

Chapter 2

Bounded and Unbounded Linear Operators

In this chapter, unless otherwise mentioned, $(\mathcal{B}, \|\cdot\|)$ and $(\mathcal{B}', \|\cdot\|')$ stand for Banach spaces over the same field $\mathbb{F}$. Similarly, $\mathcal{H}$ will denote a Hilbert space equipped with the norm $\|\cdot\|$ and the inner product $\langle \cdot, \cdot \rangle$. Further, I and O stand respectively for the identity and zero operators of $\mathcal{B}$ defined by $Ix = x$ and $Ox = O$ for all $x \in \mathcal{B}$.

2.1 Introduction

This chapter is devoted to the basic material on operator theory, semigroups, evolution families, interpolation spaces, intermediate spaces, and their basic properties needed in the sequel. In each section, illustrative examples will be discussed in-depth. The technical Lemma 2.2 (Diagana et al. [52]) and Lemma 2.4 (Diagana [62]) will play a key role throughout the book. Detailed proofs of these lemmas will be discussed at the very end of this chapter.

2.2 Linear Operators

2.2.1 Bounded Operators

A linear operator $A : \mathcal{B} \rightarrow \mathcal{B}'$ is a transformation which maps linearly $\mathcal{B}$ in $\mathcal{B}'$, that is, $A(\alpha u + \beta v) = \alpha Au + \beta Av$ for all $u, v \in \mathcal{B}$ and $\alpha, \beta \in \mathbb{F}$.

Definition 2.1. A linear operator $A : \mathcal{B} \rightarrow \mathcal{B}'$ is said to be bounded if there exists $K \geq 0$ such that

$$\|Au\|' \leq K \|u\| \quad \text{for each } u \in \mathcal{B}. \quad (2.1)$$

If $A : \mathcal{B} \rightarrow \mathcal{B}'$ is a bounded linear operator, then its norm $\|A\|$ is the smallest K for which (2.1) holds, that is,

$$\|A\| := \sup_{u \neq 0} \frac{\|Au\|'}{\|u\|}. \quad (2.2)$$

The collection of all bounded linear operators from $\mathcal{B}$ into $\mathcal{B}'$ is denoted by $B(\mathcal{B}, \mathcal{B}')$. In particular, $B(\mathcal{B}, \mathcal{B})$ is denoted by $B(\mathcal{B})$. It can be shown that $B(\mathcal{B}, \mathcal{B}')$ equipped with the operator topology given above is a Banach space.

Example 2.1. Let $\mathcal{B} = C[0, 1]$ be the collection of all continuous functions from $[0, 1]$ in the complex plane $\mathbb{C}$ equipped with its corresponding sup norm defined for each function $f \in C[0, 1]$ by

$$\|f\|_\infty := \max_{t \in [0, 1]} |f(t)|.$$

Define the integral operator A by setting for each $f \in C[0, 1]$,

$$Af = \int_0^1 K(t, \tau) f(\tau) d\tau$$

where K is a jointly continuous function.

Clearly, the operator A is linear. For the continuity, it suffices to see that

$$\|Af\|_\infty \leq \|f\|_\infty \max_{t \in [0, 1]} \left(\int_0^1 |K(t, \tau)| d\tau \right).$$

It can be shown that (see for instance [79])

$$\|A\| = \max_{t \in [0, 1]} \left(\int_0^1 |K(t, \tau)| d\tau \right).$$

Example 2.2. Let $\mathcal{B} = BC([0, \infty), \mathbb{C})$ be the collection of all bounded continuous functions from $[0, \infty)$ in $\mathbb{C}$ equipped with its corresponding supnorm $\|\cdot\|_\infty$. Define the transformation

$$(Af)(t) = \frac{1}{t} \int_0^t f(s) ds.$$

Using the L'Hôpital rule, it can be easily seen that $\lim_{t \rightarrow 0} (Af)(t) = f(0)$. Clearly, A is linear. Moreover $\|Af\|_\infty \leq \|f\|_\infty$, that is, A is a bounded linear operator.

Theorem 2.1. *If $A : \mathcal{B} \rightarrow \mathcal{B}'$ is a linear operator, then the following statements are equivalent:*

- (i) A is continuous;
- (ii) A is continuous at 0;
- (iii) there exists $K > 0$ such that $\|Au\|' \leq K \cdot \|u\|$ for each $u \in \mathcal{B}$.

Proof. Clearly, (i) yields (ii). Suppose (ii) holds. Hence there exists $\eta > 0$ such that $\|Ax\|' \leq 1$ whenever $\|x\| \leq \eta$. Now for each nonzero $x \in \mathcal{B}$,

$$\left\| \frac{\eta x}{\|x\|} \right\| = \eta.$$

Now

$$1 \geq \left\| A \left(\frac{\eta x}{\|x\|} \right) \right\|' = \frac{\eta \|Ax\|'}{\|x\|},$$

and hence $\|Ax\|' \leq \eta^{-1} \|x\|$ and (iii) holds.

Now if (iii) holds, it is then clear that

$$\|Ax - Ax_0\|' = \|A(x - x_0)\|' \leq K \|x - x_0\|.$$

Consequently, for each $\varepsilon > 0$ there exists $\eta = \frac{\varepsilon}{K}$ such that $\|Ax - Ax_0\|' < \varepsilon$ whenever $\|x - x_0\| \leq \eta$. Therefore, A is continuous at x_0 . Since $x_0 \in \mathcal{B}$ was arbitrary, it follows that A is continuous everywhere in $\mathcal{B}$.

Proposition 2.1. *If A, B are bounded linear operators on $\mathcal{B}$ and if $\lambda \in \mathbb{C}$, then $A + B, \lambda A$, and AB are also bounded operators. Moreover,*

$$(i) \|A + B\| \leq \|A\| + \|B\|;$$

$$(ii) \|\lambda A\| = |\lambda| \cdot \|A\|;$$

$$(iii) \|AB\| \leq \|A\| \|B\|.$$

Proof. (i) We make use of the inequality $\|Ax\| \leq \|A\| \|x\|$ for each $x \in \mathcal{B}$, which can be easily deduced from the definition of the norm $\|A\|$ of A . Let $x \neq 0$. We have

$$\begin{aligned} \|(A + B)x\| &\leq \|Ax\| + \|Bx\| \\ &\leq \|A\| \|x\| + \|B\| \|x\|, \end{aligned}$$

and hence

$$\frac{\|(A + B)x\|}{\|x\|} \leq \|A\| + \|B\|,$$

and therefore,

$$\|A + B\| \leq \|A\| + \|B\|.$$

$$(ii) \|\lambda A\| = \sup_{0 \neq x \in \mathcal{B}} \frac{\|\lambda Ax\|}{\|x\|} = |\lambda| \cdot \sup_{0 \neq x \in \mathcal{B}} \frac{\|Ax\|}{\|x\|} = |\lambda| \cdot \|A\|.$$

$$(iii) \|AB\| = \sup_{0 \neq x \in \mathcal{B}} \frac{\|ABx\|}{\|x\|} \leq \|A\| \cdot \sup_{0 \neq x \in \mathcal{B}} \frac{\|Bx\|}{\|x\|} = \|A\| \cdot \|B\|.$$

2.2.1.1 Adjoint For Bounded Operators

Let $A \in B(\mathcal{H})$. Clearly, the quantity $\langle Ax, y \rangle$ is linear in x , conjugate-linear in y and bounded. Therefore, according to the Riesz representation theorem [159], there

exists a unique $A^* \in B(\mathcal{H})$ such that

$$\langle Ax, y \rangle = \langle x, A^*y \rangle$$

for all $x, y \in \mathcal{H}$.

The transformation $y \mapsto A^*y$ is called the adjoint of the linear operator A .

Proposition 2.2. *If $A : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator, then $A^* \in B(\mathcal{H})$. Furthermore, $\|A\| = \|A^*\|$.*

Proof. We first show that A^* is a bounded linear operator. Note that

$$\begin{aligned} \langle u, A^*(\alpha u + \beta w) \rangle &= \langle Au, \alpha u + \beta w \rangle \\ &= \overline{\alpha} \langle Au, u \rangle + \overline{\beta} \langle Au, w \rangle \\ &= \overline{\alpha} \langle u, A^*u \rangle + \overline{\beta} \langle u, A^*w \rangle \\ &= \langle u, \alpha A^*u + \beta A^*w \rangle \end{aligned}$$

and hence A^* is linear.

Now

$$\begin{aligned} \|A^*u\|^2 &= \langle A^*u, A^*u \rangle \\ &= \langle AA^*u, u \rangle \\ &\leq \|AA^*u\| \cdot \|u\| \\ &\leq \|A\| \cdot \|A^*u\| \cdot \|u\| \end{aligned}$$

and hence $\|A^*u\| \leq \|A\| \cdot \|u\|$, that is, $\|A^*\| \leq \|A\|$.

Similarly, $\|(A^*)^*\| \leq \|A^*\|$. Now using the fact $(A^*)^* = A$ it follows that $\|A\| \leq \|A^*\|$, which completes the proof.

Corollary 2.1. *If $A : \mathcal{H} \rightarrow \mathcal{H}$ is a bounded linear operator, then*

$$\|AA^*\| = \|A^*A\| = \|A^*\|^2 = \|A\|^2.$$

Proof. Using Propositions 2.2 and 2.1 it follows that $\|A^*A\| \leq \|A^*\| \cdot \|A\| = \|A\| \cdot \|A\| = \|A\|^2$. It is also clear that $\|A\|^2 \leq \|A^*A\|$, which completes the proof.

Proposition 2.3. *If A, B are bounded linear operators on $\mathcal{H}$ and if $\lambda \in \mathbb{C}$, then*

$$I^* = I, \tag{2.3}$$

$$O^* = O, \tag{2.4}$$

$$(A + B)^* = A^* + B^*, \tag{2.5}$$

$$(\lambda A)^* = \overline{\lambda} A^*, \quad (2.6)$$

$$(AB)^* = A^* B^*. \quad (2.7)$$

Example 2.3. Let $\mathcal{H} = L^2([\alpha, \beta])$ and let $A : L^2([\alpha, \beta]) \rightarrow L^2([\alpha, \beta])$ be the bounded linear operator defined by

$$A\phi(s) = \int_{\alpha}^{\beta} V(s, t)\phi(t)dt, \quad \forall \phi \in L^2([\alpha, \beta]),$$

where $V : [\alpha, \beta] \times [\alpha, \beta] \rightarrow \mathbb{C}$ is continuous.

It can be easily shown that the adjoint A^* of A is defined by

$$A^*\psi(s) = \int_{\alpha}^{\beta} \overline{V(t, s)}\psi(t)dt, \quad \forall \psi \in L^2([\alpha, \beta]).$$

Definition 2.2. A bounded linear operator $A : \mathcal{H} \mapsto \mathcal{H}$ is called self-adjoint or symmetric if $A = A^*$.

Example 2.4. Consider the integral operator given in Example 2.3. Assuming that V satisfies $V(s, t) = \overline{V(t, s)}$ for all $t, s \in [\alpha, \beta]$, one can easily see that A is symmetric.

Let $A : \mathcal{H} \mapsto \mathcal{H}$ be a bounded linear selfadjoint operator. Then the following properties hold. Their proofs are left as an exercise for the reader:

- (i) $\langle Ax, x \rangle \in \mathbb{R}$ for all $x \in \mathcal{H}$.
- (ii) $\|A\| = \sup_{x \neq 0} \frac{|\langle Ax, x \rangle|}{\|x\|^2}$.
- (iii) If $B \in B(\mathcal{B})$ is also self-adjoint and if $AB = BA$, then AB is also self-adjoint.

2.2.1.2 The Inverse Operator

Definition 2.3. An operator $A \in B(\mathcal{B})$ is called invertible if there exists $B \in B(\mathcal{B})$ such that $AB = BA = I$. In that event, the operator B is called the inverse operator of A and denoted by $B = A^{-1}$.

Theorem 2.2. If $A \in B(\mathcal{B})$ is a linear operator such that $\|A\| < 1$, then the operator $I - A$ is invertible.

Proof. Note that $(I - A)(I + A^2 + \dots + A^n) = I - A^{n+1}$, and $\|A^{n+1}\| \leq \|A\|^{n+1} \mapsto 0$ as $n \mapsto \infty$, since $\|A\| < 1$. Consequently,

$$\lim_{n \rightarrow \infty} (I - A)(I + A^2 + \dots + A^n) = I \text{ in } B(\mathcal{B}).$$

Now since $\|A\| < 1$ and that $B(\mathcal{B})$ is a Banach algebra, $S := \lim_{n \rightarrow \infty} (I + A^2 + \dots + A^n)$ does exist, hence $(I - A)S = I$. In fact,

$$(I - A)S = I + (I - A) [S - (I + A^2 + \dots + A^n)].$$

On the other hand,

$$\begin{aligned} \|(I - A) [S - (I + A^2 + \dots + A^n)]\| &\leq \|I - S\| \\ &\quad + \|S - (I + A^2 + \dots + A^n)\| \\ &\mapsto 0 \text{ as } n \mapsto \infty, \end{aligned}$$

hence $(I - A)S = I$.

In summary, $(I - A)$ is invertible and $(I - A)^{-1} = S$, where

$$S = \sum_{k=0}^{\infty} A^k \quad (A^0 \text{ being } I).$$

Remark 2.1. Note that if $A, B \in B(\mathcal{B})$ are invertible, so is their composition AB . Moreover, $(AB)^{-1} = B^{-1}A^{-1}$.

Similarly, if $A \in B(\mathcal{B})$ is invertible and if $B \in B(\mathcal{B})$ is such that $\|A - B\| < \frac{1}{\|A^{-1}\|}$, then B is invertible. Indeed, write

$$B = A[I - A^{-1}(A - B)].$$

Since $\|A^{-1}(A - B)\| < 1$, using Theorem 2.2, it follows that $I - A^{-1}(A - B)$ is invertible. Now since A is invertible it follows that $A[I - A^{-1}(A - B)]$ is invertible, too.

Definition 2.4. If $A : \mathcal{B} \rightarrow \mathcal{B}$ is a bounded linear operator, $N(A)$, $R(A)$, $\sigma(A)$, and $\rho(A)$ stand for the kernel, range, spectrum, and the resolvent of A , respectively, defined by

$$N(A) = \{u \in \mathcal{B} : Au = 0\},$$

$$R(A) = \{Au : u \in \mathcal{B}\},$$

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\},$$

and $\rho(A)$ is the collection of all $\lambda \in \mathbb{C}$ such that the operator $A - \lambda I$ is one-to-one ($N(A - \lambda I) = 0$), onto ($R(A - \lambda I) = \mathcal{B}$), and bounded.

Remark 2.2. Note that $\lambda \in \sigma(A)$ if and only if at least one of the following assertions holds true:

- (i) $R(\lambda I - A) \neq \mathcal{B}$;
- (ii) $\lambda I - A$ is not one-to-one.

Note that if (ii) of Remark 2.2 holds, λ is called an eigenvalue of the operator A with corresponding eigenspace $N(A - \lambda I)$. Therefore, if $0 \neq u \in N(\lambda I - A)$ is an eigenvalue then $Au = \lambda u$.

Example 2.5. Let $q : [\alpha, \beta] \rightarrow \mathbb{C}$ be a continuous function. Define the bounded linear operator M_q on $\mathcal{B} = L^2([\alpha, \beta])$ by

$$(M_q \phi)(s) = q(s)\phi(s), \quad \forall s \in [\alpha, \beta].$$

It can be shown that $\lambda I - M_q$ is invertible on $L^2([\alpha, \beta])$ if and only if

$$\lambda - q(s) \neq 0, \quad \forall s \in [\alpha, \beta]. \quad (2.8)$$

The inverse $(\lambda I - M_q)^{-1}$ of $\lambda I - M_q$ is defined by

$$[(\lambda I - M_q)^{-1}] \psi(s) = \left[\frac{1}{\lambda - q(s)} \right] \psi(s)$$

with the following estimate:

$$\|(\lambda I - M_q)^{-1}\| \leq \max_{s \in [\alpha, \beta]} \left[\frac{1}{|\lambda - q(s)|} \right].$$

The spectrum $\sigma(M_q)$ of M_q is given by

$$\sigma(M_q) = \{q(s) : s \in [\alpha, \beta]\}.$$

2.2.1.3 Compact Operators

Definition 2.5. A bounded linear operator A on $\mathcal{B}$ is said to be compact if it maps the unit ball U ($U = \{x \in \mathcal{B} : \|x\| \leq 1\}$) into a set whose closure is compact.

Equivalently,

Definition 2.6. A bounded linear operator A on $\mathcal{B}$ is said to be compact if for each sequence $(x_n)_{n \in \mathbb{N}}$ in $\mathcal{B}$ with $\|x_n\| \leq 1$ for each $n \in \mathbb{N}$, the sequence $(Ax_n)_{n \in \mathbb{N}}$ has a subsequence which converges in $\mathcal{B}$.

The collection of all compact operators on $\mathcal{B}$ is denoted $\mathcal{K}(\mathcal{B})$. The next theorem shows that $\mathcal{K}(\mathcal{B})$ is a two-sided ideal of $B(\mathcal{B})$. Moreover, $\mathcal{K}(\mathcal{B})$ is closed for the operator norm.

Theorem 2.3. If $A, B \in B(\mathcal{B})$ are compact linear operators, then

- (i) αA is compact;
- (ii) $A + B$ is compact;
- (iii) if $C \in B(\mathcal{B})$, then AC and CA are compact.

Proof. (i) is straightforward.

(ii) Let $(u_n) \in \mathcal{B}$ with $\|u_n\| \leq 1$. Since A is compact, $(Au_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(Au_{n_k})_{k \in \mathbb{N}}$.

Similarly, $(Bu_n)_{n \in \mathbb{N}}$ has a convergent subsequence $(Bu_{n_k})_{k \in \mathbb{N}}$. Therefore, $((A + B)u_{n_k})_{k \in \mathbb{N}}$ converges.

(iii) Let $(v_n)_{n \in \mathbb{N}} \subset \mathcal{B}$ with $\|v_n\| \leq 1$ for each $n \in \mathbb{N}$. Thus $(Cv_n)_{n \in \mathbb{N}}$ is bounded. Now since A is compact, it is clear $(ACv_n)_{n \in \mathbb{N}}$ has a convergent subsequence.

Let $(w_n)_{n \in \mathbb{N}} \subset \mathbb{H}$ with $\|w_n\| \leq 1$ for each $n \in \mathbb{N}$. Now since A is compact, $(Aw_n)_{n \in \mathbb{N}}$ has a convergent subsequence, say $(Aw_{n_k})_{k \in \mathbb{N}}$. Now by the continuity of C it follows that $(CAw_{n_k})_{k \in \mathbb{N}}$ converges.

Example 2.6. In $\mathcal{B} = L^2[a, b]$, define the integral operator A by

$$(Af)(t) := \int_a^b V(t, \tau) f(\tau) d\tau \text{ for each } f \in L^2[a, b].$$

Assuming that $V \in L^2([a, b] \times [a, b])$, it can be shown that A is compact.

Remark 2.3. (i) A bounded linear operator is of finite-rank if its image is a finite-dimensional Banach space.

(ii) A finite-rank operator is compact since all balls are pre-compact in a finite-dimensional Banach space.

(iii) Compact operators Hilbert spaces are uniform operator norm limits of finite-rank operators, and conversely.

(iv) If a sequence of compact operators converges to some bounded operator for the operator norm, then the limit is also a compact operator.

(v) If A is a compact operator, so is its adjoint A^* .

2.2.1.4 Hilbert–Schmidt Operators

An important subclass of compact operators consists of the so-called Hilbert–Schmidt operators. Here we study basic properties of Hilbert–Schmidt operators.

Definition 2.7. Let $(e_n)_{n=1,2,\dots}$ be an orthonormal basis for the Hilbert space $\mathcal{H}$. An operator $A \in B(\mathcal{H})$ is called a Hilbert–Schmidt if

$$\|A\|_2 := \left(\sum_{n=1}^{\infty} \|Ae_n\|^2 \right)^{1/2} < \infty. \quad (2.9)$$

If Eq. (2.9) holds, the number $\|A\|_2$ is called the Hilbert–Schmidt norm of A . We denote the class of Hilbert–Schmidt linear operators on $\mathcal{H}$ by $\mathbb{L}_2(\mathcal{H})$. More generally, if $\mathcal{H}'$ is another Hilbert space, we denote the collection of all Hilbert–Schmidt operators from $\mathcal{H}$ to $\mathcal{H}'$ by $\mathbb{L}_2(\mathcal{H}, \mathcal{H}')$.

Example 2.7. Let $(e_n)_{n \geq 1}$ be the canonical orthonormal basis for the Hilbert space $\mathcal{H} = l^2$ and let A be the operator defined by

$$Ax = \sum_{n=1}^{\infty} \frac{1}{n} \langle x, e_n \rangle e_n.$$

Clearly, $\|A\|_2 = \frac{\pi}{\sqrt{6}}$, and hence A is a Hilbert–Schmidt operator.

More generally:

Example 2.8. Let $\mathcal{H}$ be a Hilbert space and let A be the diagonal operator defined by

$$Au = \sum_{n=1}^{\infty} \alpha_n \langle u, e_n \rangle e_n, \quad \forall u \in \mathbb{H},$$

where $(e_n)_{n \geq 1}$ is an orthonormal basis for $\mathcal{H}$.

Clearly, $Ae_n = \alpha_n e_n$, $\forall n = 1, 2, \dots$, and $\|A\|_2 = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{1/2}$. Hence, A is Hilbert–Schmidt if and only if

$$\|A\|_2 = \left(\sum_{n=1}^{\infty} |\alpha_n|^2 \right)^{1/2} < \infty.$$

For instance, the operator B defined on $\mathbb{H}$ by

$$Bu = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \langle u, e_n \rangle e_n, \quad \forall u \in \mathcal{H}$$

is not Hilbert–Schmidt while the operator C ,

$$Cu = \sum_{n=1}^{\infty} \frac{1}{n^2} \langle u, e_n \rangle e_n, \quad \forall u \in \mathcal{H}$$

is since $\|C\|_2 = \frac{\pi^2}{3\sqrt{10}}$.

Remark 2.4. Let us notice that the Hilbert–Schmidt norm $\|\cdot\|_2$ is independent of the orthonormal basis $(e_n)_{n \geq 1}$ considered in the Definition 2.7.

Proposition 2.4. *A bounded linear operator A on $\mathcal{H}$ is Hilbert–Schmidt if and only if its adjoint A^* is. Furthermore, $\|A\| \leq \|A\|_2$, and $\|A\|_2 = \|A^*\|_2$.*

Proof. Since A is a Hilbert–Schmidt operator then there exists an orthonormal basis $(e_n)_{n \geq 1}$ for $\mathcal{H}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$.

If $(f_n)_{n \geq 1}$ is another orthonormal basis for $\mathcal{H}$, then

$$\begin{aligned} \sum_{n \geq 1} \|A^* f_n\|^2 &= \sum_{n \geq 1} \sum_{m \geq 1} |\langle A^* f_n, e_m \rangle|^2 \\ &= \sum_{m \geq 1} \sum_{n \geq 1} |\langle f_n, Ae_m \rangle|^2 \\ &= \sum_{n \geq 1} \|Ae_n\|^2 \end{aligned}$$

and hence A^* is a Hilbert–Schmidt operator with $\|A^*\|_2 = \|A\|_2$.

The converse can be proved using similar arguments as above.

Now

$$\begin{aligned}\|Au\|^2 &= \sum_{m \geq 1} |\langle f_m, Au \rangle|^2 \\ &\leq \|u\|^2 \sum_{m \geq 1} \|A^* f_m\|^2\end{aligned}$$

and hence $\|A\| \leq \|A^*\|_2 = \|A\|_2$.

Proposition 2.5. *Let $A, B \in B(\mathcal{H})$. Suppose that $A \in \mathbb{L}_2(\mathcal{H})$, then both AB and BA are in $\mathbb{L}_2(\mathcal{H})$.*

Proof. Since A is a Hilbert–Schmidt operator then there exists an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ for $\mathcal{H}$ such that $\sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty$. We have

$$\sum_{n=1}^{\infty} \|BAe_n\|^2 \leq \|B\|^2 \sum_{n=1}^{\infty} \|Ae_n\|^2 < \infty,$$

hence $BA \in \mathbb{L}_2(\mathcal{H})$.

To complete the proof it remains to show that AB is a Hilbert–Schmidt operator. Indeed, $AB = (B^*A^*)^* \in \mathbb{L}_2(\mathcal{H})$.

Theorem 2.4. *Every Hilbert–Schmidt operator is compact and is the limit in $\|\cdot\|_2$ -norm of a sequence of operators of finite-rank.*

Proof. We refer the reader to [176].

Proposition 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and let $A \in B(L^2(\Omega))$, then $A \in \mathbb{L}_2(L^2(\Omega))$ if and only if there exists a kernel $V \in L^2(\Omega \times \Omega)$ such that*

$$A\phi(x) = \int_{\Omega} V(x, y)\phi(y)dy \quad (2.10)$$

for all $x \in \Omega$ and $\phi \in L^2(\Omega)$.

The adjoint A^* of A is the integral operator with kernel W defined by $W(x, y) := \overline{V(y, x)}$. Thus,

$$A^*\psi(x) = \int_{\Omega} W(x, y)\psi(y)dy \quad (2.11)$$

for all $x \in \Omega$ and $\psi \in L^2(\Omega)$.

Proof. See for instance [176].

Remark 2.5. Let $(e_n)_{n \in \mathbb{N}}$ be an orthonormal basis for a separable Hilbert space $\mathcal{H}$ and let A and B be Hilbert–Schmidt operators on $\mathcal{H}$, respectively. Define

$$\langle\langle A, B \rangle\rangle = \sum_{n=1}^{\infty} \langle Ae_n, Be_n \rangle. \quad (2.12)$$

It can be easily shown that $\langle\langle A, B \rangle\rangle$ is an inner product on $\mathbb{L}_2(\mathcal{H})$ and that $|A| = (\langle\langle A, A \rangle\rangle)^{\frac{1}{2}}$. In particular, $(\mathbb{L}_2(\mathcal{H}), \langle\langle \cdot, \cdot \rangle\rangle)$ is a Hilbert space.

2.2.2 Unbounded Linear Operators

Definition 2.8. An unbounded linear operator A from $\mathcal{B}$ in $\mathcal{B}'$ is a pair $(D(A), A)$ consisting of a subspace $D(A) \subset \mathcal{B}$ (called the domain of A) and a (possibly not continuous) linear transformation $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}'$.

2.2.3 Examples of Unbounded Operators

Example 2.9. [51, 130] Set $\mathcal{B} = \mathcal{B}' = L^2(\mathbb{R})$ and consider the one-dimensional Laplace operator defined by

$$D(A) = W^{2,2}(\mathbb{R}) = H^2(\mathbb{R}) \quad \text{and} \quad Au = -u''$$

for all $u \in H^2(\mathbb{R})$.

Recall that $L^2(\mathbb{R})$ is endowed with the norm defined by

$$\|\psi\|_2^2 := \int_{-\infty}^{+\infty} |\psi(t)|^2 dt, \quad \forall \psi \in L^2(\mathbb{R}).$$

Now, consider the sequence of functions defined by $\psi_n(t) = e^{-n|t|}$, $n = 1, 2, \dots$. Clearly, for each $n = 1, 2, \dots$, $\psi_n \in D(A) = H^2(\mathbb{R})$. Furthermore,

$$\|\psi_n\|_2^2 = \int_{-\infty}^{+\infty} e^{-2n|t|} dt = \frac{1}{n}$$

and

$$\|A\psi_n\|_2^2 = \int_{-\infty}^{+\infty} n^4 e^{-2n|t|} dt = n^3.$$

Therefore, $\frac{\|A\psi_n\|_2}{\|\psi_n\|_2} = n \rightarrow \infty$ as n goes to ∞ , that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

Example 2.10. [51, 130] Set $\mathcal{B} = \mathcal{B}' = L^2(0, 1)$ and consider the derivative operator defined by

$$D(A) = C^1(0, 1) \quad \text{and} \quad Au = u'$$

for all $u \in C^1(0, 1)$, where $C^1(0, 1)$ is the collection of continuously differentiable functions over $(0, 1)$.

Consider the sequence of functions defined by $\phi_n(t) = t^n$, $n = 1, 2, \dots$. Clearly, for each $n = 1, 2, \dots$, $\phi_n \in C^1(0, 1)$. Furthermore,

$$\|\phi_n\|_2^2 = \int_0^1 t^{2n} dt = \frac{1}{2n+1},$$

and

$$\|A\phi_n\|_2^2 = \int_0^1 n^2 t^{2n-2} dt = \frac{n^2}{2n-1}.$$

Here again,

$$\frac{\|A\phi_n\|_2}{\|\phi_n\|_2} = n \sqrt{\frac{2n+1}{2n-1}} \rightarrow \infty$$

as n goes to ∞ , that is, A is an unbounded linear operator on $L^2(\mathbb{R})$.

Example 2.11. (Multiplication Operator) Let $\mathcal{O} \subset \mathbb{R}$ be an arbitrary interval and let $C_0(\mathcal{O})$ denote the collection of all continuous functions $u : \mathcal{O} \rightarrow \mathbb{C}$ satisfying $\forall \varepsilon > 0$ then there exists a compact interval $I_\varepsilon \subset \mathcal{O}$ such that

$$|u(s)| < \varepsilon, \quad \forall s \in \mathcal{O} \setminus I_\varepsilon.$$

Then define the multiplication operator M_γ on $C_0(\mathcal{O})$ by

$$\begin{cases} D(M_\gamma) = \{u \in C_0(\mathcal{O}) : \gamma u \in C_0(\mathcal{O})\}, \\ M_\gamma u = \gamma(x)u, \quad \forall u \in D(M_\gamma), \end{cases}$$

where $\gamma : \mathcal{O} \rightarrow \mathbb{C}$ is continuous.

In view of the above, M_γ is an unbounded linear operator on $C_0(\mathcal{O})$. Moreover, one can show that M_γ is bounded if and only if γ is bounded. In that event,

$$\|M_\gamma\| = \|\gamma\|_\infty := \sup_{s \in \mathcal{O}} |\gamma(s)|.$$

Definition 2.9. If $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is an unbounded linear operator on $\mathcal{B}$, then its graph is defined by

$$\mathcal{G}(A) = \{(x, Ax) \in \mathcal{B} \times \mathcal{B} : x \in D(A)\}.$$

Definition 2.10. If A, B are unbounded linear operators on $\mathcal{B}$, then A is said to be an extension of B if $D(B) \subset D(A)$ and $Au = Bu$ for all $u \in D(B)$. In that event, we denote it by $B \subset A$. Moreover, $B \subset A$ if and only if $\mathcal{G}(B) \subset \mathcal{G}(A)$.

The notion of graph of an operator is very important as it enables us to deal with the closure of an operator.

2.2.3.1 Closed and Closable Linear Operators

Definition 2.11. A linear operator $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is called closed if its graph $\mathcal{G}(A) \subset \mathcal{B} \times \mathcal{B}$ is closed.

The closedness of an unbounded linear operator A can be characterized as follows: if $u_n \in D(A)$ such that $u_n \rightarrow u$ and $Au_n \rightarrow v$ in $\mathcal{B}$ as $n \rightarrow \infty$, then $u \in D(A)$ and $Au = v$.

Example 2.12. Every bounded linear operator $A : \mathcal{B} \rightarrow \mathcal{B}$ is closed.

Proof. Suppose $(u_n)_{n \in \mathbb{N}} \in D(A)$ such that $u_n \rightarrow u$ with $Au_n \rightarrow v$ in $\mathcal{B}$ as $n \rightarrow \infty$. Now since A is bounded, therefore $D(A) = \mathcal{B}$. Again, from the continuity of A it is clear that $u \in \mathcal{B}$ and $Au = v$.

Example 2.13. Let $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ be a closed linear operator and let $B \in B(\mathcal{B})$, then $A + B$ is closed.

Proof. Suppose $(u_n)_{n \in \mathbb{N}} \in D(A + B) = D(A)$ such that $u_n \rightarrow u$ and $(A + B)u_n \rightarrow v$ in $\mathcal{B}$ as $n \rightarrow \infty$.

Now since B is bounded it follows that $Au_n \rightarrow v - Bu$ in $\mathcal{B}$ as $n \rightarrow \infty$. Since A is closed, then $u \in D(A)$ and $Au = v - Bu$.

Example 2.13 can be illustrated as follows: Let $\mathcal{B} = L^2(\mathbb{R}^n)$ and define A and B by

$$D(A) = W^{2,2}(\mathbb{R}^n) = H^2(\mathbb{R}^n) \quad \text{and} \quad Au = -\Delta u, \quad \forall u \in H^2(\mathbb{R}^n),$$

and

$$D(B) = \left\{ u \in L^2(\mathbb{R}^n) : \gamma(x)u \in L^2(\mathbb{R}^n) \right\} \quad \text{and} \quad Bu = \gamma(x)u, \quad \forall u \in D(B),$$

where Δ is the n -dimensional Laplace operator defined by

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$$

and that $\gamma \in L^\infty(\mathbb{R}^n)$.

It is then clear that $B = M_\gamma$ is a bounded linear operator and therefore $-\Delta + \gamma$ is closed. Furthermore, $D(-\Delta + \gamma) = H^2(\mathbb{R}^n)$.

Example 2.14. The multiplication operator M_γ given in Example 2.11 is closed.

Definition 2.12. An operator $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is said to be closable if it has a closed extension.

It is well-known that A is closable if $\overline{\mathcal{G}(A)}$ is a graph. Equivalently, $u_n \in D(A)$, $u_n \rightarrow 0$ and $Au_n \rightarrow v$ then $v = 0$.

If A is closable, then its smallest closed extension is called the closure of A and is denoted by $\bar{A}$. The operator $\bar{A}$ is defined by

$$\begin{cases} D(\bar{A}) = \{u \in \mathcal{B} : \exists u_n \in D(A), u_n \rightarrow u, Au_n \text{ converges}\}, \\ \bar{A}u = \lim Au_n, \quad \forall u \in D(\bar{A}). \end{cases}$$

Moreover, the closure $\bar{A}$ of A satisfies $\mathcal{G}(\bar{A}) = \overline{\mathcal{G}(A)}$.

Example 2.15. Suppose $\mathcal{B} = L^2(\mathbb{R}^n)$ and consider the linear operator A defined by

$$D(A) = C_0^\infty(\mathbb{R}^n) \text{ and } Au = -\Delta u, \text{ for all } u \in C_0^\infty(\mathbb{R}^n).$$

Clearly, A is closable and its closure is given by

$$D(\bar{A}) = H^2(\mathbb{R}^n) \text{ and } \bar{A}u = -\Delta u, \text{ for all } u \in H^2(\mathbb{R}^n).$$

2.2.3.2 Spectral Theory for Unbounded Linear Operators

If $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is a closed linear operator on $\mathcal{B}$, then $\rho(A)$ the resolvent set of A is defined by

$$\rho(A) = \left\{ \lambda \in \mathbb{C} : \lambda I - A \text{ is one-to-one, and } (\lambda I - A)^{-1} \in B(\mathcal{B}) \right\},$$

and $\sigma(A)$ the spectrum of A is the complement of the resolvent set $\rho(A)$ in $\mathbb{C}$.

Now, if $\lambda \in \rho(A)$, then the operator-valued function $R(\lambda, A) := (\lambda I - A)^{-1} : \rho(A) \mapsto B(\mathcal{B})$ is called the resolvent of the operator A . It should be mentioned that $\rho(A) \neq \emptyset$ if A is closed.

As for bounded linear operators, the spectrum of an unbounded operator can be divided into three disjoint subsets of the complex plane, that is,

$$\sigma(A) = \sigma_c(A) \cup \sigma_r(A) \cup \sigma_p,$$

where $\sigma_c(A), \sigma_r(A), \sigma_p(A)$ are respectively the continuous spectrum, the residual spectrum, and the point spectrum of the operator A defined by:

- (i) $\lambda \in \sigma_c(A)$ if $\lambda \in \mathbb{C}$, $\lambda I - A$ is one-to-one, and $\overline{R(\lambda I - A)} = \mathcal{B}$;
- (ii) $\lambda \in \sigma_r(A)$ if $\lambda \in \mathbb{C}$, $\lambda I - A$ is one-to-one, and $\overline{R(\lambda I - A)} \neq \mathcal{B}$; and
- (iii) $\lambda \in \sigma_p(A)$ if $\lambda \in \mathbb{C}$, and $\lambda I - A$ is not one-to-one.

Example 2.16. Fix $\Theta, L > 0$. In $\mathcal{B} := L^2(0, L)$ equipped with its natural topology $\|\cdot\|_2$, define the operator A by

$$A\varphi := -\Theta u'', \quad \forall \varphi \in D(A),$$

where $D(A) := H_0^1(0, L) \cap H^2(0, L)$.

The resolvent and spectrum of the linear operator A are respectively given by

$$\rho(A) = \mathbb{C} - \left\{ \frac{\Theta \pi^2}{L^2} n^2 : n = 1, 2, 3, \dots \right\}$$

and

$$\sigma(A) = \sigma_p(A) = \left\{ \frac{\Theta \pi^2}{L^2} n^2 : n = 1, 2, 3, \dots \right\}.$$

Proposition 2.7. *If $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is a closed linear operator and if $\lambda, \mu \in \rho(A)$, then for any $\lambda \in \rho(A)$, the operator $R(\lambda, \mathcal{B})$ is a bounded linear operator $\mathcal{B}$.*

Proof. Let $\lambda \in \rho(A)$. Clearly, $R(\lambda I - A) = D((\lambda I - A)^{-1})$ is dense in $\mathcal{B}$ and there exists $K > 0$ such that

$$\|(\lambda I - A)u\| \geq K\|u\| \quad \text{for all } u \in D(A).$$

To complete the proof, we have to show that $R(\lambda I - A) = \mathcal{B}$. Indeed, let $(u_n)_{n \in \mathbb{N}} \subset D(A)$ and suppose that $(\lambda I - A)u_n \rightarrow v$ as $n \rightarrow \infty$. Using the above-mentioned inequality it follows that there exists some $u \in \mathcal{B}$ such that $u_n \rightarrow u$ as $n \rightarrow \infty$. Since A is closed it follows that $u \in D(A)$ and $(\lambda I - A)u = v$. Consequently, by the density assumption $R(\lambda I - A) = \mathcal{B}$, we must have $R(\lambda I - A) = \mathcal{B}$.

Proposition 2.8. *Let A and B be two (possibly unbounded) closed linear operators on $\mathcal{B}$.*

(i) *If $\lambda, \mu \in \rho(A)$, then*

$$R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A). \quad (2.13)$$

Furthermore, $R(\lambda, A)$ and $R(\mu, A)$ commute.

(ii) *If $D(A) \subset D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have*

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B). \quad (2.14)$$

(iii) *If $D(A) = D(B)$, then for all $\lambda \in \rho(A) \cap \rho(B)$ we have*

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B) = R(\lambda, B)(A - B)R(\lambda, A). \quad (2.15)$$

Proof. (i) Write

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= R(\lambda, A)[(\mu I - A) - (\lambda I - A)]R(\mu, A) \\ &= (\mu - \lambda)R(\lambda, A)R(\mu, A). \end{aligned}$$

Now, the second statement is obtained from the first one by

$$\begin{aligned} R(\lambda, A)R(\mu, A) &= \frac{1}{\mu - \lambda}[R(\lambda, A) - R(\mu, A)] \\ &= \frac{1}{\lambda - \mu}[R(\mu, A) - R(\lambda, A)] \\ &= R(\mu, A)R(\lambda, A). \end{aligned}$$

(ii) Write

$$\begin{aligned} R(\lambda, A) - R(\lambda, B) &= R(\lambda, A)[(\lambda I - B) - (\lambda I - A)]R(\lambda, B) \\ &= R(\lambda, A)(A - B)R(\lambda, B). \end{aligned}$$

(iii) Write

$$\begin{aligned} R(\lambda, A) - R(\lambda, B) &= R(\lambda, A)[(\lambda I - B) - (\lambda I - A)]R(\lambda, B) \\ &= R(\lambda, A)(A - B)R(\lambda, B) \\ &= R(\lambda, B)(A - B)R(\lambda, A). \end{aligned}$$

Theorem 2.5. *If $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{B}$ is a closed linear operator, then $\rho(A)$ is an open subset of $\mathbb{F}$. Therefore, $\sigma(A)$ is closed. Namely, if $\lambda \in \rho(A)$, then $\mu \in \rho(A)$ for all $\mu \in \mathbb{F}$ such that $|\lambda - \mu| < \|R(\lambda, A)\|^{-1}$ and for those μ , the following holds:*

$$R(\mu, A) = \sum_{n \in \mathbb{N}} (\lambda - \mu)^n R(\lambda, A)^{n+1}.$$

If $A \in B(\mathcal{H})$, then $\{\mu \in \mathbb{F} : |\mu| > \|A\|\} \subset \rho(A)$. Moreover, the spectrum of A is compact, and

$$R(\mu, A) = \sum_{n \in \mathbb{N}} \lambda^{-n-1} A^n \text{ for all } |\mu| > \|A\|.$$

Proof. See [186] for details.

2.2.3.3 Symmetric and Self-Adjoint Linear Operators

Definition 2.13. If $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a densely defined linear operator, then its adjoint denoted A^* is defined in a unique fashion by

$$D(A^*) = \left\{ v \in \mathcal{H} : u \mapsto \langle Au, v \rangle \text{ is } \mathcal{H} \text{ -- continuous over } D(A) \right\},$$

and

$$\langle Au, v \rangle = \langle u, A^*v \rangle, \quad \forall u \in D(A), \quad v \in D(A^*).$$

Define the mappings $U : \mathcal{H} \times \mathcal{H} \mapsto \mathcal{H} \times \mathcal{H}$ and $V : \mathcal{H} \times \mathcal{H} \mapsto \mathcal{H} \times \mathcal{H}$ by setting $U(x, y) = (y, -x)$ and $V(x, y) = (y, x)$ for all $(x, y) \in \mathcal{H} \oplus \mathcal{H}$. Clearly, U and V are isomorphisms from $\mathcal{H} \oplus \mathcal{H}$ onto $\mathcal{H} \oplus \mathcal{H}$. Furthermore, their inverses are defined by

$$U^{-1}(x, y) = (-y, x)$$

and

$$V^{-1}(x, y) = (y, x)$$

for all $x, y \in \mathcal{H}$.

If A is a densely defined (possibly unbounded) linear operator, that is, $\overline{D(A)} = \mathcal{H}$, one can easily see that

$$\mathcal{G}(A^*) = U \left(\mathcal{G}(A)^\perp \right) = \left(U \mathcal{G}(A) \right)^\perp. \quad (2.16)$$

Proposition 2.9. [176] *If $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a densely defined $\overline{D(A)} = \mathcal{H}$ unbounded linear operator, then*

- (i) A^* is closed;
- (ii) A is closable if and only if A^* is densely defined; in this case $\bar{A} = (A^*)^*$; and
- (iii) if A is closable, then $(\bar{A})^* = A^*$.

Proof. (i) Using the identity Eq. (2.16) it follows that $\mathcal{G}(A^*) = \left(U\mathcal{G}(A) \right)^\perp$. Consequently, $\mathcal{G}(A^*)$ is closed.

(ii) Using the fact

$$\begin{aligned} \overline{\mathcal{G}(A)} &= \mathcal{G}(A)^{\perp\perp} \\ &= \left(U^{-1}\mathcal{G}(A^*) \right)^\perp \\ &= \{ (u, v) \in \mathcal{H} \oplus \mathcal{H} : \langle u, A^*z \rangle - \langle v, z \rangle = 0 \text{ for all } z \in D(A^*) \} \end{aligned}$$

it follows that $(0, v) \in \overline{\mathcal{G}(A)}$ if and only if $v \in D(A^*)^\perp$. And hence $(0, v) \in \overline{\mathcal{G}(A)}$ yields $v = 0$ if and only if $D(A^*) = \mathcal{H}$. Therefore, $\mathcal{G}(A)$ is a graph if and only if the linear operator A^* is densely.

Now, if $D(A^*)$ is dense, then

$$\mathcal{G}(A^{**}) = \left(U^{-1}\mathcal{G}(A^*)^\perp \right) = \left(U^{-1}U\mathcal{G}(A)^{\perp\perp} \right) = \overline{\mathcal{G}(A)} = \mathcal{G}(\bar{A}).$$

(iii) Suppose A is closable. Now,

$$\mathcal{G}(A^*) = U \left(\mathcal{G}(A)^\perp \right) = U \left(\overline{\mathcal{G}(A)}^\perp \right) = U \left(\mathcal{G}(\bar{A}) \right)^\perp = \mathcal{G}(A^*),$$

and hence $A^* = (\bar{A})^*$.

Proposition 2.10. [176] *If A, B are densely defined unbounded linear operators on $\mathcal{H}$, then*

- (i) $A^*B^* \subset (BA)^*$;
- (ii) if $B \in B(\mathcal{H})$, then $A^*B^* = (BA)^*$; and
- (iii) if $A + B$ is densely defined, we have $(A + B)^* \supset A^* + B^*$.

Proof. (i) Let us show that the operators A^*B^* and BA are adjoint to each other. Indeed, let $u \in D(A^*B^*)$ and $v \in D(BA)$. Clearly, $u \in D(B^*)$ such that $B^*u \in D(A^*)$. Similarly, $v \in D(A)$ such that $Av \in D(B)$. Using the definition of the adjoint it follows

$$\langle A^*B^*u, v \rangle = \langle B^*u, Av \rangle = \langle u, BAv \rangle.$$

(ii) Using (i) it is enough to show that $D(BA)^* \subset D(A^*B^*)$. Indeed, let $u \in D(BA)^*$. Now since B^* is bounded it follows that for all $v \in D(BA) = D(A)$, we obtain

$$\langle (BA)^*u, v \rangle = \langle u, BAv \rangle = \langle B^*u, v \rangle,$$

and hence $B^*u \in D(A^*)$, that is, $u \in D(A^*B^*)$.

(iii) Let $u \in D(A^* + B^*) = D(A^*) \cap D(B^*)$. Clearly, for all $v \in D(A + B) = D(A) \cap D(B)$, we have

$$\langle (A^* + B^*)u, v \rangle = \langle A^*u, v \rangle + \langle B^*u, v \rangle = \langle u, Av \rangle + \langle u, Bv \rangle = \langle u, (A + B)v \rangle$$

and hence $u \in D((A + B)^*)$ and $(A + B)^*u = A^*u + B^*u$.

Definition 2.14. If $A : D(A) \subset \mathcal{H} \rightarrow \mathcal{H}$ is a densely defined operator on $\mathcal{H}$, then

- (i) A is symmetric if $A \subset A^*$.
- (ii) A is self-adjoint if $A = A^*$.

Example 2.17. Let $\mathcal{H} = L^2[0, 1]$ and define the linear operator A by

$$D(A) = \{u \in L^2[0, 1] : u \in C^1[0, 1], u(0) = u(1) = 0\}$$

and

$$Au = iu' \text{ for all } u \in D(A).$$

It is not hard to see that

$$A^*u = iu' \text{ for all } u \in D(A^*)$$

where

$$D(A^*) = \{u : u \text{ is absolutely continuous, } u' \in L^2[0, 1]\}.$$

Therefore, $A \subset A^*$, that is, A is symmetric. It should also be noted that A is not closed. It is obviously closable and has a closure $\bar{A}$ defined by

$$\bar{A}u = iu' \text{ for all } u \in D(\bar{A})$$

where

$$D(\bar{A}) = \{u : u \text{ is absolutely continuous, } u' \in L^2[0, 1], u(0) = u(1) = 0\}.$$

Example 2.18. Let $\mathcal{H} = L^2[0, 1]$ and define the linear operator B by

$$D(B) = \{u : u \text{ is absolutely continuous, } u' \in L^2[0, 1], u(0) = u(1) = 0\}$$

and

$$Au = iu' \text{ for all } u \in D(B).$$

It is not hard to see that $B \subset B^*$, that is, B is symmetric. Moreover, it can be shown that $B = B^*$.

Proposition 2.11. Every symmetric operator A on $\mathcal{H}$ is closable.

Proof. Clearly A is closable since $A \subset A^*$ and A^* is closed by Proposition 2.9 (i). Now for all $u, v \in D(A)$ one can find sequences $u_n, v_n \subset D(A)$ such that $u_n \rightarrow u$ and $v_n \rightarrow v$ and $Au_n \rightarrow \bar{A}u$, $Av_n \rightarrow \bar{A}v$ as $n \rightarrow \infty$. Since A is symmetric it follows that

$$\langle \bar{A}u, v \rangle = \lim_{n \rightarrow \infty} \langle Au_n, v_n \rangle = \lim_{n \rightarrow \infty} \langle u_n, Av_n \rangle = \langle u, \bar{A}v \rangle.$$

Since $D(\bar{A})$ is dense it follows that $\bar{A}$ is symmetric, too.

Remark 2.6. Notice that a symmetric operator A is called essentially self-adjoint if it has a unique self-adjoint extension.

Theorem 2.6. *Let $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$ be a self-adjoint operator, then*

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A).$$

Proof. We refer the reader to [69].

Theorem 2.7. [69] *Let $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ be a spectral family of orthoprojections E_λ , that is, $E_\lambda \leq E_\mu$ for $\lambda \leq \mu$ and $E_\lambda \rightarrow 0$ as $\lambda \rightarrow -\infty$, $E_\lambda \rightarrow I$ as $\lambda \rightarrow \infty$ (in the strong sense) and $E_{\lambda+0} = E_\lambda$. Now let A be the operator defined by*

$$D(A) = \{u \in \mathcal{H} : \int_{-\infty}^{\infty} \lambda^2 d\langle E_\lambda x, x \rangle < \infty\}$$

and

$$A = \int_{-\infty}^{\infty} \lambda dE_\lambda$$

that is for each $u \in D(A)$, we have

$$Au = \int_{-\infty}^{\infty} \lambda dE_\lambda u.$$

Then A is a self-adjoint linear operator on $\mathcal{H}$ and

$$\|Au\|^2 = \int \lambda^2 d\langle E_\lambda u, u \rangle.$$

2.3 Sectorial Linear Operators

An important class of (unbounded) linear operators is that of sectorial linear operators. Such a class of operators will play an important role throughout this book.

2.3.1 Basic Definitions

Definition 2.15. A linear operator $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ (not necessarily densely defined) is said to be sectorial if the following hold: There exist constants $\zeta \in \mathbb{R}$, $\theta \in (\frac{\pi}{2}, \pi)$, and $M > 0$ such that

(i) $\rho(A) \supset S_{\theta, \zeta} := \{\lambda \in \mathbb{C} : \lambda \neq \zeta, |\arg(\lambda - \zeta)| < \theta\}$, and

(ii) $\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \zeta|}$ for each $\lambda \in S_{\theta, \zeta}$.

Let us notice that since the resolvent of a sectorial operator A is nonempty, then A is closed. Therefore, the space $(D(A), \|\cdot\|_A)$ where

$$\|x\|_A = \|x\| + \|Ax\|$$

for each $x \in D(A)$, is a Banach space.

Note that the norm $\|\cdot\|_A$ which depends heavily on the operator A is called the graph norm of A .

Proposition 2.12. [129] *Let A be a linear operator on $\mathcal{B}$ such that $\rho(A)$ contains the half-plane $\{\lambda \in \mathbb{C} : \Re \lambda \geq \zeta\}$, and*

$$\|\lambda R(\lambda, A)\| \leq M, \quad \Re \lambda \geq \zeta,$$

with $\zeta \in \mathbb{R}$ and $M > 0$. Then A is sectorial.

2.3.2 Examples of Sectorial Operators

Example 2.19. In $\mathcal{B} = L^p(0, 1)$ ($p \geq 1$) equipped with its natural norm, define the linear operator A by

$$Au = u'' \text{ for each } u \in D(A) = \{u \in W^{2,p}(0, 1) : u(0) = u(1) = 0\}.$$

Then the linear operator A defined above is sectorial.

Example 2.20. In $\mathcal{B} = C[0, 1]$ equipped with the sup norm, define the linear operator A by

$$Au = u'' \text{ for each } u \in D(A) = \{u \in C^2[0, 1] : u(0) = u(1) = 0\}.$$

Then the linear operator A defined above is sectorial.

Example 2.21. Let $\mathcal{O} \subset \mathbb{R}^n$ be a bounded open subset with C^2 boundary $\partial\mathcal{O}$. Let $\mathcal{B} = L^2(\mathcal{O})$ and define the second-order differential operator

$$Au = \Delta u, \quad \forall u \in D(A) = W_0^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}).$$

It can be shown that A is sectorial.

Example 2.22. Let $\mathcal{O} \subset \mathbb{R}^N$ be a bounded open subset whose boundary $\partial\mathcal{O}$ is of class C^2 . Let $n(x)$ denote the outer normal to $\mathcal{O}$ for each $x \in \partial\mathcal{O}$.

Consider the differential operator defined by

$$A_0 u(x) = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x)$$

where the coefficients a_{ij} and b_i and c are real, bounded, and continuous on $\overline{\mathcal{O}}$. Moreover, we suppose that for each $x \in \overline{\mathcal{O}}$, the matrix $[a_{ij}(x)]$ is symmetric and strictly positive definite, that is,

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \omega |\xi|^2 \text{ for all } x \in \overline{\mathcal{O}}, \xi \in \mathbb{R}^N.$$

Theorem 2.8. (S. Agmon [5] and Lunardi et al. [131]) Let $p > 1$.

(i) Let $A_p : W^{2,p}(\mathbb{R}^N) \mapsto L^p(\mathbb{R}^N)$ be the linear operator defined by $A_p u = A_0 u$. Then the operator A_p is sectorial in $L^p(\mathbb{R}^N)$ and the domain $D(A_p)$ is dense in $L^p(\mathbb{R}^N)$.

(ii) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = W^{2,p}(\mathcal{O}) \cap W_0^{1,p}(\mathcal{O}), \quad A_p u = A_0 u.$$

Then the linear operator A_p is sectorial in $L^p(\Omega)$. Moreover, $D(A_p)$ is dense in $L^p(\mathcal{O})$.

(iii) Let A_0 be defined as above and let A_p be the linear operator defined by

$$D(A_p) = \{u \in W^{2,p}(\mathcal{O}) : Bu|_{\partial\mathcal{O}} = 0\}, \quad A_p u = A_0 u, \quad u \in D(A_p)$$

where

$$Bu(x) = b_0 u(x) + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i}$$

with the coefficients b_i ($i = 1, \dots, N$) are in $C^1(\overline{\mathcal{O}})$ and the condition

$$\sum_{i=1}^N b_i(x) n_i(x) \neq 0 \quad x \in \partial\mathcal{O}$$

holds. Then A_p is sectorial in $L^p(\mathcal{O})$ and $D(A_p)$ is dense in $L^p(\mathcal{O})$.

2.4 Semigroups of Linear Operators

2.4.1 Basic Definitions

Definition 2.16. Let $(\mathcal{B}, \|\cdot\|)$ be a Banach space. The family of bounded operators $(T(t))_{t \in \mathbb{R}^+} : \mathcal{B} \rightarrow \mathcal{B}$ is said to be a semigroup or one-parameter semigroup if the following statements hold true:

(i) $T(0) = I$; and

(ii) $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$.

Moreover if

(iii) $\lim_{t \searrow 0} \|T(t) - I\| = 0$, then the semigroup $T(t)$ is said to be uniformly continuous.

Remark 2.7. If $(T(t))_{t \in \mathbb{R}^+} : \mathcal{B} \rightarrow \mathcal{B}$ is a semigroup of bounded linear operator, one can associate with it an operator $(D(A), A)$ called the infinitesimal generator of the semigroup defined by

$$D(A) := \left\{ u \in \mathcal{B} : \lim_{t \searrow 0} \frac{T(t)u - u}{t} \text{ exists} \right\}, \quad (2.17)$$

and

$$Au := \lim_{t \searrow 0} \frac{T(t)u - u}{t}, \text{ for every } u \in D(A). \quad (2.18)$$

Remark 2.8. An operator A is the infinitesimal generator of a uniformly continuous semigroup of bounded linear operators $(T(t))_{t \in \mathbb{R}^+}$ if and only if A is bounded. In that event, it can be shown that $T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$.

Definition 2.17. A semigroup of bounded linear operators $(T(t))_{t \in \mathbb{R}^+} : \mathcal{B} \mapsto \mathcal{B}$ is said to be a strongly continuous semigroup of bounded linear operators (or c_0 -semigroup) if $\lim_{t \searrow 0} \|T(t)x - x\| = 0$ for each $x \in \mathcal{B}$.

Example 2.23. Suppose that $\mathcal{B} = (BUC(\mathbb{R}), \|\cdot\|_{\infty})$ is the Banach space of bounded uniformly continuous functions on the real number line equipped with the sup norm. Define

$$(S(t)\phi)(\sigma) = \phi(t + \sigma), \quad \forall \phi \in BUC(\mathbb{R}).$$

Then $(S(t))_{t \in \mathbb{R}}$ is a c_0 -semigroup with $\|S(t)\| \leq 1$ for each $t \in [0, \infty)$. Moreover, its infinitesimal generator A is defined by

$$D(A) = H^1(\mathbb{R}), \text{ and } A\phi = \phi', \quad \forall \phi \in H^1(\mathbb{R}),$$

where $H^1(\mathbb{R})$ is the Sobolev space.

Example 2.24. Let $1 \leq p < \infty$ and let $\mathcal{B} = L^p(\mathbb{R})$ equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}$, and

$$(S(t))u(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{\frac{-|x-y|^2}{4t}} u(y) dy, \quad t > 0, \quad x \in \mathbb{R}.$$

Then $S(t)$ is a c_0 -semigroup satisfying

$$\|S(t)u\|_p \leq \|u\|_p$$

and whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}), \quad A_p u = u'', \quad \text{for all } u \in D(A_p).$$

Example 2.25. This is a generalization of Example 2.24. Let $1 \leq p < \infty$ and let $\mathcal{B} = L^p(\mathbb{R}^N)$ (or $BC(\mathbb{R}^N, \mathbb{C})$ equipped with the sup norm) equipped with its natural norm $\|\cdot\|_p$. Define $(S(0))u(x) = u(x)$ for all $x \in \mathbb{R}^N$, and

$$(S(t))u(x) = \frac{1}{(4\pi t)^{N/2}} \int_{-\infty}^{\infty} e^{-\frac{\|x-y\|^2}{4t}} u(y) dy, \quad t > 0, \quad x \in \mathbb{R}.$$

Then $S(t)$ is a c_0 -semigroup satisfying

$$\|S(t)u\|_p \leq \|u\|_p$$

and whose infinitesimal generator A_p is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}^N), \quad A_p u = \Delta u, \quad \text{for all } u \in D(A_p).$$

2.4.2 Basic Properties of Semigroups

Theorem 2.9. Let $(T(t))_{t \in \mathbb{R}^+} : \mathcal{B} \rightarrow \mathcal{B}$ be a semigroup of bounded linear operators, then

- (i) there are constants C, ζ such that $\|T(t)\| \leq C e^{\zeta t}$, $t \in \mathbb{R}^+$;
- (ii) the infinitesimal generator A of the semigroup $T(t)$ is a densely defined closed operator;
- (iii) the map $t \mapsto T(t)x$ which goes from $\mathbb{R}^+$ into $\mathcal{B}$ is continuous for every $x \in \mathcal{B}$;
- (iv) the differential equation given by

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax,$$

holds for every $x \in D(A)$;

- (v) for every $x \in \mathcal{B}$, then $T(t)x = \lim_{\lambda \searrow 0} (\exp(tA_\lambda))x$, with

$$A_\lambda x := \frac{T(\lambda)x - x}{\lambda},$$

where the above convergence is uniform on every compact subset of $\mathbb{R}^+$; and

- (vi) if $\lambda \in \mathbb{C}$ with $\Re \lambda > \zeta$, then the integral

$$R(\lambda, A)x := (\lambda I - A)^{-1}x = \int_0^\infty e^{-\zeta t} T(t)x dt,$$

defines a bounded linear operator $R(\lambda, A)$ on $\mathcal{B}$ whose range is $D(A)$ and

$$(\lambda I - A) R(\lambda, A) = R(\lambda, A)(\lambda I - A) = I.$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

Remark 2.9. In (i) above if $\zeta = 0$, then the corresponding semigroup is uniformly bounded. Moreover, if $C = 1$, then $(T(t))_{t \in \mathbb{R}^+}$ is said to be a c_0 -semigroup of contractions.

Theorem 2.10. (Hille–Yosida) *Let $A : D(A) \rightarrow \mathcal{B}$ be an unbounded linear operator in a Banach space $\mathbb{H}$. Then A is the infinitesimal generator of a c_0 -semigroup of contractions $(T(t))_{t \in \mathbb{R}^+}$ if and only if:*

- (i) *A is a densely defined closed operator; and*
- (ii) *the resolvent $\rho(A)$ of A contains $\mathbb{R}^+$ and*

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0. \quad (2.19)$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

Definition 2.18. Let $\mathcal{B}$ be a Banach space. The family of bounded operators $(T(t))_{t \in \mathbb{R}} : \mathcal{B} \rightarrow \mathcal{B}$ is said to be a c_0 -group if the following statements hold true:

- (i) $T(0) = I$,
- (ii) $T(t+s) = T(t)T(s)$ for every $s, t \in \mathbb{R}$,
- (iii) $\lim_{t \rightarrow 0} \|T(t)x - x\| = 0$ for $x \in \mathcal{B}$.

Remark 2.10. As for semigroups of bounded linear operators, for a given c_0 -group $(T(t))_{t \in \mathbb{R}}$ one can associate with it an infinitesimal generator A defined as in (3.3) and (3.4).

We have

Theorem 2.11. *Let $A : D(A) \rightarrow \mathcal{B}$ be a linear operator on $\mathcal{B}$. Then A is the infinitesimal generator of a c_0 -group of bounded linear operators $(T(t))_{t \in \mathbb{R}}$ satisfying $\|T(t)\| \leq C e^{\zeta|t|}$ if and only if:*

- (i) *A is a densely defined closed operator; and*
- (ii) *every $\lambda \in \mathbb{R}$ such that $|\lambda| \geq \zeta$ is in $\rho(A)$ and that for such a λ , the following holds:*

$$\|(\lambda I - A)^{-n}\| \leq \frac{C}{(|\lambda| - \zeta)^n}. \quad (2.20)$$

Proof. For the proof, we refer the reader to the book by Pazy [153].

2.4.3 Analytic Semigroups

Definition 2.19. A semigroup $T(t)$ on $\mathcal{B}$ is called analytic whenever $t \mapsto T(t)$ is analytic in $(0, \infty)$ with values in $B(\mathcal{B})$.

Let us mention that if $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ is a sectorial operator with constants $\zeta \in \mathbb{R}$, $\theta \in (\pi/2, \pi)$, and $M > 0$, then one can construct an analytic semigroup $T(t)$ associated to A by the means of the Dunford integral as follows (see Lunardi [129]):

$$T(t) = \frac{1}{2\pi i} \int_{\zeta + \Gamma_{r,s}} e^{t\lambda} R(\lambda, A) d\lambda, \quad \forall t > 0, \quad (2.21)$$

where $r > 0$, $\pi/2 < s < \theta$, and $\Gamma_{r,s}$ is the curve of the complex plane given by

$$\left\{ \lambda \in \mathbb{C} : |\arg \lambda| = s, |\lambda| \geq r \right\} \cup \left\{ \lambda \in \mathbb{C} : |\arg \lambda| \leq s, |\lambda| = r \right\},$$

that is oriented counterclockwise.

We have

Proposition 2.13. [129] *Let A be a sectorial operator and let $T(t)$ be the analytic semigroup given in (2.21). Then the following hold:*

(i) $T(t)u \in D(A^k)$ for all $t > 0$, $u \in \mathcal{B}$, $n \in \mathbb{N}$. If $D(A^n)$, then

$$A^n T(t)u = T(t)A^n u, \quad t \geq 0;$$

(ii) there exist constants $M_0, M_1, \dots$ such that

$$\|T(t)\| \leq M_0 e^{\zeta t}, \quad t > 0, \quad \text{and}$$

$$\|t^n (A - \zeta I)^n T(t)\| \leq M_n e^{\zeta t}, \quad t > 0; \quad \text{and}$$

(iii) the mapping $t \rightarrow T(t)$ belongs to $C^\infty((0, \infty), B(\mathbb{H}))$ and

$$\frac{d^n}{dt^n} T(t) = A^n T(t), \quad t > 0, \quad \forall n \in \mathbb{N}.$$

Conversely, the next proposition characterizes analytic semigroups in terms of sectorial operators.

Proposition 2.14. [129] *Let $(T(t))_{t \geq 0}$ be a family of bounded linear operators on $\mathcal{B}$ such that $t \mapsto T(t)$ is differentiable, and*

(i) $T(t+s) = T(t)T(s)$ for all $t, s > 0$;

(ii) there exist $\zeta \in \mathbb{R}$, $M_0, M_1 > 0$ such that

$$\|T(t)\| \leq M_0 e^{\zeta t}, \quad \|tT'(t)\| \leq M_1 e^{\zeta t}, \quad \forall t > 0;$$

(iii) either (a) there exists $t > 0$ such that $T(t)$ is one-to-one, or (b) for every $x \in \mathcal{B}$, $s - \lim_{t \rightarrow 0} T(t)x = x$.

Then $t \mapsto T(t)$ is analytic in $(0, \infty)$ with values in $B(\mathcal{B})$, and there exists a unique sectorial operator $A : D(A) \subset \mathbb{H} \rightarrow \mathcal{B}$ such that $(T(t))_{t \geq 0}$ is the semigroup associated with A .

Proof. For the proof, we refer the reader to the book by Lunardi [129].

2.5 Intermediate Spaces

2.5.1 Fractional Powers of Sectorial Operators

Let A be a sectorial linear operator on $\mathcal{B}$ whose associated analytic semigroup $T(t)$ satisfies the following: For all $t > 0$,

$$\|T(t)\| \leq M_0 e^{-\omega t}, \quad \|tAT(t)\| \leq M_1 e^{-\omega t},$$

where $M_0, M_1, \omega > 0$.

For each $\alpha > 0$ one defines the fractional powers of $-A$ implicitly by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt, \quad (2.22)$$

where Γ is defined by $\Gamma(x) := \int_0^{+\infty} e^{-xt} t^{x-1} dt$ for each $x > 0$.

Lemma 2.1. *For all $\alpha, \beta > 0$, the following hold:*

- (i) $(-A)^{-\alpha}(-A)^{-\beta} = A^{-(\alpha+\beta)}$.
- (ii) $\lim_{\alpha \rightarrow 0} (-A)^{-\alpha} = I$ in the strong operator topology.

Proof.

$$\begin{aligned} (-A)^{-\alpha}(-A)^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^{+\infty} t^{\alpha-1} s^{\beta-1} T(t)T(s) dt ds \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} t^{\alpha-1} \int_t^{+\infty} (u-t)^{\beta-1} T(u) du dt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^u t^{\alpha-1} (u-t)^{\beta-1} dt T(u) du \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 v^{\alpha-1} (1-v)^{\beta-1} dv \int_0^\infty u^{\alpha+\beta-1} T(u) du \\ &= \frac{1}{\Gamma(\alpha+\beta)} \int_0^{+\infty} u^{\alpha+\beta-1} T(u) du \\ &= (-A)^{-\alpha-\beta}. \end{aligned}$$

It remains to prove that $(-A)^{-\alpha} \rightarrow I$ as $\alpha \rightarrow 0$. Since $(-A)^{-\alpha}$ is one-to-one, if $v \in D(A)$, there exists $u \in \mathbb{H}$ such that $v = (-A)^{-\alpha}u$. Thus $(-A)^{-\alpha}v - v = (-A)^{-1-\alpha}u - (-A)^{-1}u \rightarrow 0$ as $\alpha \rightarrow 0$ by the fact that $(-A)^{-\alpha}$ is continuous with respect to uniform operator norm.

Remark 2.11. (i) Let $\alpha \in (0, 1)$. Using the fact that

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t) dt,$$

the formula (2.22) can be rewritten as

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^{+\infty} \lambda^{-\alpha} (\lambda I - A)^{-1} d\lambda. \quad (2.23)$$

(ii) The operator $(-A)^{-\alpha}$ is one-to-one, and hence has an inverse, which obviously is $(-A)^\alpha$. The operator $(-A)^\alpha$ is closed with domain $D((-A)^\alpha) = R((-A)^{-\alpha})$. The operators $(-A)^\alpha$ are called fractional powers of $-A$.

(iii) If $\alpha > \beta$, then $D((-A)^\alpha) \subset D((-A)^\beta)$.

(iv) $D((-A)^\alpha)$ is endowed with the norm $\|u\|_\alpha = \|(-A)^\alpha u\|$ for each $u \in D((-A)^\alpha)$.

(v) $(-A)^\alpha$ commutes with $T(t)$ on $D(-A)^\alpha$ with

$$\|T(t)\|_{B(D(-A)^\alpha)} \leq M_0 e^{-\omega t}, \quad t > 0.$$

Example 2.26. Let A be the operator given by $Au = -u''$ for all $u \in D(A)$ where the domain $D(A)$ is defined by

$$D(A) := \{u \in L^2([0, \pi]) : u'' \in L^2([0, \pi]), u(0) = u(\pi) = 0\}.$$

Clearly, the operator A has a discrete spectrum with eigenvalues of the form $n^2, n \in \mathbb{N}$, and corresponding normalized eigenfunctions given by

$$z_n(\xi) := \sqrt{\frac{2}{\pi}} \sin(n\xi).$$

In addition to the above, the following properties hold:

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis for $L^2[0, \pi]$.

(b) The operator $-A$ is the infinitesimal generator of an analytic semigroup $R(t)$ which is compact for $t > 0$. The semigroup $R(t)$ is defined for $u \in L^2[0, \pi]$ by

$$R(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n.$$

(c) The operator A can be rewritten as

$$Au = \sum_{n=1}^{\infty} n^2 \langle u, z_n \rangle z_n$$

for every $u \in D(A)$.

Moreover, it is possible to define fractional powers of A . In particular,

(d) For $u \in L^2[0, \pi]$ and $\alpha \in (0, 1)$,

$$A^{-\alpha}u = \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}} \langle u, z_n \rangle z_n.$$

(e) The operator $A^\alpha : D(A^\alpha) \subseteq L^2[0, \pi] \mapsto L^2[0, \pi]$ given by

$$A^\alpha u = \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n, \quad \forall u \in D(A^\alpha),$$

where $D(A^\alpha) = \left\{ u \in L^2[0, \pi] : \sum_{n=1}^{\infty} n^{2\alpha} \langle u, z_n \rangle z_n \in L^2[0, \pi] \right\}$.

Clearly, for all $t \geq 0$ and $0 \neq u \in L^2[0, \pi]$,

$$\begin{aligned} |R(t)u| &= \left| \sum_{n=1}^{\infty} e^{-n^2 t} \langle u, z_n \rangle z_n \right| \\ &\leq \sum_{n=1}^{\infty} e^{-t} |\langle u, z_n \rangle z_n| \\ &= e^{-t} \sum_{n=1}^{\infty} |\langle u, z_n \rangle z_n| \\ &\leq e^{-t} |u| \end{aligned}$$

and hence $\|R(t)\|_{B(L^2[0, \pi])} \leq 1$ for all $t \geq 0$.

2.5.2 The Spaces $D_A(\alpha, p)$ and $D_A(\alpha)$

Let A be a sectorial linear operator on $\mathcal{B}$ whose associated analytic semigroup $T(t)$ satisfies the following: For all $t > 0$,

$$\|T(t)\| \leq M_0 e^{-\omega t}, \quad \|tAT(t)\| \leq M_1 e^{-\omega t},$$

where $M_0, M_1, \omega > 0$.

Definition 2.20. Let $\alpha \in (0, 1)$. A Banach space $(\mathcal{B}_\alpha, \|\cdot\|_\alpha)$ is called an intermediate space between $\mathcal{B}$ and $D(A)$, or a space of class $\mathcal{J}_\alpha$, if $D(A) \subset \mathcal{B}_\alpha \subset \mathcal{B}$ and there is a constant $C > 0$ such that

$$\|u\|_\alpha \leq C \|u\|^{1-\alpha} \|u\|_A^\alpha, \quad u \in D(A). \quad (2.24)$$

Concrete examples of $\mathcal{B}_\alpha$ include $D((-A)^\alpha)$ for $\alpha \in (0, 1)$, the domains of the fractional powers of $-A$, the real interpolation spaces $D_A(\alpha, \infty)$, $\alpha \in (0, 1)$, defined as follows:

Definition 2.21. Let $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ be a sectorial operator and let $\alpha \in (0, 1)$. Define

$$D_A(\alpha, \infty) := \left\{ u \in \mathcal{B} : [u]_\alpha = \sup_{0 \leq t \leq 1} \|t^{1-\alpha} AT(t)u\| < \infty \right\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha, \infty)} = \|u\| + [u]_\alpha.$$

One should point out that $D_A(\alpha, \infty)$ is characterized by the behavior of the quantity $t \mapsto \|t^{1-\alpha} AT(t)u\|$ near $t = 0$. Moreover, all the spaces $D_A(\alpha, \infty)$ are subspaces of $\overline{D(A)}$. Namely, the following embeddings hold with equivalent norms:

$$D(A) \subset D_A(\beta, \infty) \subset D_A(\alpha, \infty) \subset \overline{D(A)}$$

for all $0 < \alpha < \beta < 1$.

If $\alpha \in (0, 1)$, it is not very hard to see that $D_A(\alpha, \infty)$ can be characterized as being the subspace of all $u \in \mathcal{B}$ such that

$$[[u]]_\alpha = \sup_{t \in (0, 1]} t^{-\alpha} \|T(t)u - u\| < \infty.$$

Furthermore, the norm defined by $u \mapsto \|u\| + [[u]]_\alpha$ is equivalent to the natural norm of $D_A(\alpha, \infty)$.

More generally, we define $D_A(\alpha, p)$ for $\alpha \in (0, 1)$ and $1 \leq p \leq \infty$ as follows:

Definition 2.22. Let $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ be a sectorial operator. Define the classes of intermediate spaces $D_A(\alpha, p)$ and $D_A(\alpha)$ between $\mathcal{B}$ and $D(A)$ (for $\alpha \in (0, 1)$ and $1 \leq p \leq \infty$) by

$$D_A(\alpha, p) := \left\{ u \in \mathcal{B} : t \mapsto v(t) = \left\| t^{1-\alpha-1/p} AT(t) \right\| \in L^p(0, 1) \right\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha, p)} = \|u\| + [u]_{D(\alpha, p)} = \|u\| + \|v\|_{L^p(0, 1)};$$

and

$$D_A(\alpha) = \left\{ u \in D(\alpha, \infty) : \lim_{t \rightarrow 0} t^{1-\alpha} AT(t)u = 0 \right\}.$$

Proposition 2.15. For $\alpha \in (0, 1)$ and $1 \leq p \leq \infty$ and for $(\alpha, p) = (1, \infty)$, then

$$D_A(\alpha, p) = (\mathcal{B}, D(A))_{\alpha, p}$$

with equivalent norms. Moreover, for $0 < \alpha < 1$, then

$$D_A(\alpha) = (\mathcal{B}, D(A))_\alpha.$$

Proof. The proof of Proposition 2.15 is too technical and so we refer the reader to Lunardi [129].

Proposition 2.16. [129] For $\alpha \in (0, 1)$, then

$$D_A(\alpha, 1) \subset D((-A)^\alpha) \subset (\mathcal{B}, D(A))_{\alpha, p}.$$

Proof. First of all, note that $D((-A)^\alpha)$ belongs to the class J_α . Notice that for each $u \in D(A)$, $(-A)^\alpha u = (-A)^{-(1-\alpha)}(-Au)$ and hence for each $\lambda > 0$,

$$\begin{aligned} \|(-A)^\alpha u\| &= \frac{1}{\Gamma(1-\alpha)} \left\| \left(\int_0^\lambda + \int_\lambda^\infty \right) t^{-\alpha} A e^{tA} u dt \right\| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left(\frac{M_0}{1-\alpha} \|Au\| \lambda^{1-\alpha} + \frac{M_1}{\alpha} \|u\| \lambda^{-\alpha} \right). \end{aligned}$$

Letting $\lambda = \frac{\|u\|}{\|Au\|}$ it follows that

$$\|(-A)^\alpha u\| \leq c \|Au\|^\alpha \|u\|^{1-\alpha}.$$

It remains to prove that $D((-A)^\alpha)$ is continuously embedded in $D_A(\alpha, \infty)$. For that let $u \in D((-A)^\alpha)$ and let $v = (-A)^\alpha u$. So for $0 < \xi \leq 1$, we have

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} u\| &= \|\xi^{1-\alpha} A e^{\xi A} (-A)^\alpha v\| \\ &\leq \frac{\xi^{1-\alpha}}{\Gamma(\alpha)} \left\| \int_0^\infty t^{\alpha-1} A e^{(\xi+t)A} v dt \right\| \\ &\leq \frac{M_1 \xi^{1-\alpha}}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{\xi+t} dt \|v\| \\ &\leq \frac{M_1}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{1+s} ds \|(-A)^\alpha u\| \end{aligned}$$

and hence $u \in D_A(\alpha, \infty)$.

Using $D_A(\alpha, \infty)$ we can define $D_A(k + \alpha, \infty)$ as follows:

Definition 2.23. Let $A : D(A) \subset \mathcal{B} \rightarrow \mathcal{B}$ be a sectorial operator. For any $k \in \mathbb{N}$ and any $\alpha \in (0, 1)$, we define

$$D_A(\alpha + k, \infty) := \left\{ u \in D(A^k) : A^k u \in D_A(\alpha, \infty) \right\}$$

equipped with the norm given by

$$\|u\|_{D(\alpha+k, \infty)} = \|u\|_{D(A^k)} + \left[A^k u \right]_\alpha.$$

Let A_α denote the part of A in $D_A(\alpha, \infty)$. It can be shown that $A_\alpha : D_A(1 + \alpha, \infty) \mapsto D_A(\alpha, \infty)$ with $A_\alpha u = Au$ is sectorial. Moreover, $\rho(A) \subset \rho(A_\alpha)$. Furthermore, the restriction of $R(\lambda, A)$ to $D_A(\alpha, \infty)$ is exactly $R(\lambda, A_\alpha)$ and

$$\|R(\lambda, A_\alpha)\|_{B(D_A(\alpha, \infty))} \leq \|R(\lambda, A)\|$$

for all $\lambda \in \rho(A)$.

Example 2.27. Let A be a realization of the Laplacian in $\mathcal{B} = BC(\mathbb{R}^N, \mathbb{C})$. Then for all $\alpha \in (0, 1)$ and $\alpha \neq 1/2$, then

$$D_A(\alpha, \infty) = C_b^{2\alpha}(\mathbb{R}^N) \quad (2.25)$$

and

$$D_A(1 + \alpha, \infty) = C_b^{2+2\alpha}(\mathbb{R}^N) \quad (2.26)$$

with equivalent norms.

For more on those spaces and related issues we refer the reader to the landmark book by Lunardi [129].

2.5.3 Hyperbolic Semigroups

Definition 2.24. Let A be a sectorial operator on $\mathcal{B}$ and let $(T(t))_{t \geq 0}$ be the analytic semigroup associated to it. The semigroup $(T(t))_{t \geq 0}$ is said to be hyperbolic if there exist a projection P and constants $M, \delta > 0$ such that each $T(t)$ commutes with P , $N(P)$ is invariant with respect to $T(t)$, $T(t) : R(Q) \mapsto R(Q)$ is invertible, and

$$\|T(t)Px\| \leq Me^{-\delta t} \|x\| \quad \text{for } t \geq 0, \quad (2.27)$$

$$\|T(t)Qx\| \leq Me^{\delta t} \|x\| \quad \text{for } t \leq 0, \quad (2.28)$$

where $Q := I - P$ and $T(t) := (T(-t))^{-1}$ for $t < 0$.

Recall that an analytic semigroup $(T(t))_{t \geq 0}$ is hyperbolic if and only if (see [70])

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \quad (2.29)$$

For the hyperbolic analytic semigroup $T(t)$, we can easily check that estimations similar to (2.27) and (2.28) hold also with norms $\|\cdot\|_\alpha$ (see Definition 2.20). In fact, as the part of A in $R(Q)$ is bounded, it follows from the inequality (2.28) that

$$\|AT(t)Qx\| \leq c'e^{\delta t} \|x\| \quad \text{for } t \leq 0.$$

In view of the above, there exists a constant $c(\alpha) > 0$ such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha)e^{\delta t} \|x\| \quad \text{for } t \leq 0. \quad (2.30)$$

Similarly,

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathcal{B}, \mathcal{B}_\alpha)} \|T(t-1)Px\| \quad \text{for } t \geq 1,$$

and then from (2.27), we obtain

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t} \|x\|, \quad t \geq 1,$$

where M' depends on α .

Clearly,

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha} \|x\|,$$

and hence there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha) t^{-\alpha} e^{-\gamma t} \|x\| \quad \text{for } t > 0. \quad (2.31)$$

We need the next lemma, which will be very crucial for our computations.

Lemma 2.2. (*Diagana [52]*) *Let $0 < \alpha, \beta < 1$. Then*

$$\|AT(t)Qx\|_\alpha \leq c e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0, \quad (2.32)$$

$$\|AT(t)Px\|_\alpha \leq c t^{\beta-\alpha-1} e^{-\gamma t} \|x\|_\beta \quad \text{for } t > 0. \quad (2.33)$$

Proof. As for (2.30), the fact that the part of A in $R(Q)$ is bounded yields

$$\|AT(t)Qx\| \leq c e^{\delta t} \|x\|_\beta, \quad \|A^2 T(t)Qx\| \leq c e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0, \quad (2.34)$$

since $\mathcal{B}_\beta \hookrightarrow \mathcal{B}$. Hence, from (2.24) there is a constant $c(\alpha) > 0$ such that

$$\|AT(t)Qx\|_\alpha \leq c(\alpha) e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0. \quad (2.35)$$

Furthermore,

$$\|AT(t)Px\|_\alpha \leq \|AT(1)\|_{B(\mathcal{B}, \mathcal{B}_\alpha)} \|T(t-1)Px\| \quad (2.36)$$

$$\leq c e^{-\delta t} \|x\|_\beta \quad \text{for } t \geq 1. \quad (2.37)$$

Now for $t \in (0, 1]$, by Proposition 2.13 (ii) and (2.24), one has

$$\|AT(t)Px\|_\alpha \leq c t^{-\alpha-1} \|x\|,$$

and

$$\|AT(t)Px\|_\alpha \leq c t^{-\alpha} \|Ax\|,$$

for each $x \in D(A)$. Thus, by the Reiteration Theorem (see [129]), it follows that

$$\|AT(t)Px\|_\alpha \leq c t^{\beta-\alpha-1} \|x\|_\beta$$

for every $x \in \mathcal{B}_\beta$ and $0 < \beta < 1$, and hence, there exist constants $M(\alpha) > 0$ and $\gamma > 0$ such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \quad \text{for } t > 0.$$

2.6 Evolution Families and Their Properties

2.6.1 Evolution Families

Let $\{A(t) : t \in \mathbb{R}\}$ be a family of closed linear operators on $\mathcal{B}$ with domain $D(A(t))$ (possibly not densely defined), which depends on $t \in \mathbb{R}$.

Definition 2.25. A family of linear operators

$$\{U(t, s) : t, s \in \mathbb{R} \text{ such that } t \geq s\}$$

on $\mathcal{B}$ associated with $A(t)$ such that $U(t, s)\mathcal{B} \subset D(A(t))$ for all $t, s \in \mathbb{R}$ with $t \geq s$, and

- (a) $U(t, s)U(s, r) = U(t, r)$ for $t, s, r \in \mathbb{R}$ such that $t \geq s \geq r$;
 - (b) $U(t, t) = I$ for $t \in \mathbb{R}$;
 - (c) $(t, s) \mapsto U(t, s) \in B(\mathcal{B})$ is continuous for $t > s$; and
 - (d) $U(\cdot, s) \in C^1((s, \infty), B(\mathcal{B}))$, $\frac{\partial U}{\partial t}(t, s) = A(t)U(t, s)$
- is called an evolution family.

For a given family of closed linear operators $\{A(t) : t \in \mathbb{R}\}$ on $\mathcal{B}$, the existence of an evolution family associated with it is not always guaranteed. However, if the family $A(t)$ satisfies the so-called Acquistapace–Terreni conditions, that is:

(AT) There exists $\lambda_0 \geq 0$ such that the linear operators $\{A(t) : t \in \mathbb{R}\}$ satisfy

$$\Sigma_\phi \cup \{0\} \subseteq \rho(A(t) - \lambda_0) \ni \lambda, \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \quad (2.38)$$

and

$$\|(A(t) - \lambda_0)R(\lambda_0, A(t) - \lambda_0) [R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\mu |\lambda|^{-\nu} \quad (2.39)$$

for $t, s \in \mathbb{R}$, $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \phi\}$, and the constants $\phi \in (\frac{\pi}{2}, \pi)$, $L, K \geq 0$, and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$, then the family of linear operators $A(t)$ has an evolution family associated to it. Moreover, the following hold:

(e)

$$\|A(t)^k U(t, s)\| \leq C(t - s)^{-k} \quad (2.40)$$

for $0 < t - s \leq 1$, $k = 0, 1$; and

(f) $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

Remark 2.12. (i) In the particular case of a constant domain $D(A(t))$, one can replace assumption (2.39) (see for instance [153]) with the following:

(AT)' There exist constants L and $0 < \mu \leq 1$ such that

$$\|(A(t) - A(s))R(\lambda_0, A(r))\| \leq L|t - s|^\mu, \quad s, t, r \in \mathbb{R}. \quad (2.41)$$

(ii) The conditions (AT) were introduced in the literature by Acquistapace–Terreni in [2, 3] for $\lambda_0 = 0$.

Definition 2.26. An evolution family $\{U(t, s) : t \geq s \text{ with } t, s \in \mathbb{R}\} \subset B(\mathcal{B})$ is said to have an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

- (i) $U(t, s)P(s) = P(t)U(t, s)$;
- (ii) the restriction $U_Q(t, s) : Q(s)\mathcal{B} \rightarrow Q(t)\mathcal{B}$ of $U(t, s)$ is invertible (we then set $U_Q(s, t) := U_Q(t, s)^{-1}$); and
- (iii) $\|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)}$ and $\|U_Q(s, t)Q(t)\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t, s \in \mathbb{R}$.

Here and throughout the rest of the book, for any projection P we set $Q = I - P$.

We recall that the following conditions are sufficient for an evolution family $\{U(t, s) : t \geq s \text{ with } t, s \in \mathbb{R}\}$ associated with $A(\cdot)$ to have exponential dichotomy:

(E₁) Let $(A(t), D(t))_{t \in \mathbb{R}}$ be generators of analytic semigroups on $\mathcal{B}$ of the same type. Suppose that $D(A(t)) \equiv D(A(0))$, $A(t)$ is invertible,

$$\sup_{t, s \in \mathbb{R}} \|A(t)A(s)^{-1}\|$$

is finite, and

$$\|A(t)A(s)^{-1} - I\| \leq L_0|t - s|^\mu$$

for $t, s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 < \mu \leq 1$.

(E₂) The semigroups $(e^{\tau A(t)})_{\tau \geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let

$$\|A(t)e^{\tau A(t)}P_t\| \leq \psi(\tau)$$

and

$$\|A(t)e^{\tau A_Q(t)}Q_t\| \leq \psi(-\tau)$$

for $\tau > 0$ and a function ψ such that $\mathbb{R} \ni s \mapsto \varphi(s) := |s|^\mu \psi(s)$ is integrable with $L_0\|\varphi\|_{L^1(\mathbb{R})} < 1$.

2.6.2 Estimates for $U(t, s)$

We need to prove some estimates related to $U(t, s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to [70], and [129] for proofs and further information on these spaces.

Let A be a sectorial operator on $\mathcal{B}$ and let $\alpha \in (0, 1)$.

Define the real interpolation space

$$\mathcal{B}_\alpha^A := \{x \in \mathcal{B} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha (A - \zeta) R(r, A - \zeta)x\| < \infty\},$$

which, by the way, is a Banach space when endowed with the norm $\|\cdot\|_\alpha^A$. For convenience we further write

$$\mathcal{B}_0^A := \mathcal{B}, \quad \|x\|_0^A := \|x\|, \quad \mathcal{B}_1^A := D(A), \quad \text{and} \quad \|x\|_1^A := \|(\zeta - A)x\|.$$

We also need the closed subspace $\hat{\mathcal{B}}^A := \overline{D(A)}$ of $\mathcal{B}$. In particular, we will frequently be using the following continuous embedding:

$$D(A) \hookrightarrow \mathcal{B}_\beta^A \hookrightarrow D((\zeta - A)^\alpha) \hookrightarrow \mathcal{B}_\alpha^A \hookrightarrow \hat{\mathcal{B}}^A \subset \mathcal{B}, \quad (2.42)$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathcal{B}_\alpha^A$ and $\mathcal{B}$. However, we have the following continuous injection:

$$\mathcal{B}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \quad (2.43)$$

for $0 < \alpha < \beta < 1$.

Given the operators $A(t)$ for $t \in \mathbb{R}$, satisfying (AT), we set

$$\mathcal{B}_\alpha^t := \mathcal{B}_\alpha^{A(t)}, \quad \hat{\mathcal{B}}^t := \hat{\mathcal{B}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in (2.42) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ and hence there is a constant $l(\alpha)$ such that

$$\|y\|_\alpha^t \leq l(\alpha) \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)). \quad (2.44)$$

We have the following fundamental estimates for the evolution family $U(t, s)$:

Proposition 2.17. [14, Baroun, Boulite, Diagana, and Maniar] For $x \in \mathcal{B}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:

(i) There is a constant $c(\alpha)$, such that

$$\|U(t, s)P(s)x\|_\alpha^t \leq c(\alpha) e^{-\frac{\delta}{2}(t-s)} (t-s)^{-\alpha} \|x\|. \quad (2.45)$$

(ii) There is a constant $m(\alpha)$, such that

$$\|\tilde{U}_Q(s, t)Q(t)x\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|. \quad (2.46)$$

Proof. (i) Using (2.44) we obtain

$$\begin{aligned} \|U(t, s)P(s)x\|_\alpha^t &\leq c(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, s)P(s)x\|^\alpha \\ &\leq c(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, t-1)U(t-1, s)P(s)x\|^\alpha \\ &\leq l(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, t-1)\|^\alpha\|U(t-1, s)P(s)x\|^\alpha \\ &\leq l(\alpha)N' e^{-\delta(t-s)(1-\alpha)} e^{-\delta(t-s-1)\alpha}\|x\| \\ &\leq c'(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)}(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)}\|x\| \end{aligned}$$

for $t-s \geq 1$ and $x \in \mathcal{B}$.

Since $(t-s)^\alpha e^{-\frac{\delta}{2}(t-s)} \rightarrow 0$ as $t \rightarrow \infty$ it easily follows that

$$\|U(t, s)P(s)x\|_\alpha^t \leq c_1(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If $0 < t-s \leq 1$, we have

$$\begin{aligned} \|U(t, s)P(s)x\|_\alpha^t &\leq l(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, s)P(s)x\|^\alpha \\ &\leq l(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, \frac{t+s}{2})U(\frac{t+s}{2}, s)P(s)x\|^\alpha \\ &\leq l(\alpha)\|U(t, s)P(s)x\|^{1-\alpha}\|A(t)U(t, \frac{t+s}{2})\|^\alpha\|U(\frac{t+s}{2}, s)P(s)x\|^\alpha \\ &\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha} e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\ &\leq l(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^\alpha(t-s)^{-\alpha} e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\ &\leq c_2(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|, \end{aligned}$$

and hence

$$\|U(t, s)P(s)x\|_\alpha^t \leq c(\alpha)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)}\|x\| \quad \text{for } t > s.$$

(ii)

$$\begin{aligned} \|\tilde{U}_Q(s, t)Q(t)x\|_\alpha^s &\leq l(\alpha)\|\tilde{U}_Q(s, t)Q(t)x\|^{1-\alpha}\|A(s)\tilde{U}_Q(s, t)Q(t)x\|^\alpha \\ &\leq l(\alpha)\|\tilde{U}_Q(s, t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\tilde{U}_Q(s, t)Q(t)x\|^\alpha \\ &\leq l(\alpha)\|\tilde{U}_Q(s, t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^\alpha\|\tilde{U}_Q(s, t)Q(t)x\|^\alpha \\ &\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^\alpha e^{-\delta(t-s)\alpha}\|x\| \\ &\leq m(\alpha)e^{-\delta(t-s)}\|x\|. \end{aligned}$$

In the last inequality we made use of the fact that $\|A(s)Q(s)\| \leq c$ for some constant $c \geq 0$, see e.g., [162, Proposition 3.18].

Remark 2.13. It should be mentioned that if $U(t, s)$ is exponentially stable, then $P(t) = I$ and $Q(t) = I - P(t) = 0$ for all $t \in \mathbb{R}$. In that case, Eq. (2.45) still holds and can be rewritten as follows: for all $x \in \mathcal{B}$,

$$\|U(t, s)x\|_\alpha^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \quad (2.47)$$

We will need the following technical lemma in Chapters 5 and 6:

Lemma 2.3. [55, Diagana] *Let $x \in \mathcal{B}$ and let $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$. Then for all $t > s$, there are constants $r(\alpha, \beta), d(\beta) > 0$ such that*

$$\|A(t)U(t, s)P(s)x\|_\alpha \leq r(\alpha, \beta)e^{-\frac{\delta}{4}(t-s)}(t-s)^{-\beta}\|x\|. \quad (2.48)$$

and

$$\|A(t)\tilde{U}_Q(t, s)Q(s)x\|_\beta \leq d(\beta)e^{-\delta(s-t)}\|x\|, \quad t \leq s. \quad (2.49)$$

Proof. Let $x \in \mathcal{B}$. First of all, note that $\|A(t)U(t, s)\|_{B(\mathcal{B}, \mathcal{B}_\beta)} \leq K(t-s)^{-(1-\beta)}$ for all t, s such that $0 < t-s \leq 1$ and $\beta \in [0, 1]$.

Suppose $t-s \geq 1$ and let $x \in \mathcal{B} \hookleftarrow \mathcal{B}_\alpha \hookleftarrow \mathcal{B}_\beta$.

$$\begin{aligned} \|A(t)U(t, s)P(s)x\|_\alpha &= \|A(t)U(t, t-1)U(t-1, s)P(s)x\|_\alpha \\ &\leq \|A(t)U(t, t-1)\|_{B(\mathcal{B}, \mathcal{B}_\alpha)}\|U(t-1, s)P(s)x\| \\ &\leq MKe^\delta e^{-\delta(t-s)}\|x\| \\ &= K_1 e^{-\delta(t-s)}\|x\| \\ &= K_1 e^{-\frac{3\delta}{4}(t-s)}(t-s)^\beta(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|. \end{aligned}$$

Now since $e^{-\frac{3\delta}{4}(t-s)}(t-s)^\beta \rightarrow 0$ as $t \rightarrow \infty$ it follows that there exists $c_4(\beta) > 0$ such that

$$\|A(t)U(t, s)P(s)x\|_\beta \leq c_4(\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|.$$

Now, let $0 < t-s \leq 1$. Using Eq. (2.45) and the fact that $2\beta > \alpha + 1$, we obtain

$$\begin{aligned} \|A(t)U(t, s)P(s)x\|_\alpha &= \|A(t)U(t, \frac{t+s}{2})U(\frac{t+s}{2}, s)P(s)x\|_\alpha \\ &\leq \|A(t)U(t, \frac{t+s}{2})\|_{B(\mathcal{B}, \mathcal{B}_\alpha)}\|U(\frac{t+s}{2}, s)P(s)x\| \\ &\leq c'k_1\|A(t)U(t, \frac{t+s}{2})\|_{B(\mathcal{B}, \mathcal{B}_\alpha)}\|U(\frac{t+s}{2}, s)P(s)x\|_\alpha \\ &\leq c'k_1K\left(\frac{t-s}{2}\right)^{\beta-1}c(\alpha)\left(\frac{t-s}{2}\right)^{-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\| \\ &= c_5(\alpha, \beta)(t-s)^{\beta-1-\alpha}e^{-\frac{\delta}{4}(t-s)}\|x\| \\ &\leq c_5(\alpha, \beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|. \end{aligned}$$

In summary, there exists $r(\alpha, \beta) > 0$ such that

$$\|A(t)U(t, s)P(s)x\|_\alpha \leq r(\alpha, \beta)(t-s)^{-\beta} e^{-\frac{\delta}{4}(t-s)} \|x\|$$

for all $t, s \in \mathbb{R}$ with $t \geq s$.

Let $x \in \mathcal{B}$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned} \|A(t)\tilde{U}_Q(t, s)Q(s)x\|_\beta &= \|A(t)A(s)^{-1}A(s)\tilde{U}_Q(t, s)Q(s)x\|_\beta \\ &\leq \|A(t)A(s)^{-1}\|_{B(\mathcal{B}, \mathcal{B}_\beta)} \|A(s)\tilde{U}_Q(t, s)Q(s)x\|_\beta \\ &\leq c_1 \|A(t)A(s)^{-1}\|_{B(\mathcal{B}, \mathcal{B}_\beta)} \|A(s)\tilde{U}_Q(t, s)Q(s)x\|_\beta \\ &\leq c_1 c_0 \|A(s)\tilde{U}_Q(t, s)Q(s)x\|_\beta \\ &\leq \tilde{c} \|\tilde{U}_Q(t, s)Q(s)x\|_\beta \\ &\leq \tilde{c} m(\beta) e^{-\delta(s-t)} \|x\| \\ &= d(\beta) e^{-\delta(s-t)} \|x\| \end{aligned}$$

for $t \leq s$ by using Eq. (2.46).

We have also the following estimates due to Diagana [62]. Here, we still assume that the Acquistapace–Terreni conditions hold and that the evolution family $U(t, s)$ associated with $A(\cdot)$ has exponential dichotomy.

Lemma 2.4. [62, Diagana] Suppose $0 \in \rho(A(t))$ for all $t \in \mathbb{R}$ such that

$$\sup_{t, s \in \mathbb{R}} \|A(s)A^{-1}(t)\|_{B(\mathcal{B}, \mathcal{B}_\alpha)} < c_0; \quad (2.50)$$

and that there exist $0 < \alpha < \beta < 1$ with $2\beta > \alpha + 1$ such that

$$\mathcal{B}_\alpha^t = \mathcal{B}_\alpha \text{ and } \mathcal{B}_\beta^t = \mathcal{B}_\beta$$

for all $t \in \mathbb{R}$, with equivalent norms. Then, there exist two constants $m(\alpha, \beta), n(\alpha, \beta) > 0$ such that

$$\|A(s)\tilde{U}_Q(t, s)Q(s)x\|_\alpha \leq m(\alpha, \beta) e^{-\delta(s-t)} \|x\|_\beta \quad \text{for } t \leq s, \quad (2.51)$$

and

$$\|A(s)U(t, s)P(s)x\|_\alpha \leq n(\alpha, \beta)(t-s)^{-\alpha} e^{-\frac{\delta}{2}(t-s)} \|x\|_\beta \quad \text{for } t > s. \quad (2.52)$$

Proof. Let $x \in \mathcal{B}_\beta$. Since the restriction of $A(s)$ to $R(Q(s))$ is a bounded linear operator it follows that

$$\begin{aligned}
\|A(s)\tilde{U}_Q(t,s)Q(s)x\|_\alpha &\leq ck(\alpha)\|\tilde{U}_Q(t,s)Q(s)x\|_\beta \\
&\leq ck(\alpha)m(\beta)e^{-\delta(s-t)}\|x\| \\
&\leq m(\alpha,\beta)e^{-\delta(s-t)}\|x\|_\beta
\end{aligned}$$

for $t \leq s$ by using (2.46).

Similarly, for each $x \in \mathcal{B}_\beta$, using (2.50), we obtain

$$\begin{aligned}
\|A(s)U(t,s)P(s)x\|_\alpha &= \|A(s)A(t)^{-1}A(t)U(t,s)P(s)x\|_\alpha \\
&\leq \|A(s)A(t)^{-1}\|_{B(\mathcal{B},\mathcal{B}_\alpha)}\|A(t)U(t,s)P(s)x\|_\alpha \\
&\leq c_0\|A(t)U(t,s)P(s)x\|_\alpha \\
&\leq c_0r(\alpha,\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\| \\
&= n(\alpha,\beta)(t-s)^{-\beta}e^{-\frac{\delta}{4}(t-s)}\|x\|
\end{aligned}$$

for $t \geq s$.

2.7 Bibliographical Notes

For the classical theory of bounded linear operators, we follow, for the most part, essentially Conway [40], Diagana [51], Gohberg, Goldberg, and Kaashoek [78], Lax [115], Eidelman, Milman, and Tsolomitis [69], Naylar and Sell [146], Rudin [159], and Weidmann [176]. Our presentation related to unbounded linear operators, their spectral theory, and invariant and reducing subspaces for unbounded linear operators are taken from Diagana [51], Eidelman, Milman, and Tsolomitis [69], Locker [130], and Weidmann [176].

The part of this chapter devoted to semigroups was taken from Pazy [153]. However, the parts on sectorial operators, analytic semigroups, and intermediate spaces were taken from Lunardi [129]. The presentation on hyperbolic semigroups is due to Engel and Nagel [70].



<http://www.springer.com/978-1-4419-9475-2>

Almost Periodic Stochastic Processes

Bezandry, P.H.; Diagana, T.

2011, XV, 235 p., Hardcover

ISBN: 978-1-4419-9475-2