Chapter 2
Analysis of Planar Linkages

In this chapter we consider assemblies of links that move in parallel planes. Any one of these planes can be used to examine the movement since the trajectories of points in any link can be projected onto this plane without changing their properties. Our focus is on linkages constructed from revolute joints with axes perpendicular to this plane and prismatic joints that move along lines parallel to it. We examine the RR, PR, and RP open chains and the closed chains constructed from them, as well as the 3R and RPR planar robots. We determine the configuration of the linkage as a function of the independent joint parameters and the physical dimensions of the links.

2.1 Coordinate Planar Displacements

A revolute joint in a planar linkage allows rotation about a point, and a prismatic joint allows translation along a line. These movements are represented by transformations of point coordinates in the plane.

Consider the rotation of a link about the revolute joint **O** located at the origin of the fixed coordinate frame F. Let $\mathbf{x} = (x, y)^T$ be the coordinates of a point measured in the frame M of the link. If the moving frame has its origin also located at **O**, and the angle between the x-axes of these two frames is θ, then the coordinates $\mathbf{X} = (X, Y)^T$ of this point in F are given by the matrix equation

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.1}$$

We introduce the extra column in this matrix to accommodate translations typical of prismatic joints as part of the matrix operation.

In particular, consider a prismatic joint that has the x-axis of F as its line of action. Let the distance between the origins of M and F along this line be s. Then

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we have

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & s \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.2}$$

A translation along a prismatic joint parallel to the y-axis is defined in the same way as

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & s \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.3}$$

The matrices in these equations define the three *coordinate displacements* of planar movement. Planar displacements are constructed from these three basic transformations.

We now introduce the notation $[Z(\theta)]$, $[X(s)]$, and $[Y(s)]$ for these coordinate displacements, so we have

$$\mathbf{X} = [Z(\theta)]\mathbf{x}, \quad \mathbf{X} = [X(s)]\mathbf{x}, \quad \text{and} \quad \mathbf{X} = [Y(s)]\mathbf{x}, \tag{2.4}$$

respectively. Notice that we do not distinguish symbolically between the coordinates $\mathbf{X}$ that are two-dimensional and those that have 1 as a third component. Some authors to refer to the former as vectors and the latter as affine points. We do not need this general distinction, and therefore will take the time to make the difference clear when needed in the context of our calculations.

2.1.1 The PR Open Chain

The benefit of this matrix formulation can be seen in considering the movement of a PR open chain. This chain consists of a link that slides along the linear guide of a P-joint relative to the ground. An end-link is attached to the slider by a revolute joint, Figure 2.1. We now determine the movement of a coordinate frame M attached to the end-link relative to a fixed frame F.

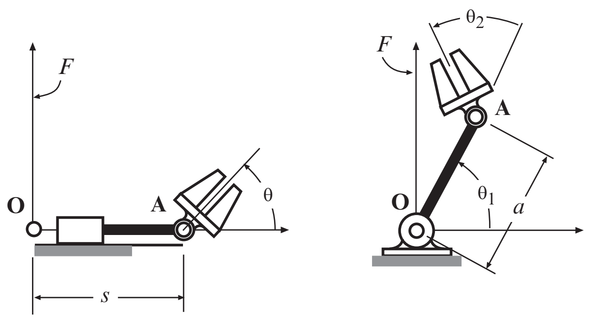

Fig. 2.1 The PR and RR open chain robots.

First, locate F so that its x-axis is parallel to the slide of the P-joint and denote its origin by **O**. Locate M in the end-link so that its origin is centered on the revolute joint, which we denote by **A**, and with its x-axis aligned initially with the x-axis of F.

The configuration of the PR chain is defined by the slide s from **O** to **A**, and the rotation angle θ about **O** measured from the x-axis of F to the x-axis of M. The transformation of coordinates from M to F is given by the matrix product

$$\mathbf{X} = [X(s)][Z(\theta)]\mathbf{x}, \tag{2.5}$$

or

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & s \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.6}$$

The set of planar displacements $[D]$, given by

$$[D] = [X(s)][Z(\theta)], \tag{2.7}$$

is the workspace of the PR open chain. This matrix equation defines the *kinematics equations* for the chain.

An important question in the analysis of an open chain is what parameter values s and θ are needed to reach a given displacement $[D]$ in the workspace of the chain. Assuming the elements of the matrix

$$[D] = \begin{bmatrix} a_{11} & a_{21} & p_x \\ a_{21} & a_{22} & p_y \\ 0 & 0 & 1 \end{bmatrix} \tag{2.8}$$

are known, equation (2.7) can be solved to determine these parameters. Notice that $p_y = 0$ is required for the displacement $[D]$ to be in the workspace reachable by the PR chain. It is now easy to see that s and θ can be determined from the elements of $[D]$ by the formulas

$$s = p_x \quad \text{and} \quad \theta = \arctan\frac{a_{21}}{a_{11}}. \tag{2.9}$$

Its important to note here that the arctan function must keep track of the signs of both a_{21} and a_{11} so the correct value for θ is obtained in the range 0 to 2π.

The arctan function in calculators often incorporates the assumption that the denominator of the fraction a_{21}/a_{11} is positive. If this denominator is actually negative, as occurs when θ is in the second and third quadrants, then π must be added to the angle returned by the calculator in order to obtain the correct result.

2.1.2 The RR Open Chain

A planar RR open chain has a fixed revolute joint **O** that connects a rotating link, or *crank*, to the ground link. A second revolute joint **A** connects the crank to the end-link, or *floating link*, Figure 2.1.

Position the fixed frame F so that its origin is the fixed pivot **O** and its x-axis is directed toward **A** when the crank **OA** is in the zero position. Introduce the moving frame M in the end-link, so that its origin is located at **A** and its x-axis is also directed, initially, along the segment **OA**.

Let θ_1 be the angle measured from F to **OA** as the linkage moves, and let θ_2 be the angle measured from **OA** to M. Then the position of M relative to F is defined by the composition of coordinate displacements

$$\mathbf{X} = [Z(\theta_1)][X(a)][Z(\theta_2)]\mathbf{x}, \tag{2.10}$$

where $a = |\mathbf{A} - \mathbf{O}|$ is the length of the crank. Expanding this equation we obtain

$$\begin{Bmatrix} X \\ Y \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & a\cos\theta_1 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & a\sin\theta_1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.11}$$

Notice that the position of the floating link of an RR chain is equivalent to a translation by the vector $\mathbf{d} = (a\sin\theta_1, a\sin\theta_1)^T$ followed by a rotation by the angle $\sigma = \theta_1 + \theta_2$.

The workspace of the RR chain is given by the set of displacements

$$[D] = [Z(\theta_1)][X(a)][Z(\theta_2)]. \tag{2.12}$$

This defines the kinematics equations of the RR chain. For a given position $[D]$ the parameter values θ_1 and θ_2 that reach it are obtained by equating (2.8) to the matrix in (2.11). The result is that the angles θ_1 and $\sigma = \theta_1 + \theta_2$ are given by

$$\theta_1 = \arctan\frac{p_y}{p_x} \quad \text{and} \quad \sigma = \arctan\frac{a_{21}}{a_{11}}. \tag{2.13}$$

Notice that the elements p_x and p_y must satisfy the relation

$$a = \sqrt{p_x^2 + p_y^2}. \tag{2.14}$$

in order for $[D]$ to be in the workspace of this chain.

2.1.3 The RPR and 3R Chains

If the distance a between the joints of an RR chain is allowed to vary, then we obtain the structure of a three-degree-of-freedom planar manipulator. This variation in length can be introduced either by a prismatic joint, forming an RPR open chain, or by a revolute joint to form a 3R open chain, Figure 2.2. The formulas for the RR chain can be used to analyze the RPR and 3R chains with minor modifications.

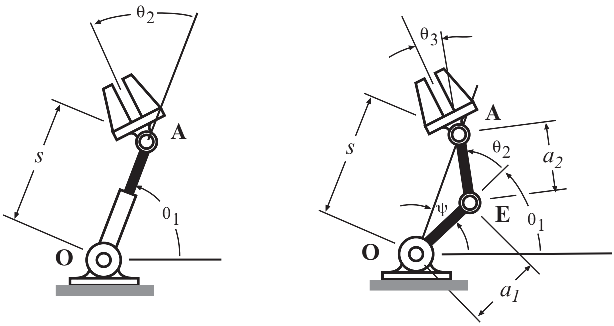

Fig. 2.2 The RPR and RRR open robots.

For the RPR, the link length a can be identified with the slide parameter s of the prismatic joint. The result is that (2.12) with $a = s$ defines the workspace of the RPR chain. Equations (2.14) and (2.13) define the values for s, θ_1, and θ_2 needed to reach a given goal displacement.

For the 3R case, we have an elbow joint **E** inserted between **O** and **A**. Let the lengths of the two links be $a_1 = |\mathbf{E}-\mathbf{O}|$ and $a_2 = |\mathbf{A}-\mathbf{E}|$. Denote the rotation angle about the elbow joint by θ_2 which is measured from **OE** counter-clockwise to **EA**. The rotation of the end-link around **A** is now denoted by θ_3. The kinematics equations of this chain become

$$[D] = [Z(\theta_1)][X(a_1)][Z(\theta_2)][X(a_2)][Z(\theta_3)]. \tag{2.15}$$

The variable length $s = |\mathbf{A}-\mathbf{O}|$ is given by the cosine law of the triangle $\triangle\mathbf{OEA}$,

$$s^2 = a_1^2 + a_2^2 + 2a_1a_2\cos\theta_2. \tag{2.16}$$

The positive sign for the cosine term in this equation arises because θ_2 is an exterior angle of the triangle $\triangle\mathbf{OEA}$. Notice that s must lie between the values $|a_2 - a_1|$ and $a_1 + a_2$.

For a given position $[D]$ of the end-link, we can determine the length s as we did for the RPR chain using (2.14). This allows us to compute the elbow joint angle as

$$\theta_2 = \arccos\frac{p_x^2 + p_y^2 - a_1^2 - a_2^2}{2a_1a_2}. \tag{2.17}$$

The arccosine function yields two values for this angle $\pm\theta_2$. We can compute the joint angle θ_1 using (2.13), however, we must account for the presence of the angle $\psi = \angle\mathbf{EOA}$, which is given by

$$\psi = \arctan\frac{a_2 \sin\theta_2}{a_1 + a_2\cos\theta_2}. \tag{2.18}$$

The result is

$$\theta_1 = \arctan\frac{p_y}{p_x} - \psi. \tag{2.19}$$

Finally, θ_3 is obtained from the fact that the rotation of the end-link is $\sigma = \theta_1 + \theta_2 + \theta_3$ in (2.13), which yields

$$\theta_3 = \arctan\frac{a_{21}}{a_{11}} - \theta_1 - \theta_2. \tag{2.20}$$

Notice that two sets of values θ_1 and θ_3 are obtained depending on the sign of θ_2. These are known as the elbow-up and elbow-down solutions.

2.2 Position Analysis of the RRRP Linkage

The RRRP linkage is called a *slider-crank* and consists of a rotating crank linked to a translating slider by a connecting rod, or *coupler*. It is a fundamental machine element found in everything from automotive engines to door-closing mechanisms. We can also view this device as a platform linkage, in which case the coupler is a workpiece supported by an RR and a PR chain.

Denote the fixed and moving pivots of the input crank by $\mathbf{O}$ and $\mathbf{A}$, respectively, and let $\mathbf{B}$ be the revolute joint attached to the slider. Position the fixed frame F so that its origin is $\mathbf{O}$ and orient it so that its x-axis is perpendicular to the direction of slide, Figure 2.3. The input crank angle θ is measured from the x-axis of F around $\mathbf{O}$ to $\mathbf{OA}$, and the travel s of the slider is measured along the y-axis to $\mathbf{B}$.

The length of the driving crank is $r = |\mathbf{A} - \mathbf{O}|$, and the length of the coupler is $L = |\mathbf{B} - \mathbf{A}|$. The distance e to the linear path of the pivot $\mathbf{B}$ is called the *offset*. Notice that the dimensions r, L, and e are always positive.

2.2.1 The Output Slide

To analyze this linkage, we determine the output slide s as a function of the input crank angle θ. The linkage moves so the pivots $\mathbf{A}$ and $\mathbf{B}$ remain the constant distance L apart. The coordinates of these pivots in F are given by

$$\mathbf{A} = \begin{Bmatrix} r\cos\theta \\ r\sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} e \\ s \end{Bmatrix}. \tag{2.21}$$

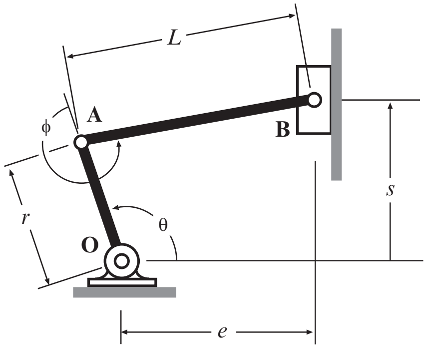

Fig. 2.3 The dimensions characterizing a slider-crank, or RRRP, linkage.

Thus, the length $L = |\mathbf{B} - \mathbf{A}|$ of the coupler provides the constraint equation

$$(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{A}) = L^2. \tag{2.22}$$

Substitute (2.21) into this expression and collect the coefficients of s to obtain the quadratic equation

$$s^2 - (2r\sin\theta)s + (r^2 + e^2 - 2er\cos\theta - L^2) = 0. \tag{2.23}$$

The quadratic formula yields the roots

$$s = r\sin\theta \pm \sqrt{L^2 - e^2 + 2er\cos\theta - r^2\cos^2\theta}. \tag{2.24}$$

Thus, for a given input crank angle θ there are two possible values of the slide s. They are, geometrically, the intersection of a circle of radius L centered on $\mathbf{A}$ with the line through $\mathbf{B}$ parallel to the y-axis of F. These two solutions define the two *assemblies* of the RRRP linkage. The positive solution generally has the slider moving above the crank, while the negative solution has it below.

2.2.2 The Range of Crank Rotation

We now consider the values of the crank angle θ for which a solution for the slider position s exists. The condition that the solution be a real number is

$$L^2 - e^2 + 2er\cos\theta - r^2\cos^2\theta \geq 0. \tag{2.25}$$

Set this to zero to obtain a quadratic equation in $\cos\theta$ that defines the minimum and maximum angular values for the crank angle θ, and obtain the roots

$$\theta_{\min} = \arccos\frac{e+L}{r} \quad \text{and} \quad \theta_{\max} = \arccos\frac{e-L}{r}. \tag{2.26}$$

Notice that the arccosine function returns two values for these limiting angles that are reflections through the x-axis of F.

If $\cos\theta_{\min} > 1$, then the lower limit $\theta_{\min}$ to the crank rotation angle does not exist. In which case the crank can reach $\theta = 0$ and pass into the lower half-plane of F. Thus, the condition that no lower limit exist is

$$S_1 = L - r + e > 0. \tag{2.27}$$

Similarly, if $\cos\theta_{\max} < -1$ then the upper limit does not exist, and the crank can reach $\theta = \pi$. This yields the condition

$$S_2 = L - r - e > 0. \tag{2.28}$$

The signs of the parameters S_1 and S_2 identify four types of slider-crank linkage depending on the input rotation of the crank:

1. **A rotatable crank:** $S_1 > 0$ and $S_2 > 0$, in which case neither limit $\theta_{\min}$ nor $\theta_{\max}$ exists, and the input crank can fully rotate.
2. **A** 0**-rocker:** $S_1 > 0$ and $S_2 < 0$, for which $\theta_{\max}$ exists but not $\theta_{\min}$, and the input crank rocks through $\theta = 0$ between the values $\pm\theta_{\max}$.
3. **A** π**-rocker:** $S_1 < 0$ and $S_2 < 0$, which means that $\theta_{\min}$ exists but not $\theta_{\max}$, and the input crank rocks through $\theta = \pi$ between the values $\pm\theta_{\min}$.
4. **A rocker:** $S_1 < 0$ and $S_2 < 0$, in which case both upper and lower limit angles exist, and the crank cannot pass through either 0 or π. Instead, it rocks in one of two separate ranges: (i) $\theta_{\min} \le \theta \le \theta_{\max}$, or (ii) $-\theta_{\max} \le \theta \le -\theta_{\min}$.

The conditions $S_1 > 0$ and $S_2 > 0$ for a fully rotatable crank can be combined to define the formula

$$S_1 S_2 = (L - r + e)(L - r - e) = (L - r)^2 - e^2 > 0. \tag{2.29}$$

Notice that because e is always positive, $L - r$ must be positive for S_2 to be positive. This allows us to conclude that

$$L - r > e \tag{2.30}$$

is the condition that ensures that the crank of the RRRP linkage can fully rotate.

The parameters S_1 or S_2 can take on zero values as well. In these cases, the pivots **O**, **A**, and **B** line up along the x-axis of F, and the slider-crank linkage is said to *fold*.

2.2.3 The Coupler Angle

Let ϕ denote the angle around the moving pivot **A** measured counterclockwise from the line extending along the crank **OA** to the segment **AB** defining the coupler. Then the coordinates of the pivot $\mathbf{B} = (e, s)^T$ are also given by the vector

$$\mathbf{B} = \begin{Bmatrix} r\cos\theta + L\cos(\theta+\phi) \\ r\sin\theta + L\sin(\theta+\phi) \end{Bmatrix}. \tag{2.31}$$

We equate the two vectors defining **B** to obtain

$$\begin{aligned} r\cos\theta + L\cos(\theta+\phi) &= e, \\ r\sin\theta + L\sin(\theta+\phi) &= s. \end{aligned} \tag{2.32}$$

These equations are called the *loop equations* of the slider-crank because they capture the fact that the linkage forms a closed loop. Solve these equations for $L\sin(\theta+\phi)$ and $L\cos(\theta+\phi)$ and use the arctan function to obtain

$$\theta + \phi = \arctan\frac{s - r\sin\theta}{e - r\cos\theta}. \tag{2.33}$$

This equation provides the value for ϕ associated with each solution for the slide s defined in (2.24).

2.2.4 The Extreme Slider Positions

The maximum translation of the slider, s_{max}, is reached when the coupler angle ϕ is equal to zero. In this instance the pivots **O**, **A**, and **B** fall on a line, so that $r+L$ forms the hypotenuse of a right triangle. This yields

$$s_{max} = \sqrt{(r+L)^2 - e^2}. \tag{2.34}$$

The crank angle θ_1 associated with s_{max} is obtained from the loop equations (2.32) as

$$\theta_1 = \arctan\frac{s_{max}}{e}. \tag{2.35}$$

Notice that the parameter s_{max} can be positive or negative, because the linkage can be assembled with the slider above over below the x-axis.

The minimum translation of the slider, s_{min}, occurs with the coupler angle ϕ is equal to π. In this configuration the pivots **A** and **B** are on opposite sides of **O** and $L-r$ is the hypotenuse of the triangle, so s_{min} is given by

$$s_{min} = \sqrt{(L-r)^2 - e^2}. \tag{2.36}$$

While s_{max} always exists, s_{min} exists only if this square root is real. There are two cases $L-r>0$ and $L-r<0$. In the first case the crank is fully rotatable and the associated crank angle is

$$\theta_2 = \pi + \arctan\frac{s_{min}}{e}. \tag{2.37}$$

The minimum slide results when the pivot **A** rotates to the position such that **O** lies between it and **B**. If $L - r < 0$ then **A** and **B** are on the same side of **O** and the crank angle is

$$\theta_2 = \arctan \frac{s_{\mathrm{min}}}{e}. \tag{2.38}$$

Notice that these extreme configurations can be reflected through the x-axis.

If the crank of the slider-crank is fully rotatable, then the angular travel of the crank as the slider moves from s_{max} to s_{min} is $|\theta_2 - \theta_1|$. The angular travel of the return from s_{min} to s_{max} is $|2\pi - (\theta_2 - \theta_1)|$. The ratio of these two ranges of travel is known as the *time ratio*

$$r_{\mathrm{t}} = \frac{|\theta_2 - \theta_1|}{|2\pi - (\theta_2 - \theta_1)|}. \tag{2.39}$$

Notice that if the offset e is nonzero then the time ratio is less than 1. This means that the crank rotates a smaller angular distance as it pulls the slider to s_{min}, than it does when it pushes it out again to s_{max}. This operation is known as *quick return* because for a constant angular velocity the slider moves slowly toward s_{max} and quickly as it returns to s_{min}.

2.2.5 The RRPR Linkage

A slider-crank linkage is often used in an inverted configuration in which the P-joint is connected to the floating link, Figure 2.4. In this form, the prismatic joint may be the piston of a linear actuator that drives the rotation of the crank **OA**. This system is analyzed as follows.

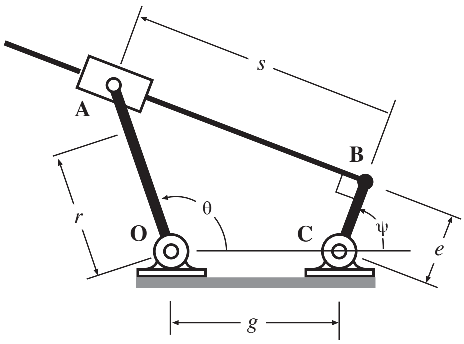

Fig. 2.4 The inverted slider-crank, or RRPR, linkage.

Let the driving RR crank be **OA** with length $r = |\mathbf{A} - \mathbf{O}|$, as before. Position the frame F with its origin at **O** and its x-axis directed toward **C**, which denotes the fixed pivot of the RP chain. The length of the ground link **OC** is $g = |\mathbf{C} - \mathbf{O}|$. Consider the line through **C** perpendicular to the direction of the slider and the line through **A** parallel this direction. Let the intersection of these two lines be the point **B**. The length $e = |\mathbf{B} - \mathbf{C}|$ is the joint offset, and $s = |\mathbf{A} - \mathbf{B}|$ is the slide distance of the prismatic joint. Denote the input crank angle by θ and let ψ be the angle measured about **C** to the segment **CB**.

These conventions allow us to introduce the intermediate parameters b and β given by

$$b = \sqrt{s^2 + e^2} \quad \text{and} \quad \tan\beta = \frac{s}{e}. \tag{2.40}$$

The cosine law for the triangle $\triangle\mathbf{COA}$ yields the relation

$$b^2 = g^2 + r^2 - 2rg\cos\theta. \tag{2.41}$$

Substitute $s^2 + e^2$ to obtain

$$s = \sqrt{g^2 + r^2 - e^2 - 2rg\cos\theta}. \tag{2.42}$$

This defines the joint slide s for a given crank angle θ. Notice that this equation can also be solved to determine θ for a given slide:

$$\cos\theta = \frac{g^2 + r^2 - e^2 - s^2}{2rg}. \tag{2.43}$$

This latter situation arises when the slider is the piston in a linear actuator driving the RR crank.

The rotation angle ψ of the RP crank is determined using the fact that the coordinates of the pivot **A** can be written in two ways

$$\mathbf{A} = \begin{Bmatrix} r\cos\theta \\ r\sin\theta \end{Bmatrix} = \begin{Bmatrix} g + b\cos(\psi + \beta) \\ b\sin(\psi + \beta) \end{Bmatrix}. \tag{2.44}$$

These equations yield the formula

$$\psi + \beta = \arctan\frac{r\sin\theta}{r\cos\theta - g}. \tag{2.45}$$

Notice that β is determined from s by (2.40).

The range of movement of the cranks and the sliding joint for this linkage can be analyzed in the same way as shown above for the RRRP linkage.

2.3 Position Analysis of the 4R Linkage

Given a planar 4R closed chain, we can identify an input RR crank and an output RR crank, Figure 2.5. Let the fixed and moving pivots of the input crank be **O** and **A**, respectively, and that the fixed and moving pivots of the output crank be **C** and

B. The distances between these points characterize the linkage:

$$a = |\mathbf{A} - \mathbf{O}|,\, b = |\mathbf{B} - \mathbf{C}|,\, g = |\mathbf{C} - \mathbf{O}|,\, h = |\mathbf{B} - \mathbf{A}|. \tag{2.46}$$

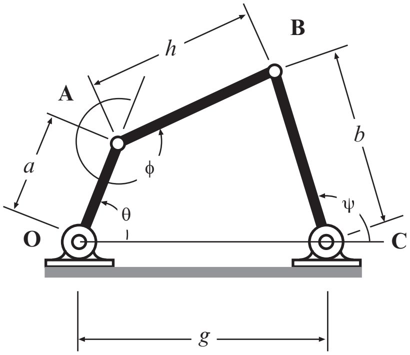

Fig. 2.5 The link lengths that define a 4R linkage.

To analyze the linkage, we locate the origin of the fixed frame F at **O**, and orient it so that the x-axis passes through the other fixed pivot **C**. Let θ be the input angle measured around **O** from the x-axis of F to **OA**. Similarly, let ψ be the angular position of the output crank **CB**.

2.3.1 Output Angle

The relationship between the input angle θ of the driving crank and the angle ψ of the driven crank is obtained from the condition that **A** and **B** remain a fixed distance apart throughout the motion of the linkage. Since $h = |\mathbf{B} - \mathbf{A}|$ is constant, we have the constraint equation

$$(\mathbf{B} - \mathbf{A}) \cdot (\mathbf{B} - \mathbf{A}) - h^2 = 0. \tag{2.47}$$

The coordinates of **A** and **B** in F are given by

$$\mathbf{A} = \begin{Bmatrix} a\cos\theta \\ a\sin\theta \end{Bmatrix} \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} g + b\cos\psi \\ b\sin\psi \end{Bmatrix}. \tag{2.48}$$

Substitute these coordinates into (2.47) to obtain

$$b^2 + g^2 + 2gb\cos\psi + a^2 - 2(a\cos\theta(g + b\cos\psi) + ab\sin\theta\sin\psi) - h^2 = 0. \tag{2.49}$$

Gathering the coefficients of $\cos\psi$ and $\sin\psi$, we obtain the constraint equation for the 4R chain as

$$A(\theta)\cos\psi + B(\theta)\sin\psi = C(\theta), \tag{2.50}$$

where

$$A(\theta) = 2ab\cos\theta - 2gb,$$
$$B(\theta) = 2ab\sin\theta,$$
$$C(\theta) = g^2 + b^2 + a^2 - h^2 - 2ag\cos\theta. \tag{2.51}$$

The solution to this equation is

$$\psi(\theta) = \arctan\left(\frac{B}{A}\right) \pm \arccos\left(\frac{C}{\sqrt{A^2+B^2}}\right). \tag{2.52}$$

Equations of the form (2.50) arise many times in the analysis of linkages, so we present its solution in Appendix A for easy reference, see (A.1).

Notice that there are two angles ψ for each angle θ. This arises because the moving pivot **B** of the output crank can be assembled above or below the diagonal joining the moving pivot **A** of the input crank to the fixed pivot **C** of the output crank. The angle $\delta = \arctan(B/A)$ defines the location of this diagonal, and $\varepsilon = \arccos(C/\sqrt{A^2+B^2})$ is the angle above and below this diagonal that locates the output crank.

The argument of the arccosine function must be in the range -1 to $+1$, which places a solvability constraint on the coefficients A, B, and C. Specifically, for a solution to exist we must have

$$A^2 + B^2 - C^2 \geq 0. \tag{2.53}$$

If this constraint is not satisfied, then the linkage cannot be assembled for the specified input crank angle θ.

2.3.2 Coupler Angle

Let ϕ denote the angle of the coupler measured about **A** relative to the segment **OA**, so $\theta + \phi$ measures the angle to **AB** from the x-axis of F. The coordinates of **B** can also be defined in terms of ϕ as

$$\mathbf{B} = \begin{Bmatrix} a\cos\theta + h\cos(\theta+\phi) \\ a\sin\theta + h\sin(\theta+\phi) \end{Bmatrix}. \tag{2.54}$$

Equating the two forms for **B**, we obtain the loop equations of the four-bar linkage

$$a\cos\theta + h\cos(\theta+\phi) = g + b\cos\psi,$$
$$a\sin\theta + h\sin(\theta+\phi) = b\sin\psi. \tag{2.55}$$

For a given value of the drive crank θ, determine ψ using (2.52) then $\cos(\theta+\phi)$ and $\sin(\theta+\phi)$ are given by

$$\cos(\theta+\phi) = \frac{g + b\cos\psi - a\cos\theta}{h} \quad \text{and} \quad \sin(\theta+\phi) = \frac{b\sin\psi - a\sin\theta}{h}. \quad (2.56)$$

Thus, the value of the coupler angle is obtained as

$$\phi = \arctan\left(\frac{b\sin\psi - a\sin\theta}{g + b\cos\psi - a\cos\theta}\right) - \theta. \quad (2.57)$$

Notice that a unique value for ϕ is associated with each of the two solutions for the output angle ψ.

2.3.2.1 An Alternative Derivation

It is useful here to present a direct calculation of the coupler angle ϕ associated with a given crank angle θ. The derivation is identical to that above for the output angle. However, our standard frame is now F', positioned with its origin at **A** and its x-axis along the vector $\mathbf{O} - \mathbf{A}$. In this coordinate frame, the pivots **B** and **C** have the coordinates

$$^{F'}\mathbf{B} = \begin{Bmatrix} h\cos(\phi-\pi) \\ h\sin(\phi-\pi) \end{Bmatrix} \quad \text{and} \quad ^{F'}\mathbf{C} = \begin{Bmatrix} a + g\cos(\pi-\theta) \\ g\sin(\pi-\theta) \end{Bmatrix}. \quad (2.58)$$

The constraint $(\mathbf{B}-\mathbf{C})\cdot(\mathbf{B}-\mathbf{C}) = b^2$ yields the equation

$$A(\theta)\cos\phi + B(\theta)\sin\phi = C(\theta), \quad (2.59)$$

where

$$\begin{aligned} A(\theta) &= 2ah - 2gh\cos\theta, \\ B(\theta) &= 2gh\sin\theta, \\ C(\theta) &= b^2 - a^2 - g^2 - h^2 + 2ag\cos\theta. \end{aligned} \quad (2.60)$$

This equation is solved in exactly the same way as before (A.1). It results in two values for ϕ for each crank angle θ. The output angle ψ associated with each of these coupler angles can be determined from the loop equations of the linkage written for **C** in F'.

This equation for the coupler angle is used in solutions for four and five position synthesis of a planar 4R linkage.

2.3.3 *Transmission Angle*

The angle ζ between the coupler and the driven crank at **B** is called the *transmission angle* of the linkage. If the only external loads on the linkage are torques on the input and output cranks, then the forces $\mathbf{F}_A$ and $\mathbf{F}_B$ acting on the coupler at the moving pivots must oppose each other along the line **AB**, Figure 2.6. Thus, the force $\mathbf{F}_B$ is directed at the angle ζ relative to the driven crank, and $\sin\zeta$ measures the component of $\mathbf{F}_B$ that is transmitted as useful output torque. The $\cos\zeta$ component is absorbed as a reaction force at the fixed pivot of the driven crank.

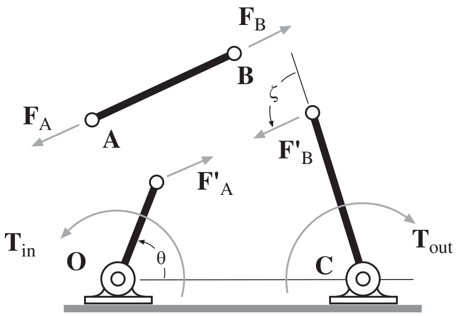

Fig. 2.6 The coupler is a two-force member connecting the input and output cranks.

To determine ζ in terms of θ, equate the cosine laws for the diagonal $d = |\mathbf{A} - \mathbf{C}|$ for the triangles $\triangle\mathbf{COA}$ and $\triangle\mathbf{ABC}$. Since ζ is the exterior angle at **B**, we have

$$d^2 = g^2 + a^2 - 2ag\cos\theta = h^2 + b^2 + 2bh\cos\zeta. \tag{2.61}$$

The result is

$$\cos\zeta = \frac{g^2 + a^2 - h^2 - b^2 - 2ag\cos\theta}{2bh}. \tag{2.62}$$

2.3.4 Coupler Curves

As a linkage moves, points in the coupler trace curves in the fixed frame. The parameterized equation of this curve is obtained from the kinematics equations of the driving RR chain. Let $\mathbf{x} = (x, y)^T$ be the coordinates of a coupler point in the frame M located at **A** with its x-axis along **AB**. The coordinates $\mathbf{X} = (X, Y)^T$ in F are given by the matrix equation

$$\begin{Bmatrix} X(\theta) \\ Y(\theta) \\ 1 \end{Bmatrix} = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & a\cos\theta \\ \sin(\theta+\phi) & \cos(\theta+\phi) & a\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ 1 \end{Bmatrix}. \tag{2.63}$$

The coupler angle ϕ is a function of θ, thus the coupler curve is parametrized by the crank angle θ.

The algebraic equation for this curve, eliminating θ, is obtained by defining the coordinates of **X** from two points of view. Let the coupler triangle $\triangle\mathbf{XAB}$ (Figure

2.7) have lengths r and s given by

$$r = |\mathbf{X}-\mathbf{A}| = \sqrt{x^2+y^2} \quad \text{and} \quad s = |\mathbf{X}-\mathbf{B}| = \sqrt{(x-h)^2+y^2}. \tag{2.64}$$

If λ is the angle to $\mathbf{AX}$ in F, and μ is the angle to $\mathbf{BX}$, then we have

$$\mathbf{X}-\mathbf{A} = \begin{Bmatrix} r\cos\lambda \\ r\sin\lambda \end{Bmatrix} \quad \text{and} \quad \mathbf{X}-\mathbf{B} = \begin{Bmatrix} s\cos\mu \\ s\sin\mu \end{Bmatrix}. \tag{2.65}$$

Rearrange these equations to isolate $\mathbf{A}$ and $\mathbf{B}$, and substitute into the identities $\mathbf{A}\cdot\mathbf{A} = a^2$ and $(\mathbf{B}-\mathbf{C})\cdot(\mathbf{B}-\mathbf{C}) = b^2$ to obtain

$$\begin{aligned} X^2+Y^2-2Xr\cos\lambda-2Yr\sin\lambda+r^2 &= a^2, \\ X^2+Y^2-2Xs\cos\mu-2Ys\sin\mu+s^2-2gs\cos\mu-2Xg+g^2 &= b^2. \end{aligned} \tag{2.66}$$

The algebraic equation of the coupler curve is obtained by eliminating λ and μ from these two equations.

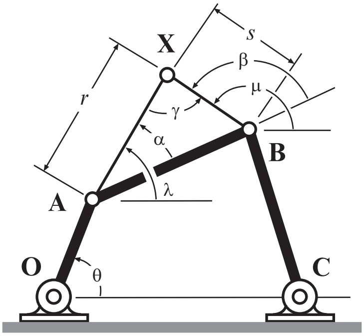

Fig. 2.7 The trajectory of a point in the floating link is known as a coupler curve of the 4R chain.

First, note that if α is the interior angle of the coupler triangle $\triangle\mathbf{XAB}$ at $\mathbf{A}$, then $\lambda = \alpha+\theta+\phi$. Similarly, if β is the exterior angle of this triangle at $\mathbf{B}$, then $\mu = \beta+\theta+\phi$, or equivalently

$$\mu-\lambda = \beta-\alpha. \tag{2.67}$$

The angle $\gamma = \beta-\alpha$ is the interior angle of the coupler triangle at $\mathbf{X}$ given by the cosine law as

$$\cos\gamma = \frac{r^2+s^2-h^2}{2rs}. \tag{2.68}$$

Substitute $\mu = \lambda+\gamma$ into (2.66) and rearrange these equations to obtain

$$A_1 \cos\lambda + B_1 \sin\lambda = C_1,$$
$$A_2 \cos\lambda + B_2 \sin\lambda = C_2, \tag{2.69}$$

where

$$A_1 = 2rX, \qquad A_2 = 2s(\cos\gamma(X-g) + Y\sin\gamma),$$
$$B_1 = 2rY, \qquad B_2 = 2s(-\sin\gamma(X-g) + Y\cos\gamma),$$
$$C_1 = X^2 + Y^2 + r^2 - a^2, \qquad C_2 = (X-g)^2 + Y^2 - b^2 + s^2. \tag{2.70}$$

Eliminate λ in these equations by solving linearly for $x = \cos\lambda$ and $y = \sin\lambda$. Then impose the condition $x^2 + y^2 = 1$. The result is

$$(C_1B_2 - C_2B_1)^2 + (A_2C_1 - A_1C_2)^2 - (A_1B_2 - A_2B_1)^2 = 0. \tag{2.71}$$

Notice that A_i and B_i are linear in the coordinates X and Y, and C_i are quadratic. Therefore, this equation defines a curve of degree six. See Hunt [50] for a detailed study of this curve, known as a tricircular sextic, and a description of its properties.

2.4 Range of Movement

2.4.1 Limits on the Input Crank Angle

The formula that defines the output angle ψ for a given input angle θ has a solution only when $A^2 + B^2 - C^2 \geq 0$. When this condition is violated, the crank is rotated to a positioned in which the mechanism cannot be assembled. The maximum and minimum values for θ are obtained by setting this condition to zero, which yields the quadratic equation in $\cos\theta$

$$4a^2g^2\cos^2\theta - 4ag(g^2 + a^2 - h^2 - b^2)\cos\theta$$
$$+\big((g^2 + a^2) - (h+b)^2\big)\big((g^2 + a^2) - (h-b)^2\big) = 0. \tag{2.72}$$

The roots of this equation are the upper and lower limiting angles $\theta_{\max}$ and $\theta_{\min}$ that define the range of movement of the input crank,

$$\cos\theta_{\min} = \frac{(g^2 + a^2) - (h-b)^2}{2ag}, \cos\theta_{\max} = \frac{(g^2 + a^2) - (h+b)^2}{2ag}. \tag{2.73}$$

These equations are the cosine laws for the two ways that the triangle $\triangle$**AOC** can be formed with the coupler **AB** aligned with the output crank **CB**, Figure 2.8. This alignment is what limits rotation of the input crank. The cosine function does not distinguish between $\pm\theta$ so there are actually two limits for each case $\pm\theta_{\min}$ and $\pm\theta_{\max}$ above and below **OC**.

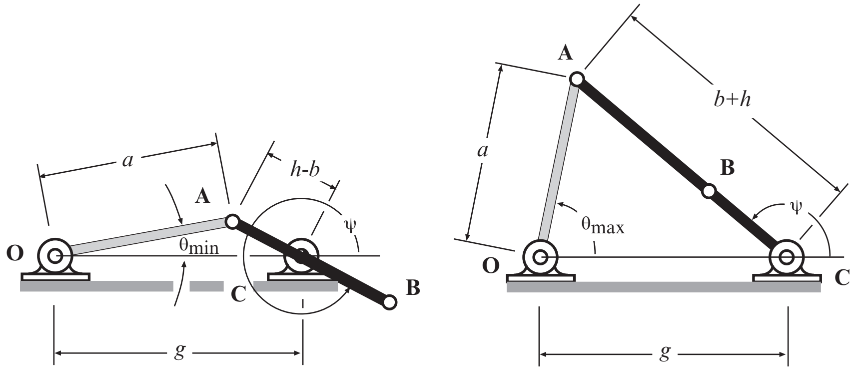

Fig. 2.8 The angles θ_{min} and θ_{max} are the limits to the range of movement of the input link.

The arccosine function yields a real angle only if its argument is between -1 and 1. This provides conditions that determine whether these crank limits exist.

2.4.1.1 The Lower Limit: θ_{min}

If θ_{min} does not exist, then the crank has no lower limit to its movement and it rotated through $\theta = 0$ to reach negative values below the segment **OC**. Thus, $\cos\theta_{\text{min}} > 1$ is the condition that there is no lower limit to the input crank rotation, that is,

$$\frac{(g^2+a^2)-(h-b)^2}{2ag} > 1. \tag{2.74}$$

This simplifies to yield

$$(g-a)^2-(h-b)^2 > 0. \tag{2.75}$$

Factor the difference of two squares to obtain

$$\begin{aligned}(g-a+h-b)(g-a-h+b) &> 0,\\ T_1T_2 &> 0,\end{aligned} \tag{2.76}$$

where

$$T_1 = g-a+h-b \quad \text{and} \quad T_2 = g-a-h+b. \tag{2.77}$$

Thus, T_1 and T_2 must both be either positive or negative for there to be no lower limit to the rotation of the input crank.

2.4.1.2 The Upper Limit: $\theta_{\max}$

If $\theta_{\max}$ does not exist, then the crank has no upper limit to its movement and it will be able to rotate through $\theta = \pi$. Thus, $\cos\theta_{\max} < -1$, or

$$\frac{(g^2+a^2)-(h+b)^2}{2ag} < -1, \tag{2.78}$$

is the condition that this limit does not exist. This inequality simplifies to

$$(h+b)^2-(g+a)^2 > 0, \tag{2.79}$$

which factors to become

$$\begin{aligned}(h+b-g-a)(h+b+g+a) &> 0,\\ T_3T_4 &> 0,\end{aligned} \tag{2.80}$$

where

$$T_3 = h+b-g-a, \quad \text{and} \quad T_4 = h+b+g+a. \tag{2.81}$$

The sum of the link lengths T_4 is always positive. Therefore, the condition that there is no upper limit to the rotation of the input crank is $T_3 > 0$.

2.4.1.3 Input Crank Types

We can now identify four types of movement available to the input crank of a 4R linkage:

1. **A crank:** $T_1T_2 > 0$ and $T_3 > 0$, in which case neither $\theta_{\min}$ nor $\theta_{\max}$ exists, and the input crank can fully rotate.
2. **A** 0**-rocker:** $T_1T_2 > 0$ and $T_3 < 0$, for which $\theta_{\max}$ exists but not $\theta_{\min}$, and the input crank rocks through $\theta = 0$ between the values $\pm\theta_{\max}$.
3. **A π-rocker:** $T_1T_2 < 0$ and $T_3 > 0$, which means that $\theta_{\min}$ exists but not $\theta_{\max}$, and the input crank rocks through $\theta = \pi$ between the values $\pm\theta_{\min}$.
4. **A rocker:** $T_1T_2 < 0$ and $T_3 < 0$, which means that both upper and lower limiting angles exist, and the crank cannot pass through either 0 or π. Instead, it rocks in one of two separate ranges: (i) $\theta_{\min} \le \theta \le \theta_{\max}$, or (ii) $-\theta_{\max} \le \theta \le -\theta_{\min}$.

2.4.2 Limits on the Output Crank Angle

The range of movement of the output crank can be analyzed in the same way. The limiting positions occur when the input crank **OA** and coupler **AB** become aligned, see Figure 2.9. The limits $\psi_{\min}$ and $\psi_{\max}$ are defined by the equations

$$\cos\psi_{\min} = \frac{(h+a)^2-(g^2+b^2)}{2bg}, \cos\psi_{\max} = \frac{(h-a)^2-(g^2+b^2)}{2bg}. \tag{2.82}$$

Note that in this case ψ is the exterior angle, which changes the sign of the cosine term in the cosine law formula.

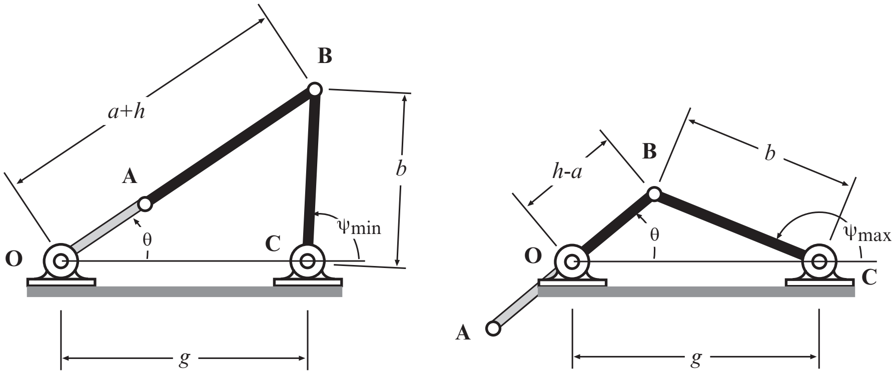

Fig. 2.9 The angles $\psi_{\min}$ and $\psi_{\max}$ are the limits to the range of motion of the output link.

Examining the existence of solutions to arccosine in (2.82), we find that the condition for no lower limit $\psi_{\min}$ is

$$\begin{aligned}(h+a-g-b)(h+a+g+b) &> 0,\\ (-T_2)(T_4) &> 0,\end{aligned} \tag{2.83}$$

where T_2 and T_4 are the same parameters used above for the input crank. Because T_4 is always greater than zero, the condition that there be no lower limit to the range of movement of the output crank is $T_2 < 0$.

Similarly, in order for there to be no upper limit $\psi_{\max}$, we have

$$\begin{aligned}(g-b-h-a)(g-b+h-a) &> 0,\\ (-T_3)(T_1) &> 0.\end{aligned} \tag{2.84}$$

Again, the parameters T_3 and T_1 are the same as were defined above and there is no upper limit to the movement of the output crank when $T_1T_3 < 0$.

2.4.2.1 Output Crank Types

We can now identify four types of movement available to the output crank of a four-bar linkage:

1. **A rocker:** $T_1T_3 > 0$ and $T_2 > 0$. In this case both limits $\psi_{\min}$ and $\psi_{\max}$ exist, and the crank cannot not pass through either 0 or π. Instead, it rocks in one of two separate ranges: (i) $\psi_{\min} \leq \psi \leq \psi_{\max}$, or (ii) $-\psi_{\max} \leq \psi \leq -\psi_{\min}$.
2. **A 0-rocker:** $T_1T_3 < 0$ and $T_2 > 0$, for which $\psi_{\max}$ exists but not $\psi_{\min}$, and the output crank rocks through $\psi = 0$ between the values $\pm\psi_{\max}$.
3. **A π-rocker:** $T_1T_3 > 0$ and $T_2 < 0$, which means that $\psi_{\min}$ exists but not $\psi_{\max}$, and the output crank rocks through $\psi = \pi$ between the values $\pm\psi_{\min}$.
4. **A crank:** $T_1T_3 < 0$ and $T_2 < 0$. Then neither limit $\psi_{\min}$ nor $\psi_{\max}$ exists, and the output crank can fully rotate.

2.4.3 The Classification of Planar 4R Linkages

A planar 4R linkage is classified by the movement of its input and output cranks. For example, a crank-rocker has a fully rotatable input link, and an output link that rocks between two limits. On the other hand a rocker-crank has an input link that rocks and an output link that fully rotates. The combinations of positive and negative signs for the parameters T_1, T_2, T_3 identify eight basic linkage types. These parameters can take zero values as well, in which case the linkage folds.

2.4.3.1 The Eight Basic Types

The link lengths a, b, g, and h for a 4R chain define the three parameters T_1, T_2, and T_3. Our classification scheme requires only the signs of these parameters, therefore we assemble the array $(\operatorname{sgn} T_1, \operatorname{sgn} T_2, \operatorname{sgn} T_3)$. The eight possible arrays identify the eight basic types of 4R linkages.

We separate the linkage types into two general classes depending upon the sign of the product $T_1T_2T_3$. If $T_1T_2T_3 > 0$ then the linkage is called *Grashof*; otherwise, it is called *nonGrashof*. There are four Grashof and four non-Grashof linkage types.

We consider the Grashof cases first:

1. $(+,+,+)$: Because $T_1T_2 > 0$ and $T_3 > 0$ the input link can fully rotate. Similarly, because $T_1T_3 > 0$ and $T_2 > 0$ the output link is a rocker with two output ranges. This linkage is a *crank-rocker*.
2. $(+,-,-)$: With $T_1T_2 < 0$ and $T_3 < 0$ the input is a rocker, and with $T_1T_3 < 0$ and $T_2 < 0$ the output is a crank. This defines the *rocker-crank* linkage.
3. $(-,-,+)$: In this case, $T_1T_2 > 0$ and $T_3 > 0$, so the input link is a crank, and $T_1T_3 < 0$ and $T_2 < 0$, which means that the output link is also a crank. This defines the *double-crank* linkage.
4. $(-,+,-)$: $T_1T_2 < 0$ and $T_3 < 0$ define the input as a rocker, and with $T_1T_3 > 0$ and $T_2 > 0$ the output is also a rocker. This defines the *Grashof double-rocker* linkage type.

Now consider the nonGrashof cases:

5. $(-,-,-)$: Here we have $T_1T_2 > 0$ and $T_3 < 0$, and the input link rocks through the value $\theta = 0$. With $T_1T_3 > 0$ and $T_2 < 0$, the output link rocks through the value $\psi = 0$. This type of linkage is termed a 00 *double-rocker*.
6. $(+,+,-)$: In this case, the input rocks through $\theta = 0$. However, with $T_1T_3 < 0$ and $T_2 > 0$ the output rocks through $\psi = \pi$. This linkage is called a 0π *double-rocker*.
7. $(+,-,+)$: With $T_1T_2 > 0$ and $T_3 > 0$ the input link rocks through π, and because $T_1T_3 < 0$ and $T_2 < 0$ the output link rocks through 0. This is the $\pi 0$ *double-rocker*.
8. $(-,+,+)$: Finally, the input again rocks through π, as does the output, defining the $\pi\pi$ *double-rocker*.

The parameters associated with these linkages are listed in Table 2.1.

Table 2.1 Basic Planar 4R Linkage types

	Linkage type	T_1	T_2	T_3
1	Crank-rocker	+	+	+
2	Rocker-crank	+	−	−
3	Double-crank	−	−	+
4	Grashof double-rocker	−	+	−
5	00 double-rocker	−	−	−
6	0π double-rocker	+	+	−
7	$\pi 0$ double-rocker	+	−	+
8	$\pi\pi$ double-rocker	−	+	+

2.4.4 Grashof Linkages

If a linkage is to be used in a continuous operation, the input crank should be able to fully rotate so that it can be driven by a rotating power source. A study of the configurations of a 4R linkage lead Grashof to conclude that, for a shortest link of length s and longest link of length l, the shortest link will fully rotate if

$$s + l < p + q, \tag{2.85}$$

where p and q are the lengths of the other two links. This is known as Grashof's criterion and linkages that have a rotatable crank are called *Grashof linkages*.

There are four linkage types that satisfy Grashof's criterion. If the input or output link is the shortest, then we have the crank-rocker or the rocker-crank, respectively. If the ground link is the shortest, then both the input and output links will fully rotate relative to the ground; this is the double-crank linkage. Finally, if the floating link is the shortest link, then the input and output links are rockers; this is the Grashof double-rocker. By examining Table 2.1, it is easy to see that these four linkage types satisfy the condition

$$T_1 T_2 T_3 > 0, \tag{2.86}$$

which can be shown to be equivalent to Grashof's criterion.

The rockers of each of the Grashof linkage types are distinguished by the fact that both upper and lower limits exist. The means that they have two distinct angular ranges of movement, one in the upper half plane and one in the lower relative to the fixed link. If the linkage is assembled so that the rocker is in one angular range, then it cannot reach the other range without disassembly. Thus, Grashof linkages have two distinct sets of configurations called *assemblies*. The linkage can move between the configurations in only one of these assemblies and cannot reach the others.

2.4.5 Folding Linkages

If any one of the parameters T_1, T_2, or T_3 has the value zero, then the linkage can take a configuration in which all four joints lie on a line. The linkage is said to fold.

If we consider the positive, negative, and zero values for the array (T_1, T_2, T_3), then we find that there are 27 types of planar 4R linkages, 19 of which fold. Furthermore, the number of parameters T_i that are zero defines the number of folding configurations of the linkage. It is often useful to have a linkage fold. However, while it is easy to drive the linkage into a folded configuration, it may be difficult to get it out of the this configuration.

Consider, for example, the *parallelogram linkage* defined by $a = b$ and $g = h$. This linkage has $T_2 = 0$ and $T_3 = 0$. Thus, it has two folding configurations, which occur for the input crank angles of $\theta = 0, \pi$. Another doubly folding example is the *kite linkage* with $g = a$ and $h = b$, which yields $T_1 = 0$ and $T_2 = 0$. This linkage folds when $\theta = 0$, at which point the output link can freely rotate because the joints **A** and **C** coincide; the second folding position occurs when $\psi = \pi$.

There is one triply folding case, the *rhombus linkage*, for which $a = b = g = h$. This linkage is a combination of the parallelogram and kite linkages. It folds at the two configurations $\theta = 0, \pi$ like the parallelogram. When $\theta = 0$ the output link is free to rotate because the joints **A** and **C** coincide as with the kite linkage. The third folding configuration occurs when $\psi = \pi$. The linkage can also reach this configuration with $\theta = \pi$, in which case it is the input crank that can freely rotate because **O** and **B** coincide.

Linkages that have small values for any of the parameters T_i are termed *near folding*. These linkages have configurations in which the joints can lie close to a line. In nearly folded configurations the transmission angle of the linkage is near 0 or π, and the output crank is difficult to move using the input crank.

2.5 Velocity Analysis

The velocity analysis of a linkage determines the angular rates of the various joint parameters as a function of the configuration of the linkage and the input joint rate. This analysis can be used in combination with the principle of virtual work to provide an important technique for determining the force and torque transmission properties of these systems.

2.5.1 Velocity of a Point in a Moving Link

Points $\mathbf{x}$ fixed in a moving link M trace *trajectories* $\mathbf{X}(t) = [T(t)]\mathbf{x}$ in the fixed frame F. The velocity of a point along its trajectory is the time derivative of its coordinate vector, that is, $\mathbf{V} = \dot{\mathbf{X}}$. Of importance to us is the relationship between this velocity and the movement of the linkage as a whole.

The usual convention in velocity calculations is to focus on the trajectories in F rather than coordinates in M of the moving point. For this reason, the trajectory of the of a segment $\mathbf{AB}$ fixed in M is defined by coordinates $\mathbf{A}$ and $\mathbf{B}$ measured in F. A general trajectory $\mathbf{X}$ of M has the property that $r = |\mathbf{X}-\mathbf{A}|$ and $\alpha = \angle\mathbf{BAX}$ are constants, because the three points $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{X}$ are part of the same link.

Let the orientation of M be defined by the angle θ of $\mathbf{AB}$ measured relative to the x-axis of F. Then we can determine the relative position vector $\mathbf{X}-\mathbf{A}$ as

$$\mathbf{X}-\mathbf{A} = \begin{Bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{Bmatrix}. \tag{2.87}$$

The time derivative of this vector yields

$$\mathbf{V} = \dot{\mathbf{X}} = \dot{\mathbf{A}} + \dot{\theta}[J](\mathbf{X}-\mathbf{A}), \quad \text{where} \quad [J] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{2.88}$$

This defines the velocity of a general point in terms of the velocity of a reference point and the rate of rotation of the body. We now show that $\dot{\theta}[J]$ is directly related to the angular velocity of this link.

Each body in a planar linkage rotates about an axis that is perpendicular to the plane of movement. Denote this direction by $\vec{k} = (0,0,1)^T$. Then the usual vector cross product yields $\vec{k}\times\vec{\imath} = \vec{\jmath}$ and $\vec{k}\times\vec{\jmath} = -\vec{\imath}$, where $\vec{\imath}$ and $\vec{\jmath}$ are unit vectors along the x- and y-axes of the fixed frame F. We now define the *angular velocity* of the link $\mathbf{AB}$ to be the vector

$$\mathbf{w}_{AB} = \dot{\theta}\vec{k}. \tag{2.89}$$

where θ defines the orientation of $\mathbf{AB}$ in F. Notice that for any vector $\mathbf{y}$, the angular velocity vector satisfies the identity

$$\mathbf{w}_{AB} \times \mathbf{y} = \dot{\theta}[J]\mathbf{y}. \tag{2.90}$$

This allows us to write equation (2.88) for the velocity of a point in the form

$$\mathbf{V} = \dot{\mathbf{A}} + \mathbf{w}_{AB} \times (\mathbf{X} - \mathbf{A}). \tag{2.91}$$

The angular velocity vector can be viewed as an operator that computes the component of velocity that arises from the rotation of the link.

Notice that if the link **AB** simply rotates about **A**, then $\dot{\mathbf{A}} = 0$, and we have

$$\mathbf{V} = \mathbf{w}_{AB} \times (\mathbf{X} - \mathbf{A}). \tag{2.92}$$

In this case, the velocity of a point **X** is directed 90° to the line joining it to **A**.

2.5.2 *Instant Center*

It interesting to note that there is a point in every moving link that has zero velocity. This point **I**, known as the *instant center*, is found by setting (2.91) to zero, that is,

$$\dot{\mathbf{A}} + \mathbf{w}_{AB} \times (\mathbf{I} - \mathbf{A}) = 0. \tag{2.93}$$

Take the cross product by $\mathbf{w}_{AB}$ and solve for **I** to obtain

$$\mathbf{I} - \mathbf{A} = \frac{\mathbf{w}_{AB} \times \dot{\mathbf{A}}}{\mathbf{w}_{AB} \cdot \mathbf{w}_{AB}}. \tag{2.94}$$

This calculation uses the vector identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

The geometric meaning of **I** is found by substituting $\dot{\mathbf{A}}$ from (2.93) into (2.91) to obtain

$$\mathbf{V} = \mathbf{w}_{AB} \times (\mathbf{X} - \mathbf{I}). \tag{2.95}$$

Compare this to (2.92) to see that the distribution of velocities in this link, at this instant, is the same as is generated by a rotation about the instant center **I**.

2.6 Velocity Analysis of an RR Chain

The kinematics equations of an open chain define the set of positions it can reach as a function of its joint parameters. If each of these parameters is given as a function of time, then we obtain a curve in its workspace that defines the trajectory of the end-link. The time derivative of the kinematics equations defines the velocity along this trajectory.

The 3×3 transform $[D] = [A, \mathbf{P}]$ for planar open chains separate into a 2×2 rotation matrix $[A]$ and a 2×1 translation vector **P**. The translation vector **P** is defined by the position of reference point in the end-link. The orientation ϕ of this link is the sum of the relative rotation angles at each joint. Thus, the velocity of any trajectory

$\mathbf{X}(t)$ of any point in the end-link is given by the equation

$$\mathbf{V} = \dot{\mathbf{P}} + \mathbf{w}_M \times (\mathbf{X} - \mathbf{P}), \tag{2.96}$$

where $\mathbf{w}_M$ is, the angular velocity vector of the end-link, is the sum of the angular velocities at each joint.

2.6.1 The Jacobian

For an RR chain let $\theta_1(t)$ and $\theta_2(t)$ be the rotation angles at each joint. We can compute

$$\mathbf{w} = (\dot{\theta}_1 + \dot{\theta}_2)\vec{k} \quad \text{and} \quad \dot{\mathbf{P}} = \begin{Bmatrix} -a\dot{\theta}_1 \sin\theta_1 \\ a\dot{\theta}_1 \cos\theta_1 \end{Bmatrix}. \tag{2.97}$$

These two equations are considered to define the velocity of the end-link as a whole, as opposed the velocity of trajectories traced by its points. The $\dot{\mathbf{P}}$ and $\dot{\phi}$ are assembled into a vector and (2.97) is written in the matrix form

$$\begin{Bmatrix} \dot{\mathbf{P}} \\ \dot{\phi} \end{Bmatrix} = \begin{bmatrix} -a\sin\theta_1 & 0 \\ a\cos\theta_1 & 0 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} \tag{2.98}$$

In robotics literature this 3×2 matrix is called the *Jacobian* of the RR chain. Given a desired velocity for the end effector, we can solve these equations to obtain the required joint rates $\dot{\theta}_1$ and $\dot{\theta}_2$.

Another form of the Jacobian is obtained by considering the trajectory of a general point $\mathbf{x}$ in M given by

$$\mathbf{X}(t) = [D(t)]\mathbf{x} = [Z(\theta_1)][X(a)][Z(\theta_2)]\mathbf{x}. \tag{2.99}$$

Compute the velocity $\mathbf{V} = \dot{\mathbf{X}}$ then eliminate the M-frame coordinates using $\mathbf{x} = [D^{-1}]\mathbf{X}$. The result is

$$\mathbf{V} = \dot{\mathbf{X}} = [\dot{D}][D^{-1}]\mathbf{X}. \tag{2.100}$$

The matrix $[S] = [\dot{D}][D^{-1}]$ can be viewed as operating on a trajectory $\mathbf{X}(t)$ to compute its velocity $\mathbf{V}$.

For the RR chain, we use (2.11) and compute

$$[S] = \begin{bmatrix} 0 & -\dot{\theta}_1 - \dot{\theta}_2 & a\dot{\theta}_2 \sin\theta_1 \\ \dot{\theta}_1 + \dot{\theta}_2 & 0 & -a\dot{\theta}_2 \cos\theta_1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{2.101}$$

The upper left 2×2 matrix is $(\dot{\theta}_1 + \dot{\theta}_2)[J]$, which is the matrix that we have associated with the angular velocity of the end-link. The third column is the velocity of the trajectory $\mathbf{Y}(t)$ that passes through the origin of F. Assemble this into the matrix equation

$$\begin{Bmatrix} \mathbf{v} \\ \dot{\phi} \end{Bmatrix} = \begin{bmatrix} 0 & a\sin\theta_1 \\ 0 & -a\cos\theta_1 \\ 1 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix}. \tag{2.102}$$

This alternative form for the Jacobian is the focus of our study in the last chapter of this text.

2.6.2 The Centrode

We now compute the instant center for the instantaneous movement of the end-link of the RR chain. From (2.94) we have

$$\mathbf{I} = \mathbf{P} + \frac{\mathbf{w}_M \times \dot{\mathbf{P}}}{\mathbf{w}_M \cdot \mathbf{w}_M}. \tag{2.103}$$

Simplify this equation and introduce the vector $\vec{e} = (\cos\theta_1, \sin\theta_1)^T$ to obtain

$$\mathbf{I} = a\left(\frac{\dot{\theta}_2}{\dot{\theta}_1 + \dot{\theta}_2}\right)\vec{e}. \tag{2.104}$$

This shows that the instant center lies on the line through the two revolute joints of the RR chain.

Equation (2.103) defines an instant center for every configuration of the chain. If the joint angles θ_1 and θ_2 are related by a function $f(\theta_1, \theta_2) = 0$, then the set of instant centers forms a curve in F known as the *centrode*.

Other planar open chains can be analyzed in the same way to relate the velocity of the end-link to the rate of change of the configuration parameters.

2.7 Velocity Analysis of a Slider-Crank

If the input crank to an RRRP linkage is driven at the rate $\dot{\theta}$, then we can determine the rotation rate $\dot{\phi}$ of the coupler link, and the linear velocity $\dot{s}$ of the slider using the *velocity loop equations*. These equations are obtained by computing the time derivative of the loop equations (2.32)

$$\dot{\theta}\begin{Bmatrix} -r\sin\theta \\ r\cos\theta \end{Bmatrix} + (\dot{\theta}+\dot{\phi})\begin{Bmatrix} -L\sin(\theta+\phi) \\ L\cos(\theta+\phi) \end{Bmatrix} = \dot{s}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{2.105}$$

Rearrange the terms so this equation takes the form

$$\begin{bmatrix} 0 & L\sin(\theta+\phi) \\ 1 & -L\cos(\theta+\phi) \end{bmatrix} \begin{Bmatrix} \dot{s} \\ \dot{\phi} \end{Bmatrix} = \dot{\theta}\begin{Bmatrix} -r\sin\theta - L\sin(\theta+\phi) \\ r\cos\theta + L\cos(\theta+\phi) \end{Bmatrix}. \tag{2.106}$$

Notice that to solve these equations we must have previously determined the parameters ϕ and s. Then Cramer's rule yields

$$\frac{\dot{s}}{\dot{\theta}} = \frac{r\sin\phi}{\sin(\theta+\phi)} \quad \text{and} \quad \frac{\dot{\phi}}{\dot{\theta}} = \frac{r\sin\theta + L\sin(\theta+\phi)}{L\sin(\theta+\phi)}. \tag{2.107}$$

It is useful to note that we can obtain the slider velocity directly from the constraint (2.23) and avoid the need to determine ϕ or $\dot{\phi}$. To do this, simply compute the time derivative of this constraint equation to obtain

$$\dot{s}(s - r\sin\theta) - \dot{\theta}r(s\cos\theta - e\sin\theta) = 0. \tag{2.108}$$

The result is

$$\frac{\dot{s}}{\dot{\theta}} = \frac{r(s\cos\theta - e\sin\theta)}{s - r\sin\theta}. \tag{2.109}$$

This equation is used to determine the mechanical advantage of this linkage.

2.7.1 Mechanical Advantage

The ratio of the static force generated at the slider to the input torque applied at the crank is known as *mechanical advantage*. We compute this using the *principle of virtual work* which states that the work done by input forces and torques must equal the work done by output forces and torques during a virtual displacement. For the RRRP linkage, we assume the weight of each link and the friction in each joint are negligible compared to the applied forces and torques. In which case, the principle of virtual work requires that the work done by the torque applied to the input crank must equal the work done by the slider on an external load during a virtual displacement.

A *virtual displacement* is a small movement of the system over which the applied forces and torques are considered to be constant. This small movement is easily defined in terms of the velocities of each link. The angular velocity of the input crank $\dot{\theta}$ acting over a small increment of time δt generates the virtual crank displacement $\delta\theta = \dot{\theta}\delta t$. Similarly, a virtual displacement of the slider is $\delta s = \dot{s}\delta t$.

Let the input torque to the crank be $\mathbf{T} = F_{\text{in}}p\vec{k}$, where F_{in} is a force applied perpendicular to the link at a distance p along it. Then the virtual work of this torque is $F_{\text{in}}p\delta\theta$. The virtual work done by the slider as it applies a force $\mathbf{F} = F_{\text{out}}\vec{\jmath}$ along its direction of movement is $F\delta s$. Thus, we have

$$F_{\text{out}}\dot{s}\delta t = F_{\text{in}}p\dot{\theta}\delta t. \tag{2.110}$$

Because the virtual time increment δt is not zero, we can equate coefficients to obtain the relationship

$$\frac{F_{\text{out}}}{F_{\text{in}}} = \frac{\dot{\theta}}{\dot{s}} = \frac{p(s - r\sin\theta)}{r(s\cos\theta - e\sin\theta)}. \tag{2.111}$$

This ratio defines the mechanical advantage of the slider-crank. Notice that it depends on the configuration of the linkage, as well as the ratio p/r, which defines the point of application of the input force F_{in}.

This formula has an interesting geometric interpretation. Let **I** be the intersection of the line through the crank **OA** and the line $y = s$ that locates the slider. We now determine the distances $|\mathbf{IA}|$ and $|\mathbf{IB}|$ from the geometry of the linkage and obtain

$$|\mathbf{IA}| = r - \frac{s}{\sin\theta} \quad \text{and} \quad |\mathbf{IB}| = e - \left(\frac{s}{\sin\theta}\right)\cos\theta. \tag{2.112}$$

Thus, we find that the mechanical advantage for the slider-crank can be written as

$$\frac{F_{\text{out}}}{F_{\text{in}}} = \frac{p|\mathbf{IA}|}{r|\mathbf{IB}|}. \tag{2.113}$$

Decreasing the distance $|\mathbf{IB}|$ increases the mechanical advantage. In fact, as $\tan\theta$ approaches s/e, the extreme position of the slider, the distance $|\mathbf{IB}|$ approaches zero, and the mechanical advantage becomes very large.

2.8 Velocity Analysis of a 4R Chain

The velocity loop equations of the 4R chain are obtained by computing the time derivative of the loop equations (2.55) to obtain

$$\dot{\theta}\begin{Bmatrix} -a\sin\theta \\ a\cos\theta \end{Bmatrix} + (\dot{\theta} + \dot{\phi})\begin{Bmatrix} -h\sin(\theta+\phi) \\ h\cos(\theta+\phi) \end{Bmatrix} = \dot{\psi}\begin{Bmatrix} -b\sin\psi \\ b\cos\psi \end{Bmatrix}. \tag{2.114}$$

For a given input angular velocity $\dot{\theta}$, these equations are linear in the angular velocities $\dot{\phi}$ and $\dot{\psi}$ of the coupler and output link. Notice that we must have already determined the angles ϕ and ψ. Assemble these equations into the matrix equation

$$\begin{bmatrix} -b\sin\psi & h\sin(\theta+\phi) \\ b\cos\psi & -h\cos(\theta+\phi) \end{bmatrix}\begin{Bmatrix} \dot{\psi} \\ \dot{\phi} \end{Bmatrix} = \dot{\theta}\begin{Bmatrix} -a\sin\theta - h\sin(\theta+\phi) \\ a\cos\theta + h\cos(\theta+\phi) \end{Bmatrix}. \tag{2.115}$$

Solve this equation to determine the velocity ratios

$$\frac{\dot{\psi}}{\dot{\theta}} = \frac{a\sin\phi}{b\sin(\theta+\phi-\psi)} \quad \text{and} \quad \frac{\dot{\phi}}{\dot{\theta}} = \frac{a\sin(\psi-\theta) - h\sin(\theta+\phi-\psi)}{h\sin(\theta+\phi-\psi)}. \tag{2.116}$$

2.8.1 Output Velocity Ratio

We now examine the velocity properties of the 4R chain in terms of the angular velocity vectors $\mathbf{w}_\mathrm{O} = \dot{\theta}\vec{k}$ and $\mathbf{w}_\mathrm{C} = \dot{\psi}\vec{k}$, where $\vec{\imath}$, $\vec{\jmath}$, and $\vec{k}$ are the unit vectors along the coordinate axes of a three dimensional frame. The time derivative of the constraint equation $(\mathbf{B}-\mathbf{A})\cdot(\mathbf{B}-\mathbf{A}) = b^2$ yields

$$(\dot{\mathbf{B}}-\dot{\mathbf{A}})\cdot(\mathbf{B}-\mathbf{A}) = 0. \tag{2.117}$$

Since $\dot{\mathbf{B}} = \mathbf{w}_\mathrm{C}\times(\mathbf{B}-\mathbf{C})$ and $\dot{\mathbf{A}} = \mathbf{w}_\mathrm{O}\times\mathbf{A}$, this can be written as

$$\big(\dot{\psi}\vec{k}\times(\mathbf{B}-\mathbf{C}) - \dot{\theta}\vec{k}\times\mathbf{A}\big)\cdot(\mathbf{B}-\mathbf{A}) = 0. \tag{2.118}$$

Interchange the dot and cross operations and expand this equation to obtain

$$\dot{\psi}\vec{k}\cdot\mathbf{B}\times(\mathbf{B}-\mathbf{A}) - \dot{\theta}\vec{k}\cdot\mathbf{A}\times(\mathbf{B}-\mathbf{A}) - \dot{\psi}\vec{k}\cdot\mathbf{C}\times(\mathbf{B}-\mathbf{A}) = 0. \tag{2.119}$$

Notice that the cross products $\mathbf{A}\times(\mathbf{B}-\mathbf{A})$ and $\mathbf{B}\times(\mathbf{B}-\mathbf{A})$ are equal, and, in fact, any point on the line L_{AB}: $\mathbf{Y}(t) = \mathbf{A}+t(\mathbf{B}-\mathbf{A})$ yields the same result. In particular, both $\mathbf{A}$ and $\mathbf{B}$ can be replaced by the point $\mathbf{I} = r\vec{\imath}$, which is the intersection of L_{AB} with the x-axis. Since $\mathbf{C} = g\vec{\imath}$, this equation takes the form

$$\vec{k}\cdot\big(\dot{\psi}(r-g) - \dot{\theta}r\big)\vec{\imath}\times(\mathbf{B}-\mathbf{A}) = 0. \tag{2.120}$$

It is now easy to see that the output velocity ratio is given by

$$\frac{\dot{\psi}}{\dot{\theta}} = \frac{-r}{g-r}. \tag{2.121}$$

The distance r to the point $\mathbf{I}$ along the x-axis can be computed by finding the parameter t that satisfies the relation $\vec{\jmath}\cdot(\mathbf{A}+t(\mathbf{B}-\mathbf{A})) = 0$. Substitute this into $r = \vec{\imath}\cdot(\mathbf{A}+t(\mathbf{B}-\mathbf{A}))$ to obtain

$$r = \frac{ab\sin(\theta-\phi)}{b\sin\phi - a\sin\theta}. \tag{2.122}$$

Notice that the velocity ratio between the output and input links can be viewed as instantaneously equivalent to the speed ratio between two gears in contact at the instant center $\mathbf{I}$ that have the radii $g-r$ and r respectively, see Figure 2.10.

2.8.2 Coupler Velocity Ratio

A similar relationship for the coupler velocity ratio is obtained by computing the velocity of $\mathbf{B}$ in the fixed frame using vector operations. Combining this with the fact that $\dot{\mathbf{B}}\cdot(\mathbf{B}-\mathbf{C}) = 0$, we obtain a geometric representation of the velocity ratio.

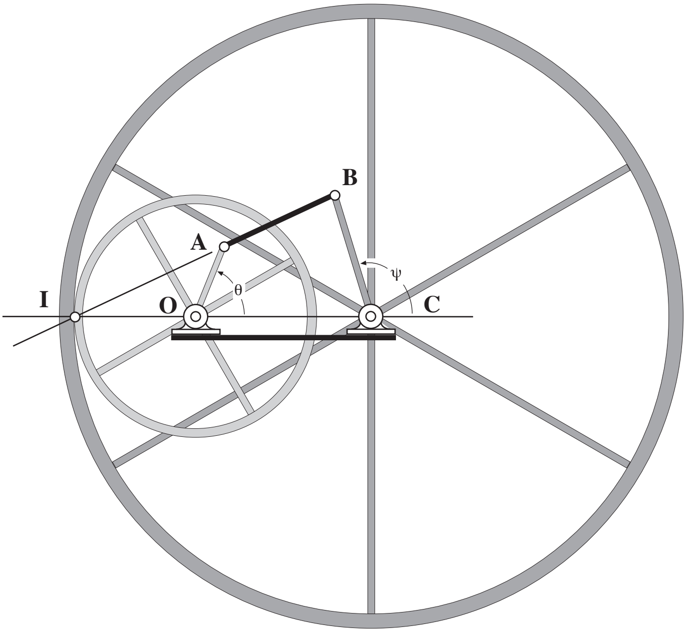

Fig. 2.10 The angular velocities of the input and output links are instantaneously equivalent to gears in contact at the instant center **I**.

The coupler has the angular velocity $\mathbf{w}_{\mathrm{A}} = (\dot{\phi} + \dot{\theta})\vec{k}$, so the velocity of **B** is given by

$$\dot{\mathbf{B}} = \dot{\mathbf{A}} + \mathbf{w}_A \times (\mathbf{B} - \mathbf{A}). \tag{2.123}$$

Since $\dot{\mathbf{A}} = \dot{\theta}\vec{k} \times \mathbf{A}$, this equation becomes

$$\dot{\mathbf{B}} = \dot{\theta}\vec{k} \times \mathbf{A} + (\dot{\phi} + \dot{\theta})\vec{k} \times (\mathbf{B} - \mathbf{A}) = \dot{\theta}\vec{k} \times \mathbf{B} + \dot{\phi}\vec{k} \times (\mathbf{B} - \mathbf{A}). \tag{2.124}$$

Substitute this into the condition $\dot{\mathbf{B}} \cdot (\mathbf{B} - \mathbf{C}) = 0$ to obtain

$$\big(\dot{\theta}\vec{k} \times \mathbf{B} + \dot{\phi}\vec{k} \times (\mathbf{B} - \mathbf{A})\big) \cdot (\mathbf{B} - \mathbf{C}) = 0. \tag{2.125}$$

Notice that **B** can be replaced by any point on the line L_{CB}: $\mathbf{Y}(t) = \mathbf{B} + t(\mathbf{B} - \mathbf{C})$ because $\vec{k} \times t(\mathbf{B} - \mathbf{C}) \cdot (\mathbf{B} - \mathbf{C}) = 0$. In particular, consider the point **J** that is the intersection of L_{CB} with the line L_{OA} that joins **O** and **A**. Let $\vec{e}$ be the unit vector in the direction **A**, so $\mathbf{A} = a\vec{e}$ and $\mathbf{J} = r\vec{e}$. Substitute this into (2.125) and obtain

$$\big(\dot{\theta}r + \dot{\phi}(r - a)\big)\vec{k} \times \vec{e} \cdot (\mathbf{B} - \mathbf{C}) = 0. \tag{2.126}$$

For this equation to be zero, the coupler velocity ratio must satisfy the relation

$$\frac{\dot{\phi}}{\dot{\theta}} = \frac{r}{a-r}. \tag{2.127}$$

Thus, the angular velocity ratio between the coupler and input link is instantaneously equivalent to the speed ratio of two gears in contact at **J** with radii of $a-r$ and a, respectively, Figure 2.11.

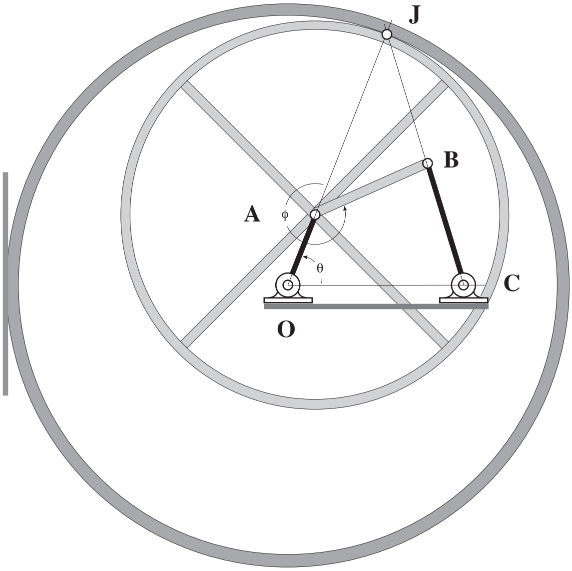

Fig. 2.11 The angular velocities of the input and coupler links are instantaneously equivalent to gears in contact at the instant center **J**.

The value of r defining **J** along the line L_{OA} is obtained by solving for t such that $\vec{e}^{\perp} \cdot (\mathbf{B} + t(\mathbf{B} - \mathbf{C})) = 0$. Note that $\vec{e}^{\perp} = (-\sin\theta, \cos\theta)^T$ is the unit vector perpendicular to $\vec{e}$. Then, substitute the result into the relation $r = \vec{e} \cdot (\mathbf{B} + t(\mathbf{B} - \mathbf{A}))$ to compute r.

2.8.3 Kennedy's Theorem

We have seen that the output velocity ratio of a 4R linkage can be viewed as generated instantaneously by a pair of gears connecting the input and output links. The points in contact along the pitch circles of the two gears, **Q** on the input link and **P** on the output link, must have the same velocity, that is, $\dot{\mathbf{Q}} = \dot{\mathbf{P}}$. The point in F that coincides with these two points is the instant center **I**. We now show that an instant center with this property exists for any two links moving relative to a ground frame F.

Consider the movement of two independent links and let their instant centers in F be $\mathbf{O}$ and $\mathbf{C}$. We now ask whether there are points, $\mathbf{Q}$ on one and $\mathbf{P}$ on the other, that have both the same coordinates $\mathbf{I} = (X,Y)^T$ in F and the same velocity.

Let $g = |\mathbf{C} - \mathbf{O}|$ be the distance between the instant centers, and let $\mathbf{Q} - \mathbf{O} = (r\cos\theta, r\sin\theta)^T$ and $\mathbf{P} - \mathbf{C} = (g + k\cos\psi, k\sin\psi)^T$ be the relative vectors locating $\mathbf{Q}$ and $\mathbf{P}$. The velocities of these points are

$$\dot{\mathbf{Q}} = \dot{\theta}\begin{Bmatrix} -r\sin\theta \\ r\cos\theta \end{Bmatrix}, \quad \dot{\mathbf{P}} = \dot{\psi}\begin{Bmatrix} -k\sin\psi \\ k\cos\psi \end{Bmatrix}. \tag{2.128}$$

If $\mathbf{Q} = \mathbf{P} = (X,Y)^T$, then $Y = r\sin\theta = k\sin\psi$, and to have the same velocity

$$\dot{\mathbf{P}} - \dot{\mathbf{Q}} = \begin{Bmatrix} -\dot{\psi}Y + \dot{\theta}Y \\ \dot{\psi}(r\cos\theta - g) - \dot{\theta}r\cos\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{2.129}$$

The first component of this equation shows that $Y = 0$. Set $\theta = \psi = 0$ and let r and k take positive and negative values so that $X = r = k + g$. From the second component of (2.129) we find that r must satisfy the equation

$$\dot{\psi}(r - g) - \dot{\theta}r = 0,$$

that is,

$$\frac{\dot{\psi}}{\dot{\theta}} = \frac{r}{r - g}. \tag{2.130}$$

Thus, $\mathbf{I} = (r,0)^T$ is the desired instant center. The fact that the point $\mathbf{I}$ must lie on the line joining the $\mathbf{O}$ and $\mathbf{C}$ is known as *Kennedy's theorem*.

2.8.4 Mechanical Advantage in a 4R Linkage

The relationship between an applied input torque and the torque generated at the output crank of a 4R linkage is easily determined by considering the equivalent set of gears and the principle of virtual work. From the velocity ratios determined above, we have the virtual displacement of the output crank defined as

$$\delta\psi = \dot{\psi}\delta t = \dot{\theta}\frac{r}{r - g}\delta t, \tag{2.131}$$

where r is the distance to the instant center $\mathbf{I}$ from the fixed pivot $\mathbf{O}$ of the drive crank.

The virtual work of the input torque $\mathbf{T}_\mathrm{O} = T_\mathrm{O}\vec{k}$ is $T_\mathrm{O}\delta\theta$. Similarly, the virtual work of the output torque, $\mathbf{T}_\mathrm{C} = T_\mathrm{C}\vec{k}$, is $T_\mathrm{C}\delta\psi$. From the principle of virtual work we obtain

$$T_\mathrm{O}\delta\theta = T_\mathrm{C}\delta\psi, \quad \text{or} \quad T_\mathrm{O}\dot{\theta}\delta t = T_\mathrm{C}\frac{r}{r - g}\dot{\theta}\delta t. \tag{2.132}$$

The virtual time increment δt is nonzero, so we equate coefficients to obtain the relationship

$$\frac{T_{\mathrm{C}}}{T_{\mathrm{O}}} = \frac{r-g}{r}. \tag{2.133}$$

Note that the distance r has a sign associated with its direction along the x-axis from **O**. The conclusion is that the torque ratio of a linkage is the inverse of its velocity ratio, which is exactly the torque ratio of the equivalent gear train. Notice that the value of this torque ratio changes with the configuration of the linkage.

Let the input torque be generated by a *couple*, which is a pair of forces in opposite directions but of equal magnitude F_{O} separated by the perpendicular distance a, so $M_{\mathrm{O}} = aF_{\mathrm{O}}$. Similarly, let the output torque result in a couple with magnitude $M_{\mathrm{C}} = bF_{\mathrm{C}}$. Then, the ratio of output force F_{C} to input force F_{O} is obtained from (2.133) as

$$\frac{F_{\mathrm{C}}}{F_{\mathrm{O}}} = \frac{a}{b}\left(\frac{r-g}{r}\right). \tag{2.134}$$

This ratio is called the *mechanical advantage* of the linkage. For a given set of dimensions a and b the mechanical advantage is directly proportional to the velocity ratio of the input and output links.

2.9 Analysis of Multiloop Planar Linkages

The study of mulitiloop planar linkages is covered in detail in a later chapter. Here we summarize how to use Dixon's determinant to solve for the configuration angles that define the assemblies of the linkage (Wampler [148]).

2.9.1 Complex Loop Equations

Given an input angle θ_0, the vector equations for each independent loop L of a planar linkage can be written as

$$\mathscr{F}_k : \alpha_k + \sum_{j=1}^{2L} \beta_{kj} \cos\theta_j + \sum_{j=1}^{2L} \gamma_{kj} \sin\theta_j = 0, \quad k = 1, \ldots, 2L. \tag{2.135}$$

Here, α_k, β_{kj}, and γ_{kj} are real quantities that depend on the dimensions of the links, and θ_j denotes the rotation angle of link j. Because the input angle θ_0 is known, we absorb it into the coefficients of α_k, and these $2L$ equations are solved for $2L$ configuration angles θ_j.

Combine the loop equations (2.135) into a single complex equation by using complex vectors $\mathbf{x} = x + iy$, where $i^2 = -1$, rather than vectors $\mathbf{x} = (x, y)$. Introduce the complex vector $\Theta_j = e^{i\theta_j}$, $j = 1, \ldots, 2L$, so the $2L$ loop equations become

$$\mathscr{C}_k: \quad c_{k0} + \sum_{j=1}^{2L} c_{k,j}\Theta_j = 0, \quad k = 1, \ldots, L. \tag{2.136}$$

We obtain a second set of loop equations by computing the complex conjugates

$$\mathscr{C}_k^\star: \quad c_{k0}^\star + \sum_{j=1}^{2L} c_{k,j}^\star \Theta_j^{-1} = 0, \quad k = 1, \ldots, L. \tag{2.137}$$

These equations combine to provide a set of $2L$ complex equations for $2L$ complex configuration angles Θ_j that are solved using the Dixon determinant.

2.9.2 The Dixon Determinant

Suppress the joint angle Θ_{2L}, so we have $2L$ complex equations in $2L-1$ variables Θ_j, labeled $\mathscr{C}_k$ and $\mathscr{C}_k^\star$. These equations form the first row of the Dixon determinant. The second row consists of the same functions but with the variable θ_1 replaced by α_1. Similarly, row three has Θ_1 and Θ_2 replaced by α_1 and α_2. This continues for the remaining rows in the determinant, so that we obtain

$$\begin{aligned} &\Delta(\Theta_1, \Theta_2, \ldots, \Theta_{2L-1}) \\ &\quad = \begin{vmatrix} \mathscr{C}_1(\Theta_1, \Theta_2, \ldots, \Theta_{2L-1}) & \ldots & \mathscr{C}_L^\star(\Theta_1, \Theta_2, \ldots, \Theta_{2L-1}) \\ \mathscr{C}_1(\alpha_1, \Theta_2, \ldots, \Theta_{2L-1}) & \ldots & \mathscr{C}_L^\star(\alpha_1, \Theta_2, \ldots, \Theta_{2L-1}) \\ \vdots & & \\ \mathscr{C}_1(\alpha_1, \alpha_2, \ldots, \alpha_{2L-1}) & \ldots & \mathscr{C}_L^\star(\alpha_1, \alpha_2, \ldots, \alpha_{2L-1}) \end{vmatrix}. \end{aligned} \tag{2.138}$$

his determinant is zero when $\Theta_1, \Theta_2, \ldots, \Theta_{2L-1}$ satisfy the loop equations, because the elements in the first row become zero.

Insight into the structure of the determinant Δ is obtained by noting that the complex equations for each loop k have the form

$$\mathscr{C}_k: c_{k0} + c_{k,2L}x + \sum_{j=1}^{2L-1} c_{k,j}\Theta_j \quad \text{and} \quad \mathscr{C}_k^\star: c_{k0}^\star + c_{k,2L}^\star x^{-1} + \sum_{j=1}^{2L-1} c_{k,j}^\star \Theta_j^{-1}, \tag{2.139}$$

where x denotes the suppressed variable Θ_{2L}. These equations maintain this form when α_j replaces Θ_j. Thus, we can row reduce Δ by subtracting the second row from the first row, then the third from the second, the fourth from the third, and so on to obtain

$$\begin{vmatrix} c_{1,1}(\Theta_1-\alpha_1) & c^*_{1,1}(\Theta_1^{-1}-\alpha_1^{-1}) & \dots & c^*_{L,1}(\Theta_1^{-1}-\alpha_1^{-1}) \\ c_{1,2}(\Theta_2-\alpha_2) & c^*_{1,2}(\Theta_2^{-1}-\alpha_2^{-1}) & \dots & c^*_{L,2}(\Theta_2^{-1}-\alpha_2^{-1}) \\ \vdots & \vdots & & \vdots \\ c_{1,2L-1}(\Theta_{2L-1}-\alpha_{2L-1}) & c^*_{1,2L-1}(\Theta_{2L-1}^{-1}-\alpha_{2L-1}^{-1}) & \dots & c^*_{L,2L-1}(\Theta_{2L-1}^{-1}-\alpha_{2L-1}^{-1}) \\ \mathscr{C}_1(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) & \mathscr{C}^\star_1(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) & \dots & \mathscr{C}^\star_L(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) \end{vmatrix}. \tag{2.140}$$

Because $\Theta_j-\alpha_j=-\Theta_j\alpha_j(\Theta_j^{-1}-\alpha_j^{-1})$, we can divide out these extraneous roots $(\Theta_j^{-1}-\alpha_j^{-1})$ to define the determinant

$$\delta=\frac{\Delta(\Theta_1,\Theta_2,\dots,\Theta_{2L-1})}{\prod_{k=1}^{2L-1}(\Theta_k^{-1}-\alpha_k^{-1})}, \tag{2.141}$$

that is,

$$\delta = \begin{vmatrix} -c_{1,1}\Theta_1\alpha_1 & c^*_{1,1} & \dots & c^*_{L,1} \\ -c_{1,2}\Theta_2\alpha_2 & c^*_{1,2} & \dots & c^*_{L,2} \\ \vdots & & & \\ -c_{1,2L-1}\Theta_{2L-1}\alpha_{2L-1} & c^*_{1,2L-1} & \dots & c^*_{L,2L-1} \\ \mathscr{C}_1(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) & \mathscr{C}^\star_1(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) & \dots & \mathscr{C}^\star_L(\alpha_1,\alpha_2,\dots,\alpha_{2L-1}) \end{vmatrix}. \tag{2.142}$$

This determinant is a polynomial of the form

$$\delta=\mathbf{a}^T[W]\mathbf{t}=0, \tag{2.143}$$

where $\mathbf{a}=(\mathbf{a}_1,\mathbf{a}_2)^T$ contains the monomials formed from α_j, $\mathbf{t}=(\mathbf{t}_1,\mathbf{t}_2)^T$ are formed from monomials of Θ_j, and $[W]$ is the $2L\times 2L$ matrix is given by

$$[W]=\begin{bmatrix} D_1x+D_2 & A^T \\ A & s(D_1^*x^{-1}+D_2^*) \end{bmatrix}, \tag{2.144}$$

where D_1 and D_2 are diagonal matrices and the elements of A obey the relations a_{ij} $= sa^\star_{ji}$ and $s=(-1)^{L-1}$.

The vectors of monomials $\mathbf{a}=(\mathbf{a}_1,\mathbf{a}_2)^T$ and $\mathbf{t}=(\mathbf{t}_1,\mathbf{t}_2)^T$ in (2.143) are generated as follows. Starting with $\mathbf{a}$, find all combinations $\binom{2L-1}{L-1}$ of distinct variables of degree $L-1$ from the set $(\alpha_1,\alpha_2,\dots,\alpha_{2L-1})^T$. Assemble these into the vector $\mathbf{a}_1$, and then form $\mathbf{a}_2$ using the complement of degree L corresponding to each monomial in $\mathbf{a}_1$. The vector $\mathbf{t}$ is obtained in the same way.

Values Θ_j that satisfy the loop equations (2.136) and (2.137) also yield $\delta=0$ for arbitrary values of the auxiliary variables α_j. Thus, solutions for these loop equations must also satisfy the matrix equation

$$[W]\mathbf{t}=0. \tag{2.145}$$

This equation has nonzero solutions only if $\det[W]=0$. Expand this determinant to obtain a polynomial in $x=\Theta_{2L}$.

The structure of $[W]$ yields

$$[W]\mathbf{t} = \left[\begin{pmatrix} D_1 & 0 \\ A & -D_2^* \end{pmatrix} x - \begin{pmatrix} -D_2 & -A^T \\ 0 & D_1^* \end{pmatrix}\right]\mathbf{t} = [Mx - N]\mathbf{t} = 0. \tag{2.146}$$

Notice that the values of x that satisfy $\det[W] = 0$ are the roots of the characteristic polynomial $p(x) = \det(Mx - N)$ of the generalized eigenvalue problem

$$N\mathbf{t} = xM\mathbf{t}. \tag{2.147}$$

Each value of $x = \Theta_{2L}$ has an associated eigenvector $\mathbf{t}$, which yields the values of the remaining joint angles Θ_j, $j = 1, 2, \dots, 2L-1$.

2.9.3 Tangent Sorting

For each value of the input angle Θ_0, the solution of the Dixon determinant yields multiple roots for the configuration angles $\vec{\Theta} = (\Theta_1, \dots, \Theta_{2L})$. Each root defines one way that the linkage can be assembled. For example, a six-bar linkage can have as many as six values for Θ_j, or six assemblies, for each input angle. An eight-bar linkage can have as many as 20 assemblies. Presented here is a way to sort these roots to define these assemblies.

Assume that for the kth value of the input angle Θ_0 we calculate the configuration angles $\vec{\Theta}_i^k, i = 1, \dots, m$ that define m assemblies of the linkage. When we increment Θ_0 and solve the loop equations, we to obtain $\vec{\Theta}_i^{k+1}$, and our goal is to sort these roots so to match those of $\vec{\Theta}_i^k$.

Use Newton's method to approximate the loop equations by computing the Jacobian $[\nabla\mathscr{C}]$, in order to obtain

$$[\nabla\mathscr{C}(\vec{\Theta}_i^k)](\vec{\Psi} - \vec{\Theta}_i^k) = 0, \tag{2.148}$$

where $\vec{\Psi}$ is an approximation to $\vec{\Theta}_i^{k+1}$ in the assembly that we seek. Solve these equations to obtain $\vec{\Psi}$, and select from the available roots $\vec{\Theta}_i^{k+1}$ the one that is closest to $\vec{\Psi}$, in order to match the assemblies.

The configurations traced by one assembly of a multiloop linkage is called a *circuit*. The solution of (2.148) identifies a value Ψ on the tangent to the ith circuit through $\vec{\Theta}_i^k$. This allows rapid and exact calculation of the configuration angles for each assembly of a multiloop planar linkage for a range of values of the input angle.

2.10 Summary

This chapter presented the position and velocity analysis of planar open chains and the closed chain slider-crank and four-bar linkages. Conditions on the existence of solutions to the input-output equations for the closed chains provide a classification scheme for these devices based on the range of movement of their cranks. The velocity analysis of these systems lead to the introduction of instant centers and Kennedy's theorem, which can be used to compute the mechanical advantage of the linkage. The position analysis of multiloop planar linkages using complex number coordinates and the Dixon determinant elimination method were also discussed.

2.11 References

The position and velocity analysis of planar open chains follows the approach used in robotics as found in Craig [15] and Paul [92]. The analysis of planar linkages including the study of accelerations and dynamic forces can be found in many textbooks. See, for example, Waldron and Kinzel [144], Erdman and Sandor [30], Mabie and Reinholtz [70], Mallik et al. [71], and Shigley and Uicker [114]. For further study of the dynamics of these systems see Krishnaprasad and Yang [58] and Sreenath et al. [125]. The strategy used to classify planar slider-crank and 4R linkages follows Murray and Larochelle [87]. The closed form kinematic analysis of planar multiloop mechanism was presented by Wampler [148]. Also see Nielsen and Roth [90], and Wampler [147].

Exercises

1. Consider the PRRP *elliptic trammel* formed from two PR chains connected so that the directions of the two sliders are at right angles in the ground link. Derive the coupler angle ϕ as a function of the input slider translation s and show that a general coupler curve is an ellipse.
2. *Oldham's coupling* is an RPPR linkage with the directions of the two sliders oriented at a right angle to form the coupler link. Analyze this linkage to determine the output crank angle ψ as a function of the input angle θ.
3. The *Scotch yoke* mechanism is an RRPP linkage with the ways of the sliders at right angles. Analyze this linkage to determine the output slide s as a function of the input θ.
4. Derive the algebraic equation of the coupler curve of an RRRP linkage and show that it is a quartic curve.
5. Analyze (i) Watt's linkage, (ii) Robert's linkage, (iii) Chebyshev's linkage, and determine the coupler angle ϕ as a function of the input crank angle θ. Generate the coupler curve of the point that traces an approximately straight line.
6. Derive the algebraic form of the 4R coupler curve and show that its highest degree terms are $(x^2+y^2)^3$, and that those of fifth and fourth degree contain the factors $(x^2+y^2)^2$ and x^2+y^2, respectively. These features identify this curve to as a *tricircular* sextic.

7. Select a coupler point **X** on a 4R linkage **OABC**. Construct the triangle $\triangle$**OCY** that is similar to the coupler triangle $\triangle$**ABX**. Show that the coupler curve traced by **X** has a double point at its intersections with the circle circumscribing $\triangle$**OCY**.
8. Show that the centrode for an RR chain becomes a circle when $\dot{\phi} = \mu\dot{\theta}$ and μ is constant. Because this curve lies in the fixed frame F it is called the *fixed centrode*.
9. Transform the coordinates of the centrode of an RR chain to the moving frame M by $\mathbf{m} = [T^{-1}]\mathbf{I}$. This defines a curve known as the *moving centrode*. Show that for $\dot{\phi} = \mu\dot{\theta}$ and constant μ, the moving centrode is a circle.

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