Chapter 2
The Basic Multiperiod Dynamic Model

Content

Chapter 1 outlined an approach for constructing a single-period newsvendor-type model that explicitly accounts for future outdating of the current order. In this chapter, we present the extension of this one-period model to a finite horizon dynamic model. We present only the general m-period lifetime case, but the reader may be interested in reviewing the simpler case for $m = 2$, as this requires only a single-dimensional state variable.

The problem dynamics are described by the one-period transfer function. Given the current state of the system, $\mathbf{x}$, the quantity of fresh product ordered, y, and the realization of demand t, the transfer function, $\mathbf{s}(y, \mathbf{x}, t)$, gives the vector of starting inventories of the next period. The logic behind the transfer function dynamics is very similar to the logic required to derive the expected outdating function $\int_0^y G_m(t; \mathbf{x})\mathrm{d}t$. The one-period transfer function is:

$$s_i(y, \mathbf{x}, t) = [x_{i+1} - (t - \sum_{j=1}^{i} x_j)^+]^+$$

and

$$s_{m-1}(y, \mathbf{x}, t) = \begin{cases} y - (t - x)^+ & \text{if excess demand is backordered,} \\ [y - (t - x)^+]^+ & \text{if excess demand is lost.} \end{cases}$$

Note the similarity of the form of the transfer function to the definition of the sequence of random variables $B_0, B_1, \ldots,$ defined in the previous chapter. This is, of course, not coincidental. In fact, both are just two different ways of showing the system dynamics. This is shown precisely in the following result which is central to the analysis of the dynamic problem.

S. Nahmias, *Perishable Inventory Systems*, International Series in Operations Research & Management Science 160, DOI 10.1007/978-1-4419-7999-5_2,

Theorem 2.1. $G_n(y;\mathbf{w}_{n-1}) = \int_0^\infty G_{n-1}[s(y;\mathbf{w}_{n-1},t)f(t)\mathrm{d}t.$

Proof. The proof is somewhat tedious, but conceptually straightforward. Note that by defining $G_0(t) = 1$ for all t, the theorem easily holds for $n = 1$. One proceeds by induction, assuming that the theorem is true for $n - 1$ and showing that this leads to the theorem for n. Again, the details appear in Nahmias (1972).

2.1 The Functional Equations for the General Dynamic Problem

Following the usual approach for dynamic programming analysis, define $C_n(\mathbf{x})$ as the minimum expected discounted cost when n periods remain. Then $C_n(\mathbf{x})$ satisfies the following system of functional equations:

$$C_n(\mathbf{x}) = \min_{y\geq 0}\left\{L(\mathbf{x},y) + \alpha\int_0^\infty C_{n-1}(\mathbf{s}(y,\mathbf{x},t)f(t)\mathrm{d}t\right\},$$

which we write as

$$C_n(\mathbf{x}) = \min_{y\geq 0}\{B_n(\mathbf{x},y)\}.$$

In order to establish the existence and to define the properties of an optimal policy, the key result we need to establish is the convexity of $B_n(\mathbf{x},y)$ in y for every set of starting inventories $\mathbf{x}$. In addition, the main theorem describes several important properties of the optimal order function, $y_n(\mathbf{x})$, when n periods remain in planning horizon. The main theorem requires 17 steps and is proven via a complex induction argument.

We will not present the proof of the main theorem here, but the interested reader can refer to Nahmias (1974) for an outline of the proof or Nahmias (1972) for a detailed exposition.

It is interesting to note that the traditional approach to proving convexity of $B_n(\mathbf{x},y)$ for the standard nonperishable problem breaks down in this case. Typically, one shows that $C_n(\mathbf{x})$ is convex in $\mathbf{x}$, that convexity is preserved via the transfer function, and sums of convex functions are convex, thus easily giving the required convexity of $B_n(\mathbf{x},y)$ in the decision variable, y.

Unfortunately, this straightforward approach does not work for the perishable inventory problem. Consider the case of $m = 2$. Here, the decision variable has only a single dimension. A necessary and sufficient conditions for $C_n(x)$ to be convex are that $C_n{}''(x) \geq 0$ for all $x \geq 0$. We can demonstrate that, in fact,

$C_1''(x) < 0$ for some value of x. As is shown in Nahmias and Pierskalla (1973) (and Nahmias 1972),

$$C_1'(x) = -\theta F(x)F(y_1(x))$$

giving

$$C_1''(x) = -\theta F(y_1(x))f(x) - f(y_1(x))F(x)y_1'(x),$$

where $y_1(x)$ is the optimal order quantity when x is the (one period old) on-hand inventory, and one planning period remains in the horizon. Note that the sign of $C_1''(x)$ is not obvious, since $y_1'(x) < 0$. Consider, however, the following special case. Let us assume that the periodic demand follows the negative exponential distribution with parameter λ. That is $f(x) = \lambda e^{-\lambda x}$ and $F(x) = 1 - e^{-\lambda t}$. Since $y_1(0) > 0$ and the function $y_1(x)$ is continuous, there must exist at least one value of x, say $\tilde{x} > 0$, such that $y_1(\tilde{x}) > \tilde{x}$. Because the exponential density is monotonically decreasing in the region $x \geq 0$ and the cumulative distribution function for the exponential is monotonically increasing in this same region, we have that $F(y_1(\tilde{x})) > F(\tilde{x})$ and $f(y_1(\tilde{x})) < f(\tilde{x})$. In addition, it has been shown that $y_1' \times (\tilde{x}) \geq -1$. Combining these results gives $C_1''(\tilde{x}) < 0$. Hence, we conclude that $C_1(x)$ is not convex in x. However, it turns out that the *degree* of nonconvexity (as measured by a lower bound on $C_1''(x)$) is not very great, and we can show that $B_2(x, y) = L(x, y) + \alpha \int_0^\infty C_1(s(y, x, t)f(t)\mathrm{d}t$ is convex in y. That is, the nonconvexity of $C_1(x)$ is more than compensated for by the convexity of $L(x, y)$. For the general m-period problem, it is the convexity of $B_n(\mathbf{x}, \mathrm{y})$ in y that allows us to establish the existence and basic properties of the optimal order function, $y_n(\mathbf{x})$.

We will assume the following notational convention. For any vector valued function, $g(\mathbf{x})$, $g^{(i)}(\mathbf{x})$ is the first partial derivative of g with respect to the ith variable, and $g^{(i,j)}(\mathbf{x})$ is the second partial derivative with respect to the ith and jth variables, respectively.

In the general m-period lifetime problem, the key result that allows us to prove convexity of $B_n(\mathbf{x}, y)$ is $C_n{}^{(1,1)}(\mathbf{x}) \geq -\theta G_{m-1}{}^{(1)}(\mathbf{x})$ (where the differentiation is done with respect to the first variable in the vector $\mathbf{x}$, which is x_{m-1}). Establishing the validity of this lower bound via induction is extremely complex, requiring a network of inequalities on the first and second partial derivatives of the optimal return functions, $C_n(\mathbf{x})$. To provide the reader with an appreciation of the complexity of this problem, we provide a complete statement of the theorem required to prove convexity. As noted, the proof will not be presented here.

Theorem 2.2. Assume that demands in each period form a sequence of independent identically distributed random variables (although the theorem also holds for nonstationary demands) and that:

(a) The demand distribution, F, possesses a bounded continuous density f with the property that $f(t) > 0$ if $t > 0$ and $f(t) = 0$ if $t < 0$.

(b) Future costs are discounted by a discount factor α, where $0 < \alpha < 1$. Then:

1. $B_n(\mathbf{x}, y)$ is convex in y for all $\mathbf{x} \in R_{m-1}$ and is strictly convex in a neighborhood of the global minimum.
2. $\lim\limits_{y\to 0} \frac{\partial B_n(\mathbf{x},y)}{\partial y} < 0$ and $\lim\limits_{y\to\infty} \frac{\partial B_n(\mathbf{x},y)}{\partial y} > 0$ for all $\mathbf{x}$.
3. There is a unique function $y_n(\mathbf{x})$ given as the solution to $\left.\frac{\partial B_n(\mathbf{x},y)}{\partial y}\right|_{y=y_n(\mathbf{x})} = 0$ and $0 < y_n(\mathbf{x}) < \infty$. In addition $y_n{}^{(i)}(\mathbf{x})$ exists and is continuous for all $\mathbf{x}$, $1 \le i \le m-1$.
4. $$C_n^{(i)}(\mathbf{x}) = -\theta \sum_{j=1}^{i} G_{m-j}(\mathbf{x}(m-j)) H_j(y_n(\mathbf{x}); \bar{\mathbf{x}}(m-j)) + \alpha \sum_{j=1}^{m-i} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{(i+1)}[\mathbf{z}_j(t)]$$
$$- C_{n-1}^{(1)}[\mathbf{z}_j(t)]\} f(t)\mathrm{d}t + \alpha \sum_{j=m-i+1}^{m-1} \int_{w_{j-1}}^{w_j} \{C_{n-1}^{m-j+1}[\mathbf{z}_j(t)] - C_{n-1}^{(1)}[\mathbf{z}_j(t)]\} f(t)\mathrm{d}t,$$
where $\mathbf{z}_j(t) = (y, x_{m-1}, ..., x_{j+1}, \sum_{i=1}^{j} x_i - t, 0, ...0)$ and $w_j = \sum_{i=1}^{j} x_i$, and $C_n^{(m)}(\mathbf{x}) \equiv 0$. The result holds for $1 \le i \le m-1$.
5. $-1 \le y_n^{(1)}(\mathbf{x}) \le y_n^{(2)}(\mathbf{x}) \le \cdots \le y_n^{m-1}(\mathbf{x}) < 0$.
6. (a) $C_n^{(i,k)}(\mathbf{x})$ exists and is continuous for all $\mathbf{x} \in R^{m-1}$ and $1 \le k, i \le m-1$. However, $C_n^{(1,1)}(t; \mathbf{x}(m-2))$ will be discontinuous at $t = 0$ whenever $f(t)$ is discontinuous at $t = 0$.
   (b) $C_n^i[\bar{\mathbf{x}}(m-i), \mathbf{0}] - C_n^{i-1}[\bar{\mathbf{x}}(m-i), \mathbf{0}] = 0$ for $2 \le i \le m-1$. The notation is meant to be interpreted as the last $m-i$ components being zeros.
7. (a) $C_n^{(1,j)}(\mathbf{x}) \ge -\theta G_{m-1}^{(j)}(\mathbf{x}) \quad 1 \le j \le m-1$.
   (b) $C_n^{(i,j)}(\mathbf{x}) - C_n^{(i-1,j)}(\mathbf{x}) \ge -\theta G_{m-i}^{(j+i+1)}(\mathbf{x}(m-i))[1 - \sum_{k=1}^{i-1} H_k\ (x_{m-i+k'}, \ldots, x_{m-i+1})]$ for $m-1 \ge j \ge i \ge 1$.
   (c) $C_n^{(1,i)}(\mathbf{x}) - C_n^{(1,i-1)}(\mathbf{x}) \le \theta[G_{m-1}^{(i-1)}(\mathbf{x}) - G_{m-1}^{(i)}(\mathbf{x})]$ for $m-1 \ge i \ge 2$.
   (d) $$[C_n^{(i,j)}(\mathbf{x}) - C_n^{(i-1,j)}(\mathbf{x})] - [C_n^{(i,j-1)}(\mathbf{x}) - C_n^{(i-1,j-1)}(\mathbf{x})] \le \theta[G_{m-i}^{(j-i)}(\mathbf{x}(m-i))$$
   $$-G_{m-i}^{(j-i+1)}(\mathbf{x}(m-i))][1 - \sum_{k=1}^{i-1} H_k(x_{m-i+k}, \ldots, x_{m-i+1})]$$
   for $m-1 \ge j > i \ge 2$.
8. (a) $$-\theta \sum_{j=1}^{i} G_{m-j}(\mathbf{x}(m-j))[1 - \sum_{k=1}^{j-1} H_k(x_{m-j+k}, \ldots, x_{m-j+1})] \le C_n^{(i)}(\mathbf{x}) \le 0$$
   for $1 \le i \le m-1$ and for all $\mathbf{x}$.
   (b) $C_n^{(i)}(\mathbf{x}) - C_n^{(j)}(\mathbf{x}) \le \theta \sum_{k=j+1}^{i} G_{m-k}(\mathbf{x}(m-k))$
   $[\sum_{q=k-j}^{k-1} H_q(x_{m-k+q}, \ldots, x_{m-k+1})]$ for $1 \le j < i \le m-1$ and for all $\mathbf{x}$.

(c) $C_n^{(i)}(\mathbf{x}) - C_n^{(j)}(\mathbf{x}) \geq -\theta \sum_{k=j+1}^{i} G_{m-k}(\mathbf{x}(m-k))[1 - \sum_{q=1}^{k-1} H_q(x_{m-k+q}, \ldots, x_{m-k+1})]$ for $1 \leq j < i \leq m-1$ and for all $\mathbf{x}$.

(d) $C_n^{(i)}(\mathbf{x}) = 0$ for $\mathbf{x} = (x_{m-1}, \mathbf{0})$ and $x_{m-1} \leq 0$.

9. (a) $\lim_{x_i \to \infty} y_n(\mathbf{x}) = 0 \quad 1 \leq i \leq m-1.$

(b) $\lim_{x_j \to \infty} C_n^{(i)}(\mathbf{x}) = 0 \quad 1 \leq i, j \leq m-1.$

Note that the functions H_k referred to in steps 7 and 8 are used as a convenience for representing derivatives of the outdating function, G_m. As they add nothing to the exposition, we will not discuss them further here. Aside from all of the machinery involving the derivatives of the optimal value functions, what does the theorem tell us about the behavior of the optimal policy function? The key piece of information we obtain from the theorem is step 5: $-1 \leq {y_n}^{(1)}(\mathbf{x}) \leq {y_n}^{(2)}(\mathbf{x}) \leq \cdots \leq {y_n}^{(m-1)}(\mathbf{x}) < 0$. This says two things. First, since all partial derivatives are negative, the optimal order quantity decreases as starting inventories increase. More importantly, it characterizes the sensitivity to starting stocks of different ages. The larger the derivative of $y_n(\mathbf{x})$ in absolute value, the greater the sensitivity of the optimal order function to changes in starting stock. This means that increasing the on-hand quantity of newer stock has a larger effect on optimal order quantities than increasing the on-hand quantity of older stock.

This is a fundamental property of perishable inventory systems that separates such systems from traditional nonperishable systems. One is concerned not only with the amounts of on-hand inventory, but also their ages as well. Because the dimension of the state variable is proportional to the lifetime of the stock in periods, computing an optimal policy is feasible only for relatively short lifetimes. One quickly faces the "curse of dimensionality" that plagues many dynamic programming formulations. For product lifetimes much more than two or three periods, it is unlikely one would use optimal policies. Also, implementation of optimal policies would be complicated by the fact that one needs to keep track of the age distribution of stock. Approximations that depend only on the total stock on-hand are of interest as they are easy to compute and easy to implement. Methods of finding simple approximations for the periodic review problem will be the subject of the next chapter.

An interesting question is whether or not these results hold when demand is discrete rather than continuous, as is assumed in Theorem 2.2. To try to prove convexity of the functions $B_n(\mathbf{x}, y)$ directly under discrete demand would be extremely tedious, if even possible. To circumvent these difficulties, Nahmias and Schmidt (1986) considered a very novel approach to the discrete demand problem. They considered an infinite sequence of continuous demand distributions, F_1, F_2,. . ., that converged weakly to the discrete distribution of demand, F. We know all of the results of Theorem 2.2 hold for each of the continuous distributions F_1, F_2,. . .. Without going into the mathematical details, the authors show that the essential results of Theorem 2.2 carry over in the limit for the discrete case. We do not believe that this approach has been used before or since in the context of a dynamic inventory problem.

It should be noted before closing this section that Fries (1975) also provided a rigorous analysis of the perishable inventory problem, but did not work with the outdate function $G_m(y, \mathbf{x})$. Instead, he developed a straightforward dynamic programming formulation that required m periods before the effects of outdating appeared in the optimal order function. This approach is equally valid as ours outlined here, and preferred for computing optimal policies, as the functional equations are somewhat simpler. As Nahmias (1977a) shows, the two methods give the same policy if one is sufficiently far from the end of the planning horizon, and the discount factor is adjusted in a suitable fashion.

http://www.springer.com/978-1-4419-7998-8

Perishable Inventory Systems
Nahmias, S.
2011, XII, 80 p., Hardcover
ISBN: 978-1-4419-7998-8