

# Preface

An intriguing and famous talk presented by Stanislaw M. Ulam in 1940 triggered the study of stability problems for various functional equations. In his talk, Ulam discussed a number of important unsolved mathematical problems. Among them, a question concerning the stability of homomorphisms seemed too abstract for anyone to reach any conclusion. In the following year, Donald H. Hyers was able to give a partial solution to Ulam's question that was the first significant breakthrough and step toward more solutions in this area. Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem. In particular, Themistocles M. Rassias succeeded in extending the result of Hyers's theorem by weakening the condition for the Cauchy difference. This remarkable result of Rassias led the concern of mathematicians toward the study of stability problems of functional equations.

Unfortunately, no books dealing with a comprehensive illustration of the fast developing field of nonlinear analysis had been published for the mathematicians interested in this field for more than a half century until D. H. Hyers, G. Isac and Th. M. Rassias published their book, *Stability of Functional Equations in Several Variables*, Birkhäuser, 1998.

This very book will complement the books of Hyers, Isac and Rassias and of Czerwik (*Functional Equations and Inequalities in Several Variables*, World Scientific, 2002) by presenting mainly the results applying to the Hyers–Ulam–Rassias stability. Many mathematicians have extensively investigated the subjects on the Hyers–Ulam–Rassias stability. This book covers and offers almost all classical results on the Hyers–Ulam–Rassias stability in an integrated and self-contained fashion.

- In Chapter 2, we discuss the Hyers–Ulam–Rassias stability problems and the related topics of the additive Cauchy equation. In Section 2.1, we explain the behaviors of additive functions. We, then, begin to discuss the Hyers–Ulam and the Hyers–Ulam–Rassias stability problems of additive functional equation in Sections 2.2 and 2.3. The stability on restricted domains and its applications are introduced in Section 2.4. We explain briefly the method of invariant means and the fixed point method in Sections 2.5 and 2.6. In Section 2.7, the composite functional congruences are surveyed. The stability of the Pexider equation will be proved in Section 2.8.

- The Hyers–Ulam–Rassias stability of some generalized additive functional equations is proved in Chapter 3. Moreover, we discuss the Hyers–Ulam stability problem in connection with a question of Th. M. Rassias and J. Tabor.
- Chapter 4 deals with the Hosszú’s functional equation. In Section 4.1, we prove that the Hosszú’s equation is stable in the sense of C. Borelli. The Hyers–Ulam stability problem is discussed in Section 4.2. In Section 4.3, we present that the generalized Hosszú’s equation is stable in the sense of Borelli. In the next section, we prove that the Hosszú’s equation is not stable on the unit interval. Moreover, the Hyers–Ulam stability of the Hosszú’s equation of Pexider type is proved in Section 4.5.
- We survey the stability problems of the homogeneous functional equation in Chapter 5. In Section 5.1, we prove the Hyers–Ulam–Rassias stability of the homogeneous functional equation between real Banach algebras. Section 5.2 deals with the superstability on restricted domains. The stability problem of the equation between vector spaces will be discussed in Section 5.3. Moreover, we present the Hyers–Ulam–Rassias stability of the homogeneous equation of Pexider type in Section 5.4.
- There are a number of functional equations including all the linear functions as their solutions. In Chapter 6, we introduce a few functional equations among them. We survey the superstability property of the system of functional equations  $f(x + y) = f(x) + f(y)$  and  $f(cx) = cf(x)$  in Section 6.1. Section 6.2 deals with the stability problem for the functional equation  $f(x + cy) = f(x) + cf(y)$ . In Section 6.3, we discuss stability problems of other systems which describe linear functions.
- Jensen’s functional equation is the most important equation among a number of variations of the additive Cauchy equation. The Hyers–Ulam–Rassias stability problems of Jensen’s equation are proved in Section 7.1, and the Hyers–Ulam stability on restricted domains is discussed in Section 7.2. In Section 7.3, we prove the stability of Jensen’s equation by using the fixed point method. The superstability and Ger type stability of the Lobačevskii functional equation will be surveyed in Section 7.4.
- Chapter 8 is dedicated to a survey on the stability problems for the quadratic functional equations. We prove the Hyers–Ulam–Rassias stability of the quadratic equation in Section 8.1. The stability problems on restricted domains are discussed in Section 8.2. Moreover, we prove the Hyers–Ulam–Rassias stability by using the fixed point method in Section 8.3. Section 8.4 deals with the Hyers–Ulam stability of another quadratic functional equation. We prove the stability of the quadratic equation of Pexider type in Section 8.5.
- In Chapter 9, we discuss the stability problems for the exponential functional equations. In Section 9.1, the superstability of the exponential Cauchy equation is proved. Section 9.2 deals with the stability of the exponential equation in the sense of R. Ger. Stability problems on restricted domains are discussed in Section 9.3. Another exponential functional equation  $f(xy) = f(x)^y$  is introduced in Section 9.4.

- Chapter 10 deals with the stability problems for the multiplicative functional equations. In Section 10.1, we discuss the superstability of the multiplicative Cauchy equation and a functional equation connected with the Reynolds operator. The results on  $\delta$ -multiplicative functionals on complex Banach algebras are presented in Section 10.2. We describe  $\delta$ -multiplicative functionals in connection with the AMNM algebras in Section 10.3. Another multiplicative functional equation  $f(x^y) = f(x)^y$  is discussed in Section 10.4. In Section 10.5, we prove that a new multiplicative functional equation  $f(x + y) = f(x)f(y)f(1/x + 1/y)$  is stable in the sense of Ger.
- In Chapter 11, we introduce a new functional equation  $f(x^y) = yf(x)$  with the logarithmic property. Moreover, the functional equation of Heuvers  $f(x + y) = f(x) + f(y) + f(1/x + 1/y)$  will be discussed.
- The addition and subtraction rules for trigonometric functions can be represented by using functional equations. Some of these equations are introduced and the stability problems are surveyed in Chapter 12. Sections 12.1 and 12.2 deal with the superstability phenomena of the cosine and the sine equations. In Section 12.3, some trigonometric functional equations with two unknown functions are discussed. In Section 12.4, we deal with the Hyers–Ulam stability of the Butler–Rassias functional equation following M. Th. Rassias’s solution.
- Chapter 13 deals with the Hyers–Ulam–Rassias stability of isometries. The historical background for Hyers–Ulam stability of isometries is introduced in Section 13.1. The Hyers–Ulam–Rassias stability of isometries on restricted domains is proved in Section 13.2. Section 13.3 is dedicated to the fixed point method for studying the stability problem of isometries. In Section 13.4, we discuss the Hyers–Ulam–Rassias stability of the Wigner equation on restricted domains.
- Section 14.1 deals with the superstability of the associativity equation. In Section 14.2, the Hyers–Ulam stability of a functional equation defining multiplicative derivations is proved for functions on  $(0, 1]$ . In Section 14.3, the Hyers–Ulam–Rassias stability of the gamma functional equation and a generalized beta functional equation is proved. The Hyers–Ulam stability of the Fibonacci functional equation will be proved in Section 14.4.

I would like to express my humble gratitude to Professor Themistocles M. Rassias for his valuable suggestions and comments. I am also very much indebted to my wife, Min-Soon Lee, as she has always stood by me and helped me type the manuscript in  $\LaTeX$  system.

This research was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (No. 2010-0007143).

Finally, I would also like to acknowledge the fine cooperation and assistance that Ms. Elizabeth Loew of Springer Publishing has provided in the publication of this book.

Hyers-Ulam-Rassias Stability of Functional Equations in  
Nonlinear Analysis

Jung, S.-M.

2011, XIV, 362 p., Hardcover

ISBN: 978-1-4419-9636-7