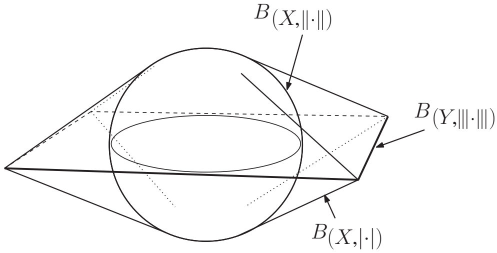
Proof: (i) We pick some $y \in C$ and consider $D := C - y$, $y_0 := x_0 - y$. Then define μ_D and f on $\mathrm{span}\{y_0\}$ as in the proof of Theorem 2.12. Let f denote the extended functional as well. We have $f(y_0) = \mu_D(y_0) \ge 1$ as $y_0 \notin D$. Also $f(x) < 1$ for $x \in D$ as D is open (Exercise 2.21), and the statement follows.

(ii) Applying (i) to the open convex set $C := A - B$ and to $x_0 := 0$ we obtain f such that $f(x) < f(0)$ $(= 0)$ for $x \in A - B$. Thus $f(a) < f(b)$ for every $a \in A$, $b \in B$. It follows that $f(a) \le \inf\{f(b) : b \in B\}$ for $a \in A$. If $a \in A$ is such that $f(a) = \inf\{f(b) : b \in B\}$, then, from the openness of A we get $f(a + h) > \inf\{f(b) : b \in B\}$ for some $a + h \in A$, a contradiction. Therefore $f(a) < \inf\{f(b) : b \in B\}$ for all $\alpha \in A$. □

Proposition 2.14 *Let $(X, \|\cdot\|)$ be a normed space. Let Y be a subspace of X and let $|||\cdot|||$ be an equivalent norm on Y. Then there is an equivalent norm $|\cdot|$ on X inducing on Y the norm $|||\cdot|||$.*

Proof: Without loss of generality, we may assume that $B_{(Y,\|\cdot\|)} \subset B_{(Y,|||\cdot|||)}$. The set $B := \operatorname{conv}\{B_{(Y,|||\cdot|||)} \cup B_{(X,\|\cdot\|)}\}$ is convex and balanced. Obviously, B is bounded and contains $B_{(X,\|\cdot\|)}$, so its Minkowski functional is an equivalent norm $|\cdot|$ on X (see Fig. 2.1).

Fig. 2.1 Extending a norm



This norm certainly induces on Y the norm $|||\cdot|||$, since $B \cap Y = B_{(Y,|||\cdot|||)}$. Indeed, $B_{(Y,|||\cdot|||)} \subset B$. On the other hand, if $y \in B \cap B_{(Y,|||\cdot|||)}$, then $y = \lambda y_1 + (1-\lambda)x$, where $0 \le \lambda \le 1$, $y_1 \in B_{(Y,|||\cdot|||)}$, and $x \in B_{(X,\|\cdot\|)}$ (see Exercise 1.12). We get $(1-\lambda)x = y - \lambda y_1 \in Y$. If $\lambda \neq 1$ then $x \in Y$, so $x \in B_{(X,\|\cdot\|)} \cap Y = B_{(Y,\|\cdot\|)} \subset B_{(Y,|||\cdot|||)}$. By convexity, $y \in B_{(Y,|||\cdot|||)}$. If, on the contrary, $\lambda = 1$, we obtain again $y \in B_{(Y,|||\cdot|||)}$. $\square$

We refer to Exercise 5.95 for an alternative proof of the Hahn–Banach theorem.

2.2 Duals of Classical Spaces

In Propositions 2.15, 2.16, 2.17, 2.18, 2.19, and 2.20, we assume the scalar field to be $\mathbb{R}$.

Proposition 2.15 (Riesz) *$c_0^* = \ell_1$ in the sense that for every $f \in c_0^*$ there is a unique $(a_i) \in \ell_1$ such that $f(x) = \sum a_i x_i$ for all $x = (x_i) \in c_0$, and the mapping $f \mapsto (a_i)$ is a linear isometry from c_0^* onto ℓ_1.*

Proof: Given $f \in c_0^*$, define $a_i = f(e_i)$, where $e_i := (0, \dots, 0, \overset{i}{1}, 0, \dots)$ are the standard unit vectors in c_0. For $n \in \mathbb{N}$ we set

$$x^n = (\operatorname{sign}(a_1), \dots, \operatorname{sign}(a_n), 0, \dots) \in c_0.$$

Then $\|x^n\|_\infty = 1$ and $f(x^n) = \sum_{i=1}^n |a_i| \le \|f\| \cdot \|x^n\|_\infty = \|f\|$. Therefore $\sum_{i=1}^\infty |a_i| \le \|f\| < \infty$, that is, the mapping $f \mapsto \big(f(e_i)\big)$ is a continuous mapping into ℓ_1. It is obviously linear.

On the other hand, if $\sum_{i=1}^{\infty}|a_i| < \infty$ then $\sum |a_i x_i| < \infty$ for every $x = (x_i) \in c_0$. Indeed, we have $\sum |a_i x_i| \le \sup |x_i| \cdot \sum |a_i| = \|(a_i)\|_1 \|(x_i)\|_\infty$. Consider the linear functional h defined on c_0 by $h(x) = \sum a_i x_i$. Then from the above estimate we have $\|h\| \le \|(a_i)\|_1$ and also $h(e_i) = a_i$, so $h \in c_0^*$ and the mapping $f \mapsto \big(f(e_i)\big)$ is thus onto. We also obtain that $\|f\| = \|\big(f(a_i)\big)\|_1$, hence the considered mapping is an isometry onto ℓ_1. □

Proposition 2.16 (Riesz) $\ell_1^* = \ell_\infty$ *in the sense that for every* $f \in \ell_1^*$ *there is a unique* $(a_i) \in \ell_\infty$ *such that* $f(x) = \sum a_i x_i$ *for all* $x = (x_0) \in \ell_1$*, and the mapping* $f \mapsto (a_i)$ *is a linear isometry from* ℓ_1^* *onto* ℓ_∞.

Proof: Given $f \in \ell_1^*$, put $a_i = f(e_i)$ for $i \in \mathbb{N}$, where e_i are the standard unit vectors in ℓ_1. Then $|a_i| \le \|f\|$, so $\|(a_i)\|_\infty \le \|f\|$. Conversely, for $(a_i) \in \ell_\infty$ consider the functional h defined on ℓ_1 by $h(x) = \sum a_i x_i$. Again, $|h(x)| \le \|(a_i)\|_\infty \|x\|_1$, hence $h \in \ell_1^*$ and $\|h\| \le \|(a_i)\|_\infty$. Similarly as in the proof of Proposition 2.15, we conclude that the mapping is a linear isometry onto ℓ_∞. □

Proposition 2.17 (Riesz) *Let* $p, q \in (1, \infty)$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $\ell_p^* = \ell_q$ *in the sense that for every* $f \in \ell_p^*$ *there exists a unique element* $(a_i) \in \ell_q$ *such that* $f(x) = \sum a_i x_i$ *for all* $x =: (x_i) \in \ell_p$*, and the mapping* $f \mapsto (a_i)$ *is a linear isometry from* ℓ_p^* *onto* ℓ_q.

Proof: For $f \in \ell_p^*$, put $a_i = f(e_i)$. Considering

$$x^n := \big(|a_1|^{q-1}\operatorname{sign}(a_1), \dots, |a_n|^{q-1}\operatorname{sign}(a_n), 0, \dots\big)$$

we see that

$$\sum_{i=1}^{n}|a_i|^q = f(x^n) \le \|f\| \cdot \|x^n\|_p = \|f\| \Big(\sum_{i=1}^{n}\big(|a_i|^{q-1}\big)^p\Big)^{\frac{1}{p}} = \|f\| \cdot \Big(\sum_{i=1}^{n}|a_i|^q\Big)^{\frac{1}{p}}.$$

This reads $\left(\sum_{i=1}^{n}|a_i|^q\right)^{\frac{1}{q}} \le \|f\|$. Hence $\|(a_i)\|_q \le \|f\| < \infty$.

If $(a_i) \in \ell_q$ and $(x_i) \in \ell_p$, then the series $\sum x_i a_i$ is convergent by the Hölder inequality (1.1) as $\sum |x_i a_i| \le \|(x_i)\|_p \|(a_i)\|_q$. Therefore the functional h defined on ℓ_p by $h(x) = \sum x_i a_i$ is well defined and $\|h\| \le \|(a_i)\|_q$. The rest of the proof is analogous to those above. □

Similarly we show that for a set Γ and $p \in [1, \infty)$ we have $c_0(\Gamma)^* = \ell_1(\Gamma)$ and $\ell_p(\Gamma)^* = \ell_q(\Gamma)$, where $\frac{1}{p} + \frac{1}{q} = 1$. This applies, in particular, for a finite set Γ.

Proposition 2.18 (Riesz) *Let* $p, q \in (1, \infty)$ *be such that* $\frac{1}{p} + \frac{1}{q} = 1$. *Then* $L_p[0,1]^* = L_q[0,1]$ *in the sense that for every* $F \in L_p^*$ *there is a unique* $f \in L_q$ *such that* $F(g) = \int_0^1 gf \,\mathrm{d}x$ *for all* $g \in L_p$*, and the mapping* $F \mapsto f$ *is a linear isometry of* L_p^* *onto* L_q.

Proof: Let $F \in L_p^*$. For $t \in [0,1]$, let $u_t = \chi_{[0,t)}$ be the characteristic function of $[0,t)$. Define $\alpha(t) = F(u_t)$. We claim that α is absolutely continuous (see the definition right before Proposition 11.13). Indeed, if $[\tau_i, t_i]$, $i = 1, \dots, n$, is a collection of non-overlapping intervals, that is, their interiors are pairwise disjoint, put $\varepsilon_i = \operatorname{sign}(\alpha(t_i) - \alpha(\tau_i))$ and estimate:

$$\begin{aligned}\sum_{i=1}^{n} |\alpha(t_i) - \alpha(\tau_i)| &= \sum_{i=1}^{n} \varepsilon_i(\alpha(t_i) - \alpha(\tau_i)) = F\Big(\sum_{i=1}^{n} \varepsilon_i(u_{t_i} - u_{\tau_i})\Big)\\ &\le \|F\|_{L_p^*} \cdot \Big\|\sum_{i=1}^{n} \varepsilon_i(u_{t_i} - u_{\tau_i})\Big\|_{L_p} = \|F\|_{L_p^*}\Big(\int_0^1 \Big|\sum_{i=1}^{n} \varepsilon_i(u_{t_i} - u_{\tau_i})\Big|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}\\ &= \|F\|\Big(\sum_{i=1}^{n} \int_{\tau_i}^{t_i} 1 \,\mathrm{d}x\Big)^{\frac{1}{p}} = \|F\| \cdot \Big(\sum_{i=1}^{n} (t_i - \tau_i)\Big)^{\frac{1}{p}}.\end{aligned}$$

Therefore α is an absolutely continuous function on $[0,1]$. By the Lebesgue fundamental theorem of calculus, we have $\alpha(t) - \alpha(0) = \int_0^t \alpha' \,\mathrm{d}x$ for every $t \in [0,1]$. Setting $f = \alpha'$ and using $\alpha(0) = F(u_0) = 0$ we get

$$F(u_t) = \alpha(t) = \int_0^t f \,\mathrm{d}x = \int_0^1 u_t f \,\mathrm{d}x.$$

Since F is linear, we also have $F(g_n) = \int_0^1 g_n f \,\mathrm{d}x$ for all step functions $g_n := \sum_{k=1}^{n} c_k\big(u_{\frac{k}{n}} - u_{\frac{k-1}{n}}\big)$.

Let g be a bounded measurable function on $[0,1]$. Then there is a sequence of step functions g_n such that $g_n \to g$ a.e. and $\{g_n\}$ is uniformly bounded. By the Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} F(g_n) = \lim_{n\to\infty} \int_0^1 g_n f \,\mathrm{d}x = \int_0^1 \lim_{n\to\infty} g_n f \,\mathrm{d}x = \int_0^1 g f \,\mathrm{d}x.$$

On the other hand, since $g_n \to g$ a.e. and g_n are uniformly bounded, the same theorem implies $\|g_n - g\|_{L_p} \to 0$ as $n \to \infty$. By the continuity of F on L_p, we thus have $F(g) = \lim\limits_{n\to\infty} F(g_n) = \int_0^1 g f \,\mathrm{d}x$. Hence $F(g) = \int_0^1 g f \,\mathrm{d}x$ for every bounded measurable function g on $[0,1]$.

We will show that $f \in L_q$ and $\|f\|_q \le \|F\|$. Consider a family of functions g_n defined by

$$g_n(x) = \begin{cases} |f(x)|^{q-1} \operatorname{sign}\big(f(x)\big) & \text{if } |f(x)| \le n,\\ 0 & \text{if } |f(x)| > n.\end{cases}$$

The functions g_n are bounded and measurable. Thus we have $F(g_n) = \int_0^1 g_n f \,\mathrm{d}x$. Note also that $|F(g_n)| \le \|F\|\,\|g_n\|_p$. On the other hand,

$$\int_0^1 |g_n|^p \,\mathrm{d}x = \int_0^1 |g_n|^{\frac{q}{q-1}} \,\mathrm{d}x = \int_0^1 |g_n(t)|\,|g_n(t)|^{\frac{1}{q-1}} \,\mathrm{d}x$$
$$\leq \int_0^1 |g_n|\,|f| \,\mathrm{d}x = \int_0^1 g_n f \,\mathrm{d}x = F(g_n) = |F(g_n)|.$$

Hence $\int_0^1 |g_n|^p \,\mathrm{d}x \leq \|F\| \cdot \|g_n\|_p = \|F\|\Big(\int_0^1 |g_n|^p \,\mathrm{d}x\Big)^{\frac{1}{p}}$, so $\Big(\int_0^1 |g_n|^p \,\mathrm{d}x\Big)^{\frac{1}{q}} \leq \|F\|$.

Since f is integrable, we have $|g_n| \to |f|^{q-1}$ a.e. By Fatou's lemma, the last inequality implies that

$$\Big(\int_0^1 |f|^q \,\mathrm{d}x\Big)^{\frac{1}{q}} = \Big(\int_0^1 |f|^{(q-1)p} \,\mathrm{d}x\Big)^{\frac{1}{q}} = \Big(\int_0^1 |g_n|^p \,\mathrm{d}x\Big)^{\frac{1}{q}} \leq \|F\|.$$

This shows that $f \in L_q$. Finally, let $g \in L_p$. There exists a sequence $\{g_n\}$ of bounded measurable functions that converges to g in L_p. Then $F(g_n) \to F(g)$ and by Hölder's inequality (1.1) we have $\int_0^1 g_n f \,\mathrm{d}x \to \int_0^1 gf \,\mathrm{d}x$. We have shown that $F(g_n) = \int_0^1 g_n f \,\mathrm{d}x$ for bounded measurable functions, so $F(g) = \int_0^1 gf \,\mathrm{d}x$ as claimed.

On the other hand, given a function $f \in L_q$, we can define a linear functional on L_p by $F(g) = \int_0^1 gf \,\mathrm{d}x$. It follows from the Hölder inequality (1.1) that F is continuous and $\|F\| \leq \|f\|_q$. □

Using similar methods, we obtain an analogous result for the space L_1.

Proposition 2.19 (Riesz) $L_1[0,1]^* = L_\infty[0,1]$ *in the sense that for every* $F \in L_1^*$ *there exists a unique* $f \in L_\infty$ *such that* $F(g) = \int_0^1 gf \,\mathrm{d}x$ *for all* $g \in L_1$*, and the mapping* $F \mapsto f$ *is a linear isometry of* L_1^* *onto* L_∞*.*

Proposition 2.20 (Riesz) *For every* $F \in C[0,1]^*$ *there exists a function* f *on* $[0,1]$ *with bounded variation such that* $F(g) = \int_0^1 g \,\mathrm{d}f$ *(Stieltjes integral) for all* $g \in C[0,1]$ *and* $\|F\| = \bigvee_0^1 f$*, where* $\bigvee_0^1 f$ *denotes the* variation *of* f *on* $[0,1]$.
On the other hand, if f *is a function of bounded variation on* $[0,1]$*, then* $F(g) := \int_0^1 g \,\mathrm{d}f$ *is a continuous linear functional on* $C[0,1]$.

Proof: Consider the space $\ell_\infty[0,1]$ of bounded functions on $[0,1]$ with the supremum norm denoted by $\|\cdot\|_\infty$. If $F \in C[0,1]^*$, we have that $|F(g)| \leq \|F\| \cdot \|g\|_\infty$ for every $g \in C[0,1]$. Since $C[0,1]$ is a subspace of $\ell_\infty[0,1]$, by the Hahn–Banach theorem we can extend F to a functional $\widetilde{F}$ on $\ell_\infty[0,1]$ such that $|\widetilde{F}(g)| \leq \|F\| \cdot \|g\|_\infty$. We will represent $\widetilde{F}$ similarly to the L_p setting above. For $t \in [0,1]$, let $u_t = \chi_{[0,t)}$, the characteristic function of $[0,t)$. Put $f(t) = \widetilde{F}(u_t)$ for $t \in [0,1]$ (note that F is not defined on u_t as u_t is not continuous). We will prove that f has bounded variation on $[0,1]$. To this end, consider $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n = 1$ and put $\varepsilon_i = \operatorname{sign}(f(t_i) - f(t_{i-1}))$. We have

$$\sum_{i=1}^{n} |f(t_i) - f(t_{i-1})| = \sum_{i=1}^{n} \varepsilon_i (f(t_i) - f(t_{i-1})) = \widetilde{F}\Big(\sum_{i=1}^{n} \varepsilon_i (u_{t_i} - u_{t_{i-1}})\Big)$$
$$\leq \|\widetilde{F}\| \cdot \Big\|\sum_{i=1}^{n} \varepsilon_i (u_{t_i} - u_{t_{i-1}})\Big\|_\infty = \|F\| \cdot 1.$$

Hence f has bounded variation on $[0, 1]$ which is bounded by $\|F\|$.

For $g \in C[0, 1]$ and $g_n := \sum_{i=1}^{n} g\big(\frac{k}{n}\big)\big(u_{\frac{k}{n}} - u_{\frac{k-1}{n}}\big)$ we have

$$\widetilde{F}(g_n) = \sum_{i=1}^{n} g\big(\tfrac{k}{n}\big)\Big(f\big(\tfrac{k}{n}\big) - f\big(\tfrac{k-1}{n}\big)\Big) = \int_0^1 g_n \, \mathrm{d}f.$$

Therefore $\lim\limits_{n\to\infty} \widetilde{F}(g_n) = \lim\limits_{n\to\infty} \sum_{k=1}^{n} g(\frac{k}{n})\big(f(\frac{k}{n}) - f(\frac{k-1}{n})\big) = \int_0^1 g \, \mathrm{d}f$. Since $\widetilde{F} \in \ell_\infty[0, 1]^*$ and $g_n \to g$ in $\|\cdot\|_\infty$, we have $\lim \widetilde{F}(g_n) = \widetilde{F}(g)$, so $\widetilde{F}(g) = \int_0^1 g \, \mathrm{d}f$. However, for $g \in C[0, 1]$ we have $\widetilde{F}(g) = F(g)$, hence $F(g) = \int_0^1 g(t) \, \mathrm{d}f(t)$.

We have already shown that $\bigvee\limits_0^1 f \leq \|F\|$. On the other hand, from the theory of Riemann–Stieltjes integral we have that given a function f of bounded variation, $F : g \mapsto \int_0^1 g \, \mathrm{d}f$ is a linear mapping and $\int_0^1 g(t) \, \mathrm{d}f(t) \leq \|g\|_\infty \bigvee\limits_0^1 f$. Therefore F is continuous and $\|F\| \leq \bigvee\limits_0^1 f$. □

In general, if K is a compact set, the space $C(K)^*$ can be identified with the space of all regular Borel measures on K of bounded variation. Every such measure μ defines a functional $F_\mu(f) := \int_K f \, \mathrm{d}\mu$, the correspondence $\mu \mapsto F_\mu$ is a linear isometry ([Rudi2, Theorem 2.14]).

Let $k \in K$. We define the corresponding *Dirac functional* (or *Dirac measure*) by $\delta_k(f) = f(k)$ for every $f \in C(K)$. Observe that δ_k is a continuous linear functional of norm one. Indeed, on one hand, $\|\delta_k\| = \sup\limits_{\|f\|\leq 1} \big(\delta_k(f)\big) = \sup\limits_{\|f\|\leq 1} \big(f(k)\big) \leq 1$. By considering the constant function $f = 1$, we obtain $\|\delta_k\| = 1$.

Proposition 2.21 *The space $C[0, 1]^*$ is not separable.*

Proof: Consider the Dirac measures δ_t for $t \in [0, 1]$. We claim that if $t_1 \neq t_2$ then $\|\delta_{t_1} - \delta_{t_2}\| = 2$. Indeed, $\|\delta_{t_1} - \delta_{t_2}\| \leq \|\delta_{t_1}\| + \|\delta_{t_2}\| = 2$. On the other hand, choose $f_0 \in C[0, 1]$ such that $f_0(t_1) = 1$, $f_0(t_2) = -1$, and $\|f_0\|_\infty = 1$. Then $\|\delta_{t_1} - \delta_{t_2}\| \geq |f_0(t_1) - f_0(t_2)| = 2$. Similarly to the case of ℓ_∞ we find that $C[0, 1]^*$ is not separable. □

Recall that the inner product on a complex Hilbert space ℓ_2, respectively L_2, is defined by $\big((x_i), (y_i)\big) = \sum x_i \bar{y}_i$, respectively $(g, f) = \int_0^1 g \bar{f} \, \mathrm{d}x$. This motivates the following identification of the dual space in case of *complex scalars*. Recall that a mapping Φ is called *conjugate linear* if $\Phi(\alpha x + y) = \bar{\alpha}\Phi(x) + \Phi(y)$ for all vectors x, y and scalars α.

Theorem 2.22 (Riesz) *Let H be a Hilbert space. For every $f \in H^*$ there is a unique $a \in H$ such that $f(x) = (x, a)$ for all $x \in H$. The mapping $f \mapsto a$ is a conjugate-linear isometry of H^* onto H.*

Proof: The uniqueness of such a is clear. Indeed, if $f(x) = (x, a_1) = (x, a_2)$ then using $x = a_1 - a_2$ we get $(a_1 - a_2, a_1) = (a_1 - a_2, a_2)$. Thus $(a_1 - a_2, a_1 - a_2) = 0$, so $a_1 = a_2$.

By the Cauchy–Schwarz inequality,

$$\|f\| = \sup_{\|x\|\le 1} |f(x)| = \sup_{\|x\|\le 1} |(x, a)| \le \sup_{\|x\|\le 1} \big(\|a\| \cdot \|x\|\big) \le \|a\|.$$

On the other hand, $\|f\| = \sup\{|f(x)| : \ \|x\| \le 1\} \ge (a/\|a\|, a) = \|a\|$. Hence $\|f\| = \|a\|$.

To obtain the representation of $0 \ne f \in H^*$, consider $N := \operatorname{Ker}(f)$. It is a proper closed subspace of H. Choose $z_0 \in N^{\perp}$ and assume without loss of generality that $f(z_0) = 1$.

We claim that $H = N \oplus \operatorname{span}\{z_0\}$. Indeed, given $h \in H$, it suffices to find a scalar α such that $h - \alpha z_0 \in N$, that is, $f(h - \alpha z_0) = 0$. This is satisfied for $\alpha = f(h)$.

We now show that $f(x) = \left(x, \frac{z_0}{\|z_0\|^2}\right)$ for every $x \in H$. Given $x := y + \alpha z_0$, where $y \in N$ and α is a scalar, we have

$$\begin{aligned} f(x) &= \alpha f(z_0) = \alpha = \alpha(z_0, z_0)/\|z_0\|^2 \\ &= (y, z_0)/\|z_0\|^2 + (\alpha z_0, z_0)/\|z_0\|^2 = \left(x, \frac{z_0}{\|z_0\|^2}\right). \end{aligned}$$

□

2.3 Banach Open Mapping Theorem, Closed Graph Theorem, Dual Operators

Definition 2.23 *Let φ be a mapping from a topological space X into a topological space Y. We say that φ is an* open mapping *if it maps open sets in X onto open sets in Y.*

Let T be an operator from a normed space X into a normed space Y. Observe that if T is an open mapping, then T is necessarily onto. Indeed, by Exercise 2.37, $\delta B_Y \subset T(B_X)$ for some $\delta > 0$ and hence by linearity, $Y \subset T(X)$. We will now establish the converse for bounded operators.

By $B_X^O(r)$ we denote the open ball with radius r centered at the origin of a Banach space X.

Lemma 2.24 (Banach) *Let X be a Banach space, Y a normed space and $T \in \mathcal{B}(X, Y)$. If $r, s > 0$ satisfy $B_Y^O(s) \subset \overline{T(B_X^O(r))}$, then $B_Y^O(s) \subset T(B_X^O(r))$.*

Proof: By considering $\frac{r}{s}T$ if necessary, we may assume that $r = s = 1$. Denote $B_X^O = B_X^O(1)$ and $B_Y^O = B_Y^O(1)$. Let $z \in B_Y^O$ be given. Choose $\delta > 0$ such that $\|z\|_Y < 1 - \delta < 1$ and put $y = (1-\delta)^{-1}z$. Note that $\|y\|_Y < 1$. We will show that $y \in (1-\delta)^{-1}T(B_X^O)$, which implies that $z \in T(B_X^O)$.

We start with $y_0 = 0$ and inductively find a sequence $y_n \in Y$ such that $\|y - y_n\|_Y < \delta^n$ and $(y_n - y_{n-1}) \in T(\delta^{n-1}B_X^O)$. Indeed, having chosen $y_0, y_1, \dots, y_{n-1} \in Y$, we have $(y - y_{n-1}) \in \delta^{n-1}B_Y^O \subset \overline{T(\delta^{n-1}B_X^O)}$, hence there is $w \in T(\delta^{n-1}B_X^O)$ such that $\|w - (y - y_{n-1})\|_Y < \delta^n$. Setting $y_n = y_{n-1} + w$ we complete the construction.

Next we find a sequence $\{x_n\}_{n=1}^\infty \subset X$ such that $\|x_n\|_X < \delta^{n-1}$ and $T(x_n) = y_n - y_{n-1}$ for $n \in \mathbb{N}$. Since the series $\sum x_i$ is absolutely convergent, we put $x = \sum_{n=1}^\infty x_n$. Then $\|x\|_X \le \sum_{n=1}^\infty \|x_n\|_X < \sum_{n=1}^\infty \delta^{n-1} = \frac{1}{1-\delta}$ and by the continuity and linearity of T,

$$T(x) = \lim_{N\to\infty} \sum_{n=1}^{N} T(x_n) = \lim_{N\to\infty} \sum_{n=1}^{N} (y_n - y_{n-1}) = \lim_{N\to\infty} y_N = y.$$

□

Note that $\overline{T(B_X^O(r))} = \overline{T(B_X(r))}$, so the conclusion of the lemma is true if we assume for instance $\delta B_Y \subset \overline{T(B_X)}$.

Theorem 2.25 (Banach open mapping principle) *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is onto Y then T is an open mapping.*

Proof: Put $G = T(B_X^O)$. Since T is linear, we only need to prove that G contains a neighborhood of the origin. Note that we have $T(B_X^O(r)) = rG$ and $\overline{rG} = r\overline{G}$ for every $r > 0$. Therefore $\overline{T(B_X^O(r))} = r\overline{G}$ for every $r > 0$. This implies that $Y = T(X) = \bigcup_{n=1}^{\infty} n\overline{G}$. By the Baire category theorem, there is $n \in \mathbb{N}$ such that $n\overline{G}$ contains an interior point, so there is $x_0 \in \overline{G}$ and $\delta > 0$ such that $\big(x_0 + B_Y^O(\delta)\big) \subset n\overline{G}$. Since $n\overline{G}$ is symmetric, we have $\big(-x_0 + B_Y^O(\delta)\big) \subset n\overline{G}$. If $x \in B_Y^O(\delta)$ then from the convexity of $n\overline{G}$ we have $x = \frac{1}{2}(x_0 + x) + \frac{1}{2}(-x_0 + x) \in n\overline{G}$. Therefore $B_Y^O(\delta) \subset \overline{T(B_X^O(n))}$ and consequently $B_Y^O\left(\frac{\delta}{n}\right) \subset \frac{1}{n}\overline{T(B_X^O(n))} = \overline{T(B_X^O)}$. By Lemma 2.24, we have $B_Y^O\left(\frac{\delta}{n}\right) \subset T(B_X^O)$ as claimed. □

It follows from the proof that if $T\colon X \to Y$ is onto, then there is $\delta > 0$ such that $\delta B_Y \subset T(B_X)$.

Note that even if $T \in \mathcal{B}(X, Y)$ is open, it does not imply that $T(M)$ is closed in Y whenever M is closed in X (Exercise 15.11).

In Exercise 2.33, a rewording of the proof of the Banach open mapping principle in the language of convex series is presented.

Corollary 2.26 *Let X, Y be Banach spaces and let $T \in \mathcal{B}(X, Y)$ be onto Y.*
(i) *If T is one-to-one, then T^{-1} is a bounded operator.*
(ii) *There is a constant $M > 0$ such that for every $y \in Y$ there is $x \in T^{-1}(y)$ satisfying $\|x\|_X \leq M\|y\|_Y$.*
(iii) *Y is isomorphic to $X/\operatorname{Ker}(T)$.*

Proof: (i) If O is open in X, then $(T^{-1})^{-1}(O) = T(O)$ is open in Y showing that T^{-1} is continuous.

(ii) By the open mapping theorem, there is $\delta > 0$ such that $\delta B_Y \subset T(B_X)$. Therefore for every $y \in Y$ such that $\|y\|_Y = \delta$, there is $x \in B_X$ such that $T(x) = y$. Thus it is enough to put $M = 1/\delta$.

(iii) Define a linear mapping $\widehat{T}$ from $X/\operatorname{Ker}(T)$ onto Y by $\widehat{T}(\hat{x}) = T(x)$ for $x \in \hat{x}$. The mapping $\widehat{T}$ is well defined. Moreover $\widehat{T}$ is one-to-one and onto Y. Let $\hat{x}_n \to 0$. Then there is $x_n \in \hat{x}_n$ such that $\|x_n\|_X < \|\hat{x}_n\| + 1/n$ and therefore $x_n \to 0$. Since T is continuous, we have $T(x_n) \to 0$ and thus $\widehat{T}(\hat{x}_n) \to 0$. Hence $\widehat{T}$ is continuous and one-to-one, so by (i) it is an isomorphism of $X/\operatorname{Ker}(T)$ onto Y. □

Theorem 2.27 (Banach closed graph theorem) *Let X, Y be Banach spaces and let T be an operator from X into Y. T is a bounded operator if and only if its graph $G := \{(x, T(x)) : x \in X\}$ is closed in $X \oplus Y$.*

Recall that the norm on $X \oplus Y$ is defined by $\|(x, y)\| = \|x\|_X + \|y\|_Y$. In particular, $(x_n, y_n) \to (x, y)$ if and only if $x_n \to x$ and $y_n \to y$ (see Definition 1.33).

Note that G, the graph of T, is a subspace of $X \oplus Y$.

Proof: If T is continuous and $(x_n, T(x_n)) \to (x_0, y_0)$, then $y_0 = T(x_0)$. Indeed, we have $x_n \to x_0$ and $T(x_n) \to y_0$, while the continuity of T implies that $T(x_n) \to T(x_0)$. This means that (x_0, y_0) $\big(= (x_0, T(x_0))\big)$ is in the graph of T, showing that G is closed.

If G is closed in $X \oplus Y$, then G is a Banach space in the norm induced from $X \oplus Y$. Consider the mapping $p\colon G \to X$ defined by $p(x, T(x)) = x$. By the definition of the norm in $X \oplus Y$ we see that p is continuous, maps G onto X, and is one-to-one. By Corollary 2.26, $p^{-1}\colon x \mapsto (x, T(x))$ is a continuous mapping from X onto G. Since also $q\colon X \oplus Y \to Y$, $q(x, y) := y$, is continuous and $T = q \circ p^{-1}$, T must be continuous. □

Definition 2.28 *Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. We define the* dual *(also called* adjoint*) operator $T^* \in \mathcal{B}(Y^*, X^*)$ for $f \in Y^*$ by $\big(T^*(f)\big)(x) = f\big(T(x)\big)$, for all $x \in X$.*

It is easy to observe that $x \mapsto f\big(T(x)\big)$ is a linear mapping. If $\|x\| \leq 1$ then $|T^*(f)(x)| = |f\big(T(x)\big)| \leq \|f\|\,\|T\|$. Thus $T^*(f)$ is also bounded, so $T^*(f) \in Y^*$ and T^* is well defined. Also the mapping $f \mapsto T^*(f)$ is linear and the above estimate shows that $\|T^*(f)\| \leq \|T\|\,\|f\|$. Consequently, T^* is a bounded operator from X^* into Y^*.

Proposition 2.29 *Let X, Y be Banach spaces. If $T \in \mathcal{B}(X, Y)$ then $\|T^*\| = \|T\|$.*

Proof: We have

$$\|T^*\| = \sup_{f \in B_{Y^*}} \|T^*(f)\|_{X^*} = \sup_{f \in B_{Y^*}} \Big\{ \sup_{x \in B_X} |T^*(f)(x)| \Big\}$$
$$= \sup_{f \in B_{Y^*}} \{ \sup_{x \in B_X} |f\big(T(x)\big)| \} = \sup_{x \in B_X} \{ \sup_{f \in B_{Y^*}} |f\big(T(x)\big)| \} = \sup_{x \in B_X} \{\|T(x)\|_Y\} = \|T\|.$$

□

Let X, Y, Z be Banach spaces and let $T \in \mathcal{B}(X, Y)$, $S \in \mathcal{B}(Y, Z)$. Then $(ST)^* = T^*S^*$. Indeed, consider $f \in Z^*$. Then for every $x \in X$ we get $(ST)^*(f)(x) = f\big(ST(x)\big) = (S^*f)\big(T(x)\big) = \big(T^*S^*(f)\big)(x)$, so $(ST)^*(f) = (T^*S^*)(f)$.

2.4 Remarks and Open Problems

Remarks

1. We mentioned in Open Problem 1 in Chapter 1 that quasi-Banach spaces behave differently from Banach spaces. This is mainly due to the fact that the Hahn–Banach theorem fails in that context, see [Kalt1].

Open Problems

1. It is an open problem if every infinite-dimensional Banach space has a separable infinite-dimensional quotient, i.e., if for every Banach space X there is an infinite-dimensional separable Banach space Y and a bounded operator from X onto Y. This problem is equivalent to the problem whether in every Banach space X there is an increasing sequence $\{E_n\}_{n=1}^{\infty}$ of distinct closed subspaces such that $\overline{\bigcup_n E_n} = X$ (see, e.g., [Muji] and [HMVZ, Chapter 4]).

Exercises for Chapter 2

2.1 Let X be a real normed space. If f is a linear functional on E that is dominated by a function $p : E \to \mathbb{R}$ (i.e., $f \leq p$), and p is continuous at 0, then f is continuous.
Hint. $-f(x) = f(-x) \leq p(-x)$, hence $-p(-x) \leq f(x) \leq p(x)$ for all $x \in E$. Since this implies that f is continuous at 0, the conclusion follows from Proposition 1.25.

2.2 Let C be a convex symmetric set in a Banach space X. Assume that a linear functional f on X is continuous at 0 when restricted to C. Show that the restriction of f to C is uniformly continuous.
Hint. Given $\varepsilon > 0$, we look for a neighborhood U of the origin in X such that $x, y \in C$ and $x - y \in U$ imply $|f(x - y)| < \varepsilon$. We have $\frac{1}{2}(x - y) \in C$, so by homogeneity of f we only need to find an open ball U centered at 0 such that $|f(w)| < \varepsilon/2$ for point $w \in C \cap U$. Such U exists by the continuity of $f\big|_C$ at 0.

2.3 Show that if X is a finite-dimensional Banach space, then every linear functional f on X is continuous on X.
Hint. Use Proposition 1.39.

2.4 Show that if X is an infinite-dimensional normed space, then X admits a discontinuous linear functional.
Hint. Let $\{e_\gamma\}$ be a Hamel basis formed by vectors of norm 1. Define a linear functional f on $\{e_\gamma\}$ so that the set $\{f(e_\gamma)\}$ is unbounded, and extend f on X linearly. Then f is not bounded on the unit ball.

2.5 Show that if $f \neq 0$ is a linear functional on a normed space X, then the codimension of $f^{-1}(0)$ in X is 1.
Hint. For $x \in X$ write $x = (x - (f(x)/f(x_0))x_0) + (f(x)/f(x_0))x_0$, where x_0 is some fixed element in X with $f(x_0) \neq 0$.

2.6 Recall that by a *hyperplane* of a normed space X we mean any subspace Y of codimension 1 (that is, $\dim(X/Y) = 1$).

Let Y be a subspace of a normed space X. Show that Y is a hyperplane if and only if there is a linear functional f, $f \neq 0$, such that $Y = f^{-1}(0)$. Show that Y is a closed hyperplane if and only if there is $f \in X^*$, $f \neq 0$, such that $Y = f^{-1}(0)$.
Hint. One direction: Exercise 2.5. Given a closed hyperplane Y, take $e \notin Y$, use Proposition 2.7 to find f. Then $Y \subset f^{-1}(0)$ and since $\mathrm{codim}(Y) = 1$, equality follows. For a general hyperplane, the proof is similar.

2.7 Let H be a hyperplane in a normed space X, and let F be a two-dimensional subspace of X. Show that $\dim(F \cap H) \geq 1$.
Hint. Use algebraic complementability of H in X.

2.8 Let H be a closed hyperplane of a Banach space X. Let $x_0 \in X \backslash H$. Prove that there is a linear and continuous projection P from X onto H *parallel* to x_0, i.e., such that $Px_0 = 0$ (for a more precise statement, see Exercise 5.7).
Hint. Exercise 2.6 gives $f \in X^*$ such that $\mathrm{Ker}\, f = H$. By scaling we may assume that $f(x_0) = 1$. Let $P : X \to X$ be defined by $P(x) = x - f(x)x_0$. This is the sought projection.

2.9 Let X be a Banach space. Show that all closed hyperplanes of X are mutually isomorphic. By induction we get that given $k \in \mathbb{N}$, all closed subspaces of X of

codimension k are mutually isomorphic. In fact, Zippin proved that the Banach–Mazur distance of two hyperplanes of the same infinite-dimensional Banach space is less than or equal to 25, see [AlKa, p. 238].
Hint. Let F and G to distinct closed hyperplanes of X. According to Exercise 2.6, there exists $f, g \in X^*$ such that $F := f^{-1}(0)$ and $G := g^{-1}(0)$. Find $e \in X$ such that $f(e) = g(e) = 1$. Let P_f (resp., P_g) be the (linear and continuous) projection of X onto F (resp., onto G) parallel to e (see Exercise 2.8). Clearly, $P_f \circ P_g(x) = x$ for every $x \in F$, and $P_g \circ P_f(y) = y$ for every $y \in G$. From this it follows that $P_g\big|_F : F \to G$ is an isomorphism. See Fig. 2.2.

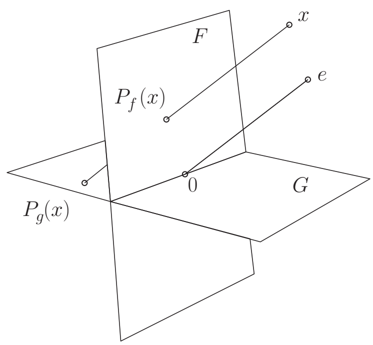


Fig. 2.2 All hyperplanes are mutually isomorphic

2.10 Let f be a linear functional on a Banach space X. Show that if f is not identically 0, the following are equivalent (see also Proposition 3.19):
 (i) f is continuous.
 (ii) f is continuous at 0.
 (iii) $f^{-1}(0)$ is closed.
 (iv) $f^{-1}(0)$ is not dense in X.
Hint. (i)$\Longrightarrow$(ii) is obvious, and (ii)$\Longrightarrow$(i) is clear from the linearity of f. (i)$\Longrightarrow$(iii) is clear. (iii)$\Longrightarrow$(iv) is obvious. (iv)$\Longrightarrow$(iii) follows from the fact that if $f^{-1}(0)$ is not closed, then $f^{-1}(0) \subsetneqq \overline{f^{-1}(0)} \subset X$ and $\overline{f^{-1}(0)}$ is a linear subspace. It is enough to use now Exercise 2.5. (iii)$\Longrightarrow$(i): Since $f^{-1}(\mathbb{R}\setminus\{0\}) \neq \emptyset$ is open, there is some ball $B = x_0 + \delta B_X$ such that $f\big|_B \neq 0$. Assume $f(x_0) > 0$, then also $f\big|_B > 0$ (connect x_0 with points of B, $f \neq 0$ on the connecting segments and f is continuous on each of those segments). Then $f\big|_{B_X} \geq -\frac{1}{\delta} f(x_0)$, so by symmetry of B_X we get $|f(x)| \leq \frac{1}{\delta} f(x_0)$ for $x \in B_X$ and f is continuous.

Another related approach is the following: two subspaces A and B of a normed space X form an (algebraic) *direct sum decomposition* of X (written $X = A \oplus B$) if $A \cap B = \{0\}$ and $A + B = X$ (see the paragraph prior to Definition 1.33). Prove first that if f is a non-zero linear functional and $x_0 \in X$ such that $f(x_0) \neq 0$, and $K := f^{-1}(0)$, then $X = K \oplus \operatorname{span}\{x_0\}$ (see Exercise 2.5). As a consequence, no subspace S of X exists such that $K \subsetneqq S \subsetneqq X$. Since $K \subset \overline{K} \subset X$, and the closure of a subspace is also a subspace (see Exercise 1.9) it follows that K is dense if it is not closed (the converse being trivially true). To finish the exercise, notice that if K is closed, then X/K is then a one-dimensional space. Let $\hat{f} : X/K \to \mathbb{K}$ a linear functional such that $\hat{f} \circ q = f$, where $q : X \to X/K$ is the canonical quotient mapping (see Exercise 2.35). Apply now Exercise 2.3.

2.11 Find a discontinuous linear mapping T from some Banach space X into X such that $\mathrm{Ker}(T)$ is closed.
Hint. Let $X = c_0$ and $T(x) = (f(x), x_1, x_2, \dots)$ for $x = (x_i)$, where f is a discontinuous linear functional on X.

2.12 Let X be a Banach space, $f \in S_{X^*}$. Show that for every $x \in X$ we have $\mathrm{dist}\big(x, f^{-1}(0)\big) = |f(x)|$.
Hint. The result is obviously true if $f = 0$. If not, put $K = f^{-1}(0)$. There exists, by Proposition 2.7, $g \in S_{X^*}$ that vanishes on K and $g(x) = \mathrm{dist}(x, K)$. Since $g^{-1}(0) = f^{-1}(0)$, we get $g = \lambda f$ for some scalar λ (see Exercise 2.5), and $|\lambda| = 1$ since both f and g belong to S_{X^*}. This proves the assertion.

2.13 (The “parallel-hyperplane lemma".) Let X be a real Banach space, $f, g \in S_{X^*}$ and $\varepsilon > 0$ be such that $|f(x)| \le \varepsilon$ for every $x \in g^{-1}(0) \cap B_X$. Prove that either $\|f - g\| \le 2\varepsilon$ or $\|f + g\| \le 2\varepsilon$ (see Fig. 2.3).

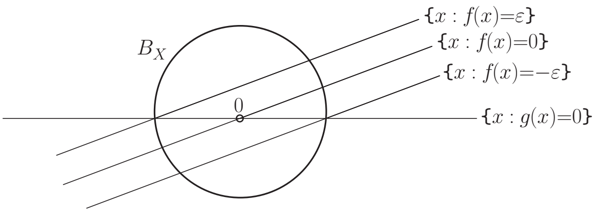


Fig. 2.3 The “parallel-hyperplane lemma”

Hint. Consider f on $g^{-1}(0)$ and extend it with the same norm (at most ε) on X, calling this extension $\tilde{f}$. Then $\tilde{f} - f = 0$ on $g^{-1}(0)$ and thus $\tilde{f} - f = \alpha g$ for some α by Lemma 3.21. Note that $|1 - |\alpha|| = \big|\|f\| - \|f - \tilde{f}\|\big| \le \|\tilde{f}\| \le \varepsilon$. Thus if $\alpha \ge 0$, then $\|g + f\| = \|(1 - \alpha)g + \tilde{f}\| \le |1 - \alpha| + \|\tilde{f}\| \le 2\varepsilon$. If $\alpha < 0$, calculate $\|g - f\|$.

2.14 If X is an infinite-dimensional Banach space, show that there are convex sets C_1 and C_2 such that $C_1 \cup C_2 = X$, $C_1 \cap C_2 = \emptyset$, and both C_1 and C_2 are dense in X.
Hint. Take a discontinuous functional f on X (Exercise 2.4), define $C_1 = \{x : f(x) \ge 0\}$ and $C_2 = \{x : f(x) < 0\}$, use Exercise 2.10.

2.15 Let X be a finite-dimensional Banach space. Let C be a convex subset of X that is dense in X. Prove that $C = X$.

Hint. We may assume that $0 \in C$. Let $\{e_1, e_2, \dots, e_n\}$ be an algebraic basis of X consisting of unit vectors. Fix $\varepsilon > 0$. For each $i \in \{1, 2, \dots, n\}$ we can find $v_i \in C$ such that $\|e_i - v_i\| < \varepsilon$. If $\varepsilon > 0$ is small enough, $\{v_1, \dots, v_n\}$ is a linearly independent set in C (look at the determinant of the matrix with columns v_i, $i = 1, 2, \dots .n$). The set $\operatorname{conv}(\{v_i : i = 1, 2, \dots, n\} \cup \{0\})$ has a nonempty interior and is contained in C, so C has a nonempty interior. If $x_0 \in X \backslash C$, then there exists a closed hyperplane H that separates $\{x_0\}$ and $\operatorname{Int}(C)$, a contradiction with the denseness of C.

2.16 Let N be a maximal ε-separated set in the unit sphere of a Banach space X (see Exercise 1.47). Show that $(1 - \varepsilon)B_X \subset \overline{\operatorname{conv}}(N)$.
Hint. Otherwise, by the separation theorem, we find $x \in X$ and $f \in S_{X^*}$ with $\|x\| \le 1 - \varepsilon$ and $f(x) > \sup_{\overline{\operatorname{conv}}(N)}(f) = \sup_N(f)$. For $\delta > 0$ choose $y \in S_X$ such that $f(y) > 1 - \delta$. By the maximality of N, there exists $z \in N$ with $\varepsilon > \|y - z\| \ge f(y) - f(z)$. Thus $\sup_N(f) \ge f(z) > f(y) - \varepsilon > 1 - \delta - \varepsilon$. This holds for any $\delta > 0$, so we have $1 - \varepsilon \le \sup_N(f) < f(x) \le \|x\| \le 1 - \varepsilon$, a contradiction.

2.17 Let $D = \{\pm e_i : i \in \mathbb{N}\} \subset \ell_2$, where e_i is the ith unit vector. The set $C := \overline{\operatorname{conv}}(D)$ has empty interior, so it coincides with its boundary. Show that 0 is not a support point of C.
Hint. If 0 is supported by some f, prove that f must be 0. That the interior of C is empty follows from the fact that C is the unit ball of ℓ_1 (use Exercise 3.36 or, more generally, Exercise 3.37).

2.18 Let C be a subset of a Banach space X and f be a Lipschitz real-valued function on C. Show that f can be extended to a Lipschitz function on X.
Hint. Assume without loss of generality that f is 1-Lipschitz. Put for $x \in X$,

$$F(x) = \inf\{f(z) + \|z - x\|; z \in C\}.$$

To see that F is finite for every $x \in X$, pick an arbitrary $z_0 \in C$. Then for any $z \in C$,

$$f(z) + \|x - z\| \ge f(z_0) - \|z - z_0\| + \|x - z\| \ge f(z_0) - \|x - z_0\|.$$

Thus

$$F(x) \ge f(z_0) - \|x - z_0\|.$$

If $x \in C$, then for every $z \in C$, $f(x) \le f(z) + \|z - x\|$. Thus $F(x) = f(x)$. To show that F is 1-Lipschitz, pick $x, y \in X$, $\varepsilon > 0$ and choose $z_0 \in C$ so that

$$f(z_0) + \|z_0 - x\| < F(x) + \varepsilon.$$

Then

$$\begin{aligned} F(y) - F(x) &\le F(y) - f(z_0) - \|z_0 - x\| + \varepsilon \\ &\le f(z_0) + \|z_0 - y\| - f(z_0) - \|z_0 - x\| + \varepsilon \\ &\le \|x - y\| + \varepsilon. \end{aligned}$$

In Exercises 2.19, 2.20, 2.21, and 2.22, μ_C denotes the Minkowski functional of a set C.

2.19 Let $(X, \|\cdot\|)$ be a Banach space. Show that $\mu_{B_X}(x) = \|x\|$.
Hint. Use continuity of the norm.

2.20 Let A, B be convex sets in a Banach space X. Show that if $A \subset B$ then $\mu_B \le \mu_A$. Show that $\mu_{cA}(x) = \frac{1}{c}\mu_A(x)$ for $c > 0$.
Hint. Follows from the definition.

2.21 Let C be a convex neighborhood of 0 in a real Banach space X (then μ_C is a non-negative positive homogeneous subadditive continuous functional on X, see Lemma 2.11). Prove the following:

(i) If C is also open, then $C = \{x : \mu_C(x) < 1\}$. If C is closed instead, then $C = \{x : \mu_C(x) \le 1\}$.

(ii) There is $c > 0$ such that $\mu_C(x) \le c\|x\|$.

(iii) If C is moreover symmetric, then μ_C is a continuous seminorm, that is, it is a continuous homogeneous subadditive functional.

(iv) If C is moreover symmetric and bounded, then μ_C is a norm that is equivalent to $\|\cdot\|_X$. In particular, it is complete, that is, (X, μ_C) is a Banach space.

Note that the symmetry condition is good only for the real case. In a complex normed space X we have to replace it by C being *balanced*.

Hint. (i) It follows from Lemma 2.11.

(ii) See Equation (2.2).

(iii) Observing that $\mu_C(-x) = \mu_C(x)$ and positive homogeneity are enough to prove $\mu_C(\lambda x) = |\lambda|\mu_C(x)$ for all $\lambda \in \mathbb{R}$, $x \in X$.

(iv) From (iii) we already have the homogeneity and the triangle inequality. We need to show that $\mu_C(x) = 0$ implies $x = 0$ (the other direction is obvious). Indeed, $\mu_C(x) = 0$ implies that $x \in \lambda C$ for all $\lambda > 0$, which by the boundedness of C only allows for $x = 0$.

In (ii) we proved $\mu_C(x) \le c\|x\|$, an upper estimate follows from $C \subset dB_X$.

The equivalence then implies completeness of the new norm.

2.22 Let K be a bounded closed convex and symmetric set in a Banach space X. Denote by Y the linear hull of K. Let $|||\cdot|||$ on Y be defined as the Minkowski functional of K. Show that $(Y, |||\cdot|||)$ is a Banach space, i.e., K is a *Banach disc*. For an extension of this result to the setting of locally convex spaces and for some of its consequences see Exercises 3.71, 3.72, 3.73, and 3.74.

Hint. If (x_n) is a Cauchy sequence in $(Y, |||\cdot|||)$ it is Cauchy in X and converges, say to x_0, in X. Given a closed ball U in $(Y, |||\cdot|||)$, note that U is closed in X. As (x_n) is Cauchy in $(Y, |||\cdot|||)$, there is $n_0 \in \mathbb{N}$ such that $x_n - x_m \in U$ for all $n, m \geq n_0$. As U is closed in X, $x_n - x_0 \in U$ for $n \geq n_o$ (in particular, $x_0 \in Y$). It follows that $x_n \to x_0$ in $(Y, |||\cdot|||)$.

2.23 Prove that, if $n \in \mathbb{N}$, the dual space of a n-dimensional Banach space is again n-dimensional. Prove that the dual space of an infinite-dimensional normed space is again infinite-dimensional.
Hint. Use Propositions 1.36 and 2.17—this last one for a finite index set. The infinite-dimensional assertion follows from this.

2.24 Show that if Y is a subspace of a Banach space X and X^* is separable then so is Y^*.
Hint. Y^* is isomorphic to the separable space $X^*/Y^\perp$.

2.25 Show that ℓ_1 is not isomorphic to a subspace of c_0.
Hint. The dual of ℓ_1 is nonseparable. Use now Exercise 2.24.

2.26 Show that c_0 is not isomorphic to $C[0, 1]$.
Hint. Check the separability of their duals—Proposition 2.21.

2.27 Let X be a Banach space.
(i) Show that in X^* we have $X^\perp = \{0\}$ and $\{0\}^\perp = X^*$. Show that in X we have $(X^*)_\perp = \{0\}$ and $\{0\}_\perp = X$.
(ii) Let $A \subset B$ be subsets of X. Show that $B^\perp$ is a subspace of $A^\perp$.
Hint. Follows from the definition.

2.28 Let X be a Banach space. Show that:
(i) $\overline{\text{span}}(A) = (A^\perp)_\perp$ for $A \subset X$.
(ii) $\overline{\text{span}}(B) \subset (B_\perp)^\perp$ for $B \subset X^*$. Note that in general we cannot put equality.
(iii) $A^\perp = \big((A^\perp)_\perp\big)^\perp$ for $A \subset X$ and $B_\perp = \big((B_\perp)^\perp\big)_\perp$ for $B \subset X^*$.
Hint. (i) Using definition, show that $A \subset (A^\perp)_\perp$. Then use that $B_\perp$ is a closed subspace for any $B \subset X^*$, proving that $\overline{\text{span}}(A) \subset (A^\perp)_\perp$. Take any $x \notin \overline{\text{span}}(A)$. Since $\overline{\text{span}}(A)$ is a closed subspace, by the separation theorem there is $f \in X^*$ such that $f(x) > 0$ and $f\big|_{\overline{\text{span}}(A)} = 0$. But then $f\big|_A = 0$, hence $f \in A^\perp$, also $f(x) > 0$, so $x \notin (A^\perp)_\perp$.
(ii) Similar to (i).
(iii) Applying (i) to $A^\perp$ we get $A^\perp \subset \big((A^\perp)_\perp\big)^\perp$. On the other hand, using $A \subset (A^\perp)_\perp$ and the previous exercise we get $\big((A^\perp)_\perp\big)^\perp \subset A^\perp$. The dual statement is proved in the same way.

2.29 Let $X = \mathbb{R}^2$ with the norm $\|x\| = (|x_1|^4 + |x_2|^4)^{1/4}$. Calculate directly the dual norm on X^* using the Lagrange multipliers.

Hint. The dual norm of $(a, b) \in X^*$ is $\sup\{ax_1 + bx_2 : x_1^4 + x_2^4 = 1\}$. Define $F(x_1, x_2, \lambda) = ax_1 + bx_2 - \lambda(x_1^4 + x_2^4 - 1)$ and multiply by x_1 and x_2, respectively, the equations you get from $\frac{\partial F}{\partial x_1} = 0$ and $\frac{\partial F}{\partial x_2} = 0$.

2.30 Let Γ be a set and let $p \in [1, \infty)$, $q \in (1, \infty]$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that $c_0(\Gamma)^* = \ell_1(\Gamma)$ and $\ell_p(\Gamma)^* = \ell_q(\Gamma)$.
Hint. See the proofs of Propositions 2.15, and 2.16, 2.17.

2.31 Show that c^* is linearly isometric to ℓ_1.
Hint. We observe that $c = c_0 \oplus \operatorname{span}\{e\}$, where $e := (1, 1, \dots)$ (express $x := (\xi_i) \in c$ in the form $x = \xi_0 e + x_0$ with $\xi_0 := \lim\limits_{i\to\infty} \xi_i$ and $x_0 \in c_0$). If $u \in c^*$, put $v_0' = u(e)$ and $v_i = u(e_i)$ for $i \geq 1$. Then we have $u(x) = u(\xi_0 e) + u(x_0) = \xi_0 v_0' + \sum_{i=1}^{\infty} v_i(\xi_i - \xi_0)$ and $(v_1, v_2, \dots) \in \ell_1$ as in Proposition 2.15. Put $\tilde{u} = (v_0, v_1, \dots)$, where $v_0 := v_0' - \sum_{i=1}^{\infty} v_i$, and write $\tilde{x} := (\xi_0, \xi_1, \dots)$. We have $u(x) = \xi_0 v_0 + \sum_{i=1}^{\infty} v_i \xi_i = \tilde{u}(\tilde{x})$.

Conversely, if $\tilde{u} \in \ell_1$ then the above rule gives a continuous linear functional u on c with $\|u\| \leq \|\tilde{u}\|$, as $|\tilde{u}(\tilde{x})| \leq \left(\sum_{i=0}^{\infty} |v_i|\right) \sup\limits_{i\geq 0} |\xi_i| = \|\tilde{u}\| \sup\limits_{i\geq 0} |\xi_i| = \|\tilde{u}\|_1 \|x\|_\infty$. The inequality $\|\tilde{u}\| \leq \|u\|$ follows like this: Let ξ_i be such that $|v_i| = \xi_i v_i$ if $v_i \neq 0$ and $\xi_i = 1$ otherwise, $i = 0, 1, \dots$. Set $x^n = (\xi_1, \dots, \xi_n, \xi_0, \xi_0, \dots)$. Then $\|x^n\|_\infty = 1$ and $|u(x^n)| = |\tilde{u}(\tilde{x}^n)| \geq |v_0| + \sum_{i=1}^{n} |v_i| - \sum_{i=n+1}^{\infty} |v_i|$. Since $|u(x^n)| \leq \|u\|$, we have $\|u\| \geq |v_0| + \sum_{i=1}^{n} |v_i| - \sum_{i=n+1}^{\infty} |v_i|$. By letting $n \to \infty$ we get $\|\tilde{u}\| \leq \|u\|$.

2.32 Let $p \in (1, \infty)$ and X_n be Banach spaces for $n \in \mathbb{N}$. By $X := \left(\sum X_n\right)_p$ we denote the normed linear space of all sequences $x = \{x_i\}_{i=1}^{\infty}$, $x_i \in X_i$, such that $\sum \|x_i\|_{X_i}^p < \infty$, with the norm $\|x\| := \left(\sum \|x_i\|_{X_i}^p\right)^{\frac{1}{p}}$.

Show that X is a Banach space and that X^* is isometric to $\left(\sum X_i^*\right)_q$ (where $\frac{1}{p} + \frac{1}{q} = 1$) in the following sense: to $f \in X^*$ we assign $\{f_i\}_{i=1}^{\infty}$ such that $f_i \in X_i^*$ and $f\left(\{x_i\}_{i=1}^{\infty}\right) = \sum f_i(x_i)$.

Remark: Sometimes the notation $\sum_{\ell_p} X_n$ will be used instead of $\left(\sum X_n\right)_p$.
Hint. Follow the proof for ℓ_p, which is the case of $X_i = \mathbb{R}$.

2.33 Prove the open mapping theorem by using the concept of convex series (Exercise 1.66) and the Baire category theorem.
Hint. Let $T : X \to Y$ be a bounded linear and onto mapping between Banach spaces. According to Exercise 1.66, B_X is a CS-compact set, so TB_X is again CS-compact, hence CS-closed. Since $Y = \bigcup_{n=1}^{\infty} n\overline{TB_X}$, the Baire category theorem ensures that $\emptyset \neq \operatorname{Int}(\overline{TB_X})$. According to Exercise 1.66, $\operatorname{Int}(TB_X) = \operatorname{Int}(\overline{TB_X})$. Again a "cone argument" (see Exercise 1.55) concludes that $0 \in \operatorname{Int}(TB_X)$, so T is an open mapping.

2.34 We proved the closed graph theorem using the open mapping theorem. Now prove the open mapping principle using the closed graph theorem.

Hint. First prove it for one-to-one mappings using the fact that $\{(y, T^{-1}(y))\}$ is closed. For the general case, note that the quotient mapping is an open mapping by the definition of the quotient topology.

2.35 Let X, Y be normed spaces, $T \in \mathcal{B}(X, Y)$. Show that $\widehat{T}\colon X/\operatorname{Ker}(T) \to Y$ defined by $\widehat{T}(\hat{x}) = T(x)$ is a bounded operator onto $T(X)$.

2.36 (i) Prove directly that if X is a Banach space and f is a non-zero linear functional on X, then f is an open mapping from X onto the scalars.

(ii) Let the operator T from c_0 into c_0 be defined by $T((x_i)) = (\frac{1}{i}x_i)$. Is T a bounded operator? Is T an open map? Does T map c_0 onto a dense subset in c_0?

Hint. (i) If $f(x) = \delta > 0$ for some $x \in B_X^O$, then $(-\delta, \delta) \subset f(B_X^O)$.

(ii) Yes. No. Yes (use finitely supported vectors).

2.37 Let T be an operator (not necessarily bounded) from a normed space X into a normed space Y. Show that the following are equivalent:

(i) T is an open mapping.

(ii) There is $\delta > 0$ such that $\delta B_Y \subset T(B_X)$.

(iii) There is $M > 0$ such that for every $y \in Y$ there is $x \in T^{-1}(y)$ satisfying $\|x\|_X \le M\|y\|_Y$.

Hint. (i)$\Longrightarrow$(ii): $T(B_X^O)$ is open and contains 0; hence it contains a closed ball centered at 0.

(ii)$\Longrightarrow$(iii): Let $0 \neq y \in Y$. We have $\delta\|y\|_Y^{-1}y \in \delta B_Y$ $(\subset T(B_X))$. We can find then $u \in B_X$ such that $\delta\|y\|_Y^{-1}y = Tu$, so $y = T(x)$, where $x := \|y\|_Y\delta^{-1}u$. Certainly $\|x\|_X \le M\|y\|_Y$, where $M := \delta^{-1}$.

(iii)$\Longrightarrow$(i): If $y \in M^{-1}B_Y$ there exists $x \in X$ such that $Tx = y$ and $\|x\|_X \le M\|y\|_Y$ (≤ 1), so $y \in T(B_X)$. This proves that $M^{-1}B_Y \subset T(B_X)$. By linearity, T is open.

2.38 Let X, Y be normed spaces, $T \in \mathcal{B}(X, Y)$. Show that if X is complete and T is an open mapping, then Y is complete.

Hint. Use (iii) in the previous exercise and Exercise 1.26.

2.39 Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Show that if T is one-to-one and $B_Y^O \subset T(B_X) \subset B_Y$, then T is an isometry onto Y.

Hint. Since $B_Y^O \subset T(B_X)$, T is onto (Exercise 2.37) and hence invertible. From $T(B_X) \subset B_Y$ we get $\|T\| \le 1$. Assume that there is $x \in S_X$ such that $\|T(x)\| < \|x\|$. Pick $\delta > 1$ such that $\delta\|T(x)\| < 1$. Then $T(\delta x) \in B_Y^O \subset T(B_X)$. Thus there must be $z \in B_X$ such that $T(z) = T(\delta x)$ but it cannot be $\delta x \notin B_X$, a contradiction with T being one-to-one.

2.40 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Show that the following are equivalent:

(i) $T(X)$ is closed.

(ii) T is an open mapping when considered as a mapping from X onto $T(X)$.

(iii) There is $M > 0$ such that for every $y \in T(X)$ there is $x \in T^{-1}(y)$ satisfying $\|x\|_X \le M\|y\|_Y$.

Hint. (i)$\Longrightarrow$(ii): Theorem 2.25. (ii)$\Longrightarrow$(iii): Exercise 2.37. (iii)$\Longrightarrow$(i): By Exercise 2.37, $T : X \to T(X)$ is an open mapping. Now use Exercise 2.38 and Fact 1.5.

2.41 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Show that if T maps bounded closed sets in X onto closed sets in Y, then $T(X)$ is closed in Y.

Hint. Assume $T(x_n) \to y \notin T(X)$. Put $M = \operatorname{Ker}(T)$, set $d_n = \operatorname{dist}(x_n, M)$ and find $w_n \in M$ such that $d_n \le \|x_n - w_n\| \le 2d_n$. If $\{x_n - w_n\}$ is bounded then $T(x_n - w_n) \to y \in T(X)$, since the closure of $\{x_n - w_n\}$ is mapped onto a closed set containing y, a contradiction. Therefore we may assume that $\|x_n - w_n\| \to \infty$. Since $T(x_n - w_n) \to y$, we have $T(\frac{x_n - w_n}{\|x_n - w_n\|}) \to 0$. By the hypothesis, M must contain a point w from the closure of $\{\frac{x_n - w_n}{\|x_n - w_n\|}\}$ as 0 lies in the closure of the image of this sequence. Fix n so that $\|\frac{x_n - w_n}{\|x_n - w_n\|} - w\| < 1/3$. Then $\big\|x_n - w_n - \|x_n - w_n\|w\big\| \le \frac{1}{3}\|x_n - w_n\| < (2/3)d_n$ and $w_n + \|x_n - w_n\|w \in M$, a contradiction.

2.42 Let X and Y be Banach spaces. Then $\mathcal{K}(X, Y)$ contains isomorphic copies of Y and X^*.

Hint. $T(x) = f^*(x)y$.

2.43 Let X and Y be normed spaces. Prove that $\mathcal{B}(X, Y)$ is an infinite-dimensional space if X is infinite-dimensional and Y is not reduced to $\{0\}$.

Hint. The space $\mathcal{B}(X, Y)$ contains an isometric copy of X^*. Use now Exercise 2.23.

2.44 Let $T \in \mathcal{B}(X, Y)$. Prove the following:

(i) $\operatorname{Ker}(T) = T^*(Y^*)_\perp$ and $\operatorname{Ker}(T^*) = T(X)^\perp$.

(ii) $\overline{T(X)} = \operatorname{Ker}(T^*)_\perp$ and $\overline{T^*(Y^*)} \subset \operatorname{Ker}(T)^\perp$.

Hint. (i) Assume $x \in T^*(Y^*)_\perp$. Then for any $g \in Y^*$ we have $g\big(T(x)\big) = T^*(g)(x) = 0$, hence $T(x) = 0$. Thus $x \in \operatorname{Ker}(T)$.

(ii) $\overline{T(X)} = \overline{\operatorname{span}}\big(T(X)\big) = (T(X)^\perp)_\perp = \operatorname{Ker}(T^*)_\perp$.

2.45 Let X, Y be normed spaces, $T \in \mathcal{B}(X, Y)$. Consider $\widehat{T}(\hat{x}) := T(x)$, where $x \in \hat{x}$, as an operator from $X/\operatorname{Ker}(T)$ into $\overline{T(X)}$. Then we get $\widehat{T}^* \colon \overline{T(X)}^* \to (X/\operatorname{Ker}(T))^*$. Using Proposition 2.6 and $\overline{T(X)}^\perp = T(X)^\perp = \operatorname{Ker}(T^*)$ we may assume that $\widehat{T}^*$ is a bounded operator from $Y^*/\operatorname{Ker}(T^*)$ into $\operatorname{Ker}(T)^\perp \subset X^*$. On the other hand, for $T^* \colon Y^* \to X^*$ we may consider $\widehat{T^*} \colon Y^*/\operatorname{Ker}(T^*) \to X^*$. Show that $\widehat{T}^* = \widehat{T^*}$.

Hint. Take any $\hat{y} \in Y^*/\operatorname{Ker}(T^*)$ and $x \in X$. Then using the above identifications we obtain

$$\widehat{T}^*(\widehat{y^*})(\hat{x}) = \widehat{y^*}\big(\widehat{T}(\hat{x})\big) = y^*\big(T(x)\big) = T^*(y^*)(x) = \widehat{T^*}(\widehat{y^*})(\hat{x}).$$

2.46 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Show that T maps X onto a dense set in Y if and only if T^* maps Y^* one-to-one into X^*.

Also, if T^* maps onto a dense set, then T is one-to-one.

Hint. If $\overline{T(X)} \neq Y$, let $f \in Y^* \setminus \{0\}$ be such that $f = 0$ on $T(X)$. Then $T^*(f) = 0$. The other implications are straightforward.

2.47 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is one-to-one, is T^* necessarily onto?

Hint. No, consider the identity mapping from ℓ_1 into ℓ_2.

2.48 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. If T is an isomorphism into Y, is T^* necessarily an isomorphism into X^*?

Hint. No, embed $\mathbb{R}$ into $\mathbb{R}^2$.

2.49 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Show that:

(i) T^* is onto if and only if T is an isomorphism into Y.

(ii) T is onto if and only if T^* is an isomorphism into X^*.

(iii) $T(X)$ is closed in Y if and only if $T^*(Y^*)$ is closed in X^*.

Hint. (i) If T^* is onto, it is an open mapping (Theorem 2.25) and by Exercise 2.37 there is $\delta > 0$ so that $\delta B_{X^*} \subset T^*(B_{Y^*})$. Then

$$\begin{aligned} \|T(x)\|_Y &= \sup_{y^* \in B_{Y^*}} y^*\big(T(x)\big) = \sup_{y^* \in B_{Y^*}} T^*(y^*)(x) = \sup_{x^* \in T^*(B_{Y^*})} \big(x^*(x)\big) \\ &\geq \sup_{x^* \in \delta B_{X^*}} \big(x^*(x)\big) = \delta \|x\|_X \end{aligned}$$

and use Exercise 1.73.

If T is an isomorphism into, then T^{-1} is a bounded operator from $T(X)$ into X. Given $x^* \in X^*$, define y^* on $T(X)$ by $y^*(y) = x^*\big(T^{-1}(y)\big)$. Clearly $y^* \in T(X)^*$, extend it to a functional in Y^*. Then $T^*(y^*) = x^*$.

(ii) If T is onto, as in (i) we find $\delta > 0$ such that $\delta B_Y \subset T(B_X)$, then $\|T^*(y^*)\|_{X^*} \geq \delta \|y^*\|_{Y^*}$ and use Exercise 1.73.

Assume T^* is an isomorphism into. By Exercise 2.37 and Lemma 2.24, it is enough to find $\delta > 0$ so that $\delta B_Y \subset \overline{T(B_X)}$. Assume by contradiction that no such δ exists. Then find $y_n \to 0$ such that $y_n \notin \overline{T(B_X)}$. The set is closed, so $d_n := \operatorname{dist}(y_n, \overline{T(B_X)}) > 0$.

Fix n, set $V_n = \bigcup\limits_{y \in T(B_X)} \big(y + B_Y^O(\frac{d_n}{2})\big)$. Then V_n is an open convex set and $y_n \notin V_n$, so by Proposition 2.13 there is $y^* \in Y^*$ such that $|y^*| < 1$ on V_n and $y^*(y_n) = 1$. Since $T(B_X) \subset V_n$, we get

$$\|T^*(y^*)\| = \sup_{x \in B_X} T^*(y^*)(x) = \sup_{x \in B_X} y^*\big(T(x)\big) = \sup_{y \in T(B_X)} \big(y^*(y)\big) \leq 1,$$

so $\|y^*\| \leq \|(T^*)^{-1}\| \, \|T^*(y^*)\| \leq \|(T^*)^{-1}\|$, $1 = y^*(y_n) \leq \|(T^*)^{-1}\| \, \|y_n\|$. This shows that $\|y_n\| \geq 1/\|(T^*)^{-1}\|$ for every n, contradicting $y_n \to 0$.

(iii) If $T(X)$ is closed and $q : X \to X/\operatorname{Ker}(T)$ is the canonical quotient mapping, then $\widehat{T}$ such that $\widehat{T} \circ q$, is an operator from $X/\operatorname{Ker}(T)$ onto a Banach space $T(X)$, hence by (ii) above, $\widehat{T}^*$ is an isomorphism into, in particular $\widehat{T}^*(Y^*/\operatorname{Ker}(T^*))$ is closed. By Exercise 2.45, $\widehat{T^*}(Y^*/\operatorname{Ker}(T^*)) = T^*(Y^*)$ is closed.

If $T^*(Y^*)$ is closed, consider $\widehat{T} \colon X \to \overline{T(X)}$. Then $\widehat{T}^*(Y^*/\operatorname{Ker}(T^*)) = \widehat{T^*}(Y^*/\operatorname{Ker}(T^*)) = T^*(Y^*)$ is closed and $\widehat{T}^*$ is one-to-one, hence it is an isomorphism into. By (ii), $\widehat{T}$ must be onto, that is, $T(X) = \overline{T(X)}$.

2.50 Show that there is no $T \in \mathcal{B}(\ell_2, \ell_1)$ such that T is an onto mapping.
Hint. By Exercise 2.49, T^* would be an isomorphism of ℓ_∞ into ℓ_2, which is impossible as ℓ_∞ is nonseparable and ℓ_2 is separable.

2.51 Let X, Y be Banach spaces, $T \in \mathcal{B}(X, Y)$. Show that:

(i) T is an isomorphism of X onto Y if and only if T^* is an isomorphism of Y^* onto X^*.

(ii) T is an isometry of X onto Y if and only if T^* is an isometry of Y^* onto X^*.
Hint. (i) Follows from Exercise 2.49.

(ii) If T is an isometry, then by (i), T^* is an isomorphism. Also $T(B_X) = B_Y$, so $\|T^*(y^*)\| = \sup\limits_{x \in B_X} T^*(y^*)(x) = \|y^*\|$. The other direction is similar.

2.52 We have $\|T\| = \|T^*\|$ for a bounded operator on a Banach space. So, if for a sequence of bounded operators T_n we have $\|T_n\| \to 0$, then $\|T_n^*\| \to 0$. Find an example of a sequence of bounded operators T_n on a Banach space X such that $\|T_n(x)\| \to 0$ for every $x \in X$ but it is not true that $\|T_n^*(x^*)\| \to 0$ for every $x^* \in X^*$.
Hint. Let $T_n(x) = (x_n, x_{n+1}, \dots)$ in ℓ_2. Then $T_n^*(x) = (0, \dots, 0, x_1, x_2, \dots)$, where x_1 is on the nth place.

2.53 Let X be a normed space with two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that X in both of them is a complete space. Assume that $\|\cdot\|_1$ is not equivalent to $\|\cdot\|_2$. Let I_1 be the identity mapping from $(X, \|\cdot\|_1)$ onto $(X, \|\cdot\|_2)$ and I_2 be the identity mapping from $(X, \|\cdot\|_2)$ onto $(X, \|\cdot\|_1)$. Show that neither I_1 nor I_2 are continuous.
Hint. The Banach open mapping theorem.

2.54 Let L be a closed subset of a compact space K. Show that $C(L)$ is isomorphic to a quotient of $C(K)$.
Hint. Let $T \colon C(K) \to C(L)$ be defined for $f \in C(K)$ by $T(f) = f\big|_L$. Then T is onto by Tietze's theorem, use Corollary 2.26.

2.55 Let X, Y be Banach spaces and $T \in \mathcal{B}(X, Y)$. Show that if Y is separable and T is onto Y, then there is a separable closed subspace Z of X such that $T(Z) = Y$.
Hint. Let $\{y_n\}$ be dense in B_Y and take $x_n \in X$ such that $T(x_n) = y_n$ and $\|x_n\| < K$ for some $K > 0$ (Corollary 2.26). Set $Z = \overline{\operatorname{span}}\{x_n\}$, clearly $T(Z) \subset Y$. By density

of $\{y_n\}$, $B_Y^O \subset \overline{T(KB_Z^O)}$, hence by Lemma 2.24 we have $B_Y^O \subset T(KB_Z^O)$. Thus $Y \subset T(Z)$.

2.56 Let Y be a closed subspace of a Banach space X. Assume that X/Y is separable. Denote by q the canonical quotient mapping of X onto X/Y. Show that there is a separable closed subspace $Z \subset X$ such that $q(Z) = X/Y$.
Hint. Apply the previous exercise.

2.57 Let X be a Banach space and let Y be a separable closed subspace of X^*. Then there is a separable closed subspace $Z \subset X$ such that Y is isometric to a subspace of Z^*.
Hint. Let $\{f_n\}$ be dense in S_{Y^*}. For every n, let $\{x_n^k\}_k \subset S_X$ be such that $f_n(x_n^k) \to 1$ as $k \to \infty$. Put $Z = \overline{\operatorname{span}}\{x_n^k : n, k \in \mathbb{N}\}$.

2.58 Let X be the normed space of all real-valued functions on $[0, 1]$ with continuous derivative, endowed with the supremum norm. Define a linear mapping T from X into $C[0, 1]$ by $T(f) = f'$. Show that T has closed graph. Prove that T is not bounded. Explain why the closed graph theorem cannot be used here.
Hint. The graph of T is closed: let $(f_n, f_n') \to (f, g)$ in $X \oplus C[0, 1]$. Then $f_n \to f$ uniformly on $[0, 1]$ and $f_n' \to g$ uniformly. Hence by a standard result of real analysis, $f' = g$.

T is not bounded: use $\{f_n\}$ bounded with $\{f_n'\}$ unbounded. The space in question is not complete.

2.59 Let X be a closed subspace of $C[0, 1]$ such that every element of X is a continuously differentiable function on $[0, 1]$. Show that X is finite-dimensional.
Hint. Let $T\colon X \to C[0, 1]$ be defined for $f \in X$ by $T(f) = f'$. The graph of T is closed (see the previous exercise). Therefore T is continuous by the closed graph theorem.

Thus for some $n \in \mathbb{N}$ we have $\|f'\|_\infty \le n$ whenever $f \in X$ satisfies $\|f\|_\infty \le 1$. Let $x_i = \frac{i}{4n}$ for $i = 0, 1, \dots, 4n$. Define an operator $S\colon X \to \mathbb{R}^{4n+1}$ by $S(f) = \{f(x_i)\}$. We claim that S is one-to-one. It is enough to show that if $\|f\|_\infty = 1$, then for some i, $S(f)(x_i) \neq 0$. Assume that this is not true. If $f(x) = 1$ and $x \in (\frac{i}{4n}, \frac{i+1}{4n})$, then by the Lagrange mean value theorem we have $|f(x) - f(\frac{i}{4n})| = |f'(\xi)||x - \frac{i}{4n}| \le n \cdot \frac{1}{4n}$, a contradiction. Therefore $\dim(X) \le 4n + 1$.

2.60 (Grothendieck) Let X be a closed subspace of $L_2[0, 1]$ whose every element belongs also to $L_\infty[0, 1]$. Show that $\dim(X) < \infty$.
Hint. The identity mapping from X to $(L_\infty[0, 1], \|\cdot\|_\infty)$ has a closed graph, so for some α we get $\|f\|_\infty \le \alpha\|f\|_2$ for every $f \in X$. Let $\{f_1, \dots, f_n\}$ be an orthonormal set in X. For every $x := \{x_1, \dots x_n\} \in \mathbb{C}^n$ we put $f_x = \sum x_k f_k$. Then $|f_x(t)| \le \alpha\|f_x\|_2 \le \alpha\|x\|_2$ for almost all $t \in [0, 1]$ and so if Λ is a countable dense set in $\mathbb{C}^n$, there exists a set of measure zero N such that $|f_x(t)| \le \alpha\|x\|_2$ for every $x \in \Lambda$ and every $t \in [0, 1]\setminus N$. Each mapping $x \mapsto f_x(t)$ from $\mathbb{C}^n$ to $\mathbb{C}$ is linear

and continuous, so $|f_x(t)| \le \alpha\|x\|_2$ for all $x \in \mathbb{C}^n$ and $t \in [0,1]\backslash N$. In particular, $|f_x(t)| \le \alpha$ for $x \in B_{\mathbb{C}^n}$ and $t \in [0,1]\backslash N$. The choice $x := (f_1(t), \ldots, f_n(t))$ gives us $\sum |f_k(t)|^2 \le \alpha^2$. Integration then gives $n = \|\sum f_k\|_2^2 = \sum \int |f_k(t)|^2 \, \mathrm{d}t \le \alpha^2$.

2.61 Show that the bounded linear one-to-one mapping ϕ from $L_1[0, 2\pi]$ into c_0 defined by $T(f) = \hat{f}(n)$, where $\hat{f}(n)$ are Fourier coefficients of f, is not onto c_0.
Hint. If T were onto c_0, then by the Banach open mapping theorem, T^{-1} would be bounded, which is not the case as the sequence $\{\chi_{\{1,\ldots,n\}}\}$ shows (note that we have $\|D_n\|_1 \to \infty$, where D_n is the Dirichlet kernel).

2.62 Show that there is a linear functional L on ℓ_∞ with the following properties:

(1) $\|L\| = 1$,

(2) if $x := (x_i) \in c$, then $L(x) = \lim\limits_{i\to\infty} x_i$,

(3) if $x := (x_i) \in \ell_\infty$ and $x_i \ge 0$ for all i, then $L(x) \ge 0$,

(4) if $x := (x_i) \in \ell_\infty$ and $x' = (x_2, x_3, \ldots)$, then $L(x) = L(x')$.

This functional is called a *Banach limit* or a *generalized limit*.

Hint. We propose several approaches.

(a) For simplicity we consider only the real scalars setting. Let M be the subspace of ℓ_∞ formed by elements $x - x'$ for $x \in \ell_\infty$ and x' as above. Let 1 denote the vector $(1, 1, \ldots)$. We claim that $\mathrm{dist}(1, M) = 1$. Note that $0 \in M$ and thus $\mathrm{dist}(1, M) \le 1$. Let $x \in \ell_\infty$. If $(x - x')_i \le 0$ for any of i then $\|1-(x-x')\|_\infty \ge 1$. If $(x-x')_i \ge 0$ for all i, then $x_i \ge x_{i+1}$ for all i, meaning that $\lim x_i$ exists. Therefore $\lim(x_i - x_i') = 0$ and thus $\|1 - (x - x')\| \ge 1$.

By the Hahn–Banach theorem, there is $L \in \ell_\infty^*$ with $\|L\| = 1$, $L(1) = 1$, and $L(m) = 0$ for all $m \in M$. This functional satisfies (1) and (4). To prove (2), it is enough to show that $c_0 \subset L^{-1}(0)$. To see this, for $x \in \ell_\infty$ we inductively define $x^{(1)} = x'$ and $x^{(n+1)} = (x^{(n)})'$ and note that by telescopic argument we have $x^{(n)} - x \in M$. Hence $L(x) = L(x^{(n)})$ for every $x \in \ell_\infty$ and every n. If $x \in c_0$ then $\|x^{(n)}\| \to 0$ and thus $L(x) = 0$. To show (3), assume that for some $x = (x_n)$ we have $x_i \ge 0$ for all i and $L(x) < 0$. By scaling, we may assume that $1 \ge x_i \ge 0$ for all i. Then $\|1 - x\|_\infty \le 1$ and $L(1 - x) = 1 - L(x) > 1$, a contradiction with $\|L\| = 1$.

(b) For $x := (x_i) \in \ell_\infty$, $k \in \mathbb{N}$ and $n_1 < \ldots < n_k$ in $\mathbb{N}$, put $\pi(x; n_1, \ldots, n_k) = \limsup_n \frac{1}{k} \sum_{i=1}^k x_{n+n_i}$ and $p(x) = \inf\{\pi(x; n_1, \ldots, n_k) : n_1 < \ldots < n_k,\ k \in \mathbb{N}\}$. Then p is a convex function on ℓ_∞. Deduce the existence of a continuous linear functional $L : \ell_\infty \to \mathbb{R}$ such that $L \le p$ and check the sought properties of L.

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