Chapter 2
Delayed Negative Feedback: A Warm-Up

Abstract A great deal about delay differential equations can be learned by a study of its simplest representative, the linear delayed negative feedback equation. We use it to illustrate features common to delay differential equations, such as the tendency of delays to give rise to oscillations that can become undamped if delays are large. The obstructions to solving delay differential equations backwards in time are readily appreciated for this simple equation. It is an unpleasant fact that even for this simple linear equation, the stability of the trivial equilibrium requires an analysis of the roots of a transcendental equation. We show how the leading root of this transcendental equation signals the oscillation in solutions of the delay differential equation.

2.1 Preliminaries

The simplest delay differential equation is given by

$$u'(t) = -u(t-\tau) \tag{2.1}$$

where $\tau > 0$ is called the delay and the negative sign on the right indicates negative feedback. When $\tau = 0$, we recover the simple ODE

$$u'(t) = -u(t) \tag{2.2}$$

whose general solution, $u(t) = u(0)e^{-t}$, decays to zero.

If we prescribe $u(t)$ for $-\tau \le t \le 0$, then Equation (2.1) should have a unique solution for $t > 0$. Suppose we set

$$u(t) = 1, -\tau \le t \le 0 \tag{2.3}$$

as "initial data" for (2.1). Then, on the interval $0 \le t \le \tau$ the argument of u on the right side satisfies $t - \tau \le 0$ so

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$$u'(t) = -u(t-\tau) = -1$$

and therefore

$$u(t) = u(0) + \int_0^t (-1)ds = 1 - t, 0 \le t \le \tau. \tag{2.4}$$

On $\tau \le t \le 2\tau$, we have $0 \le t - \tau \le \tau$ so by (2.4) we have

$$u'(t) = -u(t-\tau) = -[1-(t-\tau)]$$

and thus

$$\begin{aligned} u(t) &= u(\tau) + \int_\tau^t -[1-(s-\tau)])ds \\ &= 1 - \tau + [-s + \frac{1}{2}(s-\tau)^2|_{s=\tau}^{s=t} \\ &= 1 - t + (t-\tau)^2/2, \tau \le t \le 2\tau. \end{aligned} \tag{2.5}$$

In exercise 2.1, we ask the reader to verify that

$$u(t) = 1 + \sum_{k=1}^{n} (-1)^k \frac{[t-(k-1)\tau]^k}{k!}, (n-1)\tau \le t < n\tau, n \ge 1. \tag{2.6}$$

Thus, $u(t)$ is a polynomial of degree n on each subinterval of the form $[(n-1)\tau, n\tau)$. It follows that $u(t)$ is a smooth function, except at each $n\tau$, $n \ge 0$. The formulas (2.4), (2.5), and (2.6) imply that

(a) $u'(0-) = 0$ and $u'(0+) = -1$ so u' has a jump discontinuity at $t = 0$.
(b) $u''(\tau-) = 0$ and $u''(\tau+) = 1$ so u'' has a jump discontinuity at $t = \tau$.
(c) $u^{(n)}((n-1)\tau-) = 0$ and $u^{(n)}((n-1)\tau+) = (-1)^n$.
(d) u is n-times continuously differentiable on $((n-1)\tau, \infty)$ for $n \ge 0$.

Here, $u^{(j)}(s+)$ denotes the limit of the j-th derivative of u as $t \to s$, $t > s$ and $u^{(j)}(s-)$ denotes the limit as $t \to s$, $t < s$. A key point is that u gets smoother as t increases.

The procedure used to solve the initial-value problem (2.1) and (2.3) above is called the method of steps for obvious reasons.

Let's begin by exploring this solution numerically by using the MATLAB DDE23 package. We want to investigate the behavior of the solution on the interval $t > 0$ for different values of the delay τ. Figure 2.1 shows the results.

Notice that the case $\tau = 0.25$ looks very much like the solution of the ODE (2.2) with $u(0) = 1$, namely it decays to zero without "overshooting" zero, i.e., it does not oscillate. When $\tau = 0.6$ the solution oscillates. In fact, despite appearances, it repeatedly changes sign. One can prove that all solutions oscillate whenever $\tau > e^{-1}$. Why does $\tau > e^{-1} \approx 0.36$ result in oscillations? We answer this later in the chapter. As τ increases the oscillations appear to be more pronounced but still they are damped. That is, it appears that the amplitude is decreasing, at least until $\tau = 2$ where now the amplitude grows.

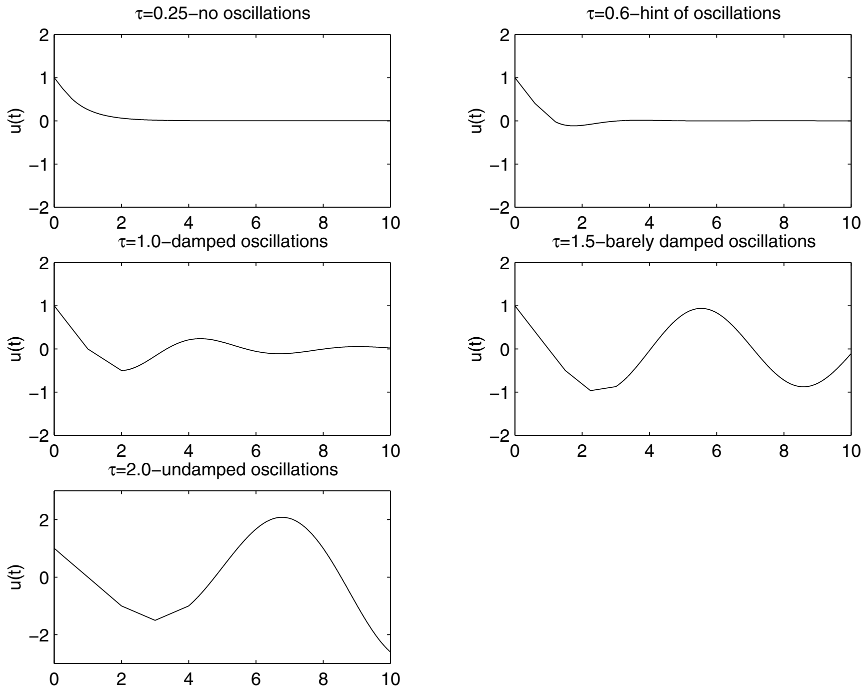

Fig. 2.1 Solution of equation (2.1) with initial data (5.1) for various τ

The reader will notice that $u \equiv 0$ is a solution of the delay differential equation (2.1); we call it a steady-state solution. We prove that it is stable when $\tau < \pi/2$ and unstable when $\tau > \pi/2 \approx 1.58$.

Equation (2.1) provides a simple illustration of what can go wrong in a negative feedback system with delays. Say you want to maintain a certain quantity u at the value $u = 0$. Imagine u satisfies a simple equation such as

$$u'(t) = c(t)$$

where you can prescribe the control $c(t)$ in order to accomplish your objective of maintaining u near zero. The system, of course, is subject to unexpected perturbations so you better be ready to handle these. If you observe that $u(t) > 0$ ($u(t) < 0$), you will want to choose $c(t) < 0$ ($c(t) > 0$). You might want to choose

$$c(t) = -\alpha u(t), \alpha > 0$$

because then no matter what value u is at time t_0, it will return to $u = 0$. For simplicity here, we take the "gain" parameter $\alpha = 1$. But consider how you would implement this feedback law. You must observe the system at time t to find $u(t)$ and then immediately respond with the control $c(t) = -u(t)$. This is clearly impossible;

there will necessarily be some time delay between the observation of the system and the implementation of the control. Therefore, an achievable control strategy would be

$$c(t) = -u(t-\tau)$$

where $\tau > 0$ is the delay between observing u and implementing the control. This results in (2.1). If this delay is too large you will not accomplish your goal. If $\tau > e^{-1}$, u will repeatedly oscillate above and below the desired setting after perturbation away from $u = 0$. But at least it will eventually get so near $u = 0$ that you would be happy. However, if the delay exceeds $\pi/2$, then your goal is unattainable with this control strategy.

2.2 The Simplest Delay Equation

We now begin a more systematic study of delayed feedback by considering the more general equation

$$u'(t) = -\alpha u(t-r) \tag{2.7}$$

where α is real and $r \geq 0$. The case $\alpha > 0$ is of most interest inasmuch as it corresponds to negative feedback but we also consider the positive feedback case $\alpha < 0$. When $r = 0$, $u = 0$ is an asymptotically stable steady state for the case of negative feedback; it is unstable for positive feedback. What happens when $r > 0$?

Scaling can reduce the number of parameters and produce simpler equations. Here, there are two natural choices. By a scaling of time: $\tau = \eta t,\ \eta > 0$, the equation for $U(\tau) = u(t)$ becomes:

$$\frac{dU}{d\tau} = \eta^{-1}\frac{du}{dt} = -\alpha\eta^{-1}u(t-r) = -\alpha\eta^{-1}U(\tau - r\eta)$$

If we take $\eta = 1/r$ and $\beta = \alpha r$ we get

$$\frac{dU}{d\tau} = -\beta U(\tau - 1). \tag{2.8}$$

We could let $\eta = |\alpha|$ and $s = r|\alpha|$, resulting in

$$\frac{dU}{d\tau} = \pm U(\tau - s)$$

where the sign is determined by that of α. Both forms are attractive but we choose to work with (2.8).

In order to determine stability of the trivial solution, we proceed exactly as for ODEs. That is, we seek (complex) values of λ such that $U(\tau) = \exp(\lambda\tau)$ is a solution of (2.8).

It is convenient to introduce the linear operator, defined on the differentiable functions, by

$$L(U) = \frac{dU}{d\tau} + \beta U(\tau - 1)$$

Then

$$L(e^{\lambda\tau}) = \lambda e^{\lambda\tau} + \beta e^{\lambda(\tau-1)} = e^{\lambda\tau}[\lambda + \beta e^{-\lambda}] \tag{2.9}$$

Clearly, we have $L(e^{\lambda\tau}) \equiv 0$ (the zero function!) if and only if λ is a root of the *characteristic equation*

$$h(\lambda) \equiv \lambda + \beta e^{-\lambda} = 0. \tag{2.10}$$

We call $\lambda \in \mathbb{C}$ a root of (2.10) of order l, where $l \geq 1$, if

$$h(\lambda) = h'(\lambda) = h''(\lambda) = \cdots = h^{(l-1)}(\lambda) = 0,\ h^{(l)}(\lambda) \neq 0.$$

Lemma 2.1. $\tau^j e^{\lambda\tau},\ j = 0, 1, \ldots, k$ *are solutions of* (2.8) *if and only if* λ *is a root of order at least* $k+1$ *of* h.

Proof. Differentiating (2.9) k times with respect to λ and using that this kth-derivative commutes with L we find, by Leibniz' rule for the derivative of a product, that

$$L(\tau^k e^{\lambda\tau}) = (\frac{\partial}{\partial\lambda})^k [e^{\lambda\tau} h(\lambda)] = e^{\lambda\tau}[\sum_{j=0}^{k} C_j^k h^{(j)}(\lambda)\tau^{k-j}]$$

where $C_j^k = k!/j!(k-j)!$ are the binomial coefficients. The result follows immediately from this observation. $\square$

Alternatively, (2.10) can be obtained by using the Laplace transform.

Based on our experience with ODEs, we have a right to expect that the trivial solution is asymptotically stable if $\Re(\lambda) < 0$ for all roots λ of the characteristic equation and that it is unstable if there is a root with positive real part. We assume this now; it is proved later.

As h is an analytic function of the complex variable λ it has the following elementary properties. See Appendix A.

(A) The set of roots can have no accumulation point in $\mathbb{C}$; therefore, for each $R > 0$, the set of roots satisfying $|\lambda| \leq R$ is finite. It follows that the set of roots is a countable (possibly finite) set.

(B) If the set of roots is infinite, denoted by $\{\lambda_n\}_{n=1}^{\infty}$, then $|\lambda_n| \to \infty$. Because $|\beta| e^{-\Re(\lambda_n)} = |\lambda_n|$, it follows that $\Re(\lambda_n) \to -\infty$. Consequently, for each $a \in \mathbb{R}$, $\Re(\lambda) \geq a$ for at most finitely many roots.

(C) If λ is a root, then it is a root of finite order.

(D) If λ is a root, so is its conjugate $\overline{\lambda}$.

Now let's focus attention on the characteristic equation (2.10). Letting $\lambda = x + iy$ and considering real and imaginary parts, we get the system

$$x = -\beta e^{-x}\cos(y) \tag{2.11}$$
$$y = \beta e^{-x}\sin(y)$$

We begin by considering real roots of (2.10).

Lemma 2.2. *The following hold.*

1. *If $\beta < 0$, then there is exactly one real root and it is positive.*
2. *If $0 < \beta < e^{-1}$, then there are exactly two real roots $x_1 < x_2$, both negative. $x_1 \to -\infty$ and $x_2 \to 0$ as $\beta \to 0$.*
3. *If $\beta = e^{-1}$, then there is a single real root of order two, namely $\lambda = -1$.*
4. *If $\beta > e^{-1}$, then there are no real roots.*

The proof is left to the exercises.

The next result summarizes important information concerning the roots in the case $\beta > 0$.

Proposition 2.1 *The following hold for* (2.10).

1. *If $0 < \beta < \pi/2$, then there exists $\delta > 0$ such that $\Re(\lambda) \leq -\delta$ for all roots.*
2. *If $\beta = \pi/2$, then $\lambda = \pm i\pi/2$ are roots of order one.*
3. *If $\beta > \pi/2$, there are roots $\lambda = x \pm iy$ with $x > 0, y \in (\pi/2, \pi)$.*

The following immediate corollary of Proposition 2.1 follows once we establish the expected result concerning the relation of the roots of (2.10) and the stability of the zero solution of (2.7).

Corollary 2.2 *The following hold for* (2.7).

1. *If $\alpha < 0$ then $u = 0$ is unstable.*
2. *If $0 < r\alpha < \pi/2$, $u = 0$ is asymptotically stable.*
3. *If $r\alpha = \pi/2$, $u = \sin(\pi\tau/2), \cos(\pi\tau/2)$ are solutions.*
4. *If $r\alpha > \pi/2$, $u = 0$ is unstable.*

Figure 2.2 depicts the stability region in the (r,α)-plane and Figure 2.3 shows two simulations, both with $\alpha = 1$. The delay r is just below $\pi/2$ for the right graph and just above it for the left graph.

Proof. **Proof of Proposition 2.1:** Because $\beta > 0$, if there is a root $x+iy$ with $x \geq 0$ and $y > 0$ of (2.11) then $\cos(y) \leq 0 < \sin(y)$ so $y \in S \equiv \cup_{n=0}^{\infty}\{[\pi/2,\pi) + 2n\pi\}$. Furthermore,

$$\frac{\sin(y)}{y} = \frac{e^x}{\beta}$$

must hold. As

$$\frac{d}{dy}\frac{\sin(y)}{y} = \frac{y\cos(y) - \sin(y)}{y^2} < 0, y \in S$$

and $\sin(y)/y = 2/\pi$ when $y = \pi/2$, we conclude that $\sin(y)/y \leq 2/\pi$ for $y \in S$ even though S is disconnected (check this). Therefore,

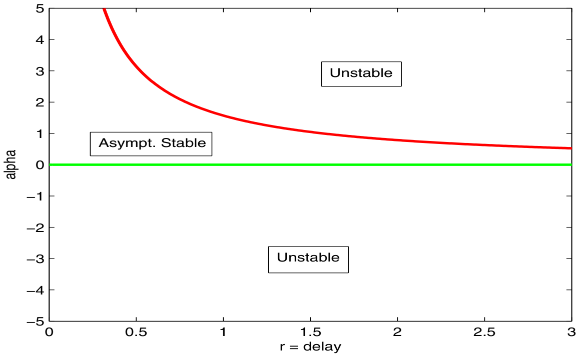

Fig. 2.2 The stability region in the (r, α)-plane for (2.7).

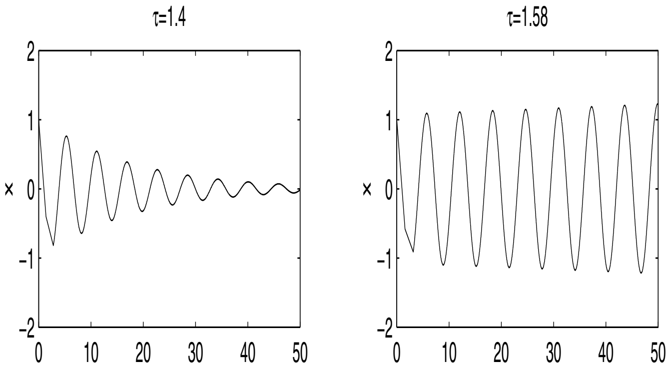

Fig. 2.3 Simulation of (2.7) for $\alpha = 1,\ r = \tau$ initial data equal one.

$$\frac{1}{\beta} \leq \frac{e^x}{\beta} = \frac{\sin(y)}{y} \leq \frac{2}{\pi}$$

so it follows that $\beta \geq \pi/2$. Thus, if $\beta < \pi/2$ then $\Re(\lambda) < 0$ for every root λ. This and the last assertion in (B) above proves (1).

Let's now turn to the final assertion (3). If we write our root as $\lambda = re^{i\theta}$, then (2.10) becomes

$$r[\cos(\theta - \pi) + i\sin(\theta - \pi)] = \beta e^{-x}[\cos(-y) + i\sin(-y)]$$

Equivalently,

$$r = \beta e^{-x} \text{ and } \theta - \pi = -y + 2k\pi$$

for some integer k. Let's search for a root in the first quadrant on the ray through the origin making angle $\theta \in (0, \pi/2)$ with the positive x-axis. Then, taking $k = 0$,

$y(\theta) = \pi - \theta > 0$ and so $x(\theta) > 0$ is determined by trigonometry because $\tan(\theta) = y/x$ (draw a picture).

We claim that:

$$\begin{aligned} x(\theta) &= (\pi - \theta)\cot(\theta) \\ y(\theta) &= \pi - \theta, \quad 0 < \theta < \pi/2 \\ \beta(\theta) &= \frac{\pi - \theta}{\sin(\theta)} e^{x(\theta)} \end{aligned} \tag{2.12}$$

is a one-parameter family of solutions of (2.10) satisfying $x > 0$ and $\pi/2 < y < \pi$. The proof of the claim is left to the exercises.

Clearly, $x(\theta), y(\theta), \beta(\theta)$ depend continuously on $\theta \in (0, \pi/2)$. Also,

$$x(\theta) \to +\infty,\ y(\theta) \to \pi,\ \beta(\theta) \to +\infty,\ \theta \to 0$$

and

$$x(\theta) \to 0,\ y(\theta) \to \pi/2,\ \beta(\theta) \to \pi/2,\ \theta \to \pi/2.$$

Inasmuch as $\beta(\theta)$ is strictly decreasing on $(0, \pi/2)$ it follows that the range of β is $(\pi/2, \infty)$. $\square$

Remark 2.3 *The map $\theta \to \beta(\theta)$ is invertible, therefore its inverse $\theta = k(\beta)$ is defined for $\beta \in (\pi/2, \infty)$ and is strictly decreasing. Inserting $\theta = k(\beta)$ into* (2.12)*, we have found a root $\lambda = \lambda(\beta) = x(\beta) + iy(\beta)$ with $x(\beta) > 0, y(\beta) \in (\pi/2, \pi)$ corresponding to $\beta > \pi/2$ which satisfies*

$$\lambda(\beta) \to i\pi/2,\ \beta \searrow \pi/2 \tag{2.13}$$

Remark 2.4 *It's easy to see from* (2.11) *that for $\beta = \pi/2 + 2n\pi$, $n = 0, 1, 2, \cdots$, then $i[\pi/2 + 2n\pi]$ is a root.*

2.3 Oscillation of Solutions

We can note that unlike the undelayed negative feedback case delayed negative feedback can result in oscillatory solutions. Let's be precise about what we mean. If $x(t)$ is a solution of (2.7) defined for $t \geq s$ for some real s, we say it is oscillatory if it has arbitrarily large zeros: for every $t_0 > s$ there exists $t_1 > t_0$ such that $x(t_1) = 0$; otherwise we say that the solution is nonoscillatory. The following is Theorem 2.1.3 in [39]; See also Theorem 1.5.1 in [37].

Theorem 2.5 *For every real α and $r > 0$ the following are equivalent.*

(a) Every solution of (2.7) *is oscillatory.*
(b) $r\alpha > 1/e$.

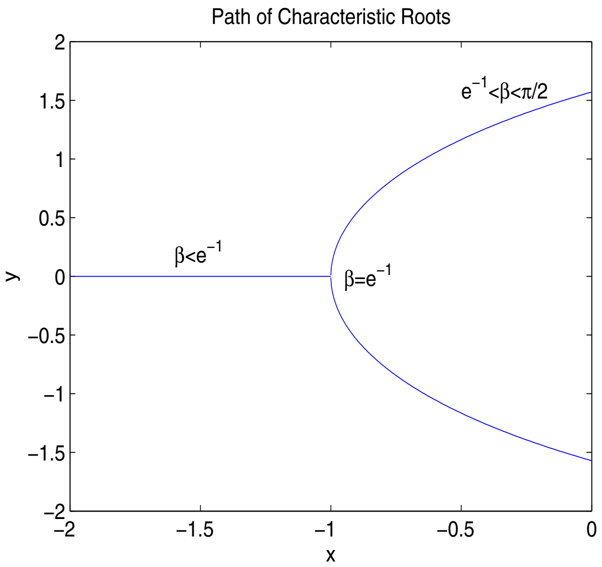

Fig. 2.4 Path of key characteristic root of (2.10) as a function of β. As β increases the root(s) move right.

Recall that by Lemma 2.2, Theorem 2.5 (b) is equivalent to there being no real roots of the characteristic equation. Therefore implication (a) implies (b) is obvious; the converse is not hard to prove. See [39], an excellent reference for the oscillatory behavior of delay differential equations. What happens at $\beta = r\alpha = e^{-1}$ that suggests solutions oscillate for larger values of β? This value of β corresponds to the double root $\lambda = -1$. See Figure 2.4 where we trace the path of the largest real root as β increases from $\beta \ll e^{-1}$ to $\beta = \pi/2$. A complex conjugate pair of roots bifurcates from the double root. This leads to oscillations.

A closer look at case (3) of Lemma 2.2 shows the following.

Lemma 2.3. *For each $\beta \in (e^{-1}, \pi/2)$ there is a complex conjugate pair of roots $\lambda = x \pm iy$ of* (2.10) *satisfying*

$$-1 < x < 0, 0 < y < \pi/2$$

This family of roots (λ, β) can be parameterized by x:

$$\beta = \beta(x), y = h(x)$$

where $\beta(x), h(x)$ are smooth, positive, monotonic increasing functions such that as $x \searrow -1$, $\lambda \to -1$, and $\beta \to e^{-1}$, and as $x \nearrow 0$, $\lambda \to (\pi/2)i$, and $\beta \to \pi/2$.

Proof. We start by changing variables $\lambda = -1 - z$, $\beta = e^{-1-\mu}$ in (2.10) so it becomes

$$z+1=e^{z-\mu}$$

or, if $z=x+iy$, in real and imaginary parts:

$$\begin{aligned} x+1 &= e^{x-\mu}\cos y \\ y &= e^{x-\mu}\sin y \end{aligned}$$

We have transformed the double root $(\lambda,\beta)=(-1,e^{-1})$ to $(z,\mu)=(0,0)$. The equations above are equivalent to

$$\begin{aligned} \frac{\tan y}{y} &= \frac{1}{x+1} \\ e^{x-\mu} &= ((x+1)^2+y^2)^{1/2} \end{aligned}$$

Function $g(y)=\tan y/y$, restricted to $(-\pi/2,\pi/2)$, is even in y, with global minimum of 1 at $y=0$, and is strictly increasing (to ∞) as $y\to\pi/2$. It follows that for each $x\in(-1,0)$, the first equation above has a unique pair of solutions $y=\pm h(x)$ where h is positive, decreasing, and with $h(-1+)=\pi/2$ and $h(0-)=0$. Parameter $\mu=\mu(x)$ is now determined by the second equation above, where $y=h(x)$. Observe that $\mu(-1+)=-1-\ln(\pi/2)$ and $\mu(0-)=0$. It remains to show that $\mu(x)$ is strictly increasing. An alternative relation satisfied by $\mu(x)$ is

$$e^{\mu(x)}=e^x\frac{\sin(h(x))}{h(x)}$$

Because $\sin(y)/y$ is strictly decreasing on $(0,\pi/2)$ and $h(x)$ is strictly decreasing, it follows that their composition is strictly increasing and therefore so is $e^{\mu(x)}$. The proof is completed by putting the results above together with the change of variables. $\square$

Lemma 2.3 and Proposition 2.1 imply that when $\beta=r\alpha>e^{-1}$, there is an oscillatory solution

$$u(t)=e^{(x+iy)t/r}$$

of (2.7). It decays to zero if $\beta\in(e^{-1},\pi/2)$, and becomes unbounded when $\beta>\pi/2$. This explains the magic number e^{-1} appearing in Theorem 2.5. Out of the double root $\lambda=-1$ at $\beta=e^{-1}$ is born a complex conjugate pair of roots that lead first to damped oscillation and then to undamped oscillation.

2.4 Solutions Backward in Time

ODEs can be solved backward in time as well as forward but delay equations are fundamentally different in this respect. As we show, for some initial data solutions in backward time exist but not for others. We have no intention here of treating the general case but rather just to see what the problems are.

Consider the initial-value problem treated in Section 1,

$$\begin{aligned} u'(t) &= -u(t-\tau) \\ u(t) &= 1, -\tau \le t \le 0 \end{aligned} \tag{2.14}$$

There, we found a solution $\hat{u}(t)$ defined for $t \ge -\tau$.

If we replace t by $t+\tau$ in the equation, we may write the equation as

$$u(t) = -u'(t+\tau) \tag{2.15}$$

Roughly, to solve for u in the past we must differentiate u in the future. Proceeding by steps, starting with $-2\tau \le t < -\tau$, and so forth leads to an extension of $\hat{u}$:

$$\hat{u}(t) = 0, t < -\tau$$

The discontinuity of $\hat{u}$ at $t = -\tau$ means that our function $\hat{u} : \mathbb{R} \to \mathbb{R}$, defined by the method of steps, does not satisfy the definition of a solution given in Chapter 1, which required continuity. But maybe that definition is too restrictive. A more serious problem is that $\hat{u}$ is not differentiable at $t = 0$ where its left-hand derivative 0 differs from its right-hand derivative -1. Still, maybe we should overlook this and not require differentiability at every point.

If instead we take initial data $u(t) = 0,\ -\tau \le t \le 0$, there is no problem: $u(t) = 0,\ t \in \mathbb{R}$ is our solution. Of course, $u = 0$ is an equilibrium (constant) solution, it is special. In fact, it is easy to see that the initial-value problem

$$\begin{aligned} u'(t) &= -u(t-\tau), t > s \\ u(t) &= 0, s-\tau \le t \le s \end{aligned}$$

has the solution $u(t) = 0,\ t \in \mathbb{R}$. But consider the implication of this observation for our backward "solution" $\hat{u}$ of the initial-value problem (2.14). Because it satisfies $\hat{u}(t) = 0,\ t < -\tau$, obviously $\hat{u}(t) = 0$ on $-3\tau \le t \le -2\tau$, so we conclude that $u(t) = 0,\ t \in \mathbb{R}$ is another solution of the corresponding initial-value problem given by the initial data $u(t) = 0$ on $-3\tau \le t \le -2\tau$. Therefore, if we accept the backward "solution" $\hat{u}$ then we are forced to accept the nonuniqueness of solutions of initial-value problems. This is a strong argument against defining the concept of "solution" so as to allow $\hat{u}$ to be one.

It seems obvious that given any continuous initial data $\phi : [-\tau, 0] \to \mathbb{R}$, we can use the method of steps to solve the initial-value problem

$$\begin{aligned} u'(t) &= -u(t-\tau) \\ u(t) &= \phi(t), -\tau \le t \le 0 \end{aligned} \tag{2.16}$$

We prove this fact in the next chapter. There exist continuous functions ϕ that are not differentiable at any point of $-\tau \le t \le 0$. Clearly, we cannot use (2.15) to find a backward-in-time solution inasmuch as ϕ is not differentiable.

Exercises

Exercise 2.1. Verify (2.6).

Exercise 2.2. Show that if λ is a root of (2.10) then so is its conjugate $\bar{\lambda}$.

Exercise 2.3. Show that all roots of (2.10) have order one except when $\beta = e^{-1}$ when $\lambda = -1$ has order two.

Exercise 2.4. Prove Lemma 2.2.

Exercise 2.5. Prove the claim in the proof of Corollary 2.2 by inserting into (2.11). Show that $x(\theta)$ and $\beta(\theta)$ are strictly decreasing in $\theta \in (0, \pi/2)$.

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