

Preface

Many problems in modern linear and nonlinear analysis are of infinite-dimensional nature. The theory of Banach spaces provides a suitable framework for the study of these areas, as it blends classical analysis, geometry, topology, and linearity. This in turn makes Banach space theory a wonderful and active research area in Mathematics.

In infinite dimensions, neighborhoods of points are not relatively compact, continuous functions usually do not attain their extrema and linear operators are not automatically continuous. By introducing weak topologies, compactness can be obtained via Tychonoff's theorem. Similarly, functions often need to be perturbed so that the problem of finding extrema is solvable. To deal with problems in linear and nonlinear analysis, a good working knowledge of Banach space theory techniques is needed. It is the purpose of this introductory text to help the reader grasp the basic principles of Banach space theory and nonlinear geometric analysis.

The text presents the basic principles and techniques that form the core of the theory. It is organized to help the reader proceed from the elementary part of the subject to more recent developments. This task is not easy. Experience shows that working through a large number of exercises, provided with hints that direct the reader, is one of the most efficient ways to master the subject. Exercises are of several levels of difficulty, ranging from simple exercises to important results or examples. They illustrate delicate points in the theory and introduce the reader to additional lines of research. In this respect, they should be considered an integral part of the text. A list of remarks and open problems ends each chapter, presenting further developments and suggesting research paths.

An effort has been made to ensure that the book can serve experts in related fields such as Optimization, Partial Differential Equations, Fixed Point Theory, Real Analysis, Topology, and Applied Mathematics, among others.

As prerequisites, basic undergraduate courses in calculus, linear algebra, and general topology, should suffice.

The text is divided into 17 chapters.

In Chapter 1 we present basic notions in Banach space theory and introduce the classical Banach spaces, in particular sequence and function spaces.

In Chapter 2 we discuss two fundamental principles of Banach space theory, namely the Hahn–Banach Theorem on extension of bounded linear functionals and the Banach Open mapping Theorem, together with some of their applications.

In Chapter 3 we discuss weak topologies and their properties related to compactness. Then we prove the third fundamental principle, namely the Banach–Steinhaus Uniform Boundedness principle. Special attention is devoted to weak compactness, in particular to the theorems of Eberlein, Šmulyan, Grothendieck and James, and the theory of reflexive Banach spaces.

In Chapter 4 we introduce Schauder bases in Banach spaces. The possibility to represent each element of the space as the sequence of its coefficients in a given Schauder basis transfers the purely geometric techniques of the elementary Banach space theory to the analytic computations of the classical analysis. Although not every separable Banach space admits a Schauder basis, the use of basic sequences and Schauder bases with additional properties is one of the main tools in the investigation of the structural properties of Banach spaces.

In Chapter 5 we continue the study of the structure of Banach spaces by adding results on extensions of operators, injectivity, and weak injectivity. The core of the chapter is the theory of separable Banach spaces not containing isomorphic copies of ℓ_1 .

Chapter 6 is an introduction to some basic results in the geometry of finite-dimensional Banach spaces and their connection to the structure of infinite-dimensional spaces. We do not discuss the deeper parts of the theory, which essentially depend on measure theoretical techniques. We introduce the notion of finite representability, and prove the principle of local reflexivity. We use the John ellipsoid to prove the Kadec–Snobar theorem and give a proof of Tzafriri’s theorem. We indicate the connection of this result with Dvoretzky’s theorem. Last part of the chapter is devoted to the Grothendieck inequality.

In Chapter 7 we present an introduction to nonlinear analysis, namely to variational principles and differentiability.

In Chapter 8 we study the interplay between differentiability of norms and the structure of separable Asplund spaces.

Chapter 9 introduces the subject of superreflexive spaces, whose structure is nicely described by the behavior of its finite-dimensional subspaces.

Chapter 10 studies the impact of the existence of higher order smooth norms on the structure of the underlying space. Special effort is devoted to countable compact spaces and ℓ_p spaces.

Chapter 11 deals with the property of dentability and results on differentiation of vector measures. We prove some basic results on Banach spaces with the Radon–Nikodým property.

Chapter 12 introduces the reader to the nonlinear geometric analysis of Banach spaces. Results on uniform and nonuniform homeomorphisms are presented, including Keller’s theorem and basic fixed points theorems (Brouwer, Schauder, etc). We discuss a proof of the homeomorphisms of Banach spaces and results on uniform, in particular Lipschitz, homeomorphisms.

Chapter 13 contains a basic study of an important class of non-separable Banach spaces, the weakly compactly generated spaces. In particular, we discuss their decompositions and renormings. We also study weakly compact operators, absolutely summing operators, and the Dunford–Pettis property.

Chapter 14 deals with results on weak topologies, focusing on special types of compacta (scattered, Eberlein, Corson, etc.).

Chapter 15 presents basic results in the spectral theory of operators. We study compact and self-adjoint operators.

Chapter 16 deals with the basic theory of tensor products. We follow the Banach space approach, focusing on the Grothendieck duality theory of tensor products, Schauder bases, applications to spaces of compact operators, etc. We include Enflo’s example of a Banach space without the approximation property.

A short appendix (Chapter 17) has been included collecting some very basic definitions and results that are used in the text, for the reader’s immediate access.

In writing the text we strived to avoid excessive technicalities, keeping each subject as elementary as reasonably possible. Each chapter ends with a brief section of Remarks and Open Questions, containing further known results and some problems in the area that are—to our best knowledge—open.

Several more specialized books and survey articles appeared recently in Banach space theory, as [AlKa], [BeLi], [BoVa], [CasGon], [DJT], [HMOVZ], [JoLi3], [Kalt4], [KaKuLP], [LPT], [MOTV2], [Wojt], among others. We hope that the present text can help both the student and the professional mathematician to get acquainted with the techniques needed in these directions. We also made an effort to make this text closer to a reference book in order to help researchers in Banach space theory.

We are grateful to many of our colleagues for suggestions, advice, and discussions on the subject of the book. We thank our Institutions: the Institute of Mathematics of the Czech Academy of Sciences, the Czech Technical University in Prague, the Department of Mathematical and Statistical Sciences at the University of Alberta, Edmonton, Canada, the Universidad Politécnica de Valencia, Spain, and its Instituto Universitario de Matemática Pura y Aplicada. This work has been supported by several Grant Agencies: The Czech National Grant Agency and the Institutional Research Plan of the Academy of Sciences (Czech Republic), NSERC Canada, the Ministerio de Educación (Spain) and the Generalitat Valenciana (Valencia, Spain). The grants involved are IAA 100 190 610, IAA 100 190 901, GAČR 201/07/0394, No. AVOZ 101 905 03 (Czech Republic), Proyecto MTM2008-03211 (Spain), BEST/2009/096 (Generalitat Valenciana) and PR2009-0267 (Ministerio de Educación), NSERC-7926 (Canada).

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We would be glad if this book inspired some young mathematicians to choose Banach Space Theory and/or Nonlinear Geometric Analysis as their field of interest.

We wish the reader a pleasant time spent over this book.

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