Chapter 2
Diffraction and Fourier Optics

Diffraction and Fourier optics are at the foundation of the theory of holographic image formation and therefore essential in the description of holographic processes and techniques. In this chapter, we review the scalar diffraction theory, which is used to describe the propagation of the optical field from an input plane to the output plane. The propagation of light through a lens is an essential part of any imaging system, and its mathematical description is relevant to holographic image formation as well.

2.1 Fourier Transform and Mathematical Background

We begin with a brief summary of basic results from Fourier analysis and related mathematical background, mostly without proof, the main purpose being establishing basic notations and collecting in one place useful expressions that are frequently used in Fourier optics [1].

2.1.1 One-Dimensional Definition

In a one-dimensional (1D) system, according to the Fourier theorem, if $f(x)$ is a reasonably well-behaved function, then it can be decomposed into a superposition of sine and cosine functions, or imaginary exponentials, of various frequencies. (Note that in this book, the term frequency will usually refer to the spatial frequencies.) The amplitudes of the decomposition are the Fourier transform of the function, thus

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}k F(k) \exp(ikx) = \mathfrak{F}^{-1}\{F(k)\}[x], \\ F(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x f(x) \exp(-ikx) = \mathfrak{F}\{f(x)\}[k]. \end{aligned} \tag{2.1}$$

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Table 2.1 Examples of Fourier transform pairs. See Fig. 2.1 for illustrations

$f(x)$	$\mathfrak{F}\{f\}[k]$	
$f(x) = \delta(x - x_0)$	$F(k) = \frac{1}{\sqrt{2\pi}} \exp(-ikx_0)$	
$f(x) = \delta(x - x_0) + \delta(x + x_0)$	$F(k) = \sqrt{\frac{2}{\pi}} \cos(kx_0)$	Fig. 2.1a
$f(x) = \exp(ik_0 x)$	$F(k) = \sqrt{2\pi}\delta(k - k_0)$	
$f(x) = \cos(k_0 x)$	$F(k) = \sqrt{\frac{\pi}{2}}\{\delta(k - k_0) + \delta(k + k_0)\}$	
$f(x) = \mathrm{rect}\left(\frac{x}{a}\right) = \begin{cases} 1; x \in [-a, a] \\ 0; \text{otherwise} \end{cases}$	$F(k) = \sqrt{\frac{2}{\pi}} a \mathrm{sinc}(ka) = \sqrt{\frac{2}{\pi}} a \frac{\sin(ka)}{ka}$	Fig. 2.1b
$f(x) = \exp\left(-\frac{x^2}{a^2}\right)$	$F(k) = \frac{a}{\sqrt{2}} \exp\left(-\frac{1}{4} a^2 k^2\right)$	Fig. 2.1c
$f(x) = \exp\left(\frac{i}{2} \alpha x^2\right)$	$F(k) = \frac{i}{\alpha} \exp\left(-\frac{i}{2} \frac{k^2}{\alpha}\right)$	Fig. 2.1d
$f(x) = \mathrm{comb}\left(\frac{x}{a}\right) = \sum_{n=-\infty}^{\infty} \delta(x - na)$	$F(k) = \frac{\sqrt{2\pi}}{a} \mathrm{comb}\left(\frac{k}{2\pi/a}\right) = \frac{\sqrt{2\pi}}{a} \sum_{n=-\infty}^{\infty} \delta\left(k - n\frac{2\pi}{a}\right)$	Fig. 2.1e

The particular notation with square brackets is used to explicitly display the variable of transform. Examples of the Fourier transform pairs are listed in Table 2.1 and illustrated in Fig. 2.1. Thus, an even (odd) pair of delta functions transforms to cosine (sine) function. A rectangle function transforms to a sinc function. The transforms of Gaussian, chirp, or comb functions transform respectively to the same type of functions.

2.1.2 *Two-Dimensional Definition*

For a two-dimensional (2D) system, the Fourier transform is written as

$$\begin{aligned} f(x, y) &= \frac{1}{2\pi} \iint \mathrm{d}k_x \mathrm{d}k_y F\left(k_x, k_y\right) \exp\left[i\left(k_x x + k_y y\right)\right] = \mathfrak{F}^{-1}\left\{F\left(k_x, k_y\right)\right\}[x, y], \\ F\left(k_x, k_y\right) &= \frac{1}{2\pi} \iint \mathrm{d}x \mathrm{d}y f(x, y) \exp\left[-i\left(k_x x + k_y y\right)\right] = \mathfrak{F}\{f(x, y)\}\left[k_x, k_y\right]. \end{aligned} \tag{2.2}$$

Generalization to N-dimensional system is straightforward, noting that the factor $\sqrt{2\pi}$ in (2.1) becomes $(2\pi)^{N/2}$.

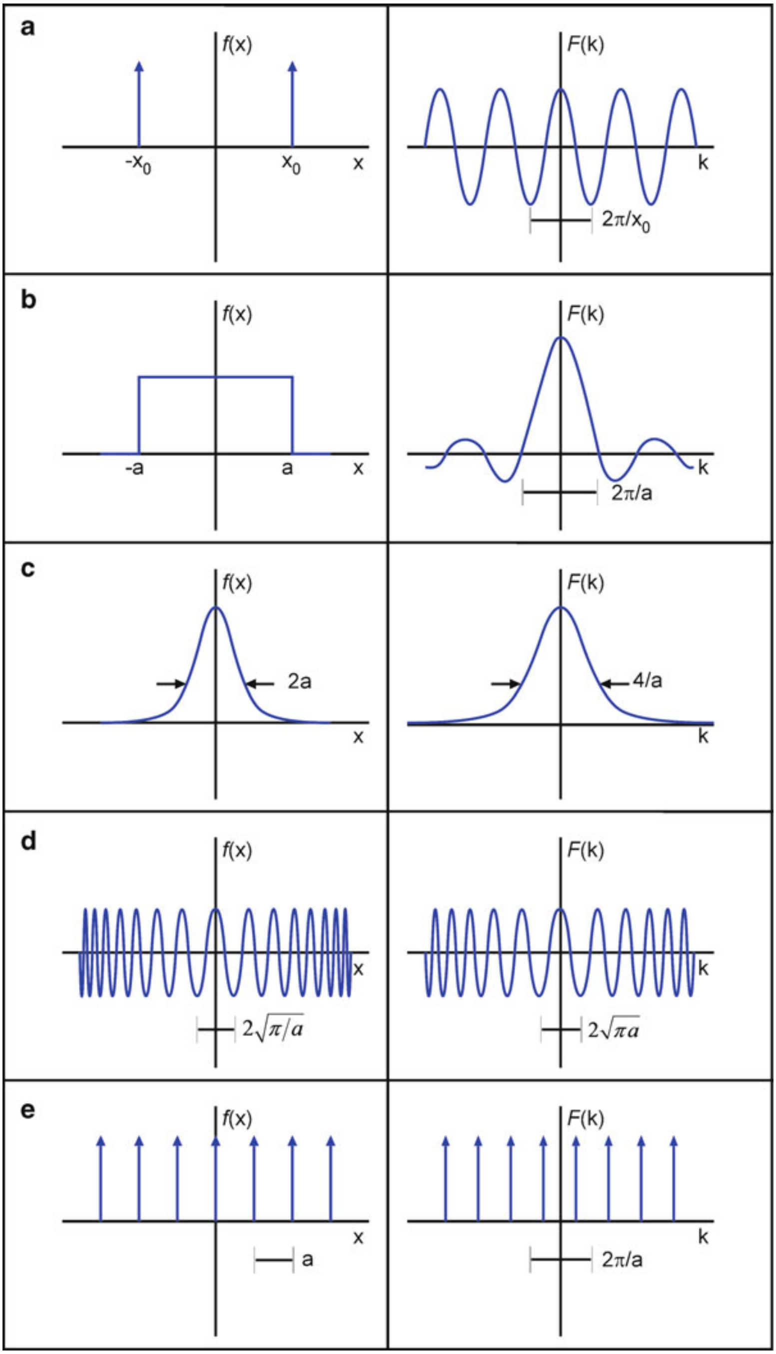


Fig. 2.1 Examples of Fourier transform pairs

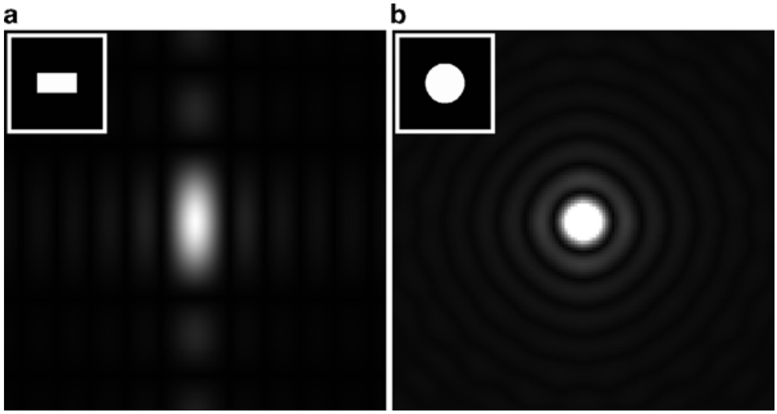


Fig. 2.2 (**a**) *Rectangular aperture* and its Fourier transform. (**b**) *Circular aperture* and its Fourier transform

2.1.3 Cartesian Geometry

If the function is separable in Cartesian coordinates, $f(x, y) = f_x(x)f_y(y)$, then so is the transform:

$$\mathfrak{F}\{f(x, y)\} = \mathfrak{F}\{f_x(x)\}\mathfrak{F}\{f_y(y)\}, \tag{2.3}$$

that is,

$$F(k_x, k_y) = F_x(k_x)F_y(k_y). \tag{2.4}$$

An important example is a rectangular aperture,

$$f(x, y) = \mathrm{rect}\left(\frac{x}{a_x}\right)\mathrm{rect}\left(\frac{y}{a_y}\right), \tag{2.5}$$

whose transform is

$$F(k_x, k_y) = \frac{2}{\pi} a_x\, a_y \mathrm{sinc}\,(k_x a_x)\mathrm{sinc}\,(k_y a_y). \tag{2.6}$$

The function $\left|F(k_x, k_y)\right|^2$ is illustrated in Fig. 2.2a with $f(x, y)$ displayed in the inset. Note that the horizontal orientation of the long side of the rectangle results in the vertical orientation of the central bright spot in the transform. This is an example of the uncertainty principle that higher localization in the spatial dimension corresponds to larger spread in the frequency dimension, and vice versa.

2.1.4 Cylindrical Geometry

If the 2D function is given in cylindrical coordinates, with

$$\begin{cases} x = \rho \cos \varphi \\ y = \rho \sin \varphi \end{cases} \quad \text{and} \quad \begin{cases} k_x = k \cos \eta \\ k_y = k \sin \eta \end{cases}, \tag{2.7}$$

then the Fourier transform is

$$F(k, \eta) = \frac{1}{2\pi} \int_0^\infty \rho \mathrm{d}\rho \int_0^{2\pi} \mathrm{d}\varphi f(\rho, \varphi) \exp[-ik\rho \cos(\varphi - \eta)]. \tag{2.8}$$

If the function has cylindrical symmetry, so that $f = f(\rho)$, then

$$F(k, \eta) = \int_0^\infty f(\rho) \rho J_0(k\rho) \mathrm{d}\rho = \mathfrak{B}\{f(\rho)\}[k], \tag{2.9}$$

which is called the Fourier–Bessel transform. An important example is a circular aperture of radius a:

$$f(\rho) \begin{cases} 1 & \rho \leqslant a \\ 0 & \rho > a \end{cases} \tag{2.10}$$

Its Fourier transform is the Airy disk, illustrated in Fig. 2.2b:

$$F(k) = \int_0^a \rho J_0(k\rho) \mathrm{d}\rho = \frac{a}{k} J_1(ka). \tag{2.11}$$

The first zero of the Bessel function $J_1(x)$ is at $x \approx 3.83$, which defines the size of the Airy disk. This is also to be compared with the first zero of sinc function $\mathrm{sinc}(x)$ at $x = \pi \approx 3.14$.

2.1.5 Basic Properties of Fourier Transforms

We list several of the well-known properties of Fourier transform in Table 2.2. The similarity property shows that if the function f is stretched in the x-direction, then its transform is shrunk in corresponding direction k_x. The shift theorem states that a shift of the spatial position of a function amounts to an additional phase oscillation in the frequency domain, which is the basis of the large field of interferometry. In the uncertainty relation, the uncertainties are defined as the root-mean-square deviation.

Table 2.2 Basic theorems of Fourier transform

Linearity:	$\mathfrak{F}\{\alpha f + \beta g\} = \alpha F + \beta G$
Similarity:	$\mathfrak{F}\{f(ax)\} = \frac{1}{\lvert a\rvert} F\left(\frac{k}{a}\right)$
	$\mathfrak{F}\{f(-x)\} = F(-k)$
Shift Property	$\mathfrak{F}\{f(x-x_0)\} = F(k)\mathrm{e}^{-ikx_0}$
Parseval's Theorem:	$\int_{-\infty}^{\infty} \lvert f(x)\rvert^2 \mathrm{d}x = \int_{-\infty}^{\infty} \lvert F(k)\rvert^2 \mathrm{d}k$
Uncertainty principle:	$\Delta x \Delta k \geq 1$
Repeated transforms:	$\mathfrak{F}\mathfrak{F}^{-1}\{f(x)\} = \mathfrak{F}^{-1}\mathfrak{F}\{f(x)\} = f(x)$
	$\mathfrak{F}\mathfrak{F}\{f(x)\} = f(-x)$

2.1.6 Convolution and Correlation

Another important result is the convolution theorem,

$$\mathfrak{F}\{f \odot g\} = \sqrt{2\pi} F\, G, \tag{2.12}$$

where the convolution of two functions is defined as

$$f \odot g(x) = \int \mathrm{d}x' f(x') g(x - x') = g \odot f(x). \tag{2.13}$$

In particular, the convolution of a function $g(x)$ with a delta function $\delta(x - x_0)$ located at $x = x_0$ copies the function to that location,

$$\delta(x - x_0) \odot g(x) = g(x - x_0). \tag{2.14}$$

Considering that any function $f(x)$ is a superposition of many delta functions with different amplitudes and positions, the convolution of $f(x)$ with a "spread function" $g(x)$ is simply a similar superposition of many copies of $g(x)$ with corresponding amplitudes and positions. This is illustrated in Fig. 2.3, where the function $f(x)$ representing a pattern of point sources in the Big Dipper is convolved with the spread function $g(x)$ to yield the "image" that includes the effect of the spread function.

Relatedly, the cross-correlation of two functions is defined as

$$f \otimes g(x) = \int \mathrm{d}x' f(x') g^*(x' - x) = f(x) \odot g^*(-x) \tag{2.15}$$

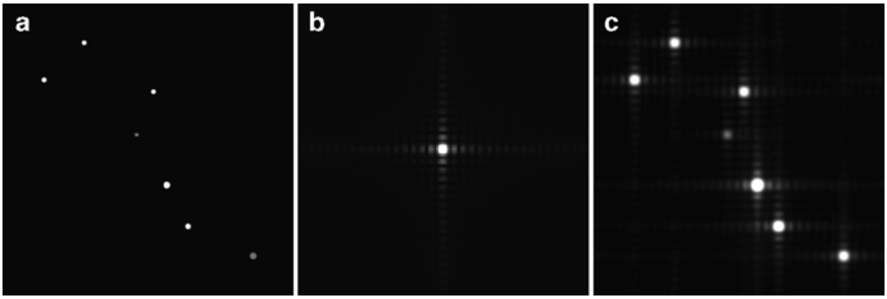


Fig. 2.3 Example of convolution. (**a**) Input image, (**b**) PSF, and (**c**) output image

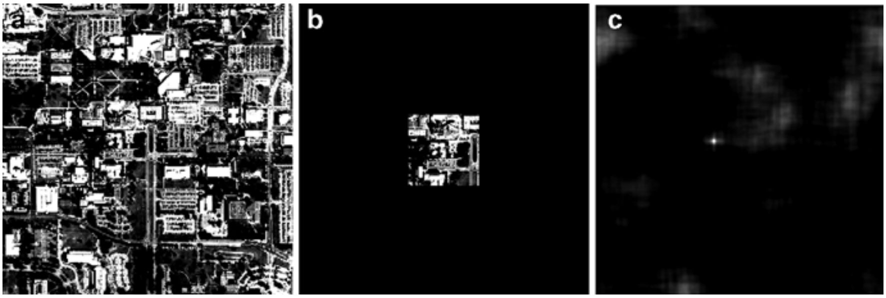


Fig. 2.4 Example of correlation. (**a**) Input image, (**b**) search image, and (**c**) result

and one can also write

$$\mathfrak{F}\{f \otimes g\} = \sqrt{2\pi} F\, G^* \tag{2.16}$$

In particular, the correlation of a function with itself is called the auto-correlation,

$$f \otimes f(x) = \int \mathrm{d}x' f(x') f^*(x' - x). \tag{2.17}$$

It is clear that if the function $f(x)$ is highly random, then $f \otimes f(0) = \int \mathrm{d}x' |f(x')|^2$, while for $x \neq 0, f \otimes f(x) \approx 0$. If $g(x)$ is a shifted copy or partial copy of $f(x)$, then the cross-correlation has a large peak corresponding to the shift, which is the basis of pattern recognition by cross-correlation. In Fig. 2.4, the cross-correlation of the map of the University of South Florida with a small area from it identifies the location of the Physics building. Roughly speaking, the lateral size of the correlation peak indicates the distance beyond which the randomness sets in.

2.1.7 Some Useful Formulas

Here we list some of the useful formulas that may come in handy throughout the discussions of diffraction and Fourier optics. Delta functions have numerous representations, including

$$\begin{aligned}
\delta(x-x_0) &= \lim_{a\to 0}\frac{1}{\pi}\frac{a}{x^2+a^2},\\
\delta(x-x_0) &= \lim_{a\to 0}\frac{1}{\sqrt{\pi}a}\exp\left[-\frac{(x-x_0)^2}{a^2}\right],\\
\delta(x-x_0) &= \lim_{a\to 0}\frac{1}{\pi a}\operatorname{sinc}\left(\frac{x-x_0}{a}\right),\\
\delta(x-x_0) &= \frac{1}{2\pi}\int \mathrm{d}k\exp[ik(x-x_0)],\\
\delta(\mathbf{r}-\mathbf{r}_0) &= \frac{1}{(2\pi)^3}\int \mathrm{d}^3k\exp[i\mathbf{k}\bullet(\mathbf{r}-\mathbf{r}_0)].
\end{aligned}\tag{2.18}$$

Note that the delta function in N-dimension has the dimensions of $(\text{length})^{-N}$.

The Gaussian integrals are needed frequently:

$$\int_{-\infty}^{\infty}\exp(-px^2)\mathrm{d}x = \sqrt{\frac{\pi}{p}},\tag{2.19}$$

$$\int_{-\infty}^{\infty}\exp(-px^2+qx)\mathrm{d}x = \sqrt{\frac{\pi}{p}}\exp\left(\frac{q^2}{4p}\right),\tag{2.20}$$

which are valid for any complex number p whose real part is nonnegative, $\mathrm{Re}\{p\}\geqslant 0$. And finally,

$$\int_{-\infty}^{\infty}\operatorname{sinc}(ax)\mathrm{d}x = \frac{\pi}{a},\tag{2.21}$$

$$\int_{-\infty}^{\infty}\frac{1}{x^2+a^2}\mathrm{d}x = \frac{\pi}{a}.\tag{2.22}$$

2.2 Scalar Diffraction Theory

The theory of propagation and diffraction of electromagnetic fields has a long history of development and has many subtleties that need to be considered carefully in order to arrive at accurate and consistent results [2]. In-depth discussions can be found in a number of authoritative textbooks such as [3, 4]. Here we take a practical approach and take the Fresnel–Kirchoff diffraction formula as the starting point, which is known to yield highly accurate results for a wide range of configurations. Referring to Fig. 2.5, a spherical wave from the point source at S illuminates the aperture Σ:

$$E_\Sigma = E_S \frac{\exp[i(kr' - \omega t)]}{kr'}. \tag{2.23}$$

The field at a point P behind the aperture is then given by

$$E_P = -\frac{i}{4\pi} E_S e^{-i\omega t} \int_\Sigma d\Sigma \frac{\exp[ik(r + r')]}{rr'} \left(\hat{\mathbf{r}} - \hat{\mathbf{r}}'\right) \bullet \hat{\mathbf{n}}, \tag{2.24}$$

where the carets (^) represent unit vectors along the relevant directions. This expression can be interpreted in terms of Huygens principle [2, 5], where the field at a point in the aperture gives rise to a secondary spherical wavelet proportional to

$$-i \frac{\exp(ikr)}{4\pi r} \left(\hat{\mathbf{r}} - \hat{\mathbf{r}}'\right) \bullet \hat{\mathbf{n}}. \tag{2.25}$$

The obliquity factor $\frac{1}{2}\left(\hat{\mathbf{r}} - \hat{\mathbf{r}}'\right) \bullet \hat{\mathbf{n}}$ becomes $\hat{\mathbf{r}} \bullet \hat{\mathbf{n}}$ or $-\hat{\mathbf{r}}' \bullet \hat{\mathbf{n}}$ in Rayleigh–Sommerfeld theory, depending on the boundary conditions imposed on the screen Σ. When the propagation is paraxial, the obliquity factor becomes close to unity in all three cases, which we take to be the case. The field at the observation point P is then

$$E_P = -\frac{i}{2\pi} \int_\Sigma d\Sigma\, E_\Sigma \frac{\exp(ikr)}{r}. \tag{2.26}$$

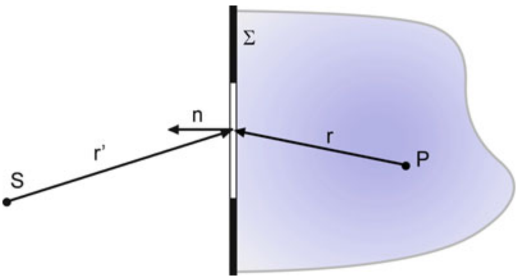


Fig. 2.5 Geometry of Fresnel–Kirchoff diffraction formula. *S* Source point, *P* observation point

2.3 Diffraction from a 2D Aperture

To be more specific, we consider the geometry of Fig. 2.6, where the input field $E_0(x_0, y_0)$ on the input plane Σ_0 propagates along the general z-direction and results in the output field $E(x, y; z)$ on the output plane Σ. Then (2.26) is written as

$$\begin{aligned} E(x,y;z) &= -\frac{ik}{2\pi}\iint_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\frac{\exp(ikr)}{r}, \\ &= -\frac{ik}{2\pi z}\iint_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left[ik\sqrt{(x-x_0)^2+(y-y_0)^2+z^2}\right], \end{aligned} \tag{2.27}$$

where we made a further approximation of $r \approx z$ in the denominator, but not in the exponent. This integral is a convolution

$$E(x,y;z) = E_0 \odot S_\mathrm{H} \tag{2.28}$$

with the kernel

$$S_\mathrm{H}(x,y;z) = -\frac{ik}{2\pi z}\exp\left[ik\sqrt{x^2+y^2+z^2}\right], \tag{2.29}$$

which is also referred to as the point spread function (PSF). (More precisely, this is a coherent spread function.) We will refer to this as the Huygens PSF, as far as the integral representing the Huygens spherical wavelet propagation.

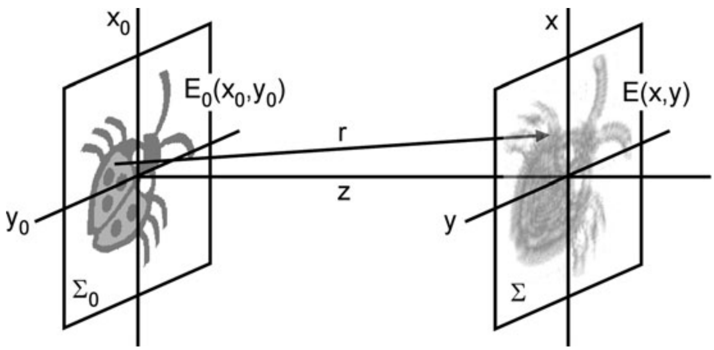


Fig. 2.6 Geometry of diffraction from a 2D aperture

2.3.1 Paraxial (Fresnel) Approximation

For theoretical developments and other purposes, it is often useful to make paraxial, or Fresnel, approximation of the PSF

$$r = \sqrt{(x-x_0)^2 + (y-y_0)^2 + z^2} \approx z + \frac{(x-x_0)^2 + (y-y_0)^2}{2z}, \tag{2.30}$$

which is valid for $z^3 \gg \frac{k}{8}\left[(x-x_0)^2 + (y-y_0)^2\right]^2_{\max}$. Then the Fresnel PSF is

$$S_{\mathrm{F}}(x,y;z) = -\frac{ik}{2\pi z}\exp(ikz)\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right], \tag{2.31}$$

where the spherical wavefront is approximated with a parabolic wavefront, or a 2D chirp function. The diffraction field is expressed with a single Fourier transform of spatial frequencies

$$k_x = k\frac{x}{z}; \quad k_y = k\frac{y}{z}. \tag{2.32}$$

Thus

$$\begin{aligned}
E(x,y;z) &= -\frac{ik}{2\pi z}\exp(ikz)\iint\limits_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left\{\frac{ik}{2z}\left[(x-x_0)^2+(y-y_0)^2\right]\right\},\\
&= -\frac{ik}{2\pi z}\exp(ikz)\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right]\\
&\quad\times\iint\limits_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left[\frac{ik}{2z}\left(x_0^2+y_0^2\right)\right]\exp\left[-\frac{ik}{z}(xx_0+yy_0)\right],\\
&= -\frac{ik}{z}\exp(ikz)\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right]\mathfrak{F}\left\{E_0(x_0,y_0)\exp\left[\frac{ik}{2z}\left(x_0^2+y_0^2\right)\right]\right\}\left[k_x,k_y\right]
\end{aligned} \tag{2.33}$$

or

$$E(x,y;z) = 2\pi\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right]\mathfrak{F}\left\{E_0(x_0,y_0)\,S_{\mathrm{F}}(x_0,y_0;z)\right\}\left[k_x,k_y\right]. \tag{2.34}$$

This is also referred to as the Fresnel transform.

2.3.2 Fraunhofer Diffraction

If we make further approximation and ignore terms of order $\frac{x_0^2+y_0^2}{z^2} \ll 1$ to write

$$r \approx z + \frac{x^2+y^2}{2z} - \frac{xx_0+yy_0}{z}, \tag{2.35}$$

then the output field is proportional to the Fourier transform of the input field:

$$\begin{aligned} E(x,y;z) &= -\frac{ik}{2\pi z}\exp(ikz)\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right]\iint_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left[-\frac{ik}{z}(xx_0+yy_0)\right], \\ &= -\frac{ik}{z}\exp(ikz)\exp\left[\frac{ik}{2z}\left(x^2+y^2\right)\right]\mathfrak{F}\{E_0(x_0,y_0)\}\left[k_x,k_y\right]. \end{aligned} \tag{2.36}$$

Therefore, for example, the Fraunhofer diffraction pattern of a rectangular aperture is a sinc function and for a circular aperture it is an Airy disk pattern, as is well known.

2.4 Propagation of Angular Spectrum

An alternative approach to describing the diffraction is given by the angular spectrum or the plane-wave decomposition. Analytically, the angular spectrum approach is shown to be equivalent to the Huygens convolution described above. On the other hand, the angular spectrum picture has the advantage of being more intuitive and free from some of the subtle difficulties of boundary conditions. It also leads to a more robust and trouble-free numerical calculations of diffraction, as we will see in later chapters.

Given an input field $E_0(x_0,y_0)$, its Fourier transform

$$A_0\left(k_x,k_y\right) = \mathfrak{F}\{E_0\} = \frac{1}{2\pi}\iint_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left[-i\left(k_xx_0+k_yy_0\right)\right] \tag{2.37}$$

describes the amplitudes of various plane-wave components that comprise the input pattern, according to the basic principle of Fourier transform, depicted in Fig. 2.7. The input field $E_0(x_0,y_0)$ is of course the inverse Fourier transform of the angular spectrum:

$$E_0(x_0,y_0) = \mathfrak{F}^{-1}\{A_0\} = \frac{1}{2\pi}\iint \mathrm{d}k_x\mathrm{d}k_y A_0\left(k_x,k_y\right)\exp\left[i\left(k_xx_0+k_yy_0\right)\right]. \tag{2.38}$$

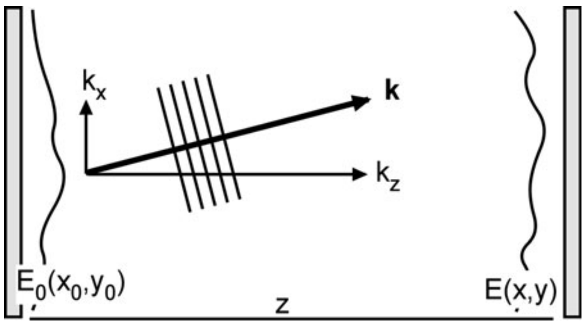


Fig. 2.7 Propagation of a plane-wave component in the angular spectrum

The complex exponential $\exp\left[i\left(k_x x_0 + k_y y_0\right)\right]$ is the projection on the (x_0, y_0)-plane of a plane wave propagating along the wave vector $\mathbf{k} = \left(k_x, k_y, k_z\right)$, where

$$k_z = \sqrt{k^2 - k_x^2 - k_y^2}. \tag{2.39}$$

Thus the input field $E_0(x_0, y_0)$ can be viewed as a projection of many plane-wave components propagating in various directions $\mathbf{k} = \left(k_x, k_y, k_z\right)$, with complex amplitude of each component given by $A_0\left(k_x, k_y\right)$. After propagation over a distance z, each plane-wave component acquires a phase factor $\exp(ik_z z)$, so that the output field is given by

$$E(x, y; z) = \frac{1}{2\pi} \iint_{\Sigma_0} \mathrm{d}k_x \mathrm{d}k_y A_0\left(k_x, k_y\right) \exp\left[i\left(k_x x + k_y y + k_z z\right)\right], \tag{2.40}$$

which is an inverse Fourier transform of $A_0\left(k_x, k_y\right) \exp(ik_z z)$:

$$\begin{aligned} E(x, y; z) &= \mathfrak{F}^{-1}\left\{A_0\left(k_x, k_y\right) \exp\left[i\sqrt{k^2 - {k_x}^2 - {k_y}^2}z\right]\right\}[x, y], \\ &= \mathfrak{F}^{-1}\left\{\mathfrak{F}\{E_0\} \exp\left[i\sqrt{k^2 - {k_x}^2 - {k_y}^2}z\right]\right\}. \end{aligned} \tag{2.41}$$

One can make several observations here. First, the square root factor in the exponent requires that

$${k_x}^2 + {k_y}^2 \leqslant k^2. \tag{2.42}$$

That is, the diffraction imposes a low-pass filtering of the input spatial frequencies. Input spatial structures finer than the wavelength do not propagate to far field. Only near field probes can access such evanescent field. Second, note that the description is based only on the fundamental properties of Fourier transform, without having to invoke particular boundary conditions. Third, the physical picture of diffraction is constructed from a set of plane waves, which by definition is well-behaved everywhere in space. On the other hand, the Huygens principle and Rayleigh–Sommerfeld theory are all built up from the behavior of spherical waves of point sources, which inherently involves singularities at the point sources. Note the factor $r \approx z$ in the denominator of (2.34), whereas the angular spectrum result (2.41) does not have such factor. These observations have important consequences when we discretize the integrals for numerical calculation of the diffraction in Chap. 4.

Still, the angular spectrum result is equivalent to the convolution result, as shown in [6]. First, expand the expressions for the Fourier and inverse transforms in (2.41)

$$\begin{aligned} E(x,y;z) &= \frac{1}{(2\pi)^2} \iint_{\Sigma_0} \mathrm{d}x_0 \mathrm{d}y_0 E_0(x_0,y_0) \iint_{\Sigma_0} \mathrm{d}k_x \mathrm{d}k_y \exp i[k_x(x-x_0)+k_y(y-y_0)] \\ &\qquad \exp\left(i\sqrt{k^2-{k_x}^2-{k_y}^2}z\right), \\ &= \frac{1}{2\pi} \iint_{\Sigma_0} \mathrm{d}x_0 \mathrm{d}y_0 E_0(x_0,y_0) \mathfrak{F}^{-1}\left\{\exp\left[i\sqrt{k^2-{k_x}^2-{k_y}^2}z\right]\right\}[x-x_0,y-y_0]. \end{aligned} \tag{2.43}$$

Noting the following Fourier transform

$$\mathfrak{F}\left\{\exp\left[ik\sqrt{x^2+y^2+z^2}\right]\right\}\left[k_x,k_y\right] = \frac{iz}{k}\exp\left[i\sqrt{k^2-{k_x}^2-{k_y}^2}z\right], \tag{2.44}$$

(2.41) is indeed seen to be the Huygens convolution,

$$E(x,y;z) = E_0 \odot S_{\mathrm{H}}. \tag{2.45}$$

If we take the paraxial approximation

$$k_z \approx k - \frac{{k_x}^2+{k_y}^2}{2k}, \tag{2.46}$$

then

$$\begin{aligned} E(x,y;z) &= \exp(ikz)\mathfrak{F}^{-1}\left\{\mathfrak{F}\{E_0\}\exp\left[-i\frac{{k_x}^2+{k_y}^2}{2k}z\right]\right\}, \\ &= E_0 \odot S_{\mathrm{F}}. \end{aligned} \tag{2.47}$$

Obviously, the angular spectrum method under paraxial approximation is equivalent to the Fresnel transform as well.

2.5 Propagation Through a Lens

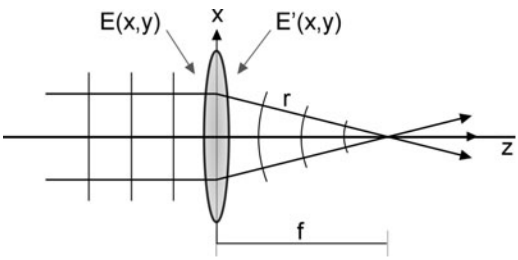
We now describe the propagation of an optical field through a lens. We use the paraxial approximation and Fresnel transform expression of diffraction, which allow us to describe the process in closed analytical forms. For a thin lens of focal length f, its effect is to introduce a quadratic phase in the transmitted optical field (Fig. 2.8),

$$E'(x,y) = E(x,y)\exp\left[-\frac{ik}{2f}\left(x^2+y^2\right)\right] = E(x,y)\psi_f(x,y). \tag{2.48}$$

Refer to Fig. 2.9 and consider the propagation of light from the input plane, Σ_0, to the output plane, Σ_1, through the lens at Σ'-plane. The three planes Σ_0, Σ', and Σ_1 are positioned at $z = 0$, z_0, and $z_0 + z_1$, respectively. The input field is $E_0(x_0, y_0)$. The field at the entrance pupil of the lens is the Fresnel transform of the input field over a distance z_0

$$E'(x',y') = -\frac{ik}{2\pi z_0}\exp(ikz_0)\iint\limits_{\Sigma_0} \mathrm{d}x_0\mathrm{d}y_0 E_0(x_0,y_0)\exp\left\{\frac{ik}{2z_0}\left[(x'-x_0)^2+(y'-y_0)^2\right]\right\}. \tag{2.49}$$

Fig. 2.8 Transmission through a thin lens



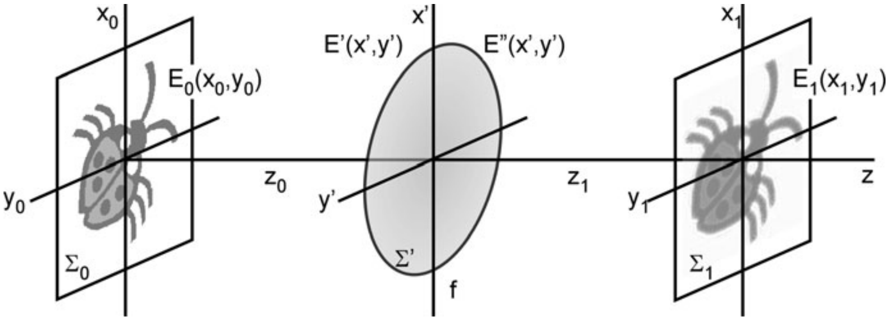


Fig. 2.9 Geometry of imaging by a lens

To save space, here and occasionally elsewhere, we will abbreviate all (x, y) terms with (x) expressions only – the missing (y) terms should be clear from the context. For example, (2.49) is abbreviated as

$$E'(x', y') = -\frac{ik}{2\pi z_0}\exp(ikz_0)\iint_{\Sigma_0} dx_0 E_0(x_0)\exp\left[\frac{ik}{2z_0}(x' - x_0)^2\right]. \tag{2.50}$$

The field at the exit pupil of the lens becomes

$$E''(x', y') = -\frac{ik}{2\pi z_0}\exp(ikz_0)\exp\left(-\frac{ik}{2f}x'^2\right)\iint_{\Sigma_0} dx_0 E_0(x_0)\exp\left[\frac{ik}{2z_0}(x' - x_0)^2\right]. \tag{2.51}$$

Further propagation over the distance z_1 yields the output field

$$\begin{aligned}
E_1(x_1, y_1) &= -\frac{k^2}{4\pi^2 z_0 z_1}\exp[ik(z_0 + z_1)]\iint_{\Sigma_0} dx_0 E_0(x_0) \\
&\quad \times \iint_{\Sigma'} dx' \exp\left[\frac{ik}{2z_0}(x' - x_0)^2 - \frac{ik}{2f}x'^2 + \frac{ik}{2z_1}(x_1 - x')^2\right], \\
&= -\frac{k^2}{4\pi^2 z_0 z_1}\exp[ik(z_0 + z_1)]\exp\left(\frac{ik}{2z_1}x_1^2\right)\iint_{\Sigma_0} dx_0 E_0(x_0)\exp\left(\frac{ik}{2z_0}x_0^2\right) \\
&\quad \times \iint_{\Sigma'} dx' \exp\left[\frac{ik}{2q}x'^2 - ik\left(\frac{x_0}{z_0} + \frac{x_1}{z_1}\right)x'\right], \\
&= -\frac{ik}{2\pi z'}\exp[ik(z_0 + z_1)]\exp\left(\frac{ik}{2z_1}x_1^2\right)\iint_{\Sigma_0} dx_0 E_0(x_0)\exp\left(\frac{ik}{2z'_0}x_0^2 - \frac{ikx_1}{z'}x_0\right),
\end{aligned} \tag{2.52}$$

where

$$\begin{cases}
\dfrac{1}{q} = \dfrac{1}{z_0} - \dfrac{1}{f} + \dfrac{1}{z_1}, \\
z' = \dfrac{z_0 z_1}{q} = z_0 + z_1 - \dfrac{z_0 z_1}{f}, \\
z'_0 = \dfrac{z_0^2}{z_0 - q}, \\
z'_1 = \dfrac{z_1^2}{z_1 - q}.
\end{cases} \tag{2.53}$$

One may note some similarity and difference with the Fresnel diffraction (2.33). We can use (2.52) to derive some of the familiar properties of the lens.

2.5.1 Fourier Transform by a Lens

Let $z_1 = f$. Then $q = z_0$ and $z'_0 \to \infty$, so that

$$E_1(x_1, y_1) = -\frac{ik}{f} \exp[ik(z_0 + f)] \exp\left[\frac{ik}{2z'_1} x_1^2\right] \mathfrak{F}\{E_0(x_0)\}[k_x] \tag{2.54}$$

with $k_x = \dfrac{kx_1}{f}$ and $k_y = \dfrac{ky_1}{f}$. If, further, $z_0 = z_1 = f$, then

$$E_1(x_1, y_1) = -\frac{ik}{f} \exp(2ikf) \mathfrak{F}\{E_0(x_0)\}[k_x] \tag{2.55}$$

and the fields at the two focal planes are Fourier transform of each other.

2.5.2 Imaging by a Lens

If $1/q = (1/z_0) + (1/z_1) - (1/f) = 0$, then re-evaluate the integrals in the second line of (2.51) as

$$\begin{aligned}
E_1(x_1, y_1) &= -\frac{k^2}{4\pi^2 z_0 z_1} \exp[ik(z_0 + z_1)] \exp\left(\frac{ik}{2z_1} x_1^2\right) \iint_{\Sigma_0} \mathrm{d}x_0 E_0(x_0) \exp\left(\frac{ik}{2z_0} x_0^2\right) \\
&\quad \times \iint_{\Sigma'} \mathrm{d}x' \exp\left[-ik\left(\frac{x_0}{z_0} + \frac{x_1}{z_1}\right) x'\right], \\
&= -\frac{z_0}{z_1} \exp[ik(z_0 + z_1)] \exp\left(\frac{ik}{2z_1} x_1^2\right) \\
&\quad \times \iint_{\Sigma_0} \mathrm{d}x_0 E_0(x_0) \exp\left(\frac{ik}{2z_0} x_0^2\right) \delta\left(x_0 + \frac{z_0}{z_1} x_1\right), \\
&= -\frac{z_0}{z_1} \exp[ik(z_0 + z_1)] \exp\left(\frac{ik}{2f} \frac{z_0}{z_1} x_1^2\right) E_0\left(-\frac{z_0}{z_1} x_1\right).
\end{aligned} \tag{2.56}$$

This expression accounts for the amplitude scaling $(-z_0/z_1)$ and the image inversion and magnification $E_0(-(z_0/z_1)x_1)$. The lateral magnification is $M_x \equiv -\frac{z_1}{z_0}$. The quadratic phase term is due to the fact that the object Σ_0 and image Σ_1 planes are not spherical from the center of the lens.

2.5.3 Lens of Finite Aperture

If the finite aperture of the lens or presence of aberrations is represented with an aperture amplitude function $A(x', y')$, so that ψ_f is replaced with $A\psi_f$ in (2.48), then, at the image position where $\frac{1}{q} = 0$,

$$\begin{aligned} E_1(x_1, y_1) &= -\frac{k^2}{4\pi^2 z_0 z_1}\left\{\exp[ik(z_0+z_1)]\exp\left(\frac{ik}{2z_1}{x_1}^2\right)\right\} \\ &\quad \times \iint_{\Sigma_0} \mathrm{d}x_0 E_0(x_0)\exp\left(\frac{ik}{2z_0}{x_0}^2\right) \iint_{\Sigma'} \mathrm{d}x' A(x')\exp\left[-ik\left(\frac{x_0}{z_0}+\frac{x_1}{z_1}\right)x'\right], \\ &= -\frac{k^2}{2\pi z_0 z_1}\{\cdots\} \iint_{\Sigma_0} \mathrm{d}x_0\, E_0(x_0)\exp\left(\frac{ik}{2z_0}{x_0}^2\right)\mathfrak{F}\{A\}\left[\frac{k}{z_0}(x_0-\tilde{x}_1)\right], \\ &= \frac{k^2}{2\pi z_0 z_1}\{\cdots\}\left\{E_0(\tilde{x}_1)\exp\left(\frac{ik}{2z_0}{\tilde{x}_1}^2\right)\right\} \odot \mathfrak{F}\{A\}\left[-\frac{k}{z_0}\tilde{x}_1\right], \\ &= \frac{k^2}{2\pi z_0 z_1}\exp[ik(z_0+z_1)]\exp\left(\frac{ik}{2z_1}{x_1}^2\right) \\ &\quad \times \left\{E_0\left(-\frac{z_0}{z_1}x_1\right)\exp\left(\frac{ik}{2z_1}\frac{z_0}{z_1}x_1^2\right)\right\} \odot \mathfrak{F}\{A\}\left[\frac{k}{z_1}x_1\right], \end{aligned} \tag{2.57}$$

where

$$\tilde{x}_0 \equiv -\frac{z_1}{z_0}x_0; \quad \tilde{x}_1 \equiv -\frac{z_0}{z_1}x_1. \tag{2.58}$$

That is, the image is convolved (i.e., smoothed) with the Fourier transform of the aperture function.

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