

Preface

This volume is intended to carry on the program initiated in **Topology, Geometry and Gauge Fields: Foundations** (henceforth, [N4]). It is written in much the same spirit and with precisely the same philosophical motivation: Mathematics and physics have gone their separate ways for nearly a century now and it is time for this to end. Neither can any longer afford to ignore the problems and insights of the other. Why are Dirac magnetic monopoles in one-to-one correspondence with the principal $U(1)$ -bundles over S^2 ? Why do Higgs fields fall into topological types? What led Donaldson, in 1980, to seek in the Yang-Mills equations of physics for the key that unlocks the mysteries of smooth 4-manifolds and what physical insights into quantum field theory led Witten, fourteen years later, to propose the vastly simpler, but equivalent Seiberg-Witten equations as an alternative? We do not presume to answer these questions here, but only to promote an atmosphere in which both mathematicians and physicists recognize the need for answers. More succinctly, we shall endeavor to provide an exposition of elementary topology and geometry that keeps one eye on the physics in which our concepts either arose independently or have been found to lead to a deeper understanding of the phenomena.

Chapter 1 provides a synopsis of the geometrical background we assume of our readers (manifolds, Lie groups, bundles, connections, etc.). While all of this material is discussed in detail in [N4], most of it is standard and it will not matter where the background has been acquired. There follows a rather long chapter on physics. The discussion here is informal and heuristic and often anticipates topological and geometrical concepts that will be introduced precisely only much later. We begin by describing a general mathematical framework for the classical gauge theories of physics and then discuss in some detail a number of specific examples. These include classical electromagnetic theory and Dirac monopoles, the Klein-Gordon and Dirac equations and $SU(2)$ Yang-Mills-Higgs theory. The real purpose here is to witness such concepts as de Rham cohomology, Chern classes and spinor structures arise of their own accord in meaningful physics.

The mathematical development resumes in Chapter 3 where we collect some basic technical tools and then study various types of frame bundles. Minkowski spacetime, and then more general spacetime manifolds are introduced and some concrete examples are discussed (the Einstein-de Sitter spacetime, de Sitter spacetime and the Einstein cylinder). With this machinery we can define precisely the notion of a spinor structure, seen in Chapter 2 to be the device required to model spin $\frac{1}{2}$ particles in the presence of gravity. Finding the topological obstruction to the existence of such a structure (the 2nd Stiefel-Whitney class) will have to wait until Chapter 6.

Chapter 4 contains a more or less standard exposition of multilinear algebra, differential forms, integration and Stokes' Theorem, but with rather more

attention paid to vector-valued forms than is customary. In particular, there is a detailed discussion of tensorial forms on principal bundles and their covariant exterior derivatives in the presence of a connection. It is these derivatives that appear in the field equations of physics.

de Rham cohomology is the subject of Chapter 5. Explicit calculations are based on the Mayer-Vietoris sequence and we introduce the Brouwer degree of a map between two compact, connected, orientable n -manifolds by showing that such a manifold has 1-dimensional n^{th} cohomology. We prove that the degree is an integer by showing how to calculate it from a critical value of the map. It is such a degree that gives rise to the topological quantum number of the Higgs field in $SU(2)$ Yang-Mills-Higgs theory. The chapter concludes with a discussion of the Hopf invariant and its explicit calculation for the complex Hopf map.

The notion of a characteristic class arises on several occasions in the physical considerations of Chapter 2 (the magnetic charge of a Dirac monopole, topological charge of an instanton, and the obstruction to the existence of a spinor structure on a spacetime manifold) and Chapter 6 takes up the subject in earnest. We construct the Chern-Weil homomorphism and from it the Chern classes of a principal bundle. We prove that $U(1)$ -bundles over S^2 are characterized (up to equivalence) by their 1st Chern class and that $SU(2)$ -bundles over S^4 are similarly characterized by the 2nd Chern class. These results complete the identification of magnetic charge and instanton number with purely topological objects. The chapter concludes with the construction of the $\mathbb{Z}_2$ -Čech cohomology of a smooth manifold from a simple cover. We build the 1st and 2nd Stiefel-Whitney classes for a semi-Riemannian manifold and prove that the former is the obstruction to orientability, while the latter, for a spacetime, is the obstruction to the existence of a spinor structure.

Seiberg-Witten gauge theory is in some sense analogous to the spin $\frac{1}{2}$ -electrodynamics discussed in Chapter 2, but sprang from quite different soil and its significance is of a very different sort. Although much of this story lies in greater depths than we have reached in the main body of the text, the profound significance of the subject for both mathematics and physics would seem to make some attempt to tell at least part of it a moral imperative. The Appendix, which is a much expanded version of the Appendix that appeared in the first edition of this work, is a modest attempt to relate as much of the story as we can with the machinery we have available and should be considered a continuation of Appendix B in [N4].

There are 228 Exercises in the book. Each is an integral part of the development and has been included (rather than the equivalent term “clearly”) to encourage active participation on the part of the reader.

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