# 2

# Fundamental Order-Theoretic Principles

In this chapter we use the Chain Generating Recursion Principle formulated in the Introduction to develop generalized iteration methods and to prove existence and comparison results for operator equations and inclusions in partially ordered sets. Algorithms are designed to solve concrete problems by appropriately constructed Maple programs.

## 2.1 Recursions and Iterations in Posets

Given a nonempty set $P$, a relation $x < y$ in $P \times P$ is called a *partial ordering*, if $x < y$ implies $y \not< x$, and if $x < y$ and $y < z$ imply $x < z$. Defining $x \le y$ if and only if $x < y$ or $x = y$, we say that $P = (P, \le)$ is a partially ordered set (poset).

An element $b$ of a poset $P$ is called an *upper bound* of a subset $A$ of $P$ if $x \le b$ for each $x \in A$. If $b \in A$, we say that $b$ is the *greatest element* of $A$, and denote $b = \max A$. A lower bound of $A$ and the smallest element $\min A$ of $A$ are defined similarly, replacing $x \le b$ above by $b \le x$. If the set of all upper bounds of $A$ has the smallest element, we call it a *supremum of* $A$ and denote it by $\sup A$. We say that $y$ is a *maximal element* of $A$ if $y \in A$, and if $z \in A$ and $y \le z$ imply that $y = z$. An infimum of $A$, $\inf A$, and a minimal element of $A$ are defined similarly. A poset $P$ is called a *lattice* if $\inf\{x, y\}$ and $\sup\{x, y\}$ exist for all $x, y \in P$. A subset $W$ of $P$ is said to be *upward directed* if for each pair $x, y \in W$ there is a $z \in W$ such that $x \le z$ and $y \le z$, and $W$ is *downward directed* if for each pair $x, y \in W$ there is a $w \in W$ such that $w \le x$ and $w \le y$. If $W$ is both upward and downward directed it is called *directed*. A set $W$ is said to be a *chain* if $x \le y$ or $y \le x$ for all $x, y \in W$. We say that $W$ is *well-ordered* if nonempty subsets of $W$ have smallest elements, and *inversely well-ordered* if nonempty subsets of $W$ have greatest elements. In both cases $W$ is a chain.

A basis to our considerations is the following Chain Generating Recursion Principle (cf. [112, Lemma 1.1], [133, Lemma 1.1.1]).

S. Carl and S. Heikkilä, *Fixed Point Theory in Ordered Sets and Applications: From Differential and Integral Equations to Game Theory*,
DOI 10.1007/978-1-4419-7585-0_2, © Springer Science+Business Media, LLC 2011

**Lemma 2.1.** *Given a nonempty poset $P$, a subset $\mathcal{D}$ of $2^P = \{A : A \subseteq P\}$ with $\emptyset \in \mathcal{D}$, where $\emptyset$ denotes the empty set, and a mapping $f : \mathcal{D} \to P$. Then there is a unique well-ordered chain $C$ in $P$ such that*

$$x \in C \quad \textit{if and only if}\; x = f(C^{<x}),\ \textit{where}\ \ C^{<x} = \{y \in C : y < x\}. \tag{2.1}$$

*If $C \in \mathcal{D}$, then $f(C)$ is not a strict upper bound of $C$.*

**Proof:** A nonempty subset $A$ of $P$ is called an *$f$-set* (with $f$ given in the lemma, and thus the proof is independent on the Axiom of Choice) if it has the following properties.

(i) $(A, <)$ is well-ordered, and if $x \in A$, then $x = f(A^{<x})$, where $A^{<x} = \{y \in A : y < x\}$.

For instance, the singleton $\{f(\emptyset)\}$ is an $f$-set. These sets possess the following property:

(a) *If $A$ and $B$ are $f$-sets and $A \not\subseteq B$, then $B = A^{<x}$ for some $x \in A$.*

Namely, according to a comparison principle for well-ordered sets (see [36]) there exists such a bijection $\varphi : B \to A^{<x}$ for some $x \in A$ that $\varphi(u) < \varphi(v)$ if and only if $u < v$ in $B$. The set $S = \{u \in B : u \neq \varphi(u)\}$ is empty, for otherwise, $y = \min S$ would exist and $B^{<y} = A^{<\varphi(y)}$, which yields a contradiction: $y \neq \varphi(y)$ and $y = f(B^{<y}) = f(A^{<\varphi(y)}) = \varphi(y)$. Thus $B = \varphi[B] = A^{<x}$, which proves (a).

Applying (a) it is then elementary to verify that the union $C$ of all $f$-sets is an $f$-set. Hence, $x = f(C^{<x})$ for all $x \in C$. Conversely, if $x \in P$ and $x = f(C^{<x})$, then $C^{<x} \cup \{x\}$ is an $f$-set, whence $x \in C$. Thus (2.1) holds for $C$.

To prove uniqueness, let $B$ be a well-ordered subset of $P$ for which $x \in B \Leftrightarrow x = f(B^{<x})$. Since $B$ is an $f$-set, so $B \subseteq C$. If $B \neq C$, then $B = C^{<x}$ by (a). But then $f(B^{<x}) = f(B) = f(C^{<x}) = x$, and $x \notin B$, which contradicts with $x \in B \Leftrightarrow x = f(B^{<x})$. Thus $B = C$, which proves the uniqueness of $C$. (the well-ordering condition is needed in this proof, since there may exist other partially ordered sets that satisfy (2.1), cf. [110]).

If $f(C)$ is defined, it cannot be a strict upper bound of $C$, for otherwise $f(C) \notin C$ and $f(C) = f(C^{<f(C)})$, so that $C \cup f(C)$ would be an $f$-set, not contained in $C$, which is the union of all $f$-sets. This proves the last assertion of the lemma. □

As a consequence of Lemma 2.1 we get the following result (cf. [116, Lemma 2]).

**Lemma 2.2.** *Given $G : P \to P$ and $c \in P$, there exists a unique well-ordered chain $C = C(G)$ in $P$, called* a w-o chain of $cG$-iterations*, satisfying*

$$x \in C \quad \textit{if and only if} \quad x = \sup\{c, G[C^{<x}]\}. \tag{2.2}$$

**Proof:** Denote $\mathcal{D} = \{W \subseteq P : W$ is well-ordered and $\sup\{c, G[W]\}$ exists$\}$. Defining $f(W) = \sup\{c, G[W]\}, \quad W \in \mathcal{D}$, we get a mapping $f : \mathcal{D} \to P$, and (2.1) is reduced to (2.2). Thus the assertion follows from Lemma 2.1. □

A subset $W$ of a chain $C$ is called an *initial segment* of $C$ if $x \in W$ and $y < x$ imply $y \in W$. The following application of Lemma 2.2 is used in the sequel.

**Lemma 2.3.** *Denote by $\mathcal{G}$ the set of all selections from $\mathcal{F} : P \to 2^P \setminus \emptyset$, i.e.,*

$$\mathcal{G} := \{G : P \to P : G(x) \in \mathcal{F}(x) \text{ for all } x \in P\}. \tag{2.3}$$

*Given $c \in P$ and $G \in \mathcal{G}$. Let $C_G$ denote the longest initial segment of the w-o chain $C(G)$ of $cG$-iterations such that the restriction $G|C_G$ of $G$ to $C_G$ is increasing (i.e., $G(x) \leq G(y)$ whenever $x \leq y$ in $C_G$). Define a partial ordering $\prec$ on $\mathcal{G}$ as follows: Let $F$, $G \in \mathcal{G}$ then*

*(O) $F \prec G$ if and only if $C_F$ is a proper initial segment of $C_G$ and $G|C_F = F|C_F$.*

*Then $(\mathcal{G}, \preceq)$ has a maximal element.*

**Proof:** Let $\mathcal{C}$ be a chain in $\mathcal{G}$. The definition (O) of $\prec$ implies that the sets $C_F$, $F \in \mathcal{C}$, form a nested family of well-ordered sets of $P$. Thus the set $C := \cup\{C_F : F \in \mathcal{C}\}$ is well-ordered. Moreover, it follows from (O) that the functions $F|C_F$, $F \in \mathcal{C}$, considered as relations in $P \times P$, are nested. This ensures that $g := \cup\{F|C_F : F \in \mathcal{C}\}$ is a function from $C$ to $P$. Since each $F \in \mathcal{C}$ is increasing in $C_F$, then $g$ is increasing, and $g(x) \in \mathcal{F}(x)$ for each $x \in C$. Let $G$ be such a selection from $\mathcal{F}$ that $G|C = g$. Then $G \in \mathcal{G}$, and $G$ is increasing on $C$. If $x \in C$, then $x \in C_F$ for some $F \in \mathcal{C}$. The definitions of $C$ and the partial ordering $\prec$ imply that $C_F$ is $C$ or its initial segment, whence $C_F^{<x} = C^{<x}$. Because $F|C_F = g|C_F = G|C_F$, then

$$x = \sup\{c, F[C_F^{<x}]\} = \sup\{c, G[C^{<x}]\}. \tag{2.4}$$

This result implies by (2.2) that $C$ is $C(G)$ or its proper initial segment. Since $G$ is increasing on $C$, then $C$ is $C_G$ or its proper initial segment. Consequently, $G$ is an upper bound of $\mathcal{C}$ in $\mathcal{G}$. This result implies by Zorn's Lemma that $\mathcal{G}$ has a maximal element. □

Let $P = (P, \leq)$ be a poset. For $z$, $w \in P$, we denote

$$[z) = \{x \in P : z \leq x\}, \ (w] = \{x \in P : x \leq w\} \text{ and } [z, w] = [z) \cap (w].$$

A poset $X$ equipped with a topology is called an *ordered topological space* if the order intervals $[z)$ and $(z]$ are closed for each $z \in X$. If the topology of $X$ is induced by a metric, we say that $X$ is an *ordered metric space*. Next we define some concepts for set-valued functions.

**Definition 2.4.** *Given posets $X$ and $P$, we say $\mathcal{F} : X \to 2^P \setminus \emptyset$ is* **increasing upward** *if $x \leq y$ in $X$ and $z \in \mathcal{F}(x)$ imply that $[z) \cap \mathcal{F}(y)$ is nonempty. $\mathcal{F}$ is* **increasing downward** *if $x \leq y$ in $X$ and $w \in \mathcal{F}(y)$ imply that $(w] \cap \mathcal{F}(x)$ is nonempty. If $\mathcal{F}$ is increasing upward and downward, we say that $\mathcal{F}$ is* **increasing**.

**Definition 2.5.** *A nonempty subset $A$ of a subset $Y$ of a poset $P$ is called* **order compact upward** *in $Y$ if for every chain $C$ of $Y$ that has a supremum in $P$ the intersection $\cap\{[y) \cap A : \ y \in C\}$ is nonempty whenever $[y) \cap A$ is nonempty for every $y \in C$. If for every chain $C$ of $Y$ that has the infimum in $P$ the intersection of all the sets $(y] \cap A$, $y \in C$ is nonempty whenever $(y] \cap A$ is nonempty for every $y \in C$, we say that $A$ is* **order compact downward** *in $Y$. If both these properties hold, we say that $A$ is* **order compact** *in $Y$. Phrase 'in $Y$' is omitted if $Y = A$.*

Every poset $P$ is order compact. If a subset $A$ of $P$ has the greatest element (respectively the smallest element), then $A$ is order compact upward (respectively downward) in any subset of $P$ that contains $A$. Thus an order compact set is not necessarily (topologically) compact, not even closed. On the other hand, every compact subset $A$ of an ordered topological space $P$ is obviously order compact in every subset of $P$ that contains $A$.

## 2.2 Fixed Point Results in Posets

In this subsection we prove existence and comparison results for fixed points of set-valued and single-valued functions defined in a poset $P = (P, \leq)$.

**Definition 2.6.** *Given a poset $P = (P \leq)$ and a set-valued function $\mathcal{F} : P \to 2^P \setminus \emptyset$, denote Fix($\mathcal{F}$) $= \{x \in P : x \in \mathcal{F}(x)\}$. Every element of Fix($\mathcal{F}$) is called a* **fixed point** *of $\mathcal{F}$. A fixed point of $\mathcal{F}$ is called* **minimal, maximal, smallest, or greatest** *if it is a minimal, maximal, smallest, or greatest element of Fix($\mathcal{F}$), respectively. For a single-valued function $G : P \to P$ replace Fix($\mathcal{F}$) by Fix($G$) $= \{x \in P : x = G(x)\}$.*

### 2.2.1 Fixed Points for Set-Valued Functions

Our first proved fixed point result is an application of Lemma 2.1.

**Lemma 2.7.** *Assume that $\mathcal{F} : P \to 2^P$ satisfies the following hypothesis.*

*($S_+$) The set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, and conditions: $C$ is a nonempty well-ordered chain in $S_+$, $G : C \to P$ is increasing and $x \leq G(x) \in \mathcal{F}(x)$ for all $x \in C$, imply that $G[C]$ has an upper bound in $S_+$.*

*Then $\mathcal{F}$ has a maximal fixed point, which is also a maximal element of $S_+$.*

**Proof:** Denote

$$\mathcal{D} = \{W \subset S_+ : W \text{ is well-ordered and has a strict upper bound in } S_+\}.$$

Because $S_+$ is nonempty by the hypothesis ($\mathrm{S}_+$), then $\emptyset \in \mathcal{D}$. Let $f : \mathcal{D} \to P$ be a function that assigns to each $W \in \mathcal{D}$ an element $y = f(W) \in [x) \cap \mathcal{F}(x)$, where $x$ is a fixed strict upper bound of $W$ in $S_+$. Lemma 2.1 ensures the existence of exactly one well-ordered chain $W$ in $P$ satisfying (2.1). By the above construction and (2.1) each element $y$ of $W$ belongs to $[x) \cap \mathcal{F}(x)$, where $x$ is a fixed strict upper bound of $W^{<y}$ in $S_+$. It is easy to verify that the set $C$ of these elements $x$ form a well-ordered chain in $S_+$; that the correspondence $x \mapsto y$ defines an increasing mapping $G : C \to P$; that $x \le G(x) \in \mathcal{F}(x)$ for all $x \in C$; and that $W = G[C]$. It then follows from the hypothesis ($\mathrm{S}_+$) that $W$ has an upper bound $x \in S_+$, which satisfies $x = \max W$. For otherwise $f(W)$ would exist, and as a strict upper bound of $W$ would contradict the last conclusion of Lemma 2.1. By the same reason $x$ is a maximal element of $S_+$.

Since $x \in S_+$, a $y \in P$ exists such that $x \le y \in \mathcal{F}(x)$. It then follows from the hypothesis ($\mathrm{S}_+$) when $C = \{x\}$ and $G(x) := y$ that $\{y\}$ has an upper bound $z$ in $S_+$. Because $x$ is a maximal element of $S_+$, then $z = y = x \in \mathcal{F}(x)$, so that $x$ is a fixed point of $\mathcal{F}$. If $z$ is a fixed point of $\mathcal{F}$ and $x \le z$, then $z \in S_+$, whence $x = z$. Thus $x$ is a maximal fixed point of $\mathcal{F}$. □

As an application of Lemma 2.7 we obtain the following result.

**Proposition 2.8.** *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is increasing upward, that the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \ne \emptyset\}$ is nonempty, that well-ordered chains of $\mathcal{F}[S_+]$ have supremums in $P$, and that the values of $\mathcal{F}$ at these supremums are order compact upward in $\mathcal{F}[S_+]$. Then $\mathcal{F}$ has a maximal fixed point, which is also a maximal element of $S_+$.*

**Proof:** It suffices to show that the hypothesis ($\mathrm{S}_+$) of Lemma 2.7 holds. Assume that $C$ is a well-ordered chain in $S_+$, that $G : C \to P$ is an increasing mapping, and that $x \le G(x) \in \mathcal{F}(x)$ for all $x \in C$. Then $G[C]$ is a well-ordered chain in $\mathcal{F}[S_+]$, so that $y = \sup G[C]$ exists. Since $\mathcal{F}$ is increasing upward, then $[x) \cap \mathcal{F}(y) \ne \emptyset$ for every $x \in G[C]$. Because $\mathcal{F}(y)$ is order compact upward in $\mathcal{F}[S_+]$, then the intersection of the sets $[x) \cap \mathcal{F}(y)$, $x \in G[C]$ contains at least one element $w$. Thus $G[C]$ has an upper bound $w$ in $\mathcal{F}(y)$. Since $y = \sup G[C]$, then $y \le w$, so that $w \in [y) \cap \mathcal{F}(y)$, i.e., $y$ belongs to $S_+$. □

The next result is the dual to Proposition 2.8.

**Proposition 2.9.** *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is increasing downward, that the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \ne \emptyset\}$ is nonempty, that inversely well-ordered chains of $\mathcal{F}[S_-]$ have infimums in $P$, and that values of $\mathcal{F}$ at these infimums are order compact downward in $\mathcal{F}[S_-]$. Then $\mathcal{F}$ has a minimal fixed point, which is also a minimal element of $S_-$.*

If the range $\mathcal{F}[P]$ has an upper bound (respectively a lower bound) in $P$, it belongs to $S_-$ (respectively to $S_+$). To derive other conditions under which the set $S_-$ or the set $S_+$ is nonempty, we introduce the following new concepts.

**Definition 2.10.** *Let $A$ be a nonempty subset of a poset $P$. The set $ocl(A)$ of all possible supremums and infimums of chains of $A$ is called the* **order closure** *of $A$. If $A = ocl(A)$, then $A$ is* **order closed**. *We say that a subset $A$ of a poset $P$ has a* **sup-center** *$c$ in $P$ if $c \in P$ and $\sup\{c, x\}$ exists in $P$ for each $x \in A$. If $\inf\{c, x\}$ exists in $P$ for each $x \in A$, we say that $c$ is an* **inf-center** *of $A$ in $P$. If $c$ has both these properties it is called an* **order center** *of $A$ in $P$. Phrase "in $P$" is omitted if $A = P$.*

If $P$ is an ordered topological space, then the order closure $ocl(A)$ of $A$ is contained in the topological closure $\overline{A}$ of $A$. If $c$ is the greatest element (respectively the smallest element) of $P$, then $c$ is an inf-center (respectively a sup-center) of $P$, and trivially $c$ is a sup-center (respectively an inf-center). Therefore, both the greatest and the smallest element of $P$ are order centers. If $P$ is a lattice, then its every point is an order center of $P$. If $P$ is a subset of $\mathbb{R}^2$, ordered coordinatewise, a necessary and sufficient condition for a point $c = (c_1, c_2)$ of $P$ to be a sup-center of a subset $A$ of $P$ in $P$ is that whenever a point $y = (y_1, y_2)$ of $A$ and $c$ are unordered, then $(y_1, c_2) \in P$ if $y_2 < c_2$ and $(c_1, y_2) \in P$ if $y_1 < c_1$.

The following result is an application of Lemma 2.3.

**Proposition 2.11.** *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is increasing upward and that its values are order compact upward in $\mathcal{F}[P]$. If well-ordered chains of $\mathcal{F}[P]$ have supremums, and if the set of these supremums has a sup-center $c$ in $P$, then the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty.*

**Proof:** Let $\mathcal{G}$ be defined by (2.3), and let the partial ordering $\prec$ be defined by (O). In view of Lemma 2.3, $(\mathcal{G}, \preceq)$ has a maximal element $G$. Let $C(G)$ be the w-o chain of $cG$-iterations, and let $C = C_G$ be the longest initial segment of $C(G)$ on which $G$ is increasing. Thus $C$ is well-ordered and $G$ is an increasing selection from $\mathcal{F}|C$. Since $G[C]$ is a well-ordered chain in $\mathcal{F}[P]$, then $w = \sup G[C]$ exists. Moreover, $\overline{x} = \sup\{c, w\}$ exists in $P$ by the choice of $c$, and it is easy to see that $\overline{x} = \sup\{c, G[C]\}$. This result and (2.2) imply that for each $x \in C$,

$$x = \sup\{c, G[C^{<x}]\} \leq \sup\{c, G[C]\} = \overline{x}.$$

This proves that $\overline{x}$ is an upper bound of $C$, and also of $G[C]$. Moreover, $\mathcal{F}$ is increasing upward and $\mathcal{F}(\overline{x})$ is order compact upward in $\mathcal{F}[P]$. Thus the proof of Proposition 2.8 implies that $G[C]$ has an upper bound $z$ in $\mathcal{F}(\overline{x})$, and $w = \sup G[C] \leq z$. To show that $\overline{x} = \max C$, assume on the contrary that $\overline{x}$ is a strict upper bound of $C$. Let $F$ be a selection from $\mathcal{F}$ whose restriction to $C \cup \{\overline{x}\}$ is $G|C \cup \{(\overline{x}, z)\}$. Since $G$ is increasing on $C$ and $F(x) = G(x) \le w \le z = F(\overline{x})$ for each $x \in C$, then $F$ is increasing on $C \cup \{\overline{x}\}$. Moreover,

$$\overline{x} = \sup\{c, G[C]\} = \sup\{c, F[C]\} = \sup\{c, F[\{y \in C \cup \{\overline{x}\} : y < \overline{x}\}]\},$$

whence $C \cup \{\overline{x}\}$ is a subset of the longest initial segment $C_F$ of the w-o chain of $cF$-iterations where $F$ is increasing. Thus $C = C_G$ is a proper subset of $C_F$, and $F|C_G = F|C_F$. By (O) this means that $G \prec F$, which, however, is impossible because $G$ is a maximal element of $(\mathcal{G}, \preceq)$. Consequently, $\overline{x} = \max C$. Since $G$ is increasing on $C$, then $\overline{x} = \sup\{c, G[C]\} = \sup\{c, G(\overline{x})\}$. In particular, $\mathcal{F}(\overline{x}) \ni G(\overline{x}) \le \overline{x}$, whence $G(\overline{x})$ belongs to the set $(\overline{x}] \cap \mathcal{F}(\overline{x})$. □

As a consequence of Propositions 2.8, 2.9, and 2.11 we obtain the following fixed point result.

**Theorem 2.12.** *Assume that $\mathcal{F} : P \to 2^P \setminus \emptyset$ is increasing, and that its values are order compact in $\mathcal{F}[P]$. If chains of $\mathcal{F}[P]$ have supremums and infimums, and if $ocl(\mathcal{F}[P])$ has a sup-center or an inf-center in $P$, then $\mathcal{F}$ has minimal and maximal fixed points.*

**Proof:** We shall give the proof in the case when $ocl(\mathcal{F}[P])$ has a sup-center in $P$, as the proof in the case of an inf-center is similar. The hypotheses of Proposition 2.11 are then valid, whence there exists a $\overline{x} \in P$ such that $(\overline{x}] \cap \mathcal{F}(\overline{x}) \ne \emptyset$. Thus the hypotheses of Proposition 2.9 hold, whence $\mathcal{F}$ has by Proposition 2.9 a minimal fixed point $x_-$. In particular $[x_-) \cap \mathcal{F}(x_-) \ne \emptyset$. The hypotheses of Proposition 2.8 are then valid, whence we can conclude that $\mathcal{F}$ has also a maximal fixed point. □

*Example 2.13.* Assume that $\mathbb{R}^m$ is ordered as follows. For all $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m) \in \mathbb{R}^m$,

$$x \le y \ \text{ if and only if } x_i \le y_i,\ i = 1, \dots, j, \text{ and } \ x_i \ge y_i,\ i = j+1, \dots, m, \tag{2.5}$$

where $j \in \{0, \dots, m\}$. Show that if $\mathcal{F} : \mathbb{R}^m \to 2^{\mathbb{R}^m} \setminus \emptyset$ is increasing, and its values are closed subsets of $\mathbb{R}^m$, and if $\mathcal{F}[\mathbb{R}^m]$ is contained in the set

$$B_R^p(c) = \{(x_1, \dots, x_m) \in \mathbb{R}^m : \sum_{i=1}^m |x_i - c_i|^p \le R^p\},\ p,\, R \in (0, \infty),$$

where $c = (c_1, \dots, c_m) \in \mathbb{R}^m$, then $\mathcal{F}$ has minimal and maximal fixed points.

**Solution:** Let $x = (x_1, \dots, x_m) \in B_R^p(c)$ be given. Since $|\max\{c_i, x_i\} - c_i| \le |x_i - c_i|$ and $|\min\{c_i, x_i\} - c_i| \le |x_i - c_i|$ for each $i = 1, \dots, m$, it follows that $\sup\{c, x\}$ and $\inf\{c, x\}$ belong to $B_R^p(c)$ for all $x \in B_R^p(c)$. Moreover, every $B_R^p(c)$ is a closed and bounded subset of $\mathbb{R}^m$, whence its monotone sequences converge in $B_R^p(c)$ with respect to the Euclidean metric of $\mathbb{R}^m$.

These results, Lemma 2.31 and the given hypotheses imply that chains of $\mathcal{F}[\mathbb{R}^m]$ have supremums and infimums, that $c$ is an order center of $ocl(\mathcal{F}[\mathbb{R}^m])$, and that the values of $\mathcal{F}$ are compact. Thus the hypotheses of Theorem 2.12 are satisfied, whence we conclude that $\mathcal{F}$ has minimal and maximal fixed points. □

### 2.2.2 Fixed Points for Single-Valued Functions

Next we present existence and comparison results for fixed points of single-valued functions. The following auxiliary result is a consequence of Proposition 2.11 and its proof. We note that the Axiom of Choice is not needed in the proof.

**Proposition 2.14.** *Assume that $G : P \to P$ is increasing, that $ocl(G[P])$ has a sup-center $c$ in $P$, and that $\sup G[C]$ exists whenever $C$ is a nonempty well-ordered chain in $P$. If $C$ is the w-o chain of $cG$-iterations, then $\overline{x} = \max C$ exists, $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$, and*

$$\overline{x} = \min\{z \in P : \sup\{c, G(z)\} \leq z\}. \tag{2.6}$$

*Moreover, $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$, and it is increasing with respect to $G$.*

**Proof:** The mapping $\mathcal{F} := G : P \to 2^P \setminus \emptyset$ is single-valued. Because $G$ is increasing, then $C$ in Lemma 2.3 is the w-o chain of $cG$-iterations. The hypotheses given for $G$ imply also that $c$ is a sup-center of $ocl(\mathcal{F}[P])$ in $P$, and that $\sup G[C]$ exists. Since $G$ is single-valued, the values of $\mathcal{F}$ are order compact in $\mathcal{F}[P]$. Thus the proof of Proposition 2.11 implies that $\overline{x} = \max C$ exists, and $\overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, G[C]\}$. To prove (2.6), let $z \in P$ satisfy $\sup\{c, G(z)\} \leq z$. Then $c = \min C \leq z$. If $x \in C$ and $\sup\{c, G(y)\} \leq z$ for each $y \in C^{<x}$, then $x = \sup\{c, G[C^{<x}]\} \leq z$. This implies by transfinite induction that $x \leq z$ for each $x \in C$. In particular $\overline{x} = \max C \leq z$. From this result and the fact that $\overline{x} = \sup\{c, G(\overline{x})\}$ we infer that $\overline{x} = x$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$, and that (2.6) holds. The last assertion is an immediate consequence of (2.6). □

The results presented in the next proposition are dual to those of Lemma 2.2 and Proposition 2.14.

**Proposition 2.15.** *Given $G : P \to P$ and $c \in P$, there exists exactly one inversely well-ordered chain $D$ in $P$, called an* inversely well-ordered (i.w-o) chain of $cG$- iterations, *satisfying*

$$x \in D \quad \textit{if and only if } x = \inf\{c, G[\{y \in D : x < y\}]\}. \tag{2.7}$$

*Assume that $G$ is increasing, that $ocl(G[P])$ has an inf-center $c$ in $P$, and that $\inf G[D]$ exists whenever $D$ is a nonempty inversely well-ordered chain* *in $P$. If $D$ is the i.w-o chain of $cG$-iterations, then* $\underline{x} = \min D$ *exists,* $\underline{x} = \inf\{c, G(\underline{x})\} = \inf\{c, G[D]\}$*, and*

$$\underline{x} = \max\{z \in P : z \leq \inf\{c, G(z)\}\}. \tag{2.8}$$

*Moreover,* $\underline{x}$ *is the greatest solution of the equation* $x = \inf\{c, G(x)\}$*, and it is increasing with respect to $G$.*

Our first fixed point result is a consequence of Propositions 2.14 and 2.15.

**Theorem 2.16.** *Let $P$ be a poset and let $G : P \to P$ be an increasing mapping.*

*(a) If* $\underline{x} \leq G(\underline{x})$*, and if* $\sup G[C]$ *exists whenever $C$ is a well-ordered chain in* $[\underline{x})$ *and $x \leq G(x)$ for every $x \in C$, then the w-o chain $C$ of* $\underline{x}G$*-iterations has a maximum $x_*$ and*

$$x_* = \max C = \sup G[C] = \min\{y \in [\underline{x}) : G(y) \leq y\}. \tag{2.9}$$

*Moreover, $x_*$ is the smallest fixed point of $G$ in* $[\underline{x})$*, and $x_*$ is increasing with respect to $G$.*

*(b) If* $G(\overline{x}) \leq \overline{x}$*, and if* $\inf G[C]$ *exists whenever $C$ is an inversely well-ordered chain* $(\overline{x}]$ *and $G(x) \leq x$ for every $x \in C$, then the i.w-o chain $D$ of* $\overline{x}G$*-iterations has a minimum $x^*$ and*

$$x^* = \min D = \inf G[D] = \max\{y \in (\overline{x}] : y \leq G(y)\}. \tag{2.10}$$

*Moreover, $x^*$ is the greatest fixed point of $G$ in* $(\overline{x}]$*, and $x^*$ is increasing with respect to $G$.*

**Proof: Ad (a)** Since $G$ is increasing and $\underline{x} \leq G(\underline{x})$, then $G[[\underline{x})] \subset [\underline{x})$. It is also easy to verify that $x \leq G(x)$ for every element $x$ of the w-o chain $C$ of $\underline{x}G$-iterations. Thus the conclusions of (a) are immediate consequences of the conclusion of Proposition 2.14 when $c = \underline{x}$ and $G$ is replaced by its restriction to $[\underline{x})$.

**Ad (b)** The proof of (b) is dual to that of (a). □

As an application of Propositions 2.14 and 2.15 and Theorem 2.16 we get the following fixed point results.

**Theorem 2.17.** *Assume that $G : P \to P$ is increasing, and that* $\sup G[C]$ *and* $\inf G[C]$ *exist whenever $C$ is a chain in $P$.*

*(a) If ocl$(G[P])$ has a sup-center or an inf-center in $P$, then $G$ has minimal and maximal fixed points.*

*(b) If ocl$(G[P])$ has a sup-center $c$ in $P$, then $G$ has the greatest fixed point $x^*$ in* $(\overline{x}]$*, where $\overline{x}$ is the smallest solution of the equation* $x = \sup\{c, G(x)\}$*. Both $\overline{x}$ and $x^*$ are increasing with respect to $G$.*

(c) *If $c$ is an inf-center of $ocl(G[P])$ in $P$, then $G$ has the smallest fixed point $x_*$ in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and $x_*$ are increasing with respect to $G$.*

Theorem 2.16, Proposition 2.8, its proof, and Proposition 2.9 imply the following results.

**Proposition 2.18.** *Assume that $G : P \to P$ is increasing.*

(a) *If the set $S_+ = \{x \in P : x \leq G(x)\}$ is nonempty, and if $\sup G[C]$ exists whenever $C$ is a well-ordered chain in $S_+$, then $G$ has a maximal fixed point. Moreover, $G$ has for every $\underline{x} \in S_+$ the smallest fixed point in $[\underline{x})$, and it is increasing with respect to $G$.*

(b) *If the set $S_- = \{x \in P : G(x) \leq x\}$ is nonempty, and if $\inf G[D]$ exists whenever $D$ is an inversely well-ordered chain in $S_-$, then $G$ has a minimal fixed point. Moreover, $G$ has for every $\overline{x} \in S_-$ the greatest fixed point in $(\overline{x}]$, and it is increasing with respect to $G$.*

*Example 2.19.* Let $\mathbb{R}_+$ be the set of nonnegative reals, and let $\mathbb{R}^m$ be ordered coordinatewise. Assume that $G : \mathbb{R}^m \to \mathbb{R}^m_+$ is increasing and maps increasing sequences of the set $S_+ = \{x \in \mathbb{R}^m_+ : x \leq G(x)\}$ to bounded sequences. Show that $G$ has the smallest fixed point and a maximal fixed point.

**Solution:** The origin is a lower bound of $G[\mathbb{R}^m]$. Let $C$ be a well-ordered chain in $S_+$. Since $G$ is increasing, then $G[C]$ is a well-ordered chain in $\mathbb{R}^m_+$. If $(y_n)$ is an increasing sequence in $G[C]$, and $x_n = \min\{x \in C : G(x) = y_n\}$, then the sequence $(x_n)$ is increasing and $y_n = G(x_n)$ for every $n$. Thus $(y_n)$ is bounded by a hypothesis, and hence converges with respect to the Euclidean metric of $\mathbb{R}^m$. This result implies by Lemma 2.31 that $\sup G[C]$ exists. Thus the assertions follow from Proposition 2.18. □

### 2.2.3 Comparison and Existence Results

In the next application of Theorem 2.16, fixed points of a set-valued function are bounded from above by a fixed point of a single-valued function.

**Theorem 2.20.** *Given a poset $X = (X, \leq)$, a subset $P$ of $X$ and $\overline{x} \in P$, assume that a function $G : P \to P$ and a set-valued function $\mathcal{F} : X \to 2^X$ have the following properties.*

(Ha) *$G$ is increasing, $G(\overline{x}) \leq \overline{x}$, and $\inf G[D]$ exists in $P$ whenever $D$ is an inversely well-ordered chain in $(\overline{x}]$.*

(Hb) *$\overline{x}$ is an upper bound of $\mathcal{F}[X] = \cup_{x \in X} \mathcal{F}(x)$, and if $x \leq p$ in $X$ and $p \in P$, then $G(p)$ is an upper bound of $\mathcal{F}(x)$.*

*Then $G$ has the greatest fixed point $x^*$ in $(\overline{x}]$, and if $x$ is any fixed point of $\mathcal{F}$, then $x \leq x^*$.*

**Proof:** Let $D$ be the i.w-o chain of $\overline{x}G$-iterations. Since $D$ is inversely well-ordered, then $x^* = \inf G[D]$ exists and belongs to $P$ by hypothesis (Ha). Moreover, $x^*$ is the greatest fixed point of $G$ in $(\overline{x}]$ by Theorem 2.16 (b). To prove that $x^*$ is an upper bound for fixed points of $\mathcal{F}$, assume on the contrary an existence of a point $x$ of $X$ such that $x \in \mathcal{F}(x)$ and $x \not\leq x^*$. Since $x^* = \min D$ and $D$ is inversely well-ordered, there exists the greatest element $p$ of $D$ such that $x \not\leq p$. Because $x \in \mathcal{F}(x)$, then $x \leq \overline{x} = \max D$ by (Hb), whence $p < \overline{x}$. If $q \in D$ and $p < q$, then $x \leq q$, so that $x \leq G(q)$ by (Hb). Thus $x$ is a lower bound of the set $G[\{q \in D : p < q\}]$. Since $\overline{x}$ is an upper bound of this set, then $p$ is by (2.7) with $c = \overline{x}$ the infimum of $G[\{q \in D : p < q\}]$. But then $x \leq p$, which contradicts with the choice of $p$. Consequently, $x \leq x^*$ for each fixed point $x$ of $\mathcal{F}$. □

Using the result of Theorem 2.20 we prove the following existence and comparison result for greatest fixed points of set-valued functions.

**Theorem 2.21.** *Given a nonempty subset $P$ of $X$ and $\mathcal{F} : X \to 2^X$, assume that*

*(H0) $\mathcal{F}[X]$ has an upper bound $\overline{x}$ in $P$.*
*(H1) If $p \in P$, then $\max \mathcal{F}(p)$ exists, belongs to $P$, and is an upper bound of $\mathcal{F}[X \cap (p]]$.*
*(H2) Inversely well-ordered chains of the set $\{\max \mathcal{F}(p) : p \in P\}$ have infimums in $P$.*

*Then $\mathcal{F}$ has a greatest fixed point, and it belongs to $P$. Assume moreover, that $\hat{\mathcal{F}} : X \to 2^X$ is another set-valued function that satisfies the following condition.*

*(H3) For each $x \in X$ and $y \in \hat{\mathcal{F}}(x)$ there exists a $z \in \mathcal{F}(x)$ such that $y \leq z$.*

*Then the greatest fixed point of $\mathcal{F}$ is an upper bound for all the fixed points of $\hat{\mathcal{F}}$.*

**Proof:** The hypothesis (H1) ensures that defining

$$G(p) := \max \mathcal{F}(p), \quad p \in P, \tag{2.11}$$

we obtain an increasing mapping $G : P \to P$. Moreover, $G(\overline{x}) \leq \overline{x}$ is by (H0), and the hypothesis (H2) means that every inversely well-ordered chain of $G[P]$ has an infimum in $P$. Thus the hypothesis (Ha) of Theorem 2.20 holds. The hypothesis (H1) and the definition (2.11) of $G$ imply that also the hypothesis (Hb) of Theorem 2.20 is valid. Thus $G$ has by Theorem 2.20 the greatest fixed point $x^*$. Because $x^* = G(x^*) = \max \mathcal{F}(x^*) \in \mathcal{F}(x^*)$, then $x^*$ is also a fixed point of $\mathcal{F}$, which is by Theorem 2.20 an upper bound all fixed points of $\mathcal{F}$. Consequently, $x^*$ is the greatest fixed point of $\mathcal{F}$, and $x^* = \max \mathcal{F}(x^*) \in P$ by (H1).

To prove the last assertion, let $\hat{\mathcal{F}} : X \to 2^X$ be such a set-valued function that (H3) holds. The hypotheses (H0) and (H3) imply that $\overline{x}$ is an upper bound of $\hat{\mathcal{F}}[X]$. Moreover, if $x \leq p$ in $X$ and $p \in P$, then for each $y \in \hat{\mathcal{F}}(x)$ there is by (H3) a $z \in \mathcal{F}(x)$ such that $y \leq z$, and $z \leq \max \mathcal{F}(p) = G(p)$ by (H1) and (2.11). Thus the hypotheses of Theorem 2.20 hold when $\mathcal{F}$ is replaced by $\hat{\mathcal{F}}$, whence $x \leq x^*$ for each fixed point $x$ of $\hat{\mathcal{F}}$. □

*Remark 2.22.* Applying Theorem 2.16 (a) we obtain obvious duals to Theorems 2.20 and 2.21.

### 2.2.4 Algorithmic Methods

Let $P$ be a poset, and let $G : P \to P$ be increasing. The first elements of the w-o chain $C$ of $cG$-iterations are: $x_0 = c$, $x_{n+1} = \sup\{c, Gx_n\}$, $n = 0, 1, \dots,$ as long as $x_{n+1}$ exists and $x_n < x_{n+1}$. Assuming that strictly monotone sequences of $G[P]$ are finite, then $C$ is a finite strictly increasing sequence $(x_n)_{n=0}^m$. If $\sup\{c, x\}$ exists for every $x \in G[P]$, then $\overline{x} = \sup\{c, G[C]\} = \max C = x_m$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$ by Proposition 2.14. In particular, $G\overline{x} \leq \overline{x}$. If $G(\overline{x}) < \overline{x}$, then first elements of the i.w-o chain $D$ of $\overline{x}G$-iterations of $\overline{x}$ are $y_0 = \overline{x} = x_m$, $y_{j+1} = Gy_j$, as long as $y_{j+1} < y_j$. Since strictly monotone sequences of $G[P]$ are finite, $D$ is a finite strictly decreasing sequence $(y_j)_{j=0}^k$, and $x^* = \inf G[D] = y_k$ is the greatest fixed point of $G$ in $(\overline{x}]$ by Theorem 2.16.

The above reasoning and its dual imply the following results.

**Corollary 2.23.** *Conclusions of Theorem 2.17 hold if* $G : P \to P$ *is increasing and strictly monotone sequences of* $G[P]$ *are finite, and if* $\sup\{c, x\}$ *and* $\inf\{c, x\}$ *exist for every* $x \in G[P]$*. Moreover,* $x^*$ *is the last element of the finite sequence determined by the following algorithm:*

*(i)* $x_0 = c$. For $n$ from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n < x_n$ else $x_{n+1} = \sup\{c, Gx_n\}$,

*and* $x_*$ *is the last element of the finite sequence determined by the following algorithm:*

*(ii)* $x_0 = c$. For $n$ from 0 while $x_n \neq Gx_n$ do: $x_{n+1} = Gx_n$ if $Gx_n > x_n$ else $x_{n+1} = \inf\{c, Gx_n\}$.

Let $G : P \to P$ satisfy the hypotheses of Theorem 2.17. The result Corollary 2.23 can be applied to approximate the fixed points $x^*$ and $x_*$ of $G$ introduced in Theorem 2.17 in the following manner. Assume that $\underline{G}$, $\overline{G} : P \to P$ satisfy the hypotheses given for $G$ in Corollary 2.23, and that

$$\underline{G}(x) \leq G(x) \leq \overline{G}(x) \quad \text{for all} \quad x \in P. \tag{2.12}$$

Since $x^*$ and $x_*$ are increasing with respect to $G$, it follows from (2.12) that $\underline{x}_* \leq x_* \leq \overline{x}_*$ and $\underline{x}^* \leq x^* \leq \overline{x}^*$, where $\underline{x}^*$ and $\overline{x}^*$ (respectively $\underline{x}_*$ and $\overline{x}_*$) are obtained by algorithm (i) (respectively (ii)) of Corollary 2.23 with $G$ replaced by $\underline{G}$ and $\overline{G}$, respectively.

Since partial ordering is the only structure needed in the proofs, the above results can be applied to problems where only ordinal scales are available. On the other hand, these results have some practical value also in real analysis. We shall demonstrate this by an example where the above described method is applied to a system of the form

$$x_i = G_i(x_1, \dots, x_m), \quad i = 1, \dots, m, \tag{2.13}$$

where the functions $G_i$ are real-valued functions of $m$ real variables.

*Example 2.24.* Approximate a solution $x^* = (x_1, y_1)$ of the system

$$x = G_1(x,y) := \frac{N_1(x,y)}{2 - |N_1(x,y)|}, \ y = G_2(x,y) := \frac{N_2(x,y)}{3 - |N_2(x,y)|}, \tag{2.14}$$

where

$$N_1(x,y) = \frac{11}{12}x + \frac{12}{13}y + \frac{1}{234} \quad \text{and} \quad N_2(x,y) = \frac{15}{16}x + \frac{14}{15}y - \frac{7}{345}, \tag{2.15}$$

by calculating upper and lower estimates of $(x_1, y_1)$ whose corresponding coordinates differ by less than $10^{-100}$.

**Solution:** The mapping $G = (G_1, G_2)$, defined by (2.14), (2.15) maps the set $P = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \le \frac{1}{2}\}$ into $P$, and is increasing on $P$. It follows from Example 2.1 that $c = (0,0)$ is an order center of $P$, and that chains of $P$ have supremums and infimums. Thus the results of Theorem 2.17 are valid.

Upper and lower estimates to the fixed point $x^* = (x_1, y_1)$ of $G$, and hence to a solution $(x_1, y_1)$ of system (2.14), (2.15), can be obtained by applying the algorithm (i) given in Corollary 2.23 to operators $\overline{G}$ and $\underline{G}$, defined by

$$\begin{cases} \overline{G}(x,y) = (10^{-101}\text{ceil}(10^{101}G_1(x,y)), 10^{-101}\text{ceil}(10^{101}G_2(x,y)), \\ \underline{G}(x,y) = (10^{-101}\text{floor}(10^{101}G_1(x,y)), 10^{-101}\text{floor}(10^{101}G_2(x,y)), \end{cases} \tag{2.16}$$

where ceil$(x)$ is the smallest integer $\ge x$ and floor$(x)$ is the greatest integer $\le x$. The so defined operators $\underline{G}$, $\overline{G}$ are increasing and map the set $P = \{(x,y) \in \mathbb{R}^2 : |x| + |y| \le \frac{1}{2}\}$ into finite subsets of $P$, and (2.12) holds. We are going to show that the required upper and lower estimates are obtained by algorithm (i) of Corollary 2.23 with $G$ replaced by $\underline{G}$ and $\overline{G}$, respectively. The following Maple program is used in calculations of the upper estimate $\overline{x}^* = (x1, y1)$.

$$(N1, N2) := (11/12 * x + 12/13 * y + 1/234, 15/16 * x + 14/15 * y - 7/345) :$$

$(z,w) := (N1/(2 - abs(N1)), N2/(3 - abs(N2)))$ :
$(G1, G2) := (ceil(10^{101}z)/10^{101}, ceil(10^{101}w)/10^{101})$ :
$(x0, y0) := (0, 0); x := x0 : y := y0 : u := G1 : v := G2 : b[0] := [x, y]$ :
for $k$ from 1 while $abs(u - x) + abs(v - y) > 0$ do :
if $u <= x$ and $v <= y$ then $(x, y) := (u, v)$
else $(x, y) := (\max\{x, u\}, \max\{y, v\}) : fi$ :
$u := G1 : v := G2 : b[k] := [x, y] : od : n := k - 1 : x1 := x; y1 = y;$

The above program yields the following results (n=1246).

$$\begin{aligned} x1 =& -0.00775318684978081165491069304103701961947143138\\ &7747172549504569995356264082732785848367182252372500 43,\\ y1 =& -0.01359961542461090148983671991312928002452425440 1\\ &289927375880599161783854868392762013556944139785572 1 \end{aligned}$$

In particular, $(x1, y1)$ is the fixed point $\overline{x}^*$ of $\overline{G}$.

Replacing 'ceil' by 'floor' in the above program, we obtain components of the fixed point $\underline{x}^* = (x2, y2)$ of $\underline{G}$ (n:=1248).

$$\begin{aligned} x2 =& -0.007753186849780811654910693041037019619471431387\\ &74717254950456999535626408273278584836718225237250 05,\\ y2 =& -0.013599615424610901489836719913129280024524254401 2\\ &899273758805991617838548683927620135569441397855721 5 \end{aligned}$$

The above calculated components of $\overline{x}^*$ and $\underline{x}^*$ are exact, and their differences are $< 10^{-100}$. According to the above reasoning the exact fixed point $x^*$ of $G$ belongs to order interval $[\underline{x}^*, \overline{x}^*]$. In particular, both $(x1, y1)$ and $(x2, y2)$ approximate an exact solution $(x_1, y_1)$ of system (2.14), (2.15) with the required precision. Moreover, $x1 \le x_1 \le y1$ and $x2 \le y_1 \le y2$. $\square$

## 2.3 Solvability of Operator Equations and Inclusions

In this section we apply the results of Sect. 2.2 to study the solvability of operator equations in the form $Lu = Nu$, where $L$ and $N$ are single-valued mappings from a poset $V = (V, \le)$ to another poset $P = (P, \le)$. The solvability of the corresponding inclusions $Lu \in \mathcal{N}u$, where $\mathcal{N} : V \to 2^P \setminus \emptyset$, is studied as well. In order to obtain solvability results applicable to implicit equations and inclusions, we make use of the so-called *graph ordering* of $V$, defined by

$$u \preceq v \quad \text{if and only if } u \le v \text{ and } \; Lu \le Lv. \tag{2.17}$$

### 2.3.1 Inclusion Problems

As an application of Theorem 2.12 and Propositions 2.8 and 2.9 we prove the following existence result for the inclusion problem $Lu \in \mathcal{N}u$.

**Theorem 2.25.** *Let $L : V \to P$ and $\mathcal{N} : V \to 2^P \setminus \emptyset$ satisfy the following hypotheses.*

*(L) The equation $Lu = x$ has for each $x \in P$ smallest and greatest solutions, and they are increasing in $x$.*
*($\mathcal{N}$1) Chains of $\mathcal{N}[V]$ have supremums and infimums in $P$, and $ocl(\mathcal{N}[V])$ has a sup-center or an inf-center in $P$.*
*($\mathcal{N}$2) $\mathcal{N}$ is increasing in $(V, \preceq)$ or in $(V, \leq)$, and its values are order compact in $\mathcal{N}[V]$.*

*Then $Lu \in \mathcal{N}u$ has minimal and maximal solutions in $(V, \preceq)$.*

**Proof:** Denote $V_- = \{\min L^{-1}\{x\} : x \in P\}$ and $L_- = L|V_-$. Define a mapping $\mathcal{F} : P \to 2^P \setminus \emptyset$ by

$$\mathcal{F}(x) := \mathcal{N}(L_-^{-1}x), \quad x \in P. \tag{2.18}$$

Assume first that $\mathcal{N}$ is increasing in $(V, \preceq)$, and that its values are order compact in $\mathcal{N}[V]$. To show that $\mathcal{F}$ is increasing, assume that $x \leq y$ in $P$. Then $u := \min L^{-1}\{x\} \leq v := \min L^{-1}\{y\}$ by condition (L), and $Lu = x \leq y = Lv$, whence $u \preceq v$. Since $\mathcal{N}$ is increasing in $(V, \preceq)$, thus $[z) \cap \mathcal{F}(y) = [z) \cap \mathcal{N}v \neq \emptyset$ for each $z \in \mathcal{N}u = \mathcal{F}(x)$ and $(w] \cap \mathcal{F}(x) = (w] \cap \mathcal{N}u \neq \emptyset$ for each $w \in \mathcal{N}v = \mathcal{F}(y)$. This proves that $\mathcal{F}$ is increasing.

As $\mathcal{F}[P]$ is contained in $\mathcal{N}[V]$ by (2.18), the chains of $\mathcal{F}[P]$ have supremums and infimums in $P$ by condition ($\mathcal{N}$), and the values of $\mathcal{F}$ are order compact in $\mathcal{F}[P]$. Moreover, $ocl(\mathcal{F}[P])$ has a sup-center or an inf-center by hypothesis.

The above proof shows that $\mathcal{F}$ satisfies the hypotheses of Theorem 2.12, whence we conclude that it has minimal and maximal fixed points. If $x$ is any fixed point of $\mathcal{F}$, and $u = L_-^{-1}x$, then $Lu = x \in \mathcal{F}(x) = \mathcal{N}u$, which shows that $u$ is a solution of the inclusion problem $Lu \in \mathcal{N}u$.

To prove the existence of a minimal solution of $Lu \in \mathcal{N}u$, let $x_-$ be a minimal fixed point of $\mathcal{F}$. Then $u_- = L_-^{-1}x_-$ is a solution of the inclusion problem $Lu \in \mathcal{N}u$. Let $v \in V$ satisfy $Lv \in \mathcal{N}v$, $v \leq u_-$ and $Lv \leq Lu_-$. Denoting $y = Lv$ and $u = L_-^{-1}y$, then $u \leq v$ and $Lu = Lv$, that is $u \preceq v$. Then $(y] \cap \mathcal{N}u \neq \emptyset$ because $\mathcal{N}$ is increasing. Since $\mathcal{N}u = \mathcal{N}L_-^{-1}y = \mathcal{F}(y)$, we have $(y] \cap \mathcal{F}(y) \neq \emptyset$. Thus $y$ belongs to the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$. Because $y = Lv \leq Lu_- = x_-$ and $x_-$ is, by Proposition 2.9, a minimal element of $S_-$, then $y = x_-$. Hence it follows $u = L_-^{-1}y = L_-^{-1}x_- = u_-$. Moreover, $u \leq v \leq u_-$, whence $v = u_-$, which proves that $u_-$ is a minimal solution of $Lu = \mathcal{N}u$ with respect to the graph ordering of $V$.

Denoting $V_+ = \{\max L^{-1}\{x\} : x \in P\}$ and $L_+ = L|V_+$, and replacing $L_-$ in (2.18) by $L_+$ we obtain another mapping $\mathcal{F} : P \to 2^P \setminus \emptyset$, which satisfies the hypotheses of Theorem 2.12. Thus $\mathcal{F}$ has minimal and maximal fixed points, and to each fixed point $x$ of $\mathcal{F}$ there corresponds a solution $u = \max L^{-1}\{x\}$ of the inclusion problem $Lu \in \mathcal{N}u$. Moreover, if $x_+$ is a maximal fixed point of $\mathcal{F}$, then by applying Proposition 2.8 one can show that $u_+ = L_+^{-1}x_+$ is a maximal solution of $Lu \in \mathcal{N}u$ in $(V, \preceq)$.

If $\mathcal{N}$ is increasing in $(V, \leq)$, it is increasing also in $(V, \preceq)$. Thus $Lu \in \mathcal{N}u$ has, by the above proof, minimal and maximal solutions in $(V, \preceq)$. □

### 2.3.2 Single-Valued Problems

Consider next the single-valued case. As an application of Theorems 2.16 and 2.25 and Propositions 2.14 and 2.15 we obtain the following existence and comparison results for the equation $Lu = Nu$.

**Theorem 2.26.** *Given posets $V$ and $P$, mappings $L, N : V \to P$, assume that $L$ satisfies the hypothesis (L), that $N$ is increasing in $(V, \preceq)$ or in $(V, \leq)$. If $ocl(N[V])$ has an order center $c$ in $P$, and if chains of $N[V]$ have supremums and infimums in $P$, then the following results hold.*

*(a) The equation $Lu = \sup\{c, Nu\}$ has the smallest solution $\overline{u}$ in $(V_+, \leq)$, where $V_+ = \{\max L^{-1}\{x\} : x \in P\}$, and the equation $Lu = \inf\{c, Nu\}$ has the greatest solution $\underline{u}$ in $(V_-, \leq)$, where $V_- = \{\min L^{-1}\{x\} : x \in P\}$.*
*(b) If $N$ is increasing in $(V, \preceq)$, then the equation $Lu = Nu$ has smallest and greatest solutions in the order interval $[\underline{u}, \overline{u}]$ of $(V, \preceq)$, and they are increasing in $(V, \leq)$ with respect to $N$.*
*(c) If $N$ is increasing in $(V, \leq)$, then the equation $Lu = Nu$ has smallest and greatest solutions in the order interval $[\underline{u}, \overline{u}]$ of $(V, \leq)$, and they are increasing in $(V, \leq)$ with respect to $N$.*
*(d) The equation $Lu = Nu$ has minimal and maximal solutions in $(V, \preceq)$.*

**Proof: Ad (a)** The given hypotheses ensure that relation

$$G(x) := NL_+^{-1}x, \quad x \in P, \text{ where } L_+ = L|V_+, \tag{2.19}$$

defines an increasing mapping $G : P \to P$. Moreover, $N[V_+] = G[P]$, which in view of the hypotheses implies that chains of $G[P]$ have supremums and infimums in $P$. Moreover, $c$ is a sup-center of $ocl(N[V_+]) = ocl(G[P])$ in $P$. Thus, by Proposition 2.14, the equation $x = \sup\{c, G(x)\}$ has the smallest solution $\overline{x}$. Denoting by $\overline{u} = L_+^{-1}\overline{x}$, then $G(\overline{x}) = N\overline{u}$, whence $L\overline{u} = \overline{x} = \sup\{c, G(\overline{x})\} = \sup\{c, N\overline{u}\}$. Thus $\overline{u}$ is a solution of the equation $Lu = \sup\{c, Nu\}$.

Assume that $v = L_+^{-1}x$, $x \in P$, and $Lv = \sup\{c, Nv\}$. Then

$$x = Lv = \sup\{c, Nv\} = \sup\{c, NL_+^{-1}x\} = \sup\{c, G(x)\}.$$

Since $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$, we get $\overline{x} \leq x$. This implies by condition (L) that $\overline{u} = L_+^{-1}\overline{x} \leq L_+^{-1}x = v$. Thus $\overline{u}$ is the smallest solution of $Lu = \sup\{c, Nu\}$ in $(V_+, \leq)$.

**Ad (b)** Assume that $N$ is increasing in $(V, \preceq)$. By the proof of (a) $\overline{u} = L_+^{-1}\overline{x}$, where $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$. Then $G(\overline{x}) \le \overline{x}$, whence $G$ has the greatest fixed point $x^*$ in $(\overline{x}]$ by Theorem 2.16. Denoting $u^* = L_+^{-1}x^*$, we see that $u^*$ is a solution of the equation $Lu = Nu$ and $u^* \le \overline{u}$ due to (L).

Let $u \in V$ be a solution of $Lu = Nu$ with $u \preceq \overline{u}$. Denoting $x = Lu$ and $v = L_+^{-1}x$, we infer $u \le v$ and $Lu = Lv$, whence $u \preceq v$. Thus $x = Lv = Lu \le L\overline{u} = \overline{x}$ and $x = Nu \le Nv = NL_+^{-1}x = G(x)$. Since $x^*$ is the greatest fixed point of $G$ in $(\overline{x}]$, it follows that $Lv = x \le x^* = Lu^*$ by (2.10). This implies by condition (L) that $v = L_+^{-1}x \le L_+^{-1}x^* = u^*$, whence $v \preceq u^*$. Since $u \preceq v$, then $u \preceq u^*$, and thus $u^*$ is the greatest solution of $Lu = Nu$ within the order interval $(\overline{u}]$ of $(V, \preceq)$.

To prove that $u^*$ is increasing with respect to $N$, assume that the hypotheses imposed on $N$ remain valid when $N$ is replaced by $\hat{N} : V \to P$, and that

$$\hat{N}u \le Nu \quad \text{for all } u \in V. \tag{2.20}$$

The above proof shows that the equation $Lu = \hat{N}u$ has the greatest solution $v$ of the form $v = L_+^{-1}y$. Applying (2.19) and (2.20) we see that $y = \hat{N}v \le Nv = G(y)$. This result and (2.10) imply that $y \le x^*$, which results in $v \le u^*$ by condition (L). This shows that $u^*$ is increasing with respect to $N$.

By dual reasoning one can show that the equation $Lu = Nu$ has the smallest solution $u_*$ within the order interval $[\underline{u})$ of $(V, \preceq)$, and that $u_*$ is increasing with respect to $N$ in $(V, \le)$. In particular, $u_*$ and $u^*$ are smallest and greatest solutions in the order interval $[\underline{u}, \overline{u}]$ of $(V, \preceq)$.

**Ad (c)** Assume that $N$ is increasing in $(V, \le)$. Then $N$ is increasing in $(V, \preceq)$. Let $u^*$ be the solution constructed in the proof of (a). By the above proof we have $u^* \le \overline{u}$. Let $u \in V$ be a solution of $Lu = Nu$ that satisfies $u \le \overline{u}$. Denoting $x = Lu$ and $v = L_+^{-1}x$, then $u \le v$ and $x = Lv = Nu \le N\overline{u} \le L\overline{u} = \overline{x}$, and $x = Lv \le Nv = NL_+^{-1}x = G(x)$. Since $x^*$ is the greatest fixed point of $G$ in $(\overline{x}]$, it follows that $x \le x^*$ by (2.10). This implies by condition (L) that $v = L_+^{-1}x \le L_+^{-1}x^* = u^*$. Because $u \le v$, we obtain $u \le u^*$. Thus $u^*$ is the greatest solution of $Lu = Nu$ within the order interval $(\overline{u}]$ of $(V, \le)$. The proof that $Lu = Nu$ has the smallest solution $u_*$ within the order interval $[\underline{u})$ of $(V, \le)$ is done in similar way. In particular, $u_*$ and $u^*$ are smallest and greatest solutions within the order interval $[\underline{u}, \overline{u}]$ of $(V, \le)$.

**Ad (d)** The hypotheses of Theorem 2.25 are valid, whence the equation $Lu = Nu$ has minimal and maximal solutions in $(V, \preceq)$. □

The hypotheses of Theorem 2.26 hold true if $L$, $N : V \to P$ and $c \in P$ fulfil the following conditions:

(L1) $L$ is a bijection and $L^{-1}$ is increasing.
(N1) $N$ is increasing, and strictly monotone sequences of $N[V]$ are finite.
(N2) $\sup\{c, x\}$ and $\inf\{c, x\}$ exist for every $x \in N[V]$.

Applying the algorithms (i) and (ii) of Corollary 2.23 to $G = N \circ L^{-1}$, we obtain the following result.

**Corollary 2.27.** *Let the hypotheses (L1), (N1), and (N2) hold for $L, N : V \to P$ and $c \in P$. Then the solutions $u^*$ and $u_*$ of the equation $Lu = Nu$ introduced in Theorem 2.26(b) are the last elements of the sequences determined by the following algorithms.*

*(iii)* $Lu_0 = c$. *For $n$ from* 0 *while* $Lu_n \neq Nu_n$ *do:* $Lu_{n+1} = Nu_n$ *if* $Nu_n < Lu_n$ *else* $Lu_{n+1} = \sup\{c, Nu_n\}$.
*(iv)* $Lu_0 = c$. *For $n$ from* 0 *while* $Lu_n \neq Nu_n$ *do:* $Lu_{n+1} = Nu_n$ *if* $Nu_n > Lu_n$ *else* $Lu_{n+1} = \inf\{c, Nu_n\}$.

The algorithms (iii) and (iv) can be used, for instance, to calculate exact or approximative solutions for the equations $Lu = Nu$ in $\mathbb{R}^m$, and hence also for systems of the form

$$L_i(u_1, \dots, u_m) = N_i(u_1, \dots, u_m), \quad i = 1, \dots, m, \tag{2.21}$$

where $L_i$ and $N_i$ are real-valued functions of $m$ real variables. Algorithms (iii) and (iv) are applied in Sect. 6.5 to calculate exact solutions of an implicit functional initial function problem.

In the case when the range of $N$ has an upper bound or a lower bound we have the following result.

**Proposition 2.28.** *Given posets $V$ and $P$ and mappings $L, N : V \to P$, assume that $L$ satisfies the hypothesis (L), and that $N$ is increasing in $(V, \leq)$ or in $(V, \preceq)$.*

*(a) If $N[V]$ has an upper bound in $P$, and if chains of $N[V]$ have infimums, then the equation $Lu = Nu$ has a minimal solution in $(V, \preceq)$. The equation has in $(V, \leq)$ the greatest solution, which is increasing with respect to $N$.*
*(b) If $N[V]$ has a lower bound in $P$, and if chains of $N[V]$ have supremums, then the equation $Lu = Nu$ has a maximal solution in $(V, \preceq)$. It has in $(V, \leq)$ the smallest solution, which is increasing with respect to $N$.*

**Proof: Ad (a)** Assume that $N[V] \subseteq (\overline{x}]$ for some $\overline{x} \in P$. Then $\overline{x}$ is an inf-center of $\mathrm{ocl}(N[V])$, so that the hypotheses of Theorem 2.25 are valid. Thus the equation $Lu = Nu$ has a minimal solution in $(V, \preceq)$.

Next we prove the existence of the greatest solution. The given hypotheses ensure that relation (2.19) defines an increasing mapping $G : P \to (\overline{x}]$. Moreover, the chains of $G[P] \subseteq N[V]$ have infimums in $P$. Thus $G$ has by Proposition 2.18 the greatest fixed point $x^*$ in $(\overline{x}]$. Denoting $u^* = L_+^{-1}x^*$, it follows that $u^*$ is a solution of the equation $Lu = Nu$.

Let $u \in V$ be a solution of $Lu = Nu$. Denoting $x = Lu$ and $v = L_+^{-1}x$, then $u \leq v$ and $Lu = Lv$, whence $u \preceq v$. Thus $x = Nu \leq Nv = NL_+^{-1}x = G(x) \leq \overline{x}$. Since $x^*$ is the greatest fixed point of $G$ in $(\overline{x}]$, we have $x \leq x^*$ by

(2.10). This implies by condition (L) that $v = L_+^{-1} x \leq L_+^{-1} x^* = u^*$. As $u \leq v$, we conclude $u \leq u^*$. Thus $u^*$ is the greatest solution of $Lu = Nu$ in $(V, \leq)$. The monotone dependence of $u^*$ with respect to $N$ can be shown in just the same way as in the proof of Theorem 2.26.

**Ad (b)** The proof of (b) is dual to the above proof. □

*Remark 2.29.* (i) If $Q : V \times P \to P$ is increasing with respect to the product ordering of $(V, \leq)$ and $(P, \leq)$, then $N := u \mapsto Q(u, Lu)$ is increasing with respect to the graph ordering $\preceq$ of $V$, defined by (2.17). Thus the result of Theorem 2.26(b) can be applied to the implicit problem $Lu = Q(u, Lu)$. Similarly, the result of Theorem 2.25 is applicable to the implicit inclusion problem $Lu \in \mathcal{Q}(u, Lu)$, where $\mathcal{Q} : V \times P \to 2^P \setminus \emptyset$.

(ii) In Sect. 8.6 we present results in the case when the functions $\mathcal{F}$, $G$, $N$, and $\mathcal{N}$ satisfy weaker monotonicity conditions as assumed above. The case when $V$ is not ordered is studied as well.

## 2.4 Special Cases

In this section we first formulate some fixed point results in ordered topological spaces derived in Sect. 2.2. Second, we present existence and comparison results for equations and inclusions in ordered normed spaces.

### 2.4.1 Fixed Point Results in Ordered Topological Spaces

Let $P = (P, \leq)$ be an ordered topological space, i.e., for each $a \in P$ the order intervals $[a) = \{x \in P : a \leq x\}$ and $(a] = \{x \in P : x \leq a\}$ are closed in the topology of $P$.

**Definition 2.30.** *A sequence $(z_n)_{n=0}^{\infty}$ of a poset is called* **increasing** *if $z_n \leq z_m$ whenever $n \leq m$,* **decreasing** *if $z_m \leq z_n$ whenever $n \leq m$, and* **monotone** *if it is increasing or decreasing. If the above inequalities are strict, the sequence $(z_n)_{n=0}^{\infty}$ is called strictly increasing, strictly decreasing, or strictly monotone, respectively.*

In what follows, we assume that $P$ has the following property:

(C) Each well-ordered chain $C$ of $P$ whose increasing sequences have limits in $P$ contains an increasing sequence that converges to $\sup C$, and each inversely well-ordered chain $C$ of $P$ whose decreasing sequences have limits in $P$ contains a decreasing sequence that converges to $\inf C$.

**Lemma 2.31.** *A second countable or metrizable ordered topological space has property (C).*

**Proof:** If $P$ is an ordered topological space that satisfies the second countability axiom, then each chain of $P$ is separable, whence $P$ has property (C) by the result of [133, Lemma 1.1.7] and its dual. If $P$ is metrizable, the assertion follows from [133, Proposition 1.1.5], and from its dual. □

The following result is a consequence of Proposition 2.18.

**Proposition 2.32.** *Given an ordered topological space $P$ with property (C), assume that $G : P \to P$ is an increasing function.*

*(a) If the set $S_+ = \{x \in P : x \leq G(x)\}$ is nonempty, and if $G$ maps increasing sequences of $S_+$ to convergent sequences, then $G$ has a maximal fixed point. Moreover, $G$ has for every $\underline{x} \in S_+$ the smallest fixed point in $[\underline{x})$, and it is increasing with respect to $G$.*
*(b) If the set $S_- = \{x \in P : G(x) \leq x\}$ is nonempty, and if $G$ maps decreasing sequences of $S_-$ to convergent sequences, then $G$ has a minimal fixed point. Moreover, $G$ has for every $\overline{x} \in S_-$ the greatest fixed point in $(\overline{x}]$, and it is increasing with respect to $G$.*

**Proof: Ad (a)** Let $C$ be a well-ordered chain in $S_+$. Since $G$ is increasing, then $G[C]$ is well-ordered. Every increasing sequence of $G[C]$ is of the form $(G(x_n))$, where $(x_n)$ is an increasing sequence in $C$. Thus the hypotheses of (a) and property (C) imply that $\sup G[C]$ exists in $P$, and, therefore, the conclusions of (a) follows from Proposition 2.18(a).

**Ad (b)** The conclusions of (b) are similar consequences of Proposition 2.18(b). □

The next result is a consequence of Theorem 2.17.

**Theorem 2.33.** *Given an ordered topological space $P$ with property (C), assume that $G : P \to P$ is increasing and maps monotone sequences of $P$ to convergent sequences.*

*(a) If $c$ is a sup-center of $\overline{G[P]}$ in $P$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the greatest fixed point $x^*$ in $(\overline{x}]$, where $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and $x^*$ are increasing with respect to $G$.*
*(b) If $c$ is an inf-center of $\overline{G[P]}$ in $P$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the smallest fixed point $x_*$ in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and $x_*$ are increasing with respect to $G$.*

As a consequence of Propositions 2.8 and 2.9 and Theorem 2.12 we obtain the following proposition.

**Proposition 2.34.** *Let $P$ be an ordered topological space with property (C), and let the values of $\mathcal{F} : P \to 2^P \setminus \emptyset$ be compact.*

*(a) If $\mathcal{F}$ is increasing upward, if the set $S_+ = \{x \in P : [x) \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, and if $(y_n)$ converges whenever it is increasing and $y_n \in \mathcal{F}(x_n)$, for every $n$, where $(x_n)$ is an increasing sequence of $S_+$, then $\mathcal{F}$ has a maximal fixed point.*

*(b) If $\mathcal{F}$ is increasing downward, if the set $S_- = \{x \in P : (x] \cap \mathcal{F}(x) \neq \emptyset\}$ is nonempty, and if $(y_n)$ converges whenever it is decreasing and $y_n \in \mathcal{F}(x_n)$, for every $n$, where $(x_n)$ is a decreasing sequence of $S_-$, then $\mathcal{F}$ has a minimal fixed point.*

*(c) If $\mathcal{F}$ is increasing, if $\overline{\mathcal{F}[P]}$ has a sup-center or an inf-center in $P$, and if $(y_n)$ converges whenever $y_n \in \mathcal{F}(x_n)$, for every $n$, and both $(x_n)$ and $(y_n)$ are either increasing or decreasing sequences of $P$, then $\mathcal{F}$ has minimal and maximal fixed points.*

The next theorem is a special case of Theorem 2.20.

**Theorem 2.35.** *Given an ordered topological space $X$ with property (C) and a subset $P$ of $X$, assume that a function $G : P \to P$ and a multifunction $\mathcal{F} : X \to 2^X$ have the following properties:*

*(ha) $G$ is increasing, $G[P]$ has an upper bound $\overline{x}$ in $P$, and $G$ maps every decreasing sequence $(x_n)$ of $P$ to a convergent sequence whose limit is in $P$.*

*(hb) $\overline{x}$ is an upper bound of $\mathcal{F}[X]$, and if $x \le p$ in $X$ and $p \in P$, then $G(p)$ is an upper bound of $\mathcal{F}(x)$.*

*Then $G$ has the greatest fixed point $x^*$, and if $x$ is any fixed point of $\mathcal{F}$, then $x \le x^*$.*

As a special case of Theorem 2.21 we formulate the following existence and comparison result for greatest fixed points of multifunctions.

**Theorem 2.36.** *Given an ordered topological space $X$ with property (C) and a subset $P$ of $X$. Assume that a multifunction $\mathcal{F} : X \to 2^X$ has the following properties:*

*(h0) $\mathcal{F}[X]$ has an upper bound in $P$.*

*(h1) If $p \in P$, then $\max \mathcal{F}(p)$ exists, belongs to $P$ and is an upper bound of $\mathcal{F}[X \cap (p]]$.*

*(h2) Decreasing sequences of $\{\max \mathcal{F}(p) : p \in P\}$ have limits in $X$ and they belong to $P$.*

*Then $\mathcal{F}$ has the greatest fixed point, and it belongs to $P$. Assume moreover, that $\hat{\mathcal{F}} : X \to 2^X$ is another multifunction that satisfies the following condition:*

*(h3) For each $x \in X$ and $y \in \hat{\mathcal{F}}(x)$ there exists a $z \in \mathcal{F}(x)$ such that $y \le z$.*

*Then the greatest fixed point of $\mathcal{F}$ majorizes all the fixed points of $\hat{\mathcal{F}}$.*

### 2.4.2 Equations and Inclusions in Ordered Normed Spaces

**Definition 2.37.** *A closed subset $E_+$ of a normed space $E$ is called an* **order cone** *if $E_+ + E_+ \subseteq E_+$, $E_+ \cap (-E_+) = \{0\}$, and $cE_+ \subseteq E_+$ for each $c \geq 0$. The space $E$, equipped with an order relation '$\leq$', defined by*

$$x \leq y \quad \text{if and only if} \quad y - x \in E_+,$$

*is called an* **ordered normed space**.

It is easy to see that the above defined order relation $\leq$ is a partial ordering in $E$.

**Lemma 2.38.** *An ordered normed space $E$ is an ordered topological space with respect to the weak and the norm topologies. Moreover, property (C) holds in both cases.*

**Proof:** It is easy to verify that the first assertion holds. To prove the second assertion, let $C$ be a well-ordered chain in $E$. If all increasing sequences of $C$ have weak limits, there is, by [44, Lemma A.3.1], an increasing sequence $(x_n)$ in $C$ that converges weakly to $x = \sup C$. If $C$ is inversely well-ordered and its decreasing sequences have weak limits, then $-C$ is a well-ordered chain whose increasing sequences have weak limits. Thus there exists an increasing sequence $(x_n)$ of $-C$ that converges weakly to $\sup(-C) = -\inf C$. Denoting $y_n = -x_n$, we obtain a decreasing sequence $(y_n)$ of $C$, which converges weakly to $\inf C$. If $E$ is equipped with norm topology, it is an ordered metric space, whence the conclusion follows from Lemma 2.31. □

The next fixed point result is a consequence of Proposition 2.32 and Lemma 2.38.

**Proposition 2.39.** *Let $P$ be a subset of an ordered normed space, and let $G : P \to P$ be increasing.*

*(a) If the set $S_+ = \{x \in P : x \leq G(x)\}$ is nonempty, and if $G$ maps increasing sequences of $S_+$ to sequences that have weak or strong limits in $P$, then $G$ has a maximal fixed point. Moreover, $G$ has for every $\underline{x} \in S_+$ the smallest fixed point in $[\underline{x})$, and it is increasing with respect to $G$.*
*(b) If the set $S_- = \{x \in P : G(x) \leq x\}$ is nonempty, and if $G$ maps decreasing sequences of $S_-$ to sequences that have weak or strong limits in $P$, then $G$ has a minimal fixed point. Moreover, $G$ has for every $\overline{x} \in S_-$ the greatest fixed point in $(\overline{x}]$, and it is increasing with respect to $G$.*

As a special case of Theorem 2.33 we obtain the following proposition.

**Proposition 2.40.** *Given a subset $P$ of an ordered normed space $E$, assume that $G : P \to P$ is increasing, and that monotone sequences of $G[P]$ have weak limits in $P$.*

*(a) If the weak closure of $G[P]$ has a sup-center $c$ in $P$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the greatest fixed point $x^*$ in $(\overline{x}]$, where $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and $x^*$ are increasing with respect to $G$.*

*(b) If the weak closure of $G[P]$ has an inf-center $c$ in $P$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the smallest fixed point $x_*$ in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and $x_*$ are increasing with respect to $G$.*

In case that $E$ is $\mathbb{R}^m$ equipped with Euclidean norm and ordered coordinatewise, we obtain the following consequence of Proposition 2.40.

**Corollary 2.41.** *Let $P$ be a closed and bounded subset of $\mathbb{R}^m$, and assume that $G : P \to P$ is increasing.*

*(a) If $P$ has a sup-center $c$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the greatest fixed point $x^*$ in $(\overline{x}]$, where $\overline{x}$ is the smallest solution of the equation $x = \sup\{c, G(x)\}$. Both $\overline{x}$ and $x^*$ are increasing with respect to $G$.*

*(b) If $P$ has an inf-center $c$, then $G$ has minimal and maximal fixed points. Moreover, $G$ has the smallest fixed point $x_*$ in $[\underline{x})$, where $\underline{x}$ is the greatest solution of the equation $x = \inf\{c, G(x)\}$. Both $\underline{x}$ and $x_*$ are increasing with respect to $G$.*

As a consequence of Proposition 2.28 and Lemma 2.38 we obtain the following proposition.

**Proposition 2.42.** *Given a poset $V$ and a subset $P$ of an ordered normed space, assume that mappings $L$, $N : V \to P$ satisfy the following hypotheses.*

*(L) The equation $Lu = x$ has for each $x \in P$ smallest and greatest solutions, and they are increasing with respect to $x$.*

*(N) $N$ is increasing in $(V, \leq)$ or in $(V, \preceq)$.*

*Then the following assertions hold.*

*(a) If $N[V]$ has an upper bound in $P$, and if decreasing sequences of $N[V]$ have weak or strong limits in $P$, then the equation $Lu = Nu$ has a minimal solution in $(V, \preceq)$. It has in $(V, \leq)$ the greatest solution, which is increasing with respect to $N$.*

*(b) If $N[V]$ has a lower bound in $P$, and if increasing sequences of $N[V]$ have weak or strong limits in $P$, then the equation $Lu = Nu$ has a maximal solution in $(V, \preceq)$. It has in $(V, \leq)$ the smallest solution, which is increasing with respect to $N$.*

In what follows, $E$ is an ordered normed space having the following properties.

(E0) Bounded and monotone sequences of $E$ have weak or strong limits.

(E1) $x^+ = \sup\{0, x\}$ exists, and $\|x^+\| \le \|x\|$ for every $x \in E$.

When $c \in E$ and $R \in [0, \infty)$, denote $B_R(c) := \{x \in E : \|x - c\| \le R\}$. Recall (cf., e.g., [227]) that if a sequence $(x_n)$ of a normed space $E$ converges weakly to $x$, then $(x_n)$ is bounded, i.e., $\sup_n \|x_n\| < \infty$, and

$$\|x\| \le \liminf_{n\to\infty} \|x_n\|. \tag{2.22}$$

The next auxiliary result is needed in the sequel.

**Lemma 2.43.** *If $c \in E$ and $R \in (0, \infty)$, then $c$ is an order center of $B_R(c)$, and for every chain $C$ of $B_R(c)$ both $\sup C$ and $\inf C$ exist and belong to $B_R(c)$.*

**Proof:** Since

$$\sup\{c, x\} = (x - c)^+ - c \quad \text{and} \quad \inf\{c, x\} = c - (c - x)^+, \quad \text{for all} \quad x \in E, \tag{2.23}$$

(E1) and (2.23) imply that

$$\|\sup\{c, x\} - c\| = \|\inf\{c, x\} - c\| = \|(x - c)^+\| \le \|x - c\| \le R$$

for every $x \in B_R(c)$. Thus both $\sup\{c, x\}$ and $\inf\{c, x\}$ belong to $B_R(c)$. Let $C$ be a chain in $B_R(c)$. Since $C$ is bounded, there is an increasing sequence $(x_n)$ in $C$ that converges weakly to $x = \sup C$ due to (E0) and Lemma 2.38. Since $\|x_n - c\| \le R$ for each $n$, it follows from (2.22) that

$$\|x - c\| \le \liminf_{n\to\infty} \|x_n - c\| \le R.$$

Thus $x = \sup C$ exists and belongs to $B_R(c)$. Similarly one can show that $\inf G[C]$ exists in $E$ and belongs to $B_R(c)$. $\square$

Applying Theorem 2.17 and Lemma 2.43 we obtain the following fixed point results.

**Theorem 2.44.** *Given a subset $P$ of $E$, assume that $G : P \to P$ is increasing, and that $G[P] \subseteq B_R(c) \subseteq P$ for some $c \in P$ and $R \in (0, \infty)$. Then $G$ has*

*(a) minimal and maximal fixed points;*
*(b) smallest and greatest fixed points $x_*$ and $x^*$ in the order interval $[\underline{x}, \overline{x}]$ of $P$, where $\underline{x}$ is the greatest solution of $x = \inf\{c, G(x)\}$, and $\overline{x}$ is the smallest solution of $x = \sup\{c, G(x)\}$.*

*Moreover, $x^*$, $x_*$, $\underline{x}$ and $\overline{x}$ are all increasing with respect to $G$.*

**Proof:** Let $C$ be a chain in $P$. Since $G[C]$ is a chain in $B_R(c)$, both $\sup G[C]$ and $\inf G[C]$ exist in $E$ and belong to $B_R(c) \subseteq P$ by Lemma 2.43. Because $c$ is an order center of $B_R(c)$ and $ocl(G[P]) \subseteq \overline{G[P]} \subseteq B_R(c) \subseteq P$, then $c$ is an order center of $ocl(G[P])$ in $P$. Thus the hypotheses of Theorem 2.17 are valid. $\square$

Next we assume also that $V = (V, \le)$ is a poset, and that $P \subseteq E$. By means of Theorem 2.26 we obtain the following results.

**Theorem 2.45.** *Assume that the hypothesis (L) holds for $L : V \to P$, that $N : V \to P$ is increasing in $(V, \leq)$ (respectively in $(V, \preceq)$), and that $N[V] \subseteq B_R(c) \subseteq P$ for some $c \in E$ and $R \in (0, \infty)$. Then the equation $Lu = Nu$ has*

*(a) minimal and maximal solutions in $(V, \preceq)$;*
*(b) smallest and greatest solutions $u_*$, $u^*$ within the order interval $[\underline{u}, \overline{u}]$ of $(V, \leq)$ (respectively $(V, \preceq)$), where $\underline{u}$ is the greatest solution of $Lu = \inf\{c, Nu\}$ in $V_- = \{\min L^{-1}\{x\} : x \in P\}$, and $\overline{u}$ is the smallest solution of $Lu = \sup\{c, Nu\}$ in $V_+ = \{\max L^{-1}\{x\} : x \in P\}$.*

*Moreover, $u_*$, $u^*$, $\underline{u}$ and $\overline{u}$ are all increasing with respect to $N$ in $(V, \leq)$.*

**Proof:** The given hypotheses imply by Lemma 2.43 (cf. the proof of Theorem 2.44) that the hypotheses of Theorem 2.26 hold. The assertions follow then from Theorem 2.26. □

Noticing that $\inf\{0, v\} = -(-v)^+$, the next result is a consequence of Theorem 2.45.

**Corollary 2.46.** *Given a poset $V$ and $R \in (0, \infty)$, assume that $L$, $N : V \to B_R(0)$ satisfy the following hypotheses.*

*(LN) $L$ is a bijection, and both $L^{-1}$ and $N$ are increasing.*

*Then the equation $Lu = Nu$ has*

*(a) minimal and maximal solutions in $(V, \preceq)$;*
*(b) smallest and greatest solutions $u_*$, $u^*$ within the order interval $[\underline{u}, \overline{u}]$ of $(V, \leq)$, where $\underline{u}$ is the greatest solution of $Lu = -(-Nu)^+$ and $\overline{u}$ is the smallest solution of $Lu = (Nu)^+$ in $(V, \leq)$.*

*Moreover, $u_*$, $u^*$, $\underline{u}$ and $\overline{u}$ are all increasing with respect to $N$ in $(V, \leq)$.*

In the set-valued case we have the following consequences of Theorems 2.12 and 2.25.

**Theorem 2.47.** *Assume that $P$ is a subset of $E$ which contains $B_R(c)$ for some $c \in E$ and $R \in (0, \infty)$. Then the following holds.*

*(a) Let $\mathcal{F} : P \to 2^P \setminus \emptyset$ be an increasing mapping whose values are weakly compact in $E$, and whose range $\mathcal{F}[P]$ is contained in $B_R(c)$. Then $\mathcal{F}$ has minimal and maximal fixed points.*
*(b) Assume that $\mathcal{N} : V \to 2^P \setminus \emptyset$ is increasing in $(V, \leq)$ or in $(V, \preceq)$, that its values are weakly compact in $E$, and that $\mathcal{N}[V] \subseteq B_R(c)$. If $L : V \to P$ satisfies the hypothesis (L), then the inclusion problem $Lu \in \mathcal{N}u$ has minimal and maximal solutions in $(V, \preceq)$.*

The next result is also a consequence of Theorem 2.45.

**Theorem 2.48.** *Given a lattice-ordered Banach space $E$ with properties (E0) and (E1) and a poset $W$, assume that mappings $\Lambda$, $F : W \to E$ satisfy the following hypotheses:*

($\Lambda$) *The equation $\Lambda u = v$ has for each $v \in E$ smallest and greatest solutions, and they are increasing with respect to $v$.*

(F) *$F$ is increasing.*

($\Lambda$F) *$\|F(u)\| \leq q(\|\Lambda u\|)$ for all $u \in W$, where $q : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing, and there exists a $R > 0$ such that $R = q(R)$, and, moreover, if $s \leq q(s)$, then $s \leq R$.*

*Then the equation*

$$\Lambda u = Fu \tag{2.24}$$

*has minimal and maximal solutions.*

**Proof:** Let $R > 0$ be the constant in the hypothesis ($\Lambda$F). Define

$$P := B_R(0), \quad \text{and} \quad V := \{u \in W : \|\Lambda u\| \leq R\}. \tag{2.25}$$

The growth condition ($\Lambda$F) implies that for each $u \in V$,

$$\|Fu\| \leq q(\|\Lambda u\|) \leq q(R) = R,$$

so that $F[V] \subseteq P$ by (2.25).

Defining $L = \Lambda_{|V}$ and $N = F_{|V}$, then $L, N : V \to P$, $L$ satisfies the hypothesis (L) and $N$ is increasing. Therefore, the results of Theorem 2.45 can be applied. In particular, it follows that the equation $Lu = Nu$ has minimal and maximal solutions. If $u \in W$ and $\Lambda u = Fu$, then

$$\|\Lambda u\| = \|Fu\| \leq q(\|\Lambda u\|)$$

by ($\Lambda$F), which implies that $\|\Lambda u\| \leq R$, i.e., $u \in V$. Thus all the solutions of (2.24) are contained in $V$, whence we conclude that (2.24) and the equation $Lu = Nu$ have the same solutions. In particular, (2.24) has minimal and maximal solutions. □

The next result is a direct consequence of Theorem 2.48

**Corollary 2.49.** *Let $E$ be a lattice-ordered Banach space $E$ with properties (E0) and (E1), let $W$ be a poset, and assume that $\Lambda : W \to E$ satisfies the hypothesis ($\Lambda$), and that $F : W \to E$ is increasing and bounded. Then the equation (2.24) has minimal and maximal solutions.*

**Proof:** Defining $q(s) \equiv R > \sup\{\|Fu\| : u \in W\}$, we see that the hypothesis ($\Lambda$F) holds. □

*Remark 2.50.* As an application of Theorem 2.45(b) one can formulate other existence results for equation (2.24). In particular, one can construct such solutions of (2.24) that are increasing with respect to $F$.

According to the hypothesis (F) the operator $F$ may be discontinuous and noncompact. Moreover, the growth condition of ($\Lambda$F) does not provide means to construct a priori upper and/or lower solutions for equation (2.24).

Thus the standard theories such as the theory of monotone operators due to Brezis and Browder, Schauder's fixed point theorem, or the method of sub- and supersolutions are, in general, not applicable under the hypotheses given above to solve (2.24).

Each of the following spaces has properties (E0) and (E1) (as for the proofs, see, e.g., [22, 44, 48, 118, 133, 152, 170]):

(a) A Sobolev space $W^{1,p}(\Omega)$ or $W_0^{1,p}(\Omega)$, $1 < p < \infty$, ordered a.e. pointwise, where $\Omega$ is a bounded domain in $\mathbb{R}^N$.
(b) A finite-dimensional normed space ordered by a cone generated by a basis.
(c) $l^p$, $1 \le p < \infty$, normed by $p$-norm and ordered coordinatewise.
(d) $L^p(\Omega)$, $1 \le p < \infty$, normed by $p$-norm and ordered a.e. pointwise, where $\Omega$ is a $\sigma$-finite measure space.
(e) A separable Hilbert space whose order cone is generated by an orthonormal basis.
(f) A weakly complete Banach lattice or a UMB-lattice (cf. [22]).
(g) $L^p(\Omega, Y)$, $1 \le p < \infty$, normed by $p$-norm and ordered a.e. pointwise, where $\Omega$ is a $\sigma$-finite measure space and $Y$ is any of the spaces (b)–(f).
(h) Newtonian spaces $N^{1,p}(Y)$, $1 < p < \infty$, ordered a.e. pointwise, where $Y$ is a metric measure space.

Thus the results of Theorems 2.44–2.48 hold if $E$ is any of the spaces listed in (a)–(g).

## 2.5 Fixed Point Results for Maximalizing Functions

In this section we prove fixed point results for a self-mapping $G$ of a poset $P$ by assuming that $G$ is **maximalizing**, i.e., $G(x)$ is a maximal element of $\{x, G(x)\}$ for all $x \in P$. Concrete examples of maximalizing functions that have or don't have fixed points are presented. The generalized iteration method introduced in Lemma 2.2 is used in the proofs.

### 2.5.1 Preliminaries

The following result helps to analyze the w-o chain of $cG$-iterations defined in (2.2).

**Lemma 2.51.** *Let $A$ and $B$ be nonempty subsets of $P$. If* $\sup A$ *and* $\sup B$ *exists, then the equation*

$$\sup(A \cup B) = \sup\{\sup A, \sup B\} \tag{2.26}$$

*is valid whenever either of its sides is defined.*

**Proof:** The sets $A \cup B$ and $\{\sup A, \sup B\}$ have the same upper bounds, which implies the assertion. □

Recall that a subset $W$ of a chain $C$ is called an *initial segment* of $C$ if $x \in W$, $y \in C$, and $y < x$ imply $y \in W$. If $W$ is well-ordered, then every element $x$ of $W$ that is not the possible maximum of $W$ has a **successor**: $Sx = \min\{y \in W : x < y\}$, in $W$.

A characterization of elements of the w-o chain of $cG$-iterations, defined by (2.2), is provided by the following lemma.

**Lemma 2.52.** *Given $G : P \to P$ and $c \in P$, and let $C$ be the w-o chain of $cG$-iterations. Then the elements of $C$ have the following properties:*

*(a)* $\min C = c$.
*(b) An element $x$ of $C$ has a successor in $C$ if and only if* $\sup\{x, G(x)\}$ *exists and $x < \sup\{x, G(x)\}$, and then $Sx = \sup\{x, G(x)\}$.*
*(c) If $W$ is an initial segment of $C$ and $y = \sup W$ exists, then $y \in C$.*
*(d) If $c < y \in C$ and $y$ is not a successor, then $y = \sup C^{<y}$.*
*(e) If $y = \sup C$ exists, then $y = \max C$.*
*(f) If $x_* = \sup\{c, G[C]\}$ exists in $P$, then $x_* = \max C$, and $G(x_*) \leq x_*$.*

**Proof: Ad (a)** $\min C = \sup\{c, G[C^{<\min C}]\} = \sup\{c, G[\emptyset]\} = \sup\{c, \emptyset\} = c$.

**Ad (b)** Assume first that $x \in C$, and that $Sx$ exists in $C$. Applying (2.2), Lemma 2.51, and the definition of $Sx$ we obtain

$$Sx = \sup\{c, G[C^{<Sx}]\} = \sup\{c, G[C^{<x}] \cup \{G(x)\}\} = \sup\{x, G(x)\}.$$

Moreover, $x < Sx$, by definition, whence $x < \sup\{x, G(x)\}$.

Assume next that $x \in C$, that $y = \sup\{x, G(x)\}$ exists, and that $x < \sup\{x, G(x)\}$. The above proof implies that

(i) there is no element $w \in C$ that satisfies $x < w < \sup\{x, G(x)\}$.

Then $\{z \in C : z \leq x\} = C^{<y}$, so that

$$\begin{aligned} x < \sup\{x, G(x)\} &= \sup\{\sup\{c, G[C^{<x}]\}, G(x)\} \\ &= \sup\{\{c\} \cup G[C^{<x}] \cup \{G(x)\}\} = \sup\{c, G[\{z \in C : z \leq x\}]\} \\ &= \sup\{c, G[C^{<y}]\}. \end{aligned}$$

Thus $y = \sup\{x, G(x)\} \in C$ by (2.2). This result and (i) imply that $y = \sup\{x, G(x)\} = \min\{z \in C : x < z\} = Sx$.

**Ad (c)** Assume that $W$ is an initial segment of $C$, and that $y = \sup W$ exists. If there is $x \in W$ that does not have the successor, then $x = \max W = y$, so that $y \in C$. Assume next that every element $x$ of $W$ has the successor $Sx$ in $W$. Since $Sx = \sup\{x, G(x)\}$ by (b), then $G(x) \leq Sx < y$. This holds for all $x \in W$. Since $c = \min C = \min W < y$, then $y$ is an upper bound of $\{c\} \cup G[W]$. If $z$ is an upper bound of $\{c\} \cup G[W]$, then $x = \sup\{c, G[C^{<x}]\} = \sup\{c, G[W^{<x}]\} \leq z$ for every $x \in W$. Thus $z$ is an upper bound of $W$, whence

$y = \sup W \le z$. But then $y = \sup\{c, G[W]\} = \sup\{c, G[C^{<y}]\}$, so that $y \in C$ by (2.2).

**Ad (d)** Assume that $c < y \in C$, and that $y$ is not a successor of any element of $C$. Obviously, $y$ is an upper bound of $C^{<y}$. Let $z$ be an upper bound of $C^{<y}$. If $x \in C^{<y}$, then also $Sx \in C^{<y}$ since $y$ is not a successor. Because $Sx = \sup\{x, G(x)\}$ by (b), then $G(x) \le Sx \in C^{<y}$. This holds for every $x \in C^{<y}$. Since also $c \in C^{<y}$, then $z$ is an upper bound of $\{c\} \cup G[C^{<y}]$. Thus $y = \sup\{c, G[C^{<y}]\} \le z$. This holds for every upper bound $z$ of $C^{<y}$, whence $y = \sup C^{<y}$.

**Ad (e)** If $y = \sup C$ exists, then $y \in C$ by (c) when $W = C$, whence $y = \max C$.

**Ad (f)** Assume that $x_* = \sup\{c, G[C]\}$ exists. If $x \in C$, then $x = \sup\{c, G[C^{<x}] \le \sup\{c, G[C]\} = x_*$, whence $x_*$ is an upper bound of $C$. If $x_*$ is a strict upper bound of $C$, then $C = C^{<x_*}$, so that $x_* = \sup\{c, G[C^{<x_*}]\}$. But then $x_* \in C$ by (2.2), and $x_*$ is a strict upper bound of $C$, a contradiction. Thus $x_* = \max C$. In particular, $G(x_*) \in G[C]$, whence $G(x_*) \le \sup\{c, G[C]\} = x_*$. □

### 2.5.2 Main Results

Let $P = (P, \le)$ be a nonempty poset. As an application of Lemma 2.52(f) we shall prove our first existence result.

**Theorem 2.53.** *A function $G : P \to P$ has a fixed point if $G$ is* maximalizing*, i.e., $G(x)$ is a maximal element of $\{x, G(x)\}$ for all $x \in P$, and if $x_* = \sup\{c, G[C]\}$ exists in $P$ for some $c \in P$ where $C$ is the w-o chain of $cG$-iterations.*

**Proof:** If $C$ is the w-o chain of $cG$-iterations, and if $x_* = \sup\{c, G[C]\}$ exists in $P$, then $x_* = \max C$ and $G(x_*) \le x_*$ by Lemma 2.52(f). Since $G$ is maximalizing, then $G(x_*) = x_*$, i.e., $x_*$ is a fixed point of $G$. □

The following proposition is a consequence of Theorem 2.53 and Lemma 2.52(b),(e).

**Proposition 2.54.** *Assume that $G : P \to P$ is maximalizing. Given $c \in P$, let $C$ be a w-o chain of $cG$-iterations. If $z = \sup C$ exists, then $z$ is a fixed point of $G$ if and only if $x_* = \sup\{z, G(z)\}$ exists.*

**Proof:** Assume that $z = \sup C$ exists. It follows from Lemma 2.52(e) that $z = \max C$. If $z$ is a fixed point of $G$, i.e., $z = G(z)$, then $x_* = \sup\{z, G(z)\} = z$.

Assume next that $\sup\{z, G(z)\}$ exists. Since $Sz$ (successor) does not exist, it follows from Lemma 2.52(b) that $z \not< \sup\{z, G(z)\}$. Thus $x_* = \sup\{z, G(z)\} = z$, so that $G(z) \le z$. Moreover, $G(z) \not< z$ because $G$ is maximalizing, whence $G(z) = z$. □

As a consequence of Proposition 2.54 we obtain the following corollary.

**Corollary 2.55.** *Assume that all nonempty chains of $P$ have supremums in $P$. If $G : P \to P$ is maximalizing, and if* $\sup\{x, G(x)\}$ *exists for all $x \in P$, then for each $c \in P$ the maximum of the w-o chain of $cG$-iterations exists and is a fixed point of $G$.*

**Proof:** Let $C$ be the w-o chain of $cG$-iterations. The given hypotheses imply that both $z = \sup C$ and $x_* = \sup\{z, G(z)\}$ exist. Thus the hypotheses of Proposition 2.54 are valid. □

For completeness we formulate the obvious duals of the above results.

**Theorem 2.56.** *A function $G : P \to P$ has a fixed point if $G$ is* minimalizing, *i.e., $G(x)$ is a minimal element of $\{x, G(x)\}$ for all $x \in P$, and if* $\inf\{c, G[W]\}$ *exists in $P$ for some $c \in P$ whenever $W$ is a non-empty chain in $P$.*

**Proposition 2.57.** *A minimalizing function $G : P \to P$ has a fixed point if every nonempty chain $P$ has the infimum in $P$, and if* $\inf\{x, G(x)\}$ *exists for all $x \in P$.*

*Remark 2.58.* The hypothesis that $G : X \to X$ is maximalizing can be weakened in Theorem 2.53 and in Proposition 2.54 to the form: $G|\{x_*\}$ is maximalizing, i.e., $G(x_*)$ is a maximal element of $\{x_*, G(x_*)\}$.

### 2.5.3 Examples and Remarks

We shall first present an example of a maximalizing mapping whose fixed point is obtained as the maximum of the w-o chain of $cG$-iterations.

*Example 2.59.* Let $P$ be a closed disc $P = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 \le 2\}$, ordered coordinatewise. Let $[u]$ denote the greatest integer $\le u$ when $u \in \mathbb{R}$. Define a function $G : P \to \mathbb{R}^2$ by

$$G(u, v) = \left(\min\{1, 1 - [u] + [v]\}, \frac{1}{2}([u] + v^2)\right), \quad (u, v) \in P. \tag{2.27}$$

It is easy to verify that $G[P] \subset P$, and that $G$ is maximalizing. To find a fixed point of $G$, choose $c = (1, 0)$. It follows from Lemma 2.52 (a) and (b) that the first elements of the w-o chain of $cG$-iterations are successive approximations

$$x_0 = c, \quad x_{n+1} = Sx_n = \sup\{x_n, G(x_n)\}, \quad n = 0, 1, \ldots, \tag{2.28}$$

as long as $Sx_n$ is defined. Denoting $x_n = (u_n, v_n)$, these successive approximations can be rewritten in the form

$$\begin{aligned} &u_0 = 1,\ u_{n+1} = \max\{u_n, \min\{1, 1 - [u_n] + [v_n]\}\}, \\ &v_0 = 0,\ v_{n+1} = \max\{v_n, \frac{1}{2}([u_n] + v_n^2)\}, \quad n = 0, 1, \ldots, \end{aligned} \tag{2.29}$$

as long as $u_n \le u_{n+1}$ and $v_n \le v_{n+1}$, and at least one of these inequalities is strict. Elementary calculations show that $u_n = 1$, for every $n \in \mathbb{N}_0$. Thus (2.29) can be rewritten as

$$u_n = 1,\ v_0 = 0,\ v_{n+1} = \max\{v_n, \frac{1}{2}(1+v_n^2)\}, \quad n = 0, 1, \dots. \tag{2.30}$$

Since the function $g(v) = \frac{1}{2}(1+v^2)$ is increasing in $\mathbb{R}_+$, then $v_n < g(v_n)$ for every $n = 0, 1, \dots$. Thus (2.30) can be reduced to the form

$$u_n = 1,\ v_0 = 0,\ v_{n+1} = g(v_n) = \frac{1}{2}(1+v_n^2), \quad n = 0, 1, \dots. \tag{2.31}$$

The sequence $(g(v_n))_{n=0}^{\infty}$ is strictly increasing, whence also $(v_n)_{n=0}^{\infty}$ is strictly increasing by (2.31). Thus the set $W = \{(1, g(v_n))\}_{n \in \mathbb{N}_0}$ is an initial segment of $C$. Moreover, $v_0 = 0 < 1$, and if $0 \le v_n < 1$, then $0 < g(v_n) < 1$. Since $(g(v_n))_{n=0}^{\infty}$ is bounded above by 1, then $v_* = \lim_n g(v_n)$ exists, and $0 < v_* \le 1$. Thus $(1, v_*) = \sup W$, and it belongs to $X$, whence $(1, v_*) \in C$ by Lemma 2.52(c). To determine $v_*$, notice that $v_{n+1} \to v_*$ by (2.31). Thus $v_* = g(v_*)$, or equivalently, $v_*^2 - 2v_* + 1 = 0$, so that $v_* = 1$. Since $\sup W = (1, v_*) = (1, 1)$, then $(1, 1) \in C$ by Lemma 2.52(c). Because $(1, 1)$ is a maximal element of $X$, then $(1, 1) = \max C$. Moreover, $G(1, 1) = (1, 1)$, so that $(1, 1)$ is a fixed point of $G$.

The first $m + 1$ elements of the w-o chain $C$ of $cG$-iterations can be estimated by the following Maple program (floor$(\cdot)$=$[\cdot]$):

$x := min\{1, 1 - \text{floor}(u) + \text{floor}(v)\} : y := (\text{floor}(u) + v^2)/2 :$

$(u, v) := (1, 0) : c[0] := (u, v) :$

for $n$ to $m$ do $(u, v) := (\max\{x, u\}, \text{evalf}(\max\{y, v\}); c[n] := (u, v)$ end do;

For instance, c[100000]=(1,0.99998).

The verification of the following properties are left to the reader:

- If $c = (u, v) \in X$, $u < 1$ and $v < 1$, then the elements of w-o chain $C$ of $cG$-iterations, after two first terms if $u < 1$, are of the form $(1, w_n)$, $n = 0, 1, \dots$, where $(w_n)_{n=0}^{\infty}$ is increasing and converges to 1. Thus $(1, 1)$ is the maximum of $C$ and a fixed point of $G$.
- If $c = (u, 1)$, $u < 1$, or $c = (1, -1)$, then $C = \{c, (1, 1)\}$.
- If $= (1, 0)$, then $G^{2k}c = (1, z_k)$ and $G^{2k+1}c = (0, y_k)$, $k \in \mathbb{N}_0$, where the sequences $(z_k)$ and $(y_k)$ are bounded and increasing. The limit $z$ of $(z_k)$ is the smaller real root of $z^4 - 8z + 4 = 0$; $z \approx 0.50834742498666121699$, and the limit $y$ of $(y_k)$ is $y = \frac{1}{2}z^2 \approx 0.12920855224528457650$. Moreover $G(1, y) = (0, z)$ and $G(0, z) = (1, y)$, whence no subsequence of the iteration $(G^n c)$ converges to a fixed point of $G$.
- For any choice of $c = (u, v) \in P \setminus \{(1, 1)\}$ the iterations $G^n c$ and $G^{n+1}c$ are not order related when $n \ge 2$. The sequence $(G^n c)$ does not converge, and no subsequence of it converges to a fixed point of $G$.

- Denote $Y = \{(u,v) \in \mathbb{R}^2_+ : u^2 + v^2 \le 2, v > 0\} \cup \{(1,0)\}$. The function $G$, defined by (2.27) satisfies $G[Y] \subset Y$, and is maximalizing. The maximum of the w-o chain of $cG$-iterations with $c = (1,0)$ is $x_* = (1,1)$, and $x_*$ is a fixed point of $G$. If $x \in Y \setminus \{x_*\}$, then $x$ and $G(x)$ are not comparable.

The following example shows that $G$ need not have a fixed point if either of the hypothesis of Theorem 2.53 is not valid.

*Example 2.60.* Denote $a = (1,y)$ and $b = (0,z)$, where $y$ and $z$ are as in Example 2.59. Choose $X = \{a,b\}$, and let $G : X \to X$ be defined by (2.27). $G$ is maximalizing, but $G$ has no fixed points, since $G(a) = b$ and $G(b) = a$. The last hypothesis of Theorem 2.53 is not satisfied.

Denoting $c = (1,z)$, then the set $X = \{a,b,c\}$ is a complete join lattice, i.e., every nonempty subset of $X$ has the supremum in $X$. Let $G : X \to X$ satisfy $G(a) = b$ and $G(b) = G(c) = a$. $G$ has no fixed points, but $G$ is not maximalizing, since $G(c) < c$.

*Example 2.61.* The components: $u = 1$, $v = 1$ of the fixed point of $G$ in Example 2.59 form also a solution of the system

$$u = \min\{1, 1 - [u] + [v]\}, \qquad v = \frac{[u] + v^2}{2}.$$

Moreover, a Maple program introduced in Example 2.59 serves a method to estimate this solution. When $m = 100000$, the estimate is: $u = 1$, $v = .99998$.

*Remark 2.62.* (i) The standard ‘solve’ and ‘fsolve’ commands of Maple 13 don’t give a solution or its approximation for the system of Example 2.61.

(ii) In Example 2.59 the mapping $G$ is non-increasing, non-extensive, non-ascending, not semi-increasing upward, and non-continuous.

(iii) The generalized iteration method presented in Lemma 2.2 is based on the w-o chain of $cG$-iterations, defined by (2.2), where $G$ is a self-mapping of a poset $P$ and $c \in P$. In the case when $c \le G(c)$, this chain equals to the w-o chain $C = C(c)$ of $G$-iterations of $c$, defined by

$$c = \min C, \text{ and } x \in C \setminus \{c\} \text{ if and only if } x = \sup G[C^{<x}]. \tag{2.32}$$

(iv) As for the use of $C(c)$ in fixed point theory and in the theory of discontinuous differential and integral equations, see, e.g., [44, 133] and the references therein.

(v) Chain $C(c)$ is compared in [175] with three other chains that generalize the sequence of ordinary iterations $(G^n(c))_{n=0}^\infty$, and which are used to prove fixed point results for $G$. These chains are: the generalized orbit $O(c)$ defined in [175] (being identical with the set $W(c)$ defined in [1]), the smallest admissible set $I(c)$ containing $c$ (cf. [33, 34, 115]), and the smallest complete $G$-chain $B(c)$

containing $c$ (cf. [96, 175]). If $G$ is extensive, and if nonempty chains of $X$ have supremums, then $C(c) = O(c) = I(c)$, and $B(c)$ is their cofinal subchain (cf. [175], Corollary 7). The common maximum $x_*$ of these four chains is a fixed point of $G$. This result implies Bourbaki's Fixed Point Theorem, cf. [28, p. 37].

(vi) On the other hand, if the hypotheses of Theorem 2.56 hold and $x \in C(c) \setminus \{c, x_*\}$, then $x$ and $G(x)$ are not necessarily comparable. The successor of such an $x$ in $C(c)$ is $\sup\{x, G(x)\}$ by [115, Prop. 5]. In such a case the chains $O(c)$, $I(c)$, and $B(c)$ attain neither $x$ nor any fixed point of $G$. For instance when $c = (0,0)$ in Example 2.59, then $C(c) = \{(0,0)\} \cup C$, where $C$ is the w-o chain of $(1,0)G$-iterations. Since $(G^n(0,0))_{n=0}^{\infty} = \{(0,0)\} \cup (G^n(1,0))_{n=0}^{\infty}$, then $B(c)$ does not exist, $O(c) = I(c) = \{(0,0),(1,0)\}$ (see [175]). Thus only $C(c)$ attains a fixed point of $G$ as its maximum. As shown in Example 2.59, the consecutive elements of the iteration sequence $(G^n(1,0))_{n=0}^{\infty}$ are unordered, and their limits are not fixed points of $G$. Hence, in these examples also finite combinations of chains $W(c_i)$ used in [15, Theorem 4.2] to prove a fixed point result are insufficient to attain a fixed point of $G$. As for other examples of such cases, see [110, Example 3], and [115, Example 16].

(vii) Neither the above mentioned four chains nor their duals are available to find fixed points of $G$ if $a$ and $G(a)$ are not ordered. For instance, they cannot be applied to prove Theorems 2.53 and 2.56 or Propositions 2.54 and 2.57.

## 2.6 Notes and Comments

In Sect. 2.1, the Chain Generating Recursion Principle is established and applied to derive generalized iteration methods. As noticed in the Introduction, the argumentation in the proof of the Chain Generating Recursion Principle is similar to that used in [231] to prove Zermelo's first well-ordering theorem. The importance of such an argumentation to set theory, to fixed point theory in posets, to theories of inductive definitions, and to computer science is studied in [147].

The recursion and iteration methods presented in Sect. 2.1 are applied to prove fixed point results and existence and comparison results for solutions of operator equations and inclusions in Sects. 2.2–2.5. The material of these sections is based on papers [52, 117, 119, 121, 125, 127].

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Fixed Point Theory in Ordered Sets and Applications
From Differential and Integral Equations to Game
Theory
Carl, S.; Heikkilä, S.
2011, XIII, 477 p., Hardcover
ISBN: 978-1-4419-7584-3