Chapter 2
Additive Cauchy Equation

The functional equation $f(x + y) = f(x) + f(y)$ is the most famous among the functional equations. Already in 1821, A. L. Cauchy solved it in the class of continuous real-valued functions. It is often called the *additive Cauchy functional equation* in honor of A. L. Cauchy. The properties of this functional equation are frequently applied to the development of theories of other functional equations. Moreover, the properties of the additive Cauchy equation are powerful tools in almost every field of natural and social sciences. In Section 2.1, the behaviors of solutions of the additive functional equation are described. The Hyers–Ulam stability problem of this equation is discussed in Section 2.2, and theorems concerning the Hyers–Ulam–Rassias stability of the equation are proved in Section 2.3. The stability on a restricted domain and its applications are introduced in Section 2.4. The method of invariant means and the fixed point method will be explained briefly in Sections 2.5 and 2.6. In Section 2.7, the composite functional congruences will be surveyed. The stability results for the Pexider equation will be treated in the last section.

2.1 Behavior of Additive Functions

The history of the study of functional equations is long. Already in 1821, A. L. Cauchy [60] noted that every continuous solution of the *additive Cauchy functional equation*

$$f(x + y) = f(x) + f(y), \tag{2.1}$$

for all $x, y \in \mathbb{R}$, is linear. Every solution of the additive Cauchy equation (2.1) is called an *additive function*.

First, we will solve the additive Cauchy equation (2.1) under some weaker conditions than that of A. L. Cauchy (ref. [59]).

Theorem 2.1. *If an additive function $f : \mathbb{R} \to \mathbb{R}$ satisfies one of the following conditions, then there exists a real constant c such that $f(x) = cx$ for all $x \in \mathbb{R}$:*

(i) f is continuous at a point;
(ii) f is monotonic on an interval of positive length;

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(iii) f *is bounded from above or below on an interval of positive length;*
(iv) f *is integrable;*
(v) f *is Lebesgue measurable;*
(vi) f *is a Borel function.*

Proof. We prove the theorem under the condition (i) only. By induction on n we first prove

$$f(nx) = nf(x) \tag{a}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Let x in $\mathbb{R}$ be arbitrary. Obviously, (a) is true for $n = 1$. Assume now that (a) is true for some n. Then, by (a), we get

$$f((n+1)x) = f(nx) + f(x) = nf(x) + f(x) = (n+1)f(x).$$

If we substitute x/n for x in (a), we obtain

$$f(x/n) = (1/n)f(x). \tag{b}$$

Following from (a) and (b) yields

$$f(qx) = qf(x) \tag{c}$$

for any $x \in \mathbb{R}$ and for all $q \in \mathbb{Q}$.

Finally, by letting $x = 1$ in (c) and considering the condition (i), we have $f(x) = cx$ for any $x \in \mathbb{R}$, where $c = f(1)$. □

As indicated in the previous theorem, if a solution of the additive Cauchy equation (2.1) additionally satisfies one of the very weak conditions (i) – (v), then it has the linearity.

Every additive function which is not linear, however, displays a very strange behavior presented in the following theorem (ref. [2]):

Theorem 2.2. *The graph of every additive function* $f : \mathbb{R} \to \mathbb{R}$ *which is not of the form* $f(x) = cx$*, for all* $x \in \mathbb{R}$*, is dense in* $\mathbb{R}^2$*.*

Proof. The graph of f is the set

$$G = \{(x, y) \in \mathbb{R}^2 \mid y = f(x)\}.$$

Choose a real number $x_1 \neq 0$. Since f is not of the form $f(x) = cx$ for any real constant c, there exists a real number $x_2 \neq 0$ such that

$$f(x_1)/x_1 \neq f(x_2)/x_2.$$

Namely,

$$\begin{vmatrix} x_1 & f(x_1) \\ x_2 & f(x_2) \end{vmatrix} \neq 0.$$

This means that the vectors $\vec{p}_1 = (x_1, f(x_1))$ and $\vec{p}_2 = (x_2, f(x_2))$ are linearly independent and thus span the whole plane $\mathbb{R}^2$. Let $\vec{p}$ be an arbitrary plane vector. Then there exist rational numbers q_1 and q_2 such that $|\vec{p} - (q_1\vec{p}_1 + q_2\vec{p}_2)| \le \varepsilon$ for any $\varepsilon > 0$, since $\mathbb{Q}^2$ is dense in $\mathbb{R}^2$. Now,

$$
\begin{aligned}
q_1\vec{p}_1 + q_2\vec{p}_2 &= q_1\big(x_1, f(x_1)\big) + q_2\big(x_2, f(x_2)\big) \\
&= \big(q_1x_1 + q_2x_2, q_1f(x_1) + q_2f(x_2)\big) \\
&= \big(q_1x_1 + q_2x_2, f(q_1x_1 + q_2x_2)\big).
\end{aligned}
$$

The last inequality follows from (c) in the proof of Theorem 2.1. Hence,

$$G_{12} = \big\{(x, y) \in \mathbb{R}^2 \mid x = q_1x_1 + q_2x_2,\ y = f(x),\ q_1, q_2 \in \mathbb{Q}\big\}$$

is dense in $\mathbb{R}^2$. From the fact $G_{12} \subset G$ we conclude that G is dense in $\mathbb{R}^2$ which completes the proof of our theorem. □

We now give some results concerning the additive complex-valued functions defined on the complex plane:

If an additive function $f : \mathbb{C} \to \mathbb{C}$ is continuous, then there exist complex constants c_1 and c_2 with $f(z) = c_1z + c_2\bar{z}$ for all $z \in \mathbb{C}$, where $\bar{z}$ denotes the complex conjugate of z.

Unlike the case of real-valued additive functions on the reals, the complex-valued continuous additive functions on the complex plane are not linear. However, every complex-valued additive function is linear if it is analytic or differentiable.

2.2 Hyers–Ulam Stability

As stated in the introduction, S. M. Ulam [354] raised the following question concerning the stability of homomorphisms:

Let G_1 and G_2 be a group and a metric group with a metric $d(\cdot, \cdot)$, respectively. Given $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d\big(h(xy), h(x)h(y)\big) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d\big(h(x), H(x)\big) < \varepsilon$ for all $x \in G_1$?

D. H. Hyers presented the first result concerning the stability of functional equations. Indeed, he obtained a celebrated theorem while he was trying to answer the question of Ulam (ref. [135]).

Theorem 2.3 (Hyers). *Let $f : E_1 \to E_2$ be a function between Banach spaces such that*

$$\|f(x + y) - f(x) - f(y)\| \le \delta \tag{2.2}$$

for some $\delta > 0$ and for all $x, y \in E_1$. Then the limit

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \tag{2.3}$$

exists for each $x \in E_1$ and $A : E_1 \to E_2$ is the unique additive function such that

$$\| f(x) - A(x)\| \le \delta \tag{2.4}$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is linear.

Proof. For any $x \in E_1$ the inequality $\| f(2x) - 2f(x)\| \le \delta$ is obvious from (2.2). Replacing x by $x/2$ in this inequality and dividing by 2 we get

$$\|(1/2) f(x) - f(x/2)\| \le (1/2)\delta$$

for any $x \in E_1$. Now, make the induction assumption

$$\left\|2^{-n} f(x) - f\left(2^{-n} x\right)\right\| \le \left(1 - 2^{-n}\right)\delta. \tag{a}$$

It then follows from the last two inequalities that

$$\left\|(1/2) f(2^{-n} x) - f\left(2^{-n-1} x\right)\right\| \le (1/2)\delta$$

and

$$\left\|2^{-n-1} f(x) - (1/2) f(2^{-n} x)\right\| \le (1/2)(1 - 2^{-n})\delta.$$

Hence,

$$\left\|2^{-n-1} f(x) - f\left(2^{-n-1} x\right)\right\| \le \left(1 - 2^{-n-1}\right)\delta.$$

Therefore, the inequality (a) is true for all $x \in E_1$ and $n \in \mathbb{N}$.

Put $q_n(x) = 2^{-n} f(2^n x)$, where $n \in \mathbb{N}$ and $x \in E_1$. Then

$$\begin{aligned} q_m(x) - q_n(x) &= 2^{-m} f(2^m x) - 2^{-n} f(2^n x) \\ &= 2^{-m}\left(f(2^{m-n} 2^n x) - 2^{m-n} f(2^n x)\right). \end{aligned}$$

Therefore, if $m < n$, we can apply the inequality (a) to the last equality and we get

$$\|q_m(x) - q_n(x)\| \le (2^{-m} - 2^{-n})\delta$$

for all $x \in E_1$. Hence, the Hyers–Ulam sequence $\{q_n(x)\}$ is a Cauchy sequence for each x, and since E_2 is complete, there exists a limit function

$$A(x) = \lim_{n\to\infty} q_n(x).$$

Let x and y be any two points of E_1. It follows from (2.2) that

$$\|f(2^n x + 2^n y) - f(2^n x) - f(2^n y)\| \leq \delta$$

for any $n \in \mathbb{N}$. Dividing by 2^n and letting $n \to \infty$ we see that A is an additive function. If we replace x by $2^n x$ in (a) and take the limit, we have the inequality (2.4).

Suppose that $A' : E_1 \to E_2$ was another additive function satisfying (2.4) in place of A, and such that $A(y) \neq A'(y)$ for some $y \in E_1$. For any integer $n > 2\delta / \|A(y) - A'(y)\|$ we see that the inequality $\|A(ny) - A'(ny)\| > 2\delta$ holds. On the other hand, this inequality contradicts the inequalities

$$\|A(ny) - f(ny)\| \leq \delta \quad \text{and} \quad \|A'(ny) - f(ny)\| \leq \delta.$$

Hence, A is the unique additive function satisfying the inequality (2.4).

Assume that f is continuous at y. If A is not continuous at a point $x \in E_1$, then there exist an integer k and a sequence $\{x_n\}$ in E_1 converging to zero such that $\|A(x_n)\| > 1/k$ for any $n \in \mathbb{N}$. Let m be an integer greater than $3k\delta$. Then

$$\|A(mx_n + y) - A(y)\| = \|A(mx_n)\| > 3\delta.$$

On the other hand,

$$\begin{aligned} \|A(mx_n + y) - A(y)\| &\leq \|A(mx_n + y) - f(mx_n + y)\| \\ &\quad + \|f(mx_n + y) - f(y)\| + \|f(y) - A(y)\| \\ &\leq 3\delta \end{aligned}$$

for sufficiently large n, since $f(mx_n + y) \to f(y)$ as $n \to \infty$. This contradiction means that the continuity of f at a point in E_1 implies the continuity of A on E_1.

For a fixed $x \in E_1$, if $f(tx)$ is continuous in t, then it follows from the above consideration that $A(tx)$ is continuous in t, hence A is linear. □

The Hyers–Ulam stability result of Theorem 2.3 remains valid if E_1 is an abelian semigroup (ref. [105]).

The following corollary has been proved in the proof of Theorem 2.3.

Corollary 2.4. *Under the hypotheses of Theorem 2.3, if f is continuous at a single point of E_1, then A is continuous everywhere in E_1.*

As we see, we can explicitly construct the unique additive function satisfying (2.4) by means of the method expressed in (2.3). D. H. Hyers was the first person to suggest this method known as a *direct method* because it allows us to construct the additive function A satisfying (2.4) directly from the given function f in Theorem 2.3. It is the most powerful tool to study the stability of several functional equations and will be frequently used to construct certain function which is a solution of a given functional equation.

2.3 Hyers–Ulam–Rassias Stability

After Hyers gave an affirmative answer to Ulam's question, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers's theorem.

There is no reason for the *Cauchy difference* $f(x+y) - f(x) - f(y)$ to be bounded as in the expression of (2.2). Toward this point, Th. M. Rassias tried to weaken the condition for the Cauchy difference and succeeded in proving what is now known to be the *Hyers–Ulam–Rassias stability* for the additive Cauchy equation (see [159, 298, 301, 306]). This terminology is justified because the theorem of Th. M. Rassias (Theorem 2.5 below) has strongly influenced mathematicians studying stability problems of functional equations. In fact, Th. M. Rassias [285] proved the following:

Theorem 2.5 (Rassias). *Let E_1 and E_2 be Banach spaces, and let $f : E_1 \to E_2$ be a function satisfying the functional inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big) \tag{2.5}$$

for some $\theta > 0$, $p \in [0,1)$, and for all $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2-2^p}\|x\|^p \tag{2.6}$$

for any $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in E_1$, then A is linear.

Proof. By induction on n, we prove that

$$\|2^{-n} f(2^n x) - f(x)\| \le \theta\|x\|^p \sum_{m=0}^{n-1} 2^{m(p-1)} \tag{a}$$

for any $n \in \mathbb{N}$. Putting $y = x$ in (2.5) and dividing by 2 yield the validity of (a) for $n = 1$. Assume now that (a) is true and we want to prove it for the case $n+1$. However, this is true because by (a) we obtain

$$\|2^{-n} f(2^n 2x) - f(2x)\| \le \theta\|2x\|^p \sum_{m=0}^{n-1} 2^{m(p-1)},$$

therefore

$$\big\|2^{-n-1} f\big(2^{n+1}x\big) - (1/2) f(2x)\big\| \le \theta\|x\|^p \sum_{m=1}^{n} 2^{m(p-1)}.$$

By the triangle inequality, we get

$$\begin{aligned}&\left\|2^{-n-1} f\left(2^{n+1} x\right)-f(x)\right\| \\ &\quad \leq\left\|2^{-n-1} f\left(2^{n+1} x\right)-(1/2) f(2 x)\right\|+\|(1/2) f(2 x)-f(x)\| \\ &\quad \leq \theta\|x\|^{p} \sum_{m=0}^{n} 2^{m(p-1)},\end{aligned}$$

which completes the proof of (a).

It then follows that

$$\left\|2^{-n} f(2^{n} x)-f(x)\right\| \leq \frac{2 \theta}{2-2^{p}}\|x\|^{p}, \tag{b}$$

since $\sum_{m=0}^{\infty} 2^{m(p-1)}$ converges to $2/(2-2^{p})$, as $0 \leq p<1$. However, for $m>n>0$, we have

$$\begin{aligned}\left\|2^{-m} f(2^{m} x)-2^{-n} f(2^{n} x)\right\| &=2^{-n}\left\|2^{-(m-n)} f(2^{m-n} 2^{n} x)-f(2^{n} x)\right\| \\ &\leq 2^{n(p-1)} \frac{2 \theta}{2-2^{p}}\|x\|^{p}.\end{aligned}$$

Therefore, the Rassias sequence $\{2^{-n} f(2^{n} x)\}$ is a Cauchy sequence for each $x \in E_1$. As E_2 is complete, we can define a function A by (2.3). It follows that

$$\left\|f\left(2^{n}(x+y)\right)-f(2^{n} x)-f(2^{n} y)\right\| \leq 2^{n p} \theta\left(\|x\|^{p}+\|y\|^{p}\right).$$

Dividing by 2^n the last expression and letting $n \to \infty$, together with (2.3), yield that A is an additive function.

The inequality (2.6) immediately follows from (b) and (2.3).

We now want to prove that A is such a unique additive function. Assume that there exists another one, denoted by $A' : E_1 \to E_2$. Then there exists a constant $\varepsilon_1 \geq 0$ and q $(0 \leq q < 1)$ with

$$\left\|A'(x)-f(x)\right\| \leq \varepsilon_{1}\|x\|^{q}. \tag{c}$$

By the triangle inequality, (2.6), and (c) we obtain

$$\begin{aligned}\left\|A(x)-A'(x)\right\| &\leq\|A(x)-f(x)\|+\left\|f(x)-A'(x)\right\| \\ &\leq \frac{2 \theta}{2-2^{p}}\|x\|^{p}+\varepsilon_{1}\|x\|^{q}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|A(x) - A'(x)\| &= (1/n)\|A(nx) - A'(nx)\| \\ &\le \frac{1}{n}\left(\frac{2\theta}{2-2^p}\|nx\|^p + \varepsilon_1\|nx\|^q\right) \\ &= n^{p-1}\frac{2\theta}{2-2^p}\|x\|^p + n^{q-1}\varepsilon_1\|x\|^q\end{aligned}$$

for all $n \in \mathbb{N}$. By letting $n \to \infty$ we get $A(x) = A'(x)$ for any $x \in E_1$.

Assume that $f(tx)$ is continuous in t for any fixed $x \in E_1$. Since $A(x+y) = A(x) + A(y)$ for each $x, y \in E_1$, $A(qx) = qA(x)$ holds true for any rational number q. Fix x_0 in E_1 and ρ in E_2^* (the dual space of E_2). Define a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \rho\big(A(tx_0)\big)$$

for all $t \in \mathbb{R}$. Then ϕ is additive. Moreover, ϕ is a Borel function because of the following reasoning: Let $\phi(t) = \lim_{n\to\infty} 2^{-n}\rho\big(f(2^n tx_0)\big)$ and set $\phi_n(t) = 2^{-n}\rho\big(f(2^n tx_0)\big)$. Then $\phi_n(t)$ are continuous functions. $\phi(t)$ is the pointwise limit of continuous functions, thus $\phi(t)$ is a Borel function. According to Theorem 2.1, ϕ is linear and hence it is continuous. Let $a \in \mathbb{R}$. Then $a = \lim_{n\to\infty} q_n$, where $\{q_n\}$ is a sequence of rational numbers. Hence,

$$\phi(at) = \phi\Big(t \lim_{n\to\infty} q_n\Big) = \lim_{n\to\infty}\phi(tq_n) = \lim_{n\to\infty} q_n\phi(t) = a\phi(t).$$

Therefore, $\phi(at) = a\phi(t)$ for any $a \in \mathbb{R}$. Thus, $A(ax) = aA(x)$ for any $a \in \mathbb{R}$. Hence, A is a linear function. □

This theorem is a remarkable generalization of Theorem 2.3 and stimulated the concern of mathematicians toward the study of the stability problems of functional equations. T. Aoki [7] has provided a proof of a special case of Th. M. Rassias's theorem just for the stability of the additive function using the direct method. Aoki did not prove the last assertion of Rassias's Theorem 2.5 which provides the stability of the linear function.

Th. M. Rassias [289] noticed that the proof of this theorem also works for $p < 0$ and asked whether such a theorem can also be proved for $p \ge 1$. Z. Gajda [112] answered the *question of Rassias* for the case of $p > 1$ by a slight modification of the expression in (2.3). His idea to prove the theorem for this case is to replace n by $-n$ in the formula (2.3).

It turns out that 1 is the only critical value of p to which Theorem 2.5 cannot be extended. Z. Gajda [112] showed that this theorem is false for $p = 1$ by constructing a counterexample:

For a fixed $\theta > 0$ and $\mu = (1/6)\theta$ define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \sum_{n=0}^{\infty} 2^{-n}\phi(2^n x),$$

where the function $\phi : \mathbb{R} \to \mathbb{R}$ is given by

$$\phi(x) = \begin{cases} \mu & (\text{for } x \in [1, \infty)), \\ \mu x & (\text{for } x \in (-1, 1)), \\ -\mu & (\text{for } x \in (-\infty, -1]). \end{cases}$$

Then the function f serves as a counterexample for $p = 1$ as presented in the following theorem.

Theorem 2.6 (Gajda). *The function f defined above satisfies*

$$|f(x+y) - f(x) - f(y)| \le \theta\big(|x| + |y|\big) \tag{2.7}$$

for all $x, y \in \mathbb{R}$, while there is no constant $\delta \ge 0$ and no additive function $A : \mathbb{R} \to \mathbb{R}$ satisfying the condition

$$|f(x) - A(x)| \le \delta |x| \tag{2.8}$$

for all $x, y \in \mathbb{R}$.

Proof. If $x = y = 0$, then (2.7) is trivially satisfied.

Now, we assume that $0 < |x| + |y| < 1$. Then there exists an $N \in \mathbb{N}$ such that

$$2^{-N} \le |x| + |y| < 2^{-(N-1)}.$$

Hence, $\left|2^{N-1}x\right| < 1$, $\left|2^{N-1}y\right| < 1$, and $\left|2^{N-1}(x+y)\right| < 1$, which implies that for each $n \in \{0, 1, \ldots, N-1\}$ the numbers $2^n x$, $2^n y$, and $2^n(x+y)$ belong to the interval $(-1, 1)$. Since ϕ is linear on this interval, we infer that

$$\phi\big(2^n(x+y)\big) - \phi(2^n x) - \phi(2^n y) = 0$$

for $n \in \{0, 1, \ldots, N-1\}$. As a result, we get

$$\begin{aligned} \frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|} &\le \sum_{n=N}^{\infty} \frac{\left|\phi\big(2^n(x+y)\big) - \phi(2^n x) - \phi(2^n y)\right|}{2^n\big(|x| + |y|\big)} \\ &\le \sum_{k=0}^{\infty} \frac{3\mu}{2^k 2^N\big(|x| + |y|\big)} \\ &\le \sum_{k=0}^{\infty} \frac{3\mu}{2^k} \\ &= \theta. \end{aligned}$$

Finally, assume $|x| + |y| \geq 1$. Then merely by means of the boundedness of f we have

$$\frac{|f(x+y) - f(x) - f(y)|}{|x| + |y|} \leq 6\mu = \theta,$$

since

$$|f(x)| \leq \sum_{n=0}^{\infty} 2^{-n}\mu = 2\mu.$$

Now, contrary to what we claim, suppose that there exist a constant $\delta \geq 0$ and an additive function $A : \mathbb{R} \to \mathbb{R}$ such that (2.8) holds true. Since f is defined by means of a uniformly convergent series of continuous functions, f itself is continuous. Hence, A is bounded on some neighborhood of zero. Then, by Theorem 2.1, there exists a real constant c such that $A(x) = cx$ for all $x \in \mathbb{R}$. Hence, it follows from (2.8) that

$$|f(x) - cx| \leq \delta|x|,$$

for any $x \in \mathbb{R}$, which implies

$$|f(x)|/|x| \leq \delta + |c|$$

for all $x \in \mathbb{R}$. On the other hand, we can choose an $N \in \mathbb{N}$ so large that $N\mu > \delta + |c|$. If we choose an $x \in \left(0, 2^{-(N-1)}\right)$, then we have $2^n x \in (0,1)$ for each $n \in \{0, 1, \ldots, N-1\}$. Consequently, for such an x we get

$$\frac{f(x)}{x} \geq \sum_{n=0}^{N-1} \frac{\phi(2^n x)}{2^n x} = \sum_{n=0}^{N-1} \frac{\mu 2^n x}{2^n x} = N\mu > \delta + |c|,$$

which leads to a contradiction. □

Similarly, Th. M. Rassias and P. Šemrl [310] introduced a simple counterexample to Theorem 2.5 for $p = 1$ as follows:

The continuous real-valued function defined by

$$f(x) = \begin{cases} x \log_2(x+1) & (\text{for } x \geq 0), \\ x \log_2|x-1| & (\text{for } x < 0) \end{cases}$$

satisfies the inequality (2.7) with $\theta = 1$ and $|f(x) - cx|/|x| \to \infty$, as $x \to \infty$, for any real number c.

Furthermore, they also investigated the behaviors of functions which satisfy the inequality (2.7).

Theorem 2.7. *Let $f : \mathbb{R}^k \to \mathbb{R}^\ell$ be a function such that $f(tx)$ is continuous in t for each fixed x, where k and ℓ are given positive integers. Assume that f satisfies* *the inequality* (2.7) *for any* $x, y \in \mathbb{R}^k$. *Then for any* $\varepsilon > 0$, *there exists a real number* M_ε *such that*

$$|f(x)| \leq \begin{cases} M_\varepsilon |x|^{1+\varepsilon} & (\text{for } |x| \geq 1), \\ M_\varepsilon |x|^{1-\varepsilon} & (\text{for } |x| \leq 1). \end{cases} \tag{2.9}$$

Proof. Applying (2.7) and induction on n, we can prove

$$\begin{aligned} &\left| f(x_1 + \cdots + x_n) - f(x_1) - \cdots - f(x_n) \right| \\ &\quad \leq \ \theta(n-1)\big(|x_1| + \cdots + |x_n|\big). \end{aligned} \tag{a}$$

Let $\{e_1, \ldots, e_k\}$ be the standard basis in $\mathbb{R}^k$. An arbitrary vector $x \in \mathbb{R}^k$ with $|x| \leq 1$ can be expressed in the form

$$x = \lambda_1 e_1 + \cdots + \lambda_k e_k,$$

where $|\lambda_i| \leq 1$ for $i \in \{1, 2, \ldots, k\}$. It follows from ($a$) that

$$\begin{aligned} &\left| f(\lambda_1 e_1 + \cdots + \lambda_k e_k) - f(\lambda_1 e_1) - \cdots - f(\lambda_k e_k) \right| \\ &\quad \leq \ \theta(k-1)\big(|\lambda_1 e_1| + \cdots + |\lambda_k e_k|\big) \\ &\quad \leq \ \theta(k-1)k. \end{aligned}$$

Then

$$\begin{aligned} \left| f(\lambda_1 e_1 + \cdots + \lambda_k e_k) \right| &\leq \theta(k-1)k + |f(\lambda_1 e_1)| + \cdots + |f(\lambda_k e_k)| \\ &\leq \theta(k-1)k + M_1 + \cdots + M_k, \end{aligned}$$

where

$$M_i = \max_{|\lambda| \leq 1} |f(\lambda e_i)|.$$

Hence, f is bounded on the unit ball in $\mathbb{R}^k$. Thus, there exists a real number c such that

$$|f(x)| \leq c|x| \tag{b}$$

for all x satisfying $1/2 \leq |x| \leq 1$.

Claim that

$$\left| 2^{-n} f(2^n x) - f(x) \right| \leq n\theta |x| \tag{c}$$

for all $n \in \mathbb{N}$. By (2.7), (c) is true for $n = 1$. Assume now that (c) is true for some $n > 0$. Using the triangle inequality and (c), we get

$$\begin{aligned} &\left| 2^{-n-1} f\left(2^{n+1} x\right) - f(x) \right| \\ &\quad \leq \left| 2^{-n-1} f(2^n 2x) - (1/2) f(2x) \right| + |(1/2) f(2x) - f(x)| \\ &\quad \leq (n+1)\theta |x|, \end{aligned}$$

which ends the proof of (c).

For any x with $|x| > 1$, we can find an integer n such that the vector $y = 2^{-n}x$ satisfies $1/2 \le |y| \le 1$. Moreover, we have $n \le \log_2 |x| + 1$. It follows from (c) that

$$|2^{-n}f(x) - f(y)| \le n\theta|y|.$$

Therefore, by (b), we obtain

$$|f(x)| \le 2^n\big(|f(y)| + n\theta|y|\big) \le 2^n|y|(c + n\theta) \le |x|\big(c + \theta(\log_2 |x| + 1)\big),$$

which proves the first part of (2.9).

A similar argument as in the proof of (c) yields

$$|2^n f(2^{-n}x) - f(x)| \le n\theta|x|$$

for any $n \in \mathbb{N}$. For any x with $|x| \le 1$, there exists an integer n such that the vector $y = 2^n x$ satisfies $1/2 \le |y| \le 1$. It follows that $n \le -\log_2 |x|$. As in the previous case, we obtain

$$|2^n f(x) - f(y)| \le n\theta|y|.$$

Thus,

$$|f(x)| \le 2^{-n}\big(|f(y)| + n\theta|y|\big) \le |x|\big(c - \theta\log_2 |x|\big).$$

Hence, the second part of (2.9) also holds true. □

Th. M. Rassias [290] asked whether (2.6) gives the best possible estimate of the difference $\|f(x) - A(x)\|$ for $p \ne 1$. Th. M. Rassias and J. Tabor [313] answered the question for $p = 1/2$, and J. Brzdęk has partially answered the question for the case of $p > 0$ ($p \ne 1$). As it is an interesting subject, we introduce the result of Brzdęk [30]:

Let $\theta > 0$ and $p > 0$ ($p \ne 1$) be given, and let $A, f : \mathbb{R} \to \mathbb{R}$ be defined by

$$A(x) = 0$$

for all $x \in \mathbb{R}$, and

$$f(x) = \operatorname{sign}(x)\,\frac{\theta}{|2^{p-1} - 1|}|x|^p$$

for all $x \in \mathbb{R}$, where sign $: \mathbb{R} \to \{-1, 0, 1\}$ is the *sign function*. Then A is an additive function and

$$|f(x) - A(x)| = \frac{\theta}{|2^{p-1} - 1|}|x|^p$$

for all $x \in \mathbb{R}$. We now aim to show that f also satisfies the inequality (2.5) for all $x, y \in \mathbb{R}$.

Let us start with the following lemma:

Lemma 2.8. *If* $f_1, f_2 : [0,1] \to \mathbb{R}$ *are functions defined by*

$$f_1(x) = |(x+1)^p - x^p - 1|/(x^p+1),$$
$$f_2(x) = |(1-x)^p + x^p - 1|/(x^p+1)$$

for some $p > 0$ $(p \neq 1)$ *and for any* $x \in [0,1]$, *then* $f_1(x) \geq f_2(x)$ *for all* $x \in [0,1]$.

Proof. Put $g_i(x) = f_i(x)(x^p+1)$ for $x \in [0,1]$ and $i \in \{1,2\}$. Note that, in the case $p > 1$,

$$(1-x)^p + x^p \leq 1 \quad \text{and} \quad (x+1)^p - x^p \geq 1 \quad \text{for} \;\; x \in [0,1]$$

and, in the case $0 < p < 1$,

$$(1-x)^p + x^p \geq 1 \quad \text{and} \quad (x+1)^p - x^p \leq 1 \quad \text{for} \;\; x \in [0,1].$$

Hence, for every $x \in [0,1]$,

$$g_1(x) = \begin{cases} (x+1)^p - x^p - 1 & (\text{for} \;\; p > 1), \\ 1 + x^p - (x+1)^p & (\text{for} \;\; 0 < p < 1) \end{cases} \tag{a}$$

and

$$g_2(x) = \begin{cases} 1 - x^p - (1-x)^p & (\text{for} \;\; p > 1), \\ (1-x)^p + x^p - 1 & (\text{for} \;\; 0 < p < 1). \end{cases}$$

Consequently, for $x \in [0,1]$,

$$g_1'(x) = \begin{cases} p(x+1)^{p-1} - px^{p-1} & (\text{for} \;\; p > 1), \\ px^{p-1} - p(x+1)^{p-1} & (\text{for} \;\; 0 < p < 1) \end{cases}$$

and

$$g_2'(x) = \begin{cases} p(1-x)^{p-1} - px^{p-1} & (\text{for} \;\; p > 1), \\ px^{p-1} - p(1-x)^{p-1} & (\text{for} \;\; 0 < p < 1). \end{cases}$$

Further, if $p > 1$, then

$$(x+1)^{p-1} - x^{p-1} \geq (1-x)^{p-1} - x^{p-1} \;\; \text{for} \;\; x \in [0,1],$$

and if $0 < p < 1$,

$$x^{p-1} - (x+1)^{p-1} > x^{p-1} - (1-x)^{p-1} \;\; \text{for} \;\; x \in (0,1).$$

Thus, $g_1'(x) \geq g_2'(x)$ for any $x \in (0,1)$. Hence, $g_1(x) \geq g_2(x)$ for all $x \in [0,1]$, since $g_1(0) = g_2(0) = 0$. That is, $f_1(x) \geq f_2(x)$ for each $x \in [0,1]$. □

Lemma 2.9. *Let f_1 be the same as in Lemma 2.8. Then*

$$\sup\{f_1(x) \mid x \in [0,1]\} = \left|2^{p-1} - 1\right|$$

for $p > 0$ and $p \neq 1$.

Proof. Suppose $f_1'(x) = 0$. Considering (a) in the proof of Lemma 2.8, it follows from the hypothesis that

$$\big(p(x+1)^{p-1} - px^{p-1}\big)\big(x^p + 1\big) - px^{p-1}\big((x+1)^p - x^p - 1\big) = 0.$$

The solution of this equation is $x = 1$ only.

In this way, we have shown that

$$\sup\{f_1(x) \mid x \in [0,1]\} = \max\{f_1(0),\ f_1(1)\} = \left|2^{p-1} - 1\right|,$$

which ends the proof. □

J. Brzdęk [30] proved that f satisfies the inequality (2.5):

Theorem 2.10 (Brzdęk). *The function f satisfies the functional inequality (2.5) for all $x, y \in \mathbb{R}$.*

Proof. First, let $x = y = 0$. Since $f(0) = 0$, it is clear that f satisfies (2.5) for this case.

Now assume $(x, y) \neq (0,0)$, and let

$$g(x,y) = \frac{\left|2^{p-1} - 1\right|}{\theta\big(|x|^p + |y|^p\big)}\left|f(x+y) - f(x) - f(y)\right|$$

for all $x, y \in \mathbb{R}$ with $x^2 + y^2 > 0$. Let us define

$$s = \sup\{g(x,y) \mid x, y \in \mathbb{R},\ x^2 + y^2 > 0\}.$$

Since $g(x,y) = g(y,x)$ for $x, y \in \mathbb{R}$ with $x^2 + y^2 > 0$, it is easily seen that

$$s = \sup\{g(x,y) \mid x, y \in \mathbb{R},\ |x| \leq |y|,\ x^2 + y^2 > 0\}.$$

Moreover, if $xy \geq 0$, $x^2 + y^2 > 0$ and $|x| \leq |y|$, then

$$g(x,y) = \big||x+y|^p - |x|^p - |y|^p\big| \big/ \big(|x|^p + |y|^p\big),$$

and if $xy < 0$ and $|x| \le |y|$, then

$$g(x, y) = \big||x + y|^p + |x|^p - |y|^p\big| / \big(|x|^p + |y|^p\big).$$

Define

$$s_1 = \sup\big\{g(x, y) \mid xy \ge 0,\ x^2 + y^2 > 0,\ |x| \le |y|\big\}$$

and

$$s_2 = \sup\big\{g(x, y) \mid xy < 0,\ |x| \le |y|\big\}.$$

Then

$$s_1 = \sup\big\{f_1(x) \mid x \in [0, 1]\big\}$$

and

$$s_2 = \sup\big\{f_2(x) \mid x \in [0, 1]\big\}.$$

Therefore, by Lemmas 2.8 and 2.9, we get

$$s = \max\big\{s_1, s_2\big\} = \big|2^{p-1} - 1\big|$$

which implies that f satisfies (2.5) for all $x, y \in \mathbb{R}$ with $x^2 + y^2 > 0$. $\square$

Until now, we have proved that $\theta\|x\|^p/|1 - 2^{p-1}|$ gives the best possible upper bound for the norm of the difference $f(x) - A(x)$ in the case of $p > 0$ $(p \ne 1)$.

Now, we return to the subject concerning the generalization of the bound condition for the norm of the Cauchy difference in (2.5).

A function $H : [0, \infty)^2 \to [0, \infty)$ is called *homogeneous* of degree p if it satisfies $H(tu, tv) = t^p H(u, v)$ for all $t, u, v \in [0, \infty)$. We can replace $\theta\big(\|x\|^p + \|y\|^p\big)$ with $H\big(\|x\|, \|y\|\big)$, where $H : [0, \infty)^2 \to [0, \infty)$ is a monotonically increasing symmetric homogeneous function of degree $p \ge 0$, $p \ne 1$, and still obtain a stability result. More precisely, Th. M. Rassias and P. Šemrl [311] generalized the result of Theorem 2.5 as follows:

Theorem 2.11 (Rassias and Šemrl). *Let E_1 and E_2 be a normed space and a real Banach space, respectively. Assume that $H : [0, \infty)^2 \to [0, \infty)$ is a monotonically increasing symmetric homogeneous function of degree p, where $p \in [0, \infty) \setminus \{1\}$. Let a function $f : E_1 \to E_2$ satisfy the inequality*

$$\|f(x + y) - f(x) - f(y)\| \le H\big(\|x\|, \|y\|\big) \tag{2.10}$$

for any $x, y \in E_1$. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \frac{H(1, 1)}{|2^p - 2|}\|x\|^p \tag{2.11}$$

for any $x \in E_1$. *Moreover, the function* A *is linear if for each fixed* $x \in E_1$ *there exists a real number* $\delta_x > 0$ *such that* $f(tx)$ *is bounded on* $[0, \delta_x]$.

Proof. The proof of the first part of the theorem is similar to Theorem 2.3 or 2.5. Therefore, we prove only the linearity of A under the condition that $f(tx)$ is locally bounded for each fixed x. So, let us assume that for every fixed $x \in E_1$ there exists a positive number δ_x such that the function $\|f(tx)\|$ is bounded on $[0, \delta_x]$. Fix $z \in E_1$ and $\varphi \in E_2^*$. Here, E_2^* denotes the dual space of E_2. Let us define

$$M_z = \sup\big\{\|f(tz)\| \mid t \in [0, \delta_z]\big\}.$$

Consider the function $\phi : \mathbb{R} \to \mathbb{R}$ defined by $\phi(t) = \varphi\big(A(tz)\big)$. It is obvious that ϕ is additive. For any real number $t \in [0, \delta_z]$, we have

$$|\phi(t)| = \big|\varphi\big(A(tz)\big)\big| \le \|\varphi\|\,\|A(tz)\| \le \|\varphi\|\big(\|A(tz) - f(tz)\| + \|f(tz)\|\big).$$

Using (2.11) we obtain

$$|\phi(t)| \le \|\varphi\|\left(\frac{H(1,1)}{|2^p - 2|}\delta_z^p\|z\|^p + M_z\right).$$

Since the additive function ϕ is bounded on an interval of positive length, in view of Theorem 2.1, it is of the form

$$\phi(t) = \phi(1)t$$

for all $t \in \mathbb{R}$. Therefore, $\varphi\big(A(tz)\big) = \varphi\big(tA(z)\big)$ for any $t \in \mathbb{R}$, and consequently A is a linear function. □

As expected, an analogue of Theorem 2.11 cannot be obtained in the case that H is a monotonically increasing symmetric homogeneous function of degree 1. It is not difficult to construct a counterexample from the following lemma presented in the paper [311].

Lemma 2.12. *Let* $h : [0, \infty) \to [0, \infty)$ *be a monotonically increasing function satisfying*

$$\lim_{t\to\infty} h(t) = \infty, \quad h(1) = 1, \quad \textit{and} \quad h(t) = th(1/t)$$

for all $t > 0$. *Then there exists a continuous monotonically increasing function* $g : [0, \infty) \to [0, \infty)$ *such that*

(i) $g(0) = 0$,
(ii) $g(t) \to \infty$ *as* $t \to \infty$,
(iii) $g(t + s) - g(t) \le h(s/t)$ *for all* $s, t \in [0, \infty)$, $t \ne 0$.

Proof. Since $h(t) \to \infty$ as $t \to \infty$, we can find a monotonically increasing sequence $\{n_k\}$ of positive integers satisfying

$$n_1 = 1 \quad \text{and} \quad h(2^{n_k}) > k. \tag{a}$$

Let $\{a_k\}$ be a sequence given by

$$a_1 = 0, \;\; a_2 = 2, \quad \text{and} \quad a_{k+1} = 2^{n_k} a_k \tag{b}$$

for $k \in \{2, 3, \ldots\}$.

We define

$$g(a_1) = 0 \quad \text{and} \quad g(a_k) = 1/3 + 1/4 + \cdots + 1/(k+1)$$

for $k \in \{2, 3, \ldots\}$. We extend g to $[0, \infty)$ such that g is piecewise linear. It is obvious that $g : [0, \infty) \to [0, \infty)$ is a continuous monotonically increasing function satisfying (i) and (ii).

Let t be a real number with $0 < t \leq 1$. Since h is a monotonically increasing function satisfying $h(1) = 1$, we have $h(1/t) \geq 1$. It follows that $th(1/t) \geq t$. Using $h(t) = th(1/t)$ we finally get that

$$h(t) \geq t \tag{c}$$

holds true for any $t \in (0, 1]$.

In order to prove (iii) let us first assume that $t \geq s$ and choose $k \in \mathbb{N}$ such that $t \in [a_k, a_{k+1}]$. The definition of g implies

$$g(t+s) - g(t) \leq \frac{s}{(k+2)(a_{k+1} - a_k)}. \tag{d}$$

For $k > 1$ we get using (b) that

$$g(t+s) - g(t) \leq \frac{s}{a_{k+1}} \frac{1}{(k+2)(1 - 2^{-n_k})} \leq \frac{s}{a_{k+1}} \leq \frac{s}{t}.$$

The relation (c) completes the proof in this case. The case, when $k = 1$, is an immediate consequence of (b), (c), and (d).

Suppose now that $s > t$ and choose $k \in \mathbb{N}$ such that $s \in [a_k, a_{k+1}]$. Let us set $a_0 = 0$. We will first consider the case that $t \geq a_{k-1}$. Then we have

$$g(t+s) - g(t) \leq g(a_{k+2}) - g(a_{k-1}) \leq 1/3 + 1/4 + 1/5 < 1.$$

The desired relation (iii) follows now from $h(1) = 1$ and $s/t > 1$.

It remains to consider the case when $t < a_{k-1}$. This implies $k \geq 3$. Therefore,

$$g(t+s) - g(t) \leq g(a_{k+2}) = 1/3 + 1/4 + \cdots + 1/(k+3) \leq k - 1.$$

Using (a) we have

$$g(t+s) - g(t) \leq h\big(a_k 2^{n_{k-1}} a_k^{-1}\big) \leq h\big(2^{n_{k-1}} s a_k^{-1}\big).$$

Applying $2^{n_{k-1}}t \le 2^{n_{k-1}}a_{k-1} = a_k$ in the previous inequality, we obtain

$$g(t+s) - g(t) \le h\big(2^{n_{k-1}}s2^{-n_{k-1}}t^{-1}\big) = h(s/t),$$

which ends the proof. □

Using the result of the last lemma, Th. M. Rassias and P. Šemrl [311] have succeeded in finding a counterexample to Theorem 2.11 for the case in which H is a monotonically increasing symmetric homogeneous function of degree 1.

Theorem 2.13. *Assume that* $H : [0,\infty)^2 \to [0,\infty)$ *is a symmetric monotonically increasing homogeneous function of degree* 1 *such that*

$$\lim_{s\to\infty} H(1,s) = \infty. \tag{2.12}$$

Then there exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ *satisfying*

$$\big|f(t+s) - f(t) - f(s)\big| \le H\big(|t|,|s|\big)$$

for all $t,s \in \mathbb{R}$ *and*

$$\sup_{t\neq 0} |f(t) - A(t)|/|t| = \infty \tag{2.13}$$

for any additive function $A : \mathbb{R} \to \mathbb{R}$.

Proof. For every real number $t > 1$ we have $H(1,t) \le H(t,t) = tH(1,1)$. This inequality yields $H(1,1) \neq 0$ using (2.12). We can assume without loss of generality that $H(1,1) = 1$. Let us define $h : [0,\infty) \to [0,\infty)$ by $h(t) = H(1,t)$. Obviously, h is a monotonically increasing function satisfying $h(t) \to \infty$, as $t \to \infty$, and $h(1) = 1$. Moreover, we have $h(t) = H(1,t) = tH(1/t,1) = tH(1,1/t) = th(1/t)$ for any $t > 0$. We choose a function g as in Lemma 2.12 and define $f(t) = (1/2)tg(t)$ for all $t \ge 0$. We extend f to $\mathbb{R}$ as an odd function.

We first prove the inequality

$$|f(t+s) - f(t) - f(s)| \le H\big(|t|,|s|\big) \tag{a}$$

for all $s,t \in \mathbb{R}$. Clearly, (a) holds true in the case that $t = 0$ or $s = 0$. Next, we consider the case in which both numbers t and s are positive. Then we have

$$|f(t+s) - f(t) - f(s)| = (1/2)\big|t\big(g(t+s) - g(t)\big) + s\big(g(t+s) - g(s)\big)\big|.$$

As g is a monotonically increasing function, Lemma 2.12 (iii) yields

$$\begin{aligned} |f(t+s) - f(t) - f(s)| &\le (1/2)\big(th(s/t) + sh(t/s)\big) \\ &= (1/2)\big(H(t,s) + H(s,t)\big) \\ &= H\big(|t|,|s|\big). \end{aligned}$$

Since f is an odd function, (a) holds true for $t,s < 0$ as well.

It remains to consider the case when $t > 0$ and $s < 0$. Let us first assume that $|t| > |s|$. The left side of (a) can be rewritten as $|f(t) - f(t+s) - f(-s)|$, but $t + s$ and $-s$ are positive real numbers. Thus,

$$|f(t+s) - f(t) - f(s)| \le H(t+s, -s) \le H\big(|t|, |s|\big).$$

The proof of (a) in the case when $|s| > |t|$ proceeds in a similar way.

Suppose now that there exists an additive function $A : \mathbb{R} \to \mathbb{R}$ such that

$$\sup_{t \neq 0} |f(t) - A(t)|/|t| < \infty. \tag{b}$$

As f is a continuous function, it is bounded on any finite interval. It follows from (b) that the additive function A is bounded on every finite interval $[a, b]$ of the real line with $0 \notin [a, b]$. This implies that A is of the form $A(t) = ct$ for some real number c. For $t \geq 0$ we have

$$|f(t) - A(t)|/|t| = |(1/2)g(t) - c|.$$

According to Lemma 2.12 (ii), this contradicts (b). □

As seen in the above proof, the condition (2.12) is essential for the construction of a function f satisfying (2.13). The above theorem is a generalization of the result of Theorem 2.6 stating that the answer to Ulam's problem is negative in the special case that $H\big(|t|, |s|\big) = |t| + |s|$.

In the following theorem, we will introduce the behavior of functions satisfying the inequality (2.10), where H is a homogeneous function of degree 1 (ref. [311]).

Theorem 2.14. *Let E_1 and E_2 be a real normed space with* $\dim E_1 > 1$ *and a real Banach space, respectively. Suppose a function $f : E_1 \to E_2$ satisfies the inequality* (2.10)*, where $H : [0, \infty)^2 \to [0, \infty)$ is a symmetric monotonically increasing homogeneous function of degree* 1*. Then the following conditions are equivalent:*

(i) $\sup_{\|x\| \le 1} \|f(x)\| < \infty$,
(ii) $\sup_{\|x\| = 1} \|f(x)\| < \infty$,
(iii) f *is continuous at* 0,
(iv) $\lim_{\|x\| \to \infty} \|f(x)\|/\|x\|^{1+\varepsilon} = 0$ *for any* $\varepsilon > 0$,
(v) $\lim_{\|x\| \to 0} \|f(x)\|/\|x\|^{1-\varepsilon} = 0$ *for any* $\varepsilon > 0$.

Proof. Let us set $H(1, 1) = \theta$. Claim that

$$\|2^{-n} f(2^n x) - f(x)\| \le (1/2)n\theta\|x\| \tag{a}$$

and

$$\|2^n f(2^{-n} x) - f(x)\| \le (1/2)n\theta\|x\| \tag{b}$$

for any $n \in \mathbb{N}$. The proof of (a) follows by induction on n. The case $n = 1$ is clear. Assume now that (a) is true for some $n > 0$ and we want to prove it for $n+1$. Using the triangle inequality and (a), we get

$$\begin{aligned}\left\|2^{-n-1} f\left(2^{n+1} x\right) - f(x)\right\| &\leq \left\|2^{-n-1} f\left(2^{n+1} x\right) - (1/2) f(2x)\right\| \\ &\quad + \|(1/2) f(2x) - f(x)\| \\ &\leq (1/2)(n+1)\theta \|x\|.\end{aligned}$$

Replacing x with $2^{-n}x$ in (a) and multiplying the resulting inequality by 2^n, we obtain (b).

The implications $(i) \Rightarrow (ii)$ and $(v) \Rightarrow (iii)$ are easily seen. In order to prove $(ii) \Rightarrow (i)$, we choose a vector z such that $\|z\| \leq 1$. Since $\dim E_1 > 1$, the intersection of the unit spheres $S(0,1) = \{x \in E_1 \mid \|x\| = 1\}$ and $S(z,1) = \{x \in E_1 \mid \|x - z\| = 1\}$ is nonempty. Choose $w \in S(0,1) \cap S(z,1)$. Clearly, $z = w + (z - w)$. Since f satisfies (2.10), we have

$$\begin{aligned}\|f(z)\| &\leq H\left(\|w\|, \|z - w\|\right) + \|f(w)\| + \|f(z-w)\| \\ &\leq \theta + 2 \sup_{\|x\|=1} \|f(x)\|.\end{aligned}$$

Claim that $(iii) \Rightarrow (i)$. It is easy to see $f(0) = 0$. From (iii) it follows that there exist positive real numbers δ and M such that $\|x\| \leq \delta$ yields $\|f(x)\| \leq M$. We fix a positive integer n_0 satisfying $2^{-n_0} \leq \delta$. For every vector $z \in E_1$, $\|z\| \leq 1$, we get using (b) that

$$\|f(z)\| \leq (1/2)n_0\theta\|z\| + 2^{n_0}\|f(2^{-n_0}z)\| \leq (1/2)n_0\theta + 2^{n_0}M.$$

Claim that $(i) \Rightarrow (iv)$. It follows from (i) that there exists a real number c such that

$$\|f(x)\| \leq c\|x\| \tag{c}$$

for all x with $1/2 \leq \|x\| \leq 1$. For any x with norm greater than 1 we can find a positive integer n such that the vector $y = 2^{-n}x$ satisfies $1/2 \leq \|y\| \leq 1$. Moreover, we have $n \leq \log_2 \|x\| + 1$. It follows from (a) that

$$\|2^{-n} f(x) - f(y)\| \leq (1/2)n\theta\|y\|.$$

Therefore,

$$\begin{aligned}\|f(x)\| &\leq 2^n\left(\|f(y)\| + (1/2)n\theta\|y\|\right) \\ &\leq 2^n\|y\|\left(c + (1/2)n\theta\right) \\ &\leq \|x\|\left(c + (1/2)\theta(\log_2 \|x\| + 1)\right),\end{aligned}$$

which completes the proof of this implication.

Claim that $(i) \Rightarrow (v)$. For any x in the unit ball, $\|x\| \le 1$, there exists an integer n such that the vector $y = 2^n x$ satisfies $1/2 \le \|y\| \le 1$. It follows that $n \le -\log_2 \|x\|$. As before we get

$$\|2^n f(x) - f(y)\| \le (1/2)n\theta\|y\|.$$

Thus,

$$\|f(x)\| \le 2^{-n}\big(\|f(y)\| + (1/2)n\theta\|y\|\big) \le \|x\|\big(c - (1/2)\theta \log_2 \|x\|\big).$$

Hence, (v) holds true.

Claim that $(iv) \Rightarrow (ii)$. It follows from (iv) with $\varepsilon = 1$ that there exist positive real numbers η and M such that $\|x\| \ge M$ implies $\|f(x)\| \le \eta\|x\|^2$. Let us fix a positive integer n_0 satisfying $2^{n_0} \ge M$. Then for every $z \in E_1$ with $\|z\| = 1$, the inequality $\|f(2^{n_0}z)\| \le 4^{n_0}\eta$ is true. A simple use of (a) completes the proof.

Applying a similar approach we can prove the implication $(v) \Rightarrow (ii)$. □

The assumption that $\dim E_1 > 1$ is indispensable in the above theorem. Every function $f : \mathbb{R} \to E_2$ satisfying (2.10), where H is a homogeneous function of degree 1, is bounded on the unit sphere $\{-1, 1\}$. However, such functions need not be bounded on the unit ball. The proof of the equivalence of the conditions (i), (iii), (iv), and (v) works also in the case that $\dim E_1 = 1$. A function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(t, s) = \begin{cases} t + s & (\text{for } t, s \in \mathbb{Q}), \\ 0 & (\text{for } t \in \mathbb{R}\backslash\mathbb{Q} \text{ or } s \in \mathbb{R}\backslash\mathbb{Q}) \end{cases}$$

is an example of a function satisfying (2.10), where H is a monotonically increasing symmetric homogeneous function of degree 1, of which point of continuity is only 0 (see [311]).

Under the additional assumption that $\sup\limits_{s \ge 0} H(1, s) < \infty$, Th. M. Rassias and P. Šemrl [311] improved the previous result as follows:

Theorem 2.15. *Let E_1 and E_2 be those in Theorem 2.14. Suppose a function $f : E_1 \to E_2$ satisfies the inequality (2.10), where $H : [0, \infty)^2 \to [0, \infty)$ is a symmetric monotonically increasing homogeneous function of degree 1. Furthermore, assume that $\sup\limits_{s \ge 0} H(1, s) < \infty$. Then the conditions (i), (ii), and (iii) given in Theorem 2.14 are equivalent to the following condition:*

(vi) there exists a real number M such that $\|f(x)\| \le M\|x\|$ for all $x \in E_1$.

Proof. All we have to do is to prove that (i) implies (vi). Let us denote

$$\sup_{s \ge 0} H(1, s) = M_1.$$

Claim that

$$\|(1/k)f(kx) - f(x)\| \leq M_1\|x\| \tag{a}$$

for any integer $k > 1$. The verification of (a) follows by induction on k. In the case $k = 2$, we have

$$\|(1/2)f(2x) - f(x)\| \leq (1/2)H\big(\|x\|, \|x\|\big) \leq (1/2)M_1\|x\|.$$

Assume now that (a) is true for some $k > 1$ and we want to prove it for $k + 1$. Using the triangle inequality and (a), we get

$$\begin{aligned}
&\big\|(k+1)^{-1}f\big((k+1)x\big) - f(x)\big\| \\
&\quad \leq \big\|(k+1)^{-1}f\big((k+1)x\big) - (k+1)^{-1}f(kx) - (k+1)^{-1}f(x)\big\| \\
&\qquad + \big\|(k+1)^{-1}f(kx) - \big(k/(k+1)\big)f(x)\big\| \\
&\quad \leq (k+1)^{-1}H\big(\|x\|, k\|x\|\big) + \big(k/(k+1)\big)\|(1/k)f(kx) - f(x)\| \\
&\quad \leq (k+1)^{-1}\big(M_1\|x\| + kM_1\|x\|\big) \\
&\quad = M_1\|x\|.
\end{aligned}$$

It follows from (i) that there exists a real number c such that (c) in the proof of Theorem 2.14 holds true for all x satisfying $1/2 \leq \|x\| \leq 1$. For any x with norm greater than 1 we can find an integer k (≥ 2) such that the vector $y = (1/k)x$ satisfies $1/2 \leq \|y\| \leq 1$. From (c) in the proof of Theorem 2.14 and (a) it follows

$$\|f(x)\| = \|f(ky)\| \leq k\|y\|(c + M_1) = \|x\|(c + M_1). \tag{b}$$

A similar argument yields $\|f(x)\| \leq \|x\|(c + M_1)$ for any x having norm smaller than $1/2$. The relations (c) in the proof of Theorem 2.14 and (b) demonstrate that the assertion of the theorem holds true with $M = c + M_1$. □

In the following corollary, Rassias and Šemrl [311] generalized the result of Theorem 2.7, i.e., it was proved that if a function $f : E_1 \to E_2$ satisfies the inequality (2.10), where H is the same as in Theorem 2.14, and some suitable condition, then it behaves like that of Theorem 2.7.

Corollary 2.16. *Let E_1 and E_2 be a finite-dimensional real normed space with* $\dim E_1 > 1$ *and a real Banach space, respectively. Suppose a function $f : E_1 \to E_2$ satisfies the inequality* (2.10)*, where $H : [0,\infty)^2 \to [0,\infty)$ is a symmetric monotonically increasing homogeneous function of degree* 1*. Moreover, assume that for every $x \in E_1$ there exists a positive real number δ_x such that the function $\|f(tx)\|$ is bounded on* $[0, \delta_x]$*. Then for every positive real number ε there exists a real number M_ε such that*

$$\|f(x)\| \leq \begin{cases} M_\varepsilon\|x\|^{1+\varepsilon} & (\text{for } \|x\| \geq 1), \\ M_\varepsilon\|x\|^{1-\varepsilon} & (\text{for } \|x\| \leq 1). \end{cases}$$

Further, assume that $\sup_{s\geq 0} H(1,s) < \infty$. *Then there exists a real number* M *such that*

$$\|f(x)\| \leq M\|x\|$$

for all $x \in E_1$.

Proof. All norms on a finite-dimensional vector space are equivalent. Thus, without loss of generality, we can assume that E_1 is a Euclidean space $\mathbb{R}^k$. Let us set $\theta = H(1,1)$. For an arbitrary pair $x, y \in E_1$ we have

$$\begin{aligned}\|f(x+y) - f(x) - f(y)\| &\leq H\big(\|x\|, \|y\|\big)\\ &\leq H\big(\|x\| + \|y\|, \|x\| + \|y\|\big)\\ &= \theta\big(\|x\| + \|y\|\big).\end{aligned}$$

Applying induction on n we can easily prove

$$\begin{aligned}&\|f(x_1 + \cdots + x_n) - f(x_1) - \cdots - f(x_n)\|\\ &\quad \leq \theta(n-1)\big(\|x_1\| + \cdots + \|x_n\|\big).\end{aligned} \tag{a}$$

Let $\{e_1, \ldots, e_k\}$ be the standard basis in $\mathbb{R}^k$. According to the hypothesis, there exist positive real numbers $M_1, \delta_1, \ldots, \delta_k$ such that $|t| \leq \delta_i$ implies $\|f(te_i)\| \leq M_1$ for $i \in \{1, \ldots, k\}$. Choose a positive number K. Using (a) in the proof of Theorem 2.14 we can find a real number M_2 such that $|t| \leq K$ implies $\|f(te_i)\| \leq M_2$ for $i \in \{1, \ldots, k\}$.

An arbitrary vector $x \in \mathbb{R}^k$ with $\|x\| \leq K$ can be expressed in the form

$$x = t_1 e_1 + \cdots + t_k e_k,$$

where $|t_i| \leq K$ for $i \in \{1, \ldots, k\}$. It follows from (a) that

$$\begin{aligned}&\|f(t_1e_1 + \cdots + t_ke_k) - f(t_1e_1) - \cdots - f(t_ke_k)\|\\ &\quad \leq \theta(k-1)\big(\|t_1e_1\| + \cdots + \|t_ke_k\|\big) \;\leq\; \theta(k-1)kK.\end{aligned}$$

Then

$$\begin{aligned}\|f(x)\| &= \|f(t_1e_1 + \cdots + t_ke_k)\|\\ &\leq \theta(k-1)kK + \|f(t_1e_1)\| + \cdots + \|f(t_ke_k)\|\\ &\leq \theta(k-1)kK + kM_2.\end{aligned}$$

Hence, we have proved that f is bounded on every bounded set in E_1. The assertion of our corollary is now a simple consequence of Theorems 2.14 and 2.15. □

G. Isac and Th. M. Rassias [142] established a different generalization of Theorem 2.5 as follows:

Theorem 2.17 (Isac and Rassias). *Let E_1 and E_2 be a real normed space and a real Banach space, respectively. Let $\psi : [0,\infty) \to [0,\infty)$ be a function satisfying the following conditions:*

(i) $\lim_{t\to\infty} \psi(t)/t = 0$,
(ii) $\psi(ts) \le \psi(t)\psi(s)$ *for all* $t, s \in [0,\infty)$,
(iii) $\psi(t) < t$ *for all* $t > 1$.

If a function $f : E_1 \to E_2$ satisfies the inequality

$$\|f(x+y) - f(x) - f(y)\| \le \theta\big(\psi(\|x\|) + \psi(\|y\|)\big) \tag{2.14}$$

for some $\theta \ge 0$ and for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \frac{2\theta}{2-\psi(2)}\psi\big(\|x\|\big) \tag{2.15}$$

for all $x \in E_1$. Moreover, if $f(tx)$ is continuous in t for each fixed x, then the function A is linear.

Proof. We will first prove that

$$\|2^{-n} f(2^n x) - f(x)\| \le \theta\psi\big(\|x\|\big)\sum_{m=0}^{n-1}\big(\psi(2)/2\big)^m \tag{a}$$

for any $n \in \mathbb{N}$ and for all $x \in E_1$. The proof of (a) follows by induction on n. The assertion for $n = 1$ is clear by (2.14). Assume now that (a) is true for $n > 0$ and we want to prove it for the case $n+1$. Replacing x with $2x$ in (a) we obtain

$$\|2^{-n} f(2^n 2x) - f(2x)\| \le \theta\psi\big(2\|x\|\big)\sum_{m=0}^{n-1}\big(\psi(2)/2\big)^m.$$

By (ii) we get

$$\big\|2^{-n} f\big(2^{n+1}x\big) - f(2x)\big\| \le \theta\psi(2)\psi\big(\|x\|\big)\sum_{m=0}^{n-1}\big(\psi(2)/2\big)^m. \tag{b}$$

Multiplying both sides of (b) by $1/2$ we have

$$\big\|2^{-n-1} f\big(2^{n+1}x\big) - (1/2)f(2x)\big\| \le \theta\psi\big(\|x\|\big)\sum_{m=1}^{n}\big(\psi(2)/2\big)^m.$$

Using the triangle inequality, we now deduce

$$\begin{aligned}
&\left\|2^{-n-1} f\left(2^{n+1}x\right) - f(x)\right\| \\
&\quad \le \left\|2^{-n-1} f\left(2^{n+1}x\right) - (1/2) f(2x)\right\| + \|(1/2) f(2x) - f(x)\| \\
&\quad \le \theta\psi\left(\|x\|\right) \sum_{m=1}^{n} \left(\psi(2)/2\right)^m + \theta\psi\left(\|x\|\right) \\
&\quad = \theta\psi\left(\|x\|\right) \sum_{m=0}^{n} \left(\psi(2)/2\right)^m,
\end{aligned}$$

which ends the proof of (a).

Thus, it follows from (a) that

$$\|2^{-n} f(2^n x) - f(x)\| \le \frac{2\theta\psi\left(\|x\|\right)}{2 - \psi(2)} \tag{c}$$

for any $n \in \mathbb{N}$.

For $m > n > 0$ we obtain

$$\begin{aligned}
\|2^{-m} f(2^m x) - 2^{-n} f(2^n x)\| &= 2^{-n} \|2^{-(m-n)} f(2^m x) - f(2^n x)\| \\
&= 2^{-n} \|2^{-r} f(2^r y) - f(y)\|,
\end{aligned}$$

where $r = m - n$ and $y = 2^n x$. Hence,

$$\begin{aligned}
\|2^{-m} f(2^m x) - 2^{-n} f(2^n x)\| &\le 2^{-n}\theta \frac{2\psi\left(\|y\|\right)}{2 - \psi(2)} \\
&\le 2^{-n}\theta \frac{2\psi(2^n)\psi\left(\|x\|\right)}{2 - \psi(2)} \\
&\le \left(\frac{\psi(2)}{2}\right)^n \theta \frac{2\psi\left(\|x\|\right)}{2 - \psi(2)}.
\end{aligned}$$

However, by (iii), the Rassias sequence $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence. Let us define

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

for all $x \in E_1$.

We will prove that A is additive. Let $x, y \in E_1$ be given. It then follows from (2.14) and (ii) that

$$\begin{aligned}
\left\| f\left(2^n (x + y)\right) - f(2^n x) - f(2^n y)\right\| &\le \theta\left(\psi\left(\|2^n x\|\right) + \psi\left(\|2^n y\|\right)\right) \\
&\le \theta\psi(2^n)\left(\psi(\|x\|) + \psi(\|y\|)\right),
\end{aligned}$$

which implies that

$$2^{-n}\big\|f\big(2^n(x+y)\big)-f(2^nx)-f(2^ny)\big\|\le\big(\psi(2)/2\big)^n\theta\big(\psi(\|x\|)+\psi(\|y\|)\big).$$

Using (iii) and letting $n\to\infty$, we conclude that A is additive.

By letting $n\to\infty$ in (c), we obtain the inequality (2.15).

Claim that A is such a unique additive function. Assume that there exists another one, denoted by $A':E_1\to E_2$, satisfying

$$\|f(x)-A'(x)\|\le\frac{2\theta'}{2-\psi'(2)}\psi'\big(\|x\|\big),\qquad (d)$$

where θ' (≥ 0) is a constant and $\psi':[0,\infty)\to[0,\infty)$ is a function satisfying (i), (ii), and (iii). By (2.15) and (d), we get

$$\begin{aligned}\|A(x)-A'(x)\|&\le\|A(x)-f(x)\|+\|f(x)-A'(x)\|\\&\le\frac{2\theta}{2-\psi(2)}\psi\big(\|x\|\big)+\frac{2\theta'}{2-\psi'(2)}\psi'\big(\|x\|\big).\end{aligned}$$

Then,

$$\begin{aligned}\|A(x)-A'(x)\|&=(1/n)\|A(nx)-A'(nx)\|\\&\le\frac{\psi(n)}{n}\frac{2\theta\psi\big(\|x\|\big)}{2-\psi(2)}+\frac{\psi'(n)}{n}\frac{2\theta'\psi'\big(\|x\|\big)}{2-\psi'(2)},\end{aligned}$$

for every integer $n>1$. In view of (i) and the last inequality, we conclude that $A(x)=A'(x)$ for all $x\in E_1$.

Because of the additivity of A it follows that $A(qx)=qA(x)$ for any $q\in\mathbb{Q}$. Using the same argument as in Theorem 2.5, we obtain that $A(ax)=aA(x)$ for all real numbers a. Hence, A is a linear function. □

G. Isac and Th. M. Rassias [142] remarked that if $\psi(t)=t^p$ with $0\le p<1$, then from the last theorem we get the result of Theorem 2.5. If $p<0$ and

$$\psi(t)=\begin{cases}0 & (\text{for } t=0),\\ t^p & (\text{for } t>0),\end{cases}$$

then from Theorem 2.17 we obtain a generalization of Theorem 2.5 for $p<0$ (ref. [112]). If $f(S)$ is bounded, where $S=\big\{x\in E_1\mid\|x\|=1\big\}$, then the A given in Theorem 2.17 is continuous. Indeed, this is a consequence of the inequalities

$$\begin{aligned}\|A(x)\|&\le\|f(x)\|+\|A(x)-f(x)\|\\&\le\|f(x)\|+\frac{2\theta}{2-\psi(2)}\psi\big(\|x\|\big)\\&\le\|f(x)\|+\frac{2\theta}{2-\psi(2)}\psi(1)\end{aligned}$$

for all $x\in S$.

The control functions H and ψ appearing in Theorems 2.11 and 2.17 were remarkably generalized and the Hyers–Ulam–Rassias stability with the generalized control function was also proved by P. Găvruta. In the following theorem, we will introduce his result [113].

Theorem 2.18 (Găvruta). *Let G and E be an abelian group and a Banach space, respectively, and let $\varphi : G^2 \to [0, \infty)$ be a function satisfying*

$$\Phi(x, y) = \sum_{k=0}^{\infty} 2^{-k-1}\varphi\big(2^k x, 2^k y\big) < \infty \tag{2.16}$$

for all $x, y \in G$. If a function $f : G \to E$ satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \le \varphi(x, y) \tag{2.17}$$

for any $x, y \in G$, then there exists a unique additive function $A : G \to E$ with

$$\|f(x) - A(x)\| \le \Phi(x, x) \tag{2.18}$$

for all $x \in G$. Moreover, if $f(tx)$ is continuous in t for each fixed $x \in G$, then A is a linear function.

Proof. Putting $y = x$ in the inequality (2.17) yields

$$\|(1/2)f(2x) - f(x)\| \le (1/2)\varphi(x, x) \tag{a}$$

for all $x \in G$. Applying an induction argument to n, we will prove

$$\|2^{-n} f(2^n x) - f(x)\| \le \sum_{k=0}^{n-1} 2^{-k-1}\varphi\big(2^k x, 2^k x\big) \tag{b}$$

for any $x \in G$. Indeed,

$$\begin{aligned}\big\|2^{-n-1} f\big(2^{n+1}x\big) - f(x)\big\| \le{}& \big\|2^{-n-1} f\big(2^{n+1}x\big) - (1/2)f(2x)\big\| \\ &+ \|(1/2)f(2x) - f(x)\|,\end{aligned}$$

and by (a) and (b), we obtain

$$\begin{aligned}&\big\|2^{-n-1} f\big(2^{n+1}x\big) - f(x)\big\| \\ &\quad\le (1/2)\sum_{k=0}^{n-1} 2^{-k-1}\varphi\big(2^{k+1}x, 2^{k+1}x\big) + (1/2)\varphi(x, x) \\ &\quad= \sum_{k=0}^{n} 2^{-k-1}\varphi\big(2^k x, 2^k x\big),\end{aligned}$$

which ends the proof of (b).

We will present that the Rassias sequence $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence. Indeed, for $n > m > 0$, we have

$$\begin{aligned}
&\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \\
&\quad = 2^{-m} \|2^{-(n-m)} f(2^{n-m} 2^m x) - f(2^m x)\| \\
&\quad \le 2^{-m} \sum_{k=0}^{n-m-1} 2^{-k-1} \varphi\big(2^{k+m} x, 2^{k+m} x\big) \\
&\quad = \sum_{k=m}^{n-1} 2^{-k-1} \varphi\big(2^k x, 2^k x\big).
\end{aligned}$$

Taking the limit as $m \to \infty$ and considering (2.16), we obtain

$$\lim_{m\to\infty} \|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| = 0.$$

Since E is a Banach space, it follows that the sequence $\{2^{-n} f(2^n x)\}$ converges. Let us denote

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x).$$

It follows from (2.17) that

$$\big\| f\big(2^n (x + y)\big) - f(2^n x) - f(2^n y)\big\| \le \varphi(2^n x, 2^n y)$$

for all $x, y \in G$. Therefore,

$$\big\|2^{-n} f\big(2^n (x + y)\big) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\big\| \le 2^{-n} \varphi(2^n x, 2^n y). \qquad (c)$$

It follows from (2.16) that

$$\lim_{n\to\infty} 2^{-n} \varphi(2^n x, 2^n y) = 0.$$

Thus, (c) implies that $A : G \to E$ is an additive function.

Taking the limit in (b) as $n \to \infty$, we obtain the inequality (2.18).

It remains to show that A is uniquely defined. Let $A' : G \to E$ be another additive function satisfying (2.18). Then, we get

$$\begin{aligned}
\|A(x) - A'(x)\| &= \|2^{-n} A(2^n x) - 2^{-n} A'(2^n x)\| \\
&\le \|2^{-n} A(2^n x) - 2^{-n} f(2^n x)\| \\
&\quad + \|2^{-n} f(2^n x) - 2^{-n} A'(2^n x)\| \\
&\le 2^{-n} \Phi(2^n x, 2^n x) + 2^{-n} \Phi(2^n x, 2^n x)
\end{aligned}$$

$$
\begin{aligned}
&= 2^{-n} \sum_{k=0}^{\infty} 2^{-k} \varphi\big(2^{k+n} x, 2^{k+n} x\big) \\
&= \sum_{k=n}^{\infty} 2^{-k} \varphi\big(2^{k} x, 2^{k} x\big)
\end{aligned}
$$

for all $n \in \mathbb{N}$. Taking the limit in the above inequality as $n \to \infty$, we get

$$A(x) = A'(x)$$

for all $x \in G$.

From the additivity of A it follows that $A(qx) = qA(x)$ for any $q \in \mathbb{Q}$. Using the same argument as in Theorem 2.5, we obtain that $A(ax) = aA(x)$ for all real numbers a. Hence, A is a linear function. □

Later, S.-M. Jung complemented Theorem 2.18 by proving a theorem which includes the following corollary as a special case (see [173, Theorem 4]).

Corollary 2.19. *Let E_1 and E_2 be a real normed space and a Banach space, respectively. Assume that a function $\varphi : E_1^2 \to [0, \infty)$ satisfies*

$$\Phi(x, y) = \sum_{k=1}^{\infty} 2^{k} \varphi\big(2^{-k} x, 2^{-k} y\big) < \infty$$

for all $x, y \in E_1$. If a function $f : E_1 \to E_2$ satisfies the inequality (2.17) for any $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq (1/2)\Phi(x, x)$$

for all $x \in E_1$.

S.-M. Jung [156] has further generalized the result of Theorem 2.18 by making use of an idea from the previous theorem. In the following theorem, let G be an abelian group and E be a Banach space. Consider a function $\varphi : G^2 \to [0, \infty)$ satisfying $\varphi(x, y) = \varphi(y, x)$ for all $x, y \in G$. For all $n \in \mathbb{N}$ and all $x, y \in G$ define $\varphi_n^1(x, y) = \varphi(nx, y)$ and $\varphi_n^2(x, y) = \varphi(x, ny)$. By $a = (a_1, a_2, \ldots)$ we denote a sequence with $a_n \in \{1, 2\}$ for all $n \in \mathbb{N}$, and we define $\psi_k^a(x, y) = \varphi_1^{a_1}(x, y) + \cdots + \varphi_k^{a_k}(x, y)$ for a fixed integer $k > 1$. Suppose that there exists a sequence $a = (a_1, a_2, \ldots)$ with $a_n \in \{1, 2\}$ for all $n \in \mathbb{N}$ such that

$$\Psi_k(x, y) = \sum_{n=1}^{\infty} k^{-n} \psi_{k-1}^a\big(k^{n-1} x, k^{n-1} y\big) < \infty \tag{2.19}$$

for all $x, y \in G$. With these notations, Jung [156] proved the following theorem.

Theorem 2.20. *Suppose $f : G \to E$ is a function satisfying*

$$\|f(x+y) - f(x) - f(y)\| \le \varphi(x, y) \tag{2.20}$$

for all $x, y \in G$. Then there exists a unique additive function $A : G \to E$ such that

$$\|f(x) - A(x)\| \le \Psi_k(x, x) \tag{2.21}$$

for all $x \in G$. Moreover, if G is a Banach space and $f(tx)$ is continuous in t for every fixed $x \in G$, then A is linear.

Proof. We first claim

$$\|(1/n)f(nx) - f(x)\| \le (1/n)\psi^a_{n-1}(x, x) \tag{a}$$

for each integer $n > 1$ and all $x \in G$. We verify it by induction on n. By putting $y = x$ in (2.20), we obtain

$$\|f(2x) - 2f(x)\| \le \varphi(x, x) = \psi^a_1(x, x).$$

This implies the validity of (a) for the case $n = 2$. Assume now that the inequality (a) is valid for $n = m$ $(m \ge 2)$, i.e.,

$$\|f(mx) - mf(x)\| \le \psi^a_{m-1}(x, x). \tag{b}$$

For the case $n = m + 1$, replacing y with mx in (2.20), we get

$$\|f(x + mx) - f(x) - f(mx)\| \le \varphi(x, mx) = \varphi(mx, x). \tag{c}$$

It follows from (b) and (c) that

$$\begin{aligned}
&\big\|f\big((m+1)x\big) - (m+1)f(x)\big\| \\
&\quad \le \big\|f\big((m+1)x\big) - f(x) - f(mx)\big\| + \|f(mx) - mf(x)\| \\
&\quad \le \psi^a_m(x, x).
\end{aligned}$$

Accordingly, the assertion (a) is true for all integers $n > 1$ and all $x \in G$.

We claim

$$\|k^{-n} f(k^n x) - f(x)\| \le \sum_{i=1}^{n} k^{-i} \psi^a_{k-1}\big(k^{i-1}x, k^{i-1}x\big) \tag{d}$$

for each $n \in \mathbb{N}$. We also prove it by induction on n. The validity of (d) for $n = 1$ follows from (a). By assuming the induction argument for $n = m$ and putting $n = 1$ and then substituting $k^m x$ for x in (d), we obtain

$$
\begin{aligned}
&\left\| k^{-m-1} f\left(k^{m+1}x\right) - f(x)\right\| \\
&\quad \le \left\| k^{-m-1} f\left(k^{m+1}x\right) - k^{-m} f(k^m x)\right\| + \|k^{-m} f(k^m x) - f(x)\| \\
&\quad \le k^{-m} \|(1/k) f(k \cdot k^m x) - f(k^m x)\| + \sum_{i=1}^{m} k^{-i} \psi_{k-1}^{a}\left(k^{i-1}x, k^{i-1}x\right) \\
&\quad \le k^{-m-1} \psi_{k-1}^{a}(k^m x, k^m x) + \sum_{i=1}^{m} k^{-i} \psi_{k-1}^{a}\left(k^{i-1}x, k^{i-1}x\right) \\
&\quad = \sum_{i=1}^{m+1} k^{-i} \psi_{k-1}^{a}\left(k^{i-1}x, k^{i-1}x\right).
\end{aligned}
$$

Hence, the inequality (d) is true for all $n \in \mathbb{N}$.

We now claim that the Rassias sequence $\{k^{-n} f(k^n x)\}$ is a Cauchy sequence. Indeed, by (d), we have

$$
\begin{aligned}
&\|k^{-n} f(k^n x) - k^{-m} f(k^m x)\| \\
&\quad = k^{-m} \left\| k^{-(n-m)} f(k^{n-m} \cdot k^m x) - f(k^m x)\right\| \\
&\quad \le k^{-m} \sum_{i=1}^{n-m} k^{-i} \psi_{k-1}^{a}\left(k^{i-1} \cdot k^m x,\ k^{i-1} \cdot k^m x\right) \\
&\quad \le \sum_{i=m+1}^{\infty} k^{-i} \psi_{k-1}^{a}\left(k^{i-1}x, k^{i-1}x\right)
\end{aligned}
$$

for $n > m$. In view of (2.19), we can make the last term as small as possible by selecting sufficiently large m. Therefore, the given sequence is a Cauchy sequence. Since E is a Banach space, the sequence $\{k^{-n} f(k^n x)\}$ converges for every $x \in G$. Thus, we may define

$$A(x) = \lim_{n\to\infty} k^{-n} f(k^n x).$$

We claim that the function A is additive. By substituting $k^n x$ and $k^n y$ for x and y in (2.20), respectively, we get

$$\left\| f\left(k^n(x+y)\right) - f(k^n x) - f(k^n y)\right\| \le \varphi(k^n x, k^n y). \tag{e}$$

However, it follows from (2.19) that

$$\lim_{n\to\infty} k^{-n} \psi_{k-1}^{a}(k^n x, k^n y) = 0.$$

Therefore, dividing both sides of (e) by k^n and letting $n \to \infty$, we conclude that A is additive.

According to (d), the inequality (2.21) holds true for all $x \in G$.

We assert that A is uniquely determined. Let $A' : G \to E$ be another additive function with the property (2.21). Since A and A' are additive functions satisfying (2.21), we have

$$\begin{aligned}
\|A(x) - A'(x)\| &= \|k^{-n} A(k^n x) - k^{-n} A'(k^n x)\| \\
&\leq k^{-n} \|A(k^n x) - f(k^n x)\| + k^{-n} \|f(k^n x) - A'(k^n x)\| \\
&\leq k^{-n} \Psi_k(k^n x, k^n x) + k^{-n} \Psi_k(k^n x, k^n x) \\
&= 2 \sum_{m=n+1}^{\infty} k^{-m} \psi_{k-1}^{a}\big(k^{m-1} x, k^{m-1} x\big).
\end{aligned}$$

In view of (2.19), we can make the last term as small as possible by selecting sufficiently large n. Hence, it follows that $A = A'$.

Finally, it can also be proved that if G is a Banach space and $f(tx)$ is continuous in t for every fixed $x \in G$, then A is a linear function in the same way as in Theorem 2.5. $\square$

It is worthwhile to note that P. Găvruta, M. Hossu, D. Popescu, and C. Căprău obtained the result of the previous theorem in the papers [118, 119] independently. Furthermore, Y.-H. Lee and K.-W. Jun generalized Theorem 2.20. Indeed, they replaced k with a rational number $a > 1$ in the condition (2.19) and proved Theorem 2.20 (see [233]).

There are still new valuable results for the Hyers–Ulam–Rassias stability of the additive Cauchy equation which were not cited above. Among them we have to state here an outstanding result of G. Isac and Th. M. Rassias [143] without proof:

Let E_1 and E_2 be a real normed space and a real Banach space, respectively. Assume that $f : E_1 \to E_2$ is a function such that $f(tx)$ is continuous in t for every fixed x in E_1. Assume that there exist $\theta \geq 0$ and $p_1, p_2 \in \mathbb{R}$ such that $p_2 \leq p_1 < 1$ or $1 < p_2 \leq p_1$. If f satisfies the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \theta\big(\|x\|^{p_1} + \|y\|^{p_2}\big)$$

for all $x, y \in E_1$, there exists a unique linear function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq \begin{cases} \theta(2 - 2^{p_1})^{-1}\big(\|x\|^{p_1} + \|x\|^{p_2}\big) & (\textit{if } p_2 \leq p_1 < 1), \\ \theta(2^{p_2} - 2)^{-1}\big(\|x\|^{p_1} + \|x\|^{p_2}\big) & (\textit{if } 1 < p_2 \leq p_1) \end{cases}$$

for any $x \in E_1$.

During the 31st International Symposium on Functional Equations, Th. M. Rassias [292] raised an open problem whether we can also expect a similar result for $p_2 < 1 < p_1$.

We here remark that J. M. Rassias [280] considered the case where the Cauchy difference $\|f(x + y) - f(x) - f(y)\|$ in (2.5) is bounded by $\theta\|x\|^p\|y\|^p$ ($\theta \geq 0$,

$0 \le p < 1/2$) and obtained a similar result to that of Theorem 2.5 except the bound for the difference $\|f(x) - A(x)\|$ in (2.6) bounded by $\theta\|x\|^{2p}/(2-2^p)$ instead of $2\theta\|x\|^p/(2-2^p)$. Furthermore, he [281] also proved the Hyers–Ulam–Rassias stability of the additive Cauchy equation for the case where the norm of the difference in (2.5) is bounded by $\theta\|x\|^p\|y\|^q$ ($\theta \ge 0$, $0 \le p+q < 1$) with the modified approximation bound $\theta\|x\|^{p+q}/(2-2^{p+q})$ for $\|f(x) - A(x)\|$ in (2.6).

Several mathematicians have remarked interesting applications of the Hyers–Ulam–Rassias stability theory to various mathematical problems. We will now present some applications of this theory to the study of nonlinear analysis, especially in fixed point theory.

In nonlinear analysis it is well-known that finding the expression of the asymptotic derivative of a nonlinear operator can be a difficult problem. In this sense, we will explain how the Hyers–Ulam–Rassias stability theory can be used to evaluate the asymptotic derivative of some nonlinear operators.

The nonlinear problems considered in this book have been extensively studied by several mathematicians (cf. [6, 54, 226, 242, 364]). G. Isac and Th. M. Rassias were the first mathematicians to introduce the use of the Hyers–Ulam–Rassias stability theory for the study of these problems (see [144]).

Let E be a Banach space. A closed subset K of E is said to be a *cone* if it satisfies the following properties:

(C1) $K + K \subset K$;
(C2) $\lambda K \subset K$ *for all* $\lambda \ge 0$;
(C3) $K \cap (-K) = \{0\}$.

By K^* we denote the *dual* of K, i.e., $K^* = \{\phi \in E^* \mid \phi(x) \ge 0 \text{ for all } x \in K\}$. It is not difficult to see that each cone $K \subset E$ induces an *ordering* on E by the hypothesis that $x \le y$ if and only if $y - x \in K$. If in E a cone is defined, then E is called an *ordered Banach space*. A cone $K \subset E$ is said to be *generating* if $E = K - K$ and it is said to be *normal* if there exists a $\delta \ge 1$ such that $\|x\| \le \delta\|x + y\|$ for all $x, y \in K$. We say that a cone $K \subset E$ is *solid* if its topological interior is nonempty. We call a point $x_0 \in K$ a *quasi-interior point* if $\phi(x_0) > 0$ for any nonzero $\phi \in K^*$. If the cone $K \subset E$ is solid, then the quasi-interior points of K coincide precisely with its interior points.

We denote by $L(E, E)$ the space of linear bounded operators from E into E. It is well-known that for every $T \in L(E, E)$ the *spectral radius* $r(T)$ is well defined, where $r(T) = \max\{|\lambda| \mid \lambda \in \sigma(T)\}$ and $\sigma(T)$ is the *spectrum* of T. We say that $T \in L(E, E)$ is *strictly monotone increasing* if for every pair $x, y \in E$ the relation $x < y$ (i.e., $x \le y$ and $x \ne y$) implies $T(y) - T(x) \in K^\circ$, where K° denotes the interior of K.

Let $D \subset E$ be a bounded set. We define $\gamma(D)$, the *measure of noncompactness* of D, to be the minimum of all positive numbers δ such that D can be covered by finitely many sets of diameter less than δ. A function $f : E \to E$ is said to be a *k-set-contraction* if it is continuous and there exists a $k \ge 0$ such that $\gamma(f(D)) \le k\gamma(D)$ for every bounded subset D of the domain of f. A function $f : E \to E$ is said to be a *strict-set-contraction* if it is a k-set-contraction for some

$k < 1$. A function $f : E \to E$ is called *compact* if it maps bounded subsets of the domain of f onto relatively compact subsets of E and f is said to be *completely continuous* if it is continuous and compact. Every completely continuous function is a strict-set-contraction.

Let K be a generating (or *total*, i.e., $E = \overline{K - K}$) cone in E. The function $f : K \to E$ is said to be *asymptotically differentiable along* K if there exists a $T \in L(E, E)$ such that

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \|f(x) - T(x)\|/\|x\| = 0.$$

In this case, T is the unique function and we call it the *derivative at infinity along* K of f. We say that a function $f : E \to E$ is *asymptotically close to zero along* K if

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \|f(x)\|/\|x\| = 0.$$

Let $\phi : [0, \infty) \to [0, \infty)$ be a function such that $\phi(t) > 0$ for all $t \geq \gamma$, where $\gamma \geq 0$. We say that $f : K \to E$ is *ϕ-asymptotically bounded along* K if there exist $b, c > 0$ such that for all $x \in K$, with $\|x\| \geq b$, we have $\|f(x)\| \leq c\phi\big(\|x\|\big)$. Every ϕ-asymptotically bounded function (along K) such that $\lim_{t \to \infty} \phi(t)/t = 0$ is asymptotically close to zero.

If K is a generating (or total) cone in E, then a function $f : K \to E$ is said to be *differentiable at $x_0 \in K$ along K* if there exists $f'(x_0) \in L(E, E)$ such that

$$\lim_{\substack{x \in K \\ x \to 0}} \|f(x_0 + x) - f(x_0) - f'(x_0)x\|/\|x\| = 0.$$

In this case, $f'(x_0)$ is the *derivative at x_0 along K* of f and it is uniquely determined.

To enlarge the class of the functions ψ such that the condition of Theorem 2.17 remains valid, we consider the following: Let F_ψ be the set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the conditions (i), (ii), and (iii) in Theorem 2.17. Let $P(\Psi)$ be the convex cone generated by the set F_ψ. We remark that a function $\psi \in P(\Psi)$ satisfies the condition (i) but generally does not satisfy the conditions (ii) and (iii) in Theorem 2.17. However, G. Isac and Th. M. Rassias [144] presented that Theorem 2.17 remains valid for each $\psi \in P(\Psi)$.

Theorem 2.21. *Let E_1 be a real normed space, E_2 a real Banach space, and $f : E_1 \to E_2$ a continuous function. Let $\psi \in P(\Psi)$ be given. If f satisfies the inequality* (2.14) *for some $\theta \geq 0$ and for all $x, y \in E_1$, then there exists a unique linear function $T : E_1 \to E_2$ satisfying the inequality* (2.15) *for all $x \in E_1$.*

Proof. We apply Theorem 2.18 for the function

$$\varphi(x, y) = \theta\big(\psi(\|x\|) + \psi(\|y\|)\big).$$

In this case, using the properties of the functions $\psi \in P(\Psi)$, we can show that $\Phi(x, y) < \infty$ for all $x, y \in E_1$, and the conclusion of our theorem follows. □

The last theorem is significant because the class of functions satisfying (2.14) with $\psi \in P(\Psi)$ is strictly larger than the class of functions defined in Theorem 2.17. G. Isac and Th. M. Rassias [144] also proved the following theorem:

Theorem 2.22. *Let E be a Banach space ordered by a generating cone K and let $f : E \to E$, $g : K \to K$ be functions such that:*

(f1) f is completely continuous, positive, and satisfies the inequality (2.14) for some $\psi \in P(\Psi)$ and $\theta > 0$ (i.e., $f(K) \subset K$);
(f2) there exist a quasi-interior point $x_0 \in K$ and $0 < \lambda_0 < 1$ such that $\lim_{n\to\infty} 2^{-n} f(2^n x_0) \leq \lambda_0 x_0$;
(g1) g is asymptotically close to zero along K;
(h1) $h = f + g$ is a strict-set-contraction from K to K.

Then $h = f + g$ has a fixed point in K.

Proof. Let $S = \{x \in E \mid \|x\| = 1\}$. By assumption $(f1)$ and Theorem 2.21, we have that $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ is well defined for every $x \in E$ and T is the unique linear operator satisfying the inequality

$$\|f(x) - T(x)\| \leq \frac{2\theta}{2 - \psi(2)} \psi\big(\|x\|\big) \tag{a}$$

for all $x \in E$. Since f is compact, we have that $f(S)$ is bounded, which implies that T is continuous. Indeed, the continuity of T is a consequence of the following inequalities:

$$\begin{aligned}
\|T(x)\| &\leq \|f(x)\| + \|T(x) - f(x)\| \\
&\leq \|f(x)\| + \frac{2\theta}{2 - \psi(2)} \psi\big(\|x\|\big) \\
&\leq \|f(x)\| + \frac{2\theta}{2 - \psi(2)} \psi(1)
\end{aligned}$$

for all $x \in S$.

From the definition of T and the fact that $f(K) \subset K$, we deduce that T is positive (i.e., $T(K) \subset K$). From (a) and the properties of ψ, it follows that

$$\lim_{\substack{x \in K \\ \|x\| \to \infty}} \|f(x) - T(x)\| / \|x\| = 0,$$

i.e., T is the asymptotic derivative of f along K. In addition, from the principal theorem of [141] or [6], T is completely continuous (and it is also a strict-set-contraction). From $(g1)$ we have

$$\begin{aligned}
&\lim_{\substack{x\in K\\ \|x\|\to\infty}} \|h(x)-T(x)\|/\|x\| \\
&\quad\le \lim_{\substack{x\in K\\ \|x\|\to\infty}} \|g(x)\|/\|x\| + \lim_{\substack{x\in K\\ \|x\|\to\infty}} \|f(x)-T(x)\|/\|x\| \\
&\quad= 0,
\end{aligned}$$

i.e., T is also the asymptotic derivative of h along K.

Since T is completely continuous, its spectrum consists of eigenvalues and zero. Suppose that $r(T) > 0$. From $(f2)$ we have that $T(x_0) \le \lambda_0 x_0$ and using the Krein–Rutman theorem ([364, Proposition 7.26]) we have that there exists $\phi_0 \in K^*\backslash\{0\}$ such that $T^*(\phi_0) = r(T)\phi_0$ and $\phi_0(x_0) > 0$ (since x_0 is a quasi-interior point of K), where we denote by T^* the adjoint of T. Hence, we deduce

$$r(T) = \frac{\big(T^*(\phi_0)\big)(x_0)}{\phi_0(x_0)} = \frac{\phi_0\big(T(x_0)\big)}{\phi_0(x_0)} \le \frac{\phi_0(\lambda_0 x_0)}{\phi_0(x_0)} = \lambda_0,$$

i.e., $r(T) < 1$.

Now, all the assumptions of [6, Theorem 1] are satisfied and hence $h = f + g$ has a fixed point in K. □

Corollary 2.23. *Let E be a Banach space ordered by a generating cone K and let $f : E \to E$ be a function satisfying the conditions (f1) and (f2). Then f has a fixed point in K.*

In Theorem 2.22 and Corollary 2.23 we can replace $(f2)$ with the following condition:

(f3) $\|f(x) - \lambda x\| > 2\theta\big(2-\psi(2)\big)^{-1}\psi\big(\|x\|\big)$ *for all* $\lambda \ge 1$ *and* $x \in K\backslash\{0\}$.

G. Isac and Th. M. Rassias [144] investigated the existence of nonzero positive fixed points. We will express the result of G. Isac and Th. M. Rassias in the following theorem.

Theorem 2.24. *Let E be a Banach space ordered by a generating cone K and let $f : E \to E$, $g : K \to K$ be functions satisfying (f1), (g1), (h1), and:*

(f4) there exist $\lambda_0 > 1$ *and* $x_0 \notin -K$ *such that* $\lim_{n\to\infty} 2^{-n} f(2^n x_0) \ge \lambda_0 x_0$;

(f5) $\|f(x) - x\| > 2\theta\big(2-\psi(2)\big)^{-1}\psi\big(\|x\|\big)$ *for all* $x \in K\backslash\{0\}$;

(h2) h is differentiable at 0 along K and $h(0) = 0$;

(h3) $h'(0)$ does not have a positive eigenvector belonging to an eigenvalue $\lambda \ge 1$.

Then $h = f + g$ has a fixed point in $K\backslash\{0\}$.

Proof. As in the proof of Theorem 2.22, h is asymptotically differentiable along K and its derivative at infinity along K is $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$ for each $x \in E$. Moreover, T is completely continuous and the inequality (a) in the proof of Theorem 2.22 is also satisfied. It follows from $(f5)$ that 1 is not an eigenvalue with corresponding positive eigenvector of T. From $(f4)$ we obtain $r(T) \geq \lambda_0$. Indeed, if $r(T) < \lambda_0$ and since $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ (Gelfand's formula), $\lambda_0^{-n}\|T^n\| \leq k^n$ for sufficiently large n and some $k < 1$. From the fact that $T(x_0) \geq \lambda_0 x_0$ we deduce $\lambda_0^{-n} T^n(x_0) \geq x_0$ (since T is positive) and if we pass to the limit in the last relation, we obtain $x_0 \leq 0$, i.e., $x_0 \in -K$, which is a contradiction. Using the Krein–Rutman theorem again, we have that $r(T)$ is an eigenvalue of T with an eigenvector in K. Thus, all the assumptions of [54, Theorem 1] are satisfied and we conclude that h has a fixed point in $K \setminus \{0\}$. □

Let E be a Banach space ordered by a cone K, and let $L, A : E \to E$ be functions. We say that $\lambda > 0$ is an *asymptotic characteristic value* of (L, A) if L and λA are *asymptotically equivalent* (with respect to K), i.e.,

$$\lim_{\substack{x \in K \\ \|x\| \to 0}} \|L(x) - \lambda A(x)\| / \|x\| = 0.$$

An asymptotic characteristic value of (L, A) is a characteristic value of (L, A) in the sense of the definition provided by M. Mininni in [242], i.e., λ is a *characteristic value* of (L, A) *in Mininni's sense*, if there exists a sequence $\{x_n\}$ of elements of K such that

$$\lim_{n\to\infty} \|x_n\| = \infty \quad \text{and} \quad \lim_{n\to\infty} \|L(x_n) - \lambda A(x_n)\| / \|x_n\| = 0.$$

The following result is a consequence of Theorem 2.24.

Corollary 2.25. *Let E be a Banach space ordered by a generating cone K $(\subset E)$, let $f : E \to E$ be a function such that $f(K) \subset K$, and let $L, A : E \to E$ be functions. Assume that the following conditions are satisfied:*

(i) f satisfies the conditions (f1), (f4), and (f5);
(ii) λ is an asymptotic characteristic value of (L, A);
(iii) $h = f + L - \lambda A$ satisfies the conditions (h1), (h2), and (h3).

Then the nonlinear eigenvalue problem, $L(x_*) + f(x_*) = x_* + \lambda A(x_*)$ with unknowns $x_* \in E \setminus \{0\}$ and $\lambda > 0$, has a solution.

It is well-known that the study of the nonlinear integral equation

$$x(u) = \int_\Omega G(u, v) f\big(v, x(v)\big) dv, \tag{α}$$

known as the *Hammerstein equation*, is of central importance in the study of several boundary-value problems (cf. [226, 364]).

In addition, some special interest is focused on the eigenvalue problem

$$x(u) = \lambda \int_{\Omega} G(u, \nu) f\big(\nu, x(\nu)\big) d\nu. \tag{β}$$

If we denote by G the linear integral operator defined by the kernel $G(u, \nu)$ and by f the Nemyckii's nonlinear operator defined by $f(\nu, x(\nu))$, i.e., $f(u)(\nu) = f(\nu, x(\nu))$, then the equation (β) takes the abstract form

$$x = \lambda G f(x). \tag{γ}$$

For the equation (γ) we consider the following hypotheses:

(H1) (E, K) *and* (F, P) *are real ordered Banach spaces. The cone* K *is normal with nonempty interior.*
(H2) *The function* $f : E \to F$ *is continuous and the operator* $G : F \to E$ *is linear, compact, and positive.*
(H3) G *is strongly positive, i.e.,* $x < y$ *implies* $G(y) - G(x) \in K^{\circ}$.

We recall that $(\lambda_*, +\infty)$ is a bifurcation from infinity of equation (α) if $\lambda_* > 0$ and there is a sequence of solutions (λ_n, x_n) of (γ) such that $\lambda_n \to \lambda_*$ and $\|x_n\| \to \infty$ as $n \to \infty$.

Isac and Rassias [144] also contributed to the following theorem.

Theorem 2.26. *Consider equation* (γ) *and suppose that* $(H1)$, $(H2)$, *and* $(H3)$ *are satisfied. In addition, assume that the function* $f : E \to F$ *satisfies the following conditions:*

(i) $f(K) \subset K$ *and* $f(S)$ *is bounded, where* $S = \big\{x \in E \mid \|x\| = 1\big\}$;
(ii) f *satisfies the inequality* (2.14) *for some* $\theta \geq 0$, $\psi \in P(\Psi)$, *and for all* $x, y \in E$;
(iii) $\lim_{n\to\infty} 2^{-n} f(2^n x) > 0$ *for all* $x \in K \setminus \{0\}$.

If we set $T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$, *for all* $x \in E$, *and* $\lambda_* = r(GT)^{-1}$, *then* $(\lambda_*, +\infty)$ *is the only bifurcation from infinity of equation* (γ).

Proof. First, we note that T is the asymptotic derivative of f along K. Since by assumption (iii) we see that T is strictly positive on K, we remark that $r(GT) > 0$. We set

$$g(x) = \begin{cases} \|x^2\| f\big(x/\|x\|\big) & (\text{for } x \neq 0), \\ 0 & (\text{for } x = 0). \end{cases}$$

We know that $g'(0) = T$ and $(\lambda, +\infty)$ is a bifurcation point of $x = \lambda G f(x)$ if and only if $(\lambda, 0)$ is a bifurcation point of $x = \lambda G g(x)$. Our theorem now follows from a theorem of [364]. $\square$

Concerning Theorem 2.26, the two-sided estimates for the spectral radius obtained by V. Y. Stetsenko [337] are very essential. Stetsenko presented that if

$A : E \to E$ is a completely continuous operator and E is a Banach space ordered by a generating closed cone K with quasi-interior points and if some special assumptions are satisfied, then we can define the numbers λ_0, ρ, the vectors u_0, ν_0, and a functional ϕ_0 such that

$$\lambda_0 - \rho \frac{\phi_0(\nu_0)}{\phi_0(u_0)} \le r(A) \le \lambda_0 - \frac{1}{\rho} \frac{\phi_0(\nu_0)}{\phi_0(u_0)}. \tag{δ}$$

G. Isac and Th. M. Rassias [144] raised an interesting problem to find new and more efficient two-sided estimates for the spectral radius of the operator GT with

$$T(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

when f satisfies the inequality (2.14) for some $\theta \ge 0$, $\psi \in P(\Psi)$, and for all $x, y \in E$.

Such an estimate of $r(GT)$, similar to the estimate (δ), is important for the approximation of the bifurcation point $(\lambda_*, +\infty)$ of the equation (γ).

2.4 Stability on a Restricted Domain

In previous sections, we have seen that the condition that a function f satisfies one of the inequalities (2.2), (2.5), (2.10), (2.14), (2.17), and (2.20) on the whole space, assures us of the existence of a unique additive function approximating f within a given distance.

It will also be interesting to study the stability problems of the additive Cauchy equation on a restricted domain. More precisely, we will study whether there exists a true additive function near a function satisfying one of those inequalities mentioned above only in a restricted domain.

F. Skof [330] and Z. Kominek [224] studied this question in the case of functions defined on certain subsets of $\mathbb{R}^N$ with values in a Banach space. First, we will introduce a lemma by Skof [330] which is necessary to prove the following propositions.

Lemma 2.27. *Let E be a Banach space. If a function $f : [0,\infty) \to E$ satisfies*

$$\|f(x+y) - f(x) - f(y)\| \le \delta$$

for some $\delta \ge 0$ and for all $x, y \ge 0$, then there exists a unique additive function $A : \mathbb{R} \to E$ such that

$$\|f(x) - A(x)\| \le \delta$$

for any $x \ge 0$.

Proof. Define a function $g : \mathbb{R} \to E$ by

$$g(x) = \begin{cases} f(x) & (\text{for } x \geq 0), \\ -f(-x) & (\text{for } x < 0). \end{cases}$$

It is not difficult to see that

$$\|g(x+y) - g(x) - g(y)\| \leq \delta$$

for all $x, y \in \mathbb{R}$. Therefore, by Theorem 2.3, there exists a unique additive function $A : \mathbb{R} \to E$ such that

$$\|g(x) - A(x)\| \leq \delta \tag{a}$$

for any $x \in \mathbb{R}$. This function A is the unique additive function which satisfies (a) for any $x \geq 0$. □

F. Skof [330] proved the following lemma for $N = 1$, and Z. Kominek [224] extended it for any positive integer N.

Lemma 2.28. *Let E be a Banach space and let N be a given positive integer. Given $c > 0$, let a function $f : [0, c)^N \to E$ satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{2.22}$$

for some $\delta \geq 0$ and for all $x, y \in [0, c)^N$ with $x + y \in [0, c)^N$. Then there exists an additive function $A : \mathbb{R}^N \to E$ such that

$$\|f(x) - A(x)\| \leq (4N - 1)\delta$$

for any $x \in [0, c)^N$.

Proof. First, we consider the case $N = 1$. We extend the function f to $[0, \infty)$ and then make use of Lemma 2.27 above. Let us represent an arbitrary $x \geq 0$ by $x = (1/2)nc + \mu$, where n is a nonnegative integer and $0 \leq \mu < (1/2)c$. Define a function $g : [0, \infty) \to E$ by

$$g(x) = f(\mu) + nf\big((1/2)c\big).$$

On the interval $[0, c)$ we have

$$\|g(x) - f(x)\| \leq \delta. \tag{a}$$

In fact, when $0 \leq x < (1/2)c$, we have $g(x) = f(x)$. When $(1/2)c \leq x < c$, we get

$$\|g(x) - f(x)\| = \big\|f(\mu) + f\big((1/2)c\big) - f\big((1/2)c + \mu\big)\big\| \leq \delta,$$

since f satisfies (2.22).

We will now show that g satisfies

$$\|g(x+y)-g(x)-g(y)\| \le 2\delta \tag{b}$$

for all $x, y \ge 0$. For given $x, y \ge 0$, let $x = (1/2)nc + \mu$ and $y = (1/2)mc + \omega$, where n and m are nonnegative integers and μ and ω belong to the interval $[0, (1/2)c)$. Assume $\mu + \omega \in [0, (1/2)c)$. Then

$$\begin{aligned}
&\|g(x+y)-g(x)-g(y)\| \\
&\quad = \big\| f(\mu+\omega) + (m+n) f\big((1/2)c\big) - f(\mu) \\
&\qquad - nf\big((1/2)c\big) - f(\omega) - mf\big((1/2)c\big)\big\| \\
&\quad = \| f(\mu+\omega) - f(\mu) - f(\omega)\| \\
&\quad \le \delta.
\end{aligned}$$

Assume now that $\mu + \omega \in [(1/2)c, c)$. Put $\mu + \omega = (1/2)c + t$. Then, by (a), we have

$$\begin{aligned}
&\|g(x+y)-g(x)-g(y)\| \\
&\quad = \big\| f\big((1/2)c\big) + f(t) - f(\mu) - f(\omega)\big\| \\
&\quad = \|g(\mu+\omega) - f(\mu+\omega) + f(\mu+\omega) - f(\mu) - f(\omega)\| \\
&\quad \le 2\delta.
\end{aligned}$$

According to Lemma 2.27, (b) implies that there exists a unique additive function $A : \mathbb{R} \to E$ such that

$$\|g(x) - A(x)\| \le 2\delta \tag{c}$$

for any $x \ge 0$. Therefore, by (a) and (c), we have

$$\|f(x) - A(x)\| \le \|f(x) - g(x)\| + \|g(x) - A(x)\| \le 3\delta$$

for all $x \in [0, c)$, which completes the proof for $N = 1$.

Assume now that $N > 1$. If we define the functions $f_i : [0, c) \to E$ for $i \in \{1, \ldots, N\}$ by

$$f_i(x_i) = f(0, \ldots, 0, x_i, 0, \ldots, 0),$$

then the functions f_i satisfy

$$\|f_i(x_i + y_i) - f_i(x_i) - f_i(y_i)\| \le \delta$$

for all $x_i, y_i \in [0, c)$ with $x_i + y_i \in [0, c)$. Thus, the first part of this proof guarantees the existence of additive functions $A_i : \mathbb{R} \to E$ such that

$$\|f_i(x_i) - A_i(x_i)\| \le 3\delta \tag{d}$$

for all $x_i \in [0, c)$.

Representing $x \in \mathbb{R}^N$ in the form $(x_1, \ldots, x_N)$ we see that the function $A : \mathbb{R}^N \to E$ given by

$$A(x) = \sum_{i=1}^{N} A_i(x_i)$$

is an additive function. Moreover, for each $x \in [0, c)^N$, by (d) and (2.22), we get

$$\begin{aligned}
&\|f(x) - A(x)\| \\
&\le \left\| f(x) - \sum_{i=1}^{N} f_i(x_i) \right\| + \sum_{i=1}^{N} \|f_i(x_i) - A_i(x_i)\| \\
&\le \left\| f(x_1, \ldots, x_{N-1}, 0) - \sum_{i=1}^{N-1} f_i(x_i) \right\| \\
&\quad + \| - f(0, \ldots, 0, x_N) - f(x_1, \ldots, x_{N-1}, 0) + f(x)\| + 3N\delta \\
&\le \left\| f(x_1, \ldots, x_{N-1}, 0) - \sum_{i=1}^{N-1} f_i(x_i) \right\| + \delta + 3N\delta \\
&\le \cdots \\
&\le (N-1)\delta + 3N\delta.
\end{aligned}$$

This ends the proof. □

Lemma 2.29. *Let E be a Banach space and let N be a positive integer. Given $c > 0$, let a function $f : (-c, c)^N \to E$ satisfy the inequality (2.22) for some $\delta \ge 0$ and for all $x, y \in (-c, c)^N$ with $x + y \in (-c, c)^N$. Then there exists an additive function $A : \mathbb{R}^N \to E$ such that*

$$\|f(x) - A(x)\| \le (5N - 1)\delta$$

for any $x \in (-c, c)^N$.

Proof. First, we prove the assertion for $N = 1$. Put

$$g(x) = (1/2)\big(f(x) - f(-x)\big) \quad \text{and} \quad h(x) = (1/2)\big(f(x) + f(-x)\big)$$

for all $x \in (-c, c)$. We note that

$$\|h(x)\| \le \delta \tag{a}$$

for $x \in (-c, c)$ and

$$\|g(x + y) - g(x) - g(y)\| \le \delta$$

for all $x, y \in (-c, c)$ with $x + y \in (-c, c)$. According to Lemma 2.28, there exists an additive function $A : \mathbb{R} \to E$ such that $\|g(x) - A(x)\| \le 3\delta$ for each $x \in [0, c)$.

Since g and A are odd, $\|g(x) - A(x)\| \leq 3\delta$ holds true for any $x \in (-c, c)$. Hence, by (a), we get

$$\|f(x) - A(x)\| \leq \|h(x)\| + \|g(x) - A(x)\| \leq 4\delta$$

for all $x \in (-c, c)$. This completes the proof for the case $N = 1$.

Assume now that $N > 1$. The proof runs in the same way as in the proof of Lemma 2.28. We define functions $f_i : (-c, c) \to E$ by

$$f_i(x_i) = f(0, \ldots, 0, x_i, 0, \ldots, 0)$$

for $i \in \{1, \ldots, N\}$. Then, the functions f_i satisfy

$$\|f_i(x_i + y_i) - f_i(x_i) - f_i(y_i)\| \leq \delta$$

for all $x_i, y_i \in (-c, c)$ with $x_i + y_i \in (-c, c)$. Thus, the first part of our proof implies that there exist additive functions $A_i : \mathbb{R} \to E$ such that

$$\|f_i(x_i) - A_i(x_i)\| \leq 4\delta \tag{b}$$

for $x_i \in (-c, c)$.

Expressing $x \in \mathbb{R}^N$ in the form $(x_1, \ldots, x_N)$ we see that the function $A : \mathbb{R}^N \to E$ defined by

$$A(x) = \sum_{i=1}^{N} A_i(x_i)$$

is additive. Let $x \in (-c, c)^N$ be given. Then, by (b) and (2.22), we have

$$\begin{aligned}
\|f(x) - A(x)\| &\leq \left\| f(x) - \sum_{i=1}^{N} f_i(x_i) \right\| + \sum_{i=1}^{N} \|f_i(x_i) - A_i(x_i)\| \\
&\leq \|f(x) - f(0, \ldots, 0, x_N) - f(x_1, \ldots, x_{N-1}, 0)\| \\
&\quad + \left\| f(x_1, \ldots, x_{N-1}, 0) - \sum_{i=1}^{N-1} f_i(x_i) \right\| + 4N\delta \\
&\leq \left\| f(x_1, \ldots, x_{N-1}, 0) - \sum_{i=1}^{N-1} f_i(x_i) \right\| + \delta + 4N\delta \\
&\leq \cdots \\
&\leq (N-1)\delta + 4N\delta,
\end{aligned}$$

which ends the proof. □

Using Corollary 2.23, Z. Kominek [224] proved a more generalized theorem concerning the stability of the additive Cauchy equation on a restricted domain.

Theorem 2.30 (Kominek). *Let E be a Banach space and let N be a positive integer. Suppose D is a bounded subset of $\mathbb{R}^N$ containing zero in its interior. Assume, moreover, that there exist a nonnegative integer n and a positive number $c > 0$ such that*

(i) $(1/2)D \subset D$,
(ii) $(-c, c)^N \subset D$,
(iii) $D \subset (-2^n c, 2^n c)^N$.

If a function $f : D \to E$ satisfies the functional inequality

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{2.23}$$

for some $\delta \geq 0$ and for all $x, y \in D$ with $x + y \in D$, then there exists an additive function $A : \mathbb{R}^N \to E$ such that

$$\|f(x) - A(x)\| \leq (2^n \cdot 5N - 1)\delta$$

for any $x \in D$.

Proof. On account of Lemma 2.29, there exists an additive function $A : \mathbb{R}^N \to E$ such that

$$\|f(x) - A(x)\| \leq (5N - 1)\delta \tag{a}$$

for any $x \in (-c, c)^N$. Taking an arbitrary $x \in D$ we observe, by (i), that $2^{-k}x \in D$ for $k \in \{1, \ldots, n\}$, and condition (iii) implies that $2^{-n}x \in (-c, c)^N$. It follows from (2.23) that for every $x \in D$ and each $k \in \{1, \ldots, n\}$

$$\left\|2^{k-1} f\left(2^{-k+1}x\right) - 2^k f\left(2^{-k}x\right)\right\| \leq 2^{k-1}\delta,$$

therefore,

$$\|f(x) - 2^n f(2^{-n}x)\| \leq (2^n - 1)\delta. \tag{b}$$

Now, by (a), (b), and the additivity of A, we get

$$\begin{aligned}
\|f(x) - A(x)\| &\leq \|f(x) - 2^n f(2^{-n}x)\| \\
&\quad + \|2^n f(2^{-n}x) - 2^n A(2^{-n}x)\| \\
&\leq (2^n - 1)\delta + 2^n(5N - 1)\delta \\
&= (2^n \cdot 5N - 1)\delta
\end{aligned}$$

for any $x \in D$, which ends the proof. □

L. Losonczi [238] proved the following theorem and applied it to the study of the Hyers–Ulam stability for Hosszú's equation (see Theorem 4.5).

Theorem 2.31. *Let E_1 and E_2 be a normed space and a Banach space, respectively. Suppose a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \tag{2.24}$$

for some $\delta \geq 0$ and for all $(x, y) \in E_1^2 \backslash B$, where B is a subset of E_1^2 such that the first (or second) coordinates of the points of B form a bounded set. Then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \leq 5\delta \tag{2.25}$$

for any $x \in E_1$.

Proof. Since the left-hand side of (2.24) is symmetric in x and y, we may assume that (2.24) holds true for all $(x, y) \in C = (E_1^2 \setminus B) \cup (E_1^2 \setminus B_s)$, where $B_s = \{(x, y) \in E_1^2 \mid (y, x) \in B\}$. Since

$$C = \left(E_1^2 \backslash B\right) \cup \left(E_1^2 \backslash B_s\right) = E_1^2 \backslash (B \cap B_s)$$

and both coordinates of the points from $B \cap B_s$ form a bounded set, we can find a number $a > 0$ such that $B \cap B_s \subset Q$, where

$$Q = \{(x, y) \in E_1^2 \mid \|x\| < a,\ \|y\| < a\}.$$

Choose a $t \in E_1$ such that $\|t\| = 2a$ and take any point $(u, v) \in B \cap B_s$. We then know that all the points

$$\big(u+v, 2t-(u+v)\big),\ \ (u, t-u),\ \ (t-v, v),\ \ (t-u, t-v),\ \ (t, t) \tag{a}$$

are in $E_1^2 \backslash Q$ and hence also in $E_1^2 \backslash (B \cap B_s)$, since the inequalities

$$\begin{aligned}
&\|2t - (u+v)\| \geq \|2t\| - \|u+v\| \geq 4a - 2a > a,\\
&\|t-u\| \geq \|t\| - \|u\| \geq a,\\
&\|t-v\| \geq \|t\| - \|v\| \geq a,\\
&\|t\| > a
\end{aligned}$$

are true. Thus, if $(u, v) \in B \cap B_s$, we get

$$\begin{aligned}
&\|f(u+v) - f(u) - f(v)\|\\
&\quad \leq \|f(u+v) + f(2t-(u+v)) - f(2t)\|\\
&\qquad + \|-f(2t-(u+v)) + f(t-u) + f(t-v)\|\\
&\qquad + \|-f(t-u) - f(u) + f(t)\|\\
&\qquad + \|-f(v) - f(t-v) + f(t)\|\\
&\qquad + \|-f(t) - f(t) + f(2t)\|\\
&\quad \leq 5\delta,
\end{aligned}$$

where we applied (2.24) for each of the points listed in (a).

If $(u, v) \in E_1^2 \backslash (B \cap B_s)$, then we obtain

$$\|f(u+v) - f(u) - f(v)\| \le \delta$$

as we have already seen. Hence, the inequality

$$\|f(x+y) - f(x) - f(y)\| \le 5\delta$$

holds true for all $x, y \in E_1$. According to Theorem 2.3, there exists a unique additive function $A : E_1 \to E_2$ such that the inequality (2.25) is true for all $x \in E_1$ because the assertion of Theorem 2.3 also holds true for the case when the domain E_1 of the function f is extended to an amenable semigroup and because each normed space is an amenable semigroup (see Corollary 2.42 below). □

F. Skof [331] proved the following theorem and applied the result to the study of an asymptotic behavior of additive functions.

Theorem 2.32 (Skof). *Let E be a Banach space, and let $a > 0$ be a given constant. Suppose a function $f : \mathbb{R} \to E$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \delta \tag{2.26}$$

for some $\delta \ge 0$ and for all $x, y \in \mathbb{R}$ with $|x| + |y| > a$. Then there exists a unique additive function $A : \mathbb{R} \to E$ such that

$$\|f(x) - A(x)\| \le 9\delta \tag{2.27}$$

for all $x \in \mathbb{R}$.

Proof. Put

$$\varphi(x, y) = f(x+y) - f(x) - f(y)$$

for $x, y \in \mathbb{R}$. Suppose real numbers x and $y \ne 0$ are given. Let m and n be integers greater than 1. We then have

$$f(nx + my) = f(nx) + f(my) + \varphi(nx, my)$$

and

$$\begin{aligned} f(nx + my) &= f\big((n-1)x + (x + my)\big) \\ &= f\big((n-1)x\big) + f(x) + f(my) \\ &\quad + \varphi(x, my) + \varphi\big((n-1)x, x + my\big), \end{aligned}$$

from which it follows that

$$\begin{aligned} &f(nx) - f\big((n-1)x\big) - f(x) \\ &\quad = \varphi(x, my) + \varphi\big((n-1)x, x + my\big) - \varphi(nx, my). \end{aligned}$$

If m is so large that $m > \big(a + |x|\big)/|y|$, the last equality implies

$$\big\| f(nx) - f\big((n-1)x\big) - f(x)\big\| \leq 3\delta \tag{a}$$

for any $x \in \mathbb{R}$ and any integer $n > 1$. The relation

$$f(nx) - nf(x) = \sum_{k=2}^{n} \big(f(kx) - f((k-1)x) - f(x)\big),$$

together with (a), yields

$$\| f(nx) - nf(x)\| \leq 3(n-1)\delta \tag{b}$$

for any real number x and $n > 1$. Obviously, it follows from (b) that

$$\begin{aligned} \|\varphi(nx, ny) - n\varphi(x, y)\| &\leq \| f(nx + ny) - nf(x + y)\| \\ &\quad + \| f(nx) - nf(x)\| + \| f(ny) - nf(y)\| \\ &\leq 9(n-1)\delta \end{aligned}$$

for all real numbers x and y with $(x, y) \neq (0, 0)$. Dividing both sides of the last inequality by n, then letting $n \to \infty$ and considering the fact that $(1/n)\varphi(nx, ny) \to 0$ as $n \to \infty$, we get

$$\|\varphi(x, y)\| \leq 9\delta \tag{c}$$

for all x, y with $(x, y) \neq (0, 0)$. Let x be a real number with $|x| > a$. Then, the inequality (2.26) with such an x and $y = 0$ yields $\| f(0)\| \leq \delta$. Hence, we obtain $\|\varphi(0, 0)\| = \| f(0)\| \leq \delta$. Therefore, (c) holds true for all $x, y \in \mathbb{R}$.

According to Theorem 2.3, there is a unique additive function $A : \mathbb{R} \to E$ such that the inequality (2.27) holds true for any x in $\mathbb{R}$. □

Analogously, we can easily generalize the last result. More precisely, we can extend the domain of the function f in the last theorem to an arbitrary normed space. Here, we remark that the domain E_1 of the function f in Theorem 2.3 can be extended to a normed space without reduction of the validity of the theorem. The proof of the following theorem given by F. Skof [331] is the same as that of the last theorem.

Theorem 2.33. *Let E_1 and E_2 be a normed space and a Banach space, respectively. Given $a > 0$, suppose a function $f : E_1 \to E_2$ satisfies*

$$\| f(x + y) - f(x) - f(y)\| \leq \delta$$

for some $\delta \geq 0$ *and for all* $x, y \in E_1$ *with* $\|x\| + \|y\| > a$. *Then there exists a unique additive function* $A : E_1 \to E_2$ *such that*

$$\|f(x) - A(x)\| \leq 9\delta$$

for all $x \in E_1$.

Using this theorem, F. Skof [331] has studied an interesting asymptotic behavior of additive functions as we see in the following theorem.

Theorem 2.34 (Skof). *Let* E_1 *and* E_2 *be a normed space and a Banach space, respectively. Suppose* z *is a fixed point of* E_1. *For a function* $f : E_1 \to E_2$ *the following two conditions are equivalent:*

(i) $f(x + y) - f(x) - f(y) \to z$ *as* $\|x\| + \|y\| \to \infty$;
(ii) $f(x + y) - f(x) - f(y) = z$ *for all* $x, y \in E_1$.

Proof. It suffices to prove the implication $(i) \Rightarrow (ii)$ only because the reverse implication is a trivial case. Define $g(x) = f(x) + z$ for all $x \in E_1$. Then the condition (i) implies that

$$g(x + y) - g(x) - g(y) \to 0 \quad \text{as} \quad \|x\| + \|y\| \to \infty. \tag{a}$$

Due to (a) there is a sequence $\{\delta_n\}$ monotonically decreasing to zero such that

$$\|g(x + y) - g(x) - g(y)\| \leq \delta_n$$

for all $x, y \in E_1$ with $\|x\| + \|y\| > n$. According to Theorem 2.33, there exists a unique additive function $A_n : E_1 \to E_2$ satisfying

$$\|g(x) - A_n(x)\| \leq 9\delta_n \tag{b}$$

for all $x \in E_1$. Let m and n be integers with $n > m > 0$. Then the additive function $A_n : E_1 \to E_2$ satisfies $\|g(x) - A_n(x)\| \leq 9\delta_m$ for all $x \in E_1$. The uniqueness argument implies $A_n = A_m$ for all integers n greater than $m > 0$. Therefore, $A_1 = A_2 = \ldots = A_n = \ldots$. If we define a function $A : E_1 \to E_2$ by $A(x) = A_n(x)$ for all $x \in E_1$ and for some $n > 1$, then A is an additive function. Letting $n \to \infty$ in (b), we conclude that g itself is an additive function. Thus,

$$0 = g(x + y) - g(x) - g(y) = f(x + y) - f(x) - f(y) - z$$

for all $x, y \in E_1$. □

The following corollary is an immediate consequence of the last theorem (see [331]).

Corollary 2.35. *The function* $f : \mathbb{R} \to \mathbb{R}$ *is an additive function if and only if* $f(x+y) - f(x) - f(y) \to 0$ *as* $|x| + |y| \to \infty$.

Skof [334] dealt with functionals on a real Banach space and proved a theorem concerning the stability of the alternative equation $|f(x+y)| = |f(x) + f(y)|$:

Let E *be a real Banach space, and let* M *be a given positive number. If a functional* $f : E \to \mathbb{R}$ *satisfies the inequality*

$$\big||f(x+y)| - |f(x) + f(y)|\big| \le \delta$$

for some $\delta \ge 0$ *and for all* $x, y \in E$ *with* $\|x\| + \|y\| > M$, *then there exists a unique additive functional* $A : E \to \mathbb{R}$ *such that*

$$|f(x) - A(x)| \le 40\delta + 11|f(0)| + 11 \max\{\delta,\ |f(0)|\}$$

for all $x \in E$.

On the basis of this result, Skof could obtain the following characterization of additive functionals (see [334]):

Let E *be a real Banach space. The functional* $f : E \to \mathbb{R}$ *is additive if and only if* $f(0) = 0$ *and* $|f(x+y)| - |f(x) + f(y)| \to 0$ *as* $\|x\| + \|y\| \to \infty$.

D. H. Hyers, G. Isac, and Th. M. Rassias [136] have proved a Hyers–Ulam–Rassias stability result of the additive Cauchy equation on a restricted domain and applied it to the study of the asymptotic derivability which is very important in nonlinear analysis.

Theorem 2.36 (Hyers, Isac, and Rassias). *Given a real normed space* E_1 *and a real Banach space* E_2, *let numbers* $M > 0$, $\theta > 0$, *and* p *with* $0 < p < 1$ *be chosen. Let a function* $f : E_1 \to E_2$ *satisfy the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big) \tag{2.28}$$

for all $x, y \in E_1$ *satisfying* $\|x\|^p + \|y\|^p > M^p$. *Then there exists an additive function* $A : E_1 \to E_2$ *such that*

$$\|f(x) - A(x)\| \le 2\theta\big(2 - 2^p\big)^{-1}\|x\|^p \tag{2.29}$$

for all $x \in E_1$ *with* $\|x\| > 2^{-1/p}M$.

Proof. When $\|x\| > 2^{-1/p}M$ or $2\|x\|^p > M^p$, we may put $y = x$ in (2.28) to obtain

$$\|(1/2)f(2x) - f(x)\| \le \theta\|x\|^p. \tag{a}$$

Since $\|2x\|$ is also greater than $2^{-1/p}M$, we can replace x with $2x$ in (a). Thus, we can use the induction argument given in Theorem 2.5 to obtain the inequality

$$\|2^{-n}f(2^n x) - f(x)\| \leq 2\theta\big(2-2^p\big)^{-1}\|x\|^p \tag{b}$$

for all $x \in E_1$ with $\|x\| > 2^{-1/p}M$ and for any $n \in \mathbb{N}$. Hence, the limit

$$g(x) = \lim_{n\to\infty} 2^{-n}f(2^n x) \tag{c}$$

exists when $\|x\| > 2^{-1/p}M$. Therefore,

$$\|g(x) - f(x)\| \leq 2\theta\big(2-2^p\big)^{-1}\|x\|^p. \tag{d}$$

Clearly, when $\|x\| > 2^{-1/p}M$, we have

$$g(2x) = \lim_{n\to\infty} 2^{-n}f\big(2^{n+1}x\big) = 2\lim_{n\to\infty} 2^{-n-1}f\big(2^{n+1}x\big),$$

so that

$$g(2x) = 2g(x) \tag{e}$$

for $\|x\| > 2^{-1/p}M$.

Assume now that $\|x\|$, $\|y\|$, and $\|x+y\|$ are all greater than $2^{-1/p}M$. Then, by (2.28), we find that

$$\big\|2^{-n}f\big(2^n(x+y)\big) - 2^{-n}f(2^n x) - 2^{-n}f(2^n y)\big\| \leq \theta 2^{-n(1-p)}\big(\|x\|^p + \|y\|^p\big)$$

for all $n \in \mathbb{N}$. It then follows from (c) that

$$g(x+y) = g(x) + g(y)$$

for all $x, y \in E_1$ with $\|x\|,\ \|y\|,\ \|x+y\| > 2^{-1/p}M$. Using an extension theorem of F. Skof [334], we will define a function $A : E_1 \to E_2$ to be an extension of the function g to the whole space E_1. Given any $x \in E_1$ with $0 < \|x\| \leq 2^{-1/p}M$, let $k = k(x)$ denote the largest integer such that

$$2^{-1/p}M < 2^k\|x\| \leq M. \tag{f}$$

Now, define a function $A : E_1 \to E_2$ by

$$A(x) = \begin{cases} 0 & (\text{for } \ x = 0), \\ 2^{-k}g\big(2^k x\big) & (\text{for } \ 0 < \|x\| \leq 2^{-1/p}M, \ \text{where } k = k(x)), \\ g(x) & (\text{for } \ \|x\| > 2^{-1/p}M). \end{cases}$$

If we take $x \in E_1$ with $0 < \|x\| \le 2^{-1/p}M$, then $k-1$ is the largest integer satisfying

$$2^{-1/p}M < \left\|2^{k-1}(2x)\right\| \le M,$$

and we have

$$A(2x) = 2^{-(k-1)}g\left(2^{k-1}2x\right) = 2^{-k}\cdot 2g\left(2^k x\right) = 2A(x)$$

for $0 < \|x\| \le 2^{-1/p}M$. From the definition of A and (e), it follows that $A(2x) = 2A(x)$ for all $x \in E_1$. Given $x \in E_1$ with $x \ne 0$, choose a positive integer m so large that $\|2^m x\| > 2^{-1/p}M$. By the definition of A, we have

$$A(x) = 2^{-m}A(2^m x) = 2^{-m}g(2^m x),$$

and by (c) this implies that

$$A(x) = \lim_{n\to\infty} 2^{-(m+n)} f(2^{m+n}x),$$

which demonstrates

$$A(x) = \lim_{s\to\infty} 2^{-s} f(2^s x) \qquad (g)$$

for $x \ne 0$. Since $A(0) = 0$, (g) holds true for all $x \in E_1$.

We will now present that

$$A(-x) = -A(x) \qquad (h)$$

for all $x \in E_1$. It is obvious for $x = 0$. Take any $x \in E_1\backslash\{0\}$ and choose an integer n so large that $\|2^n x\| > 2^{-1/p}M$. If we replace x and y in (2.28) with $2^n x$ and $-2^n x$, respectively, and divide the resulting inequality by 2^n, then we obtain

$$\|2^{-n} f(2^n x) + 2^{-n} f(-2^n x)\| \le 2\theta 2^{-n(1-p)}\|x\|^p + 2^{-n}\|f(0)\|.$$

If we let $n \to \infty$ in the last inequality, (h) follows from (g).

We note that the equation

$$A(x+y) = A(x) + A(y) \qquad (i)$$

holds true when either x or y is zero. Assume then that $x \ne 0$ and $y \ne 0$. If $x + y = 0$, then (h) implies the validity of (i). The only remaining case is when x, y, and $x + y$ are all different from zero. In this case we may choose an integer n such that $\|2^n x\|$, $\|2^n y\|$, and $\|2^n(x+y)\|$ are all greater than $2^{-1/p}M$. Then, by (2.28), we get

$$\left\| f\left(2^n(x+y)\right) - f(2^n x) - f(2^n y)\right\| \le \theta 2^{np}\left(\|x\|^p + \|y\|^p\right).$$

If we divide both sides of this inequality by 2^n and then let $n \to \infty$, we find by (g) that (i) is true, thus A is additive.

By definition we have $A(x) = g(x)$ when $\|x\| > 2^{-1/p}M$, thus (2.29) follows from (d). □

For convenience in applications, we give the following modified version of Theorem 2.36 (ref. [136]).

Theorem 2.37. *Given a real normed space E_1 and a real Banach space E_2, let numbers $m > 0$, $\theta > 0$, and p with $0 \le p < 1$ be chosen. Suppose a function $f : E_1 \to E_2$ satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in E_1$ such that $\|x\| > m$, $\|y\| > m$, and $\|x+y\| > m$. Then there exists an additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le 2\theta(2-2^p)^{-1}\|x\|^p$$

for all $x \in E_1$ with $\|x\| > m$.

Proof. Assume that $\|x\| > m$. Then as in the proof of Theorem 2.36 we obtain (a) – (e) (in the proof of Theorem 2.36) inclusive, but now all these formulas are satisfied for $\|x\| > m$. In particular,

$$g(x) = \lim_{n\to\infty} 2^{-n} f(2^n x)$$

when $\|x\| > m$. Also, if $\|x\| > m$, $\|y\| > m$, and $\|x+y\| > m$, then by hypothesis we see that

$$\big\|2^{-n} f\big(2^n(x+y)\big) - 2^{-n} f(2^n x) - 2^{-n} f(2^n y)\big\| \le \theta 2^{-n(1-p)}\big(\|x\|^p + \|y\|^p\big)$$

and

$$g(x+y) = g(x) + g(y)$$

also hold true.

To apply Skof's extension procedure in the present case, let x in E_1 be given with $0 < \|x\| \le m$ and define $k = k(x)$ to be the unique positive integer such that

$$m < 2^k\|x\| \le 2m. \tag{j}$$

Now define a function $A : E_1 \to E_2$ by

$$A(x) = \begin{cases} 0 & (\text{for } \ x = 0), \\ 2^{-k} g\big(2^k x\big) & (\text{for } \ 0 < \|x\| \le m), \\ g(x) & (\text{for } \ \|x\| > m). \end{cases}$$

The proofs, of (g) and (h), in the proof of the last theorem follow as before with the obvious changes. Indeed, we start with $x \in E_1$ satisfying $0 < \|x\| \leq m$ and let $k = k(x)$ as defined by (j), etc. Thus, (g) and (h) mentioned above hold true under the conditions of this theorem. The proof of the additivity of A also follows as before. □

We apply the result of Theorem 2.37 to the study of p-asymptotical derivatives: Let E_1 and E_2 be Banach spaces. Suppose $A : E_1 \to E_2$ is a function satisfying eventually a special property, for example, additivity, linearity, etc.

Let $0 < p < 1$ be arbitrary. A function $f : E_1 \to E_2$ is called *p-asymptotically close* to A if and only if $\|f(x) - A(x)\|/\|x\|^p \to 0$ as $\|x\| \to \infty$. Moreover, if $A \in L(E_1, E_2)$, then we say that A is a *p-asymptotical derivative* of f and if such an A exists, then f is called *p-asymptotically derivable*.

A function $f : E_1 \to E_2$ is called *p-asymptotically additive* if and only if for every $\theta > 0$ there exists a $\delta > 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in E_1$ such that $\|x\|^p,\ \|y\|^p,\ \|x+y\|^p > \delta$.

A function $A : E_1 \to E_2$ is called *additive outside a ball* if there exists an $r > 0$ such that $A(x+y) = A(x) + A(y)$ for all $x, y \in E_1$ such that $\|x\| \geq r$, $\|y\| \geq r$, and $\|x+y\| \geq r$.

Hyers, Isac, and Rassias [136] contributed to the following Theorems 2.38, 2.39, and Corollary 2.40.

Theorem 2.38. *If a function $f : E_1 \to E_2$ is p-asymptotically close to a function $A : E_1 \to E_2$ which is additive outside a ball, then f is p-asymptotically additive.*

Theorem 2.39. *If a function $f : E_1 \to E_2$ is p-asymptotically close to a function $A : E_1 \to E_2$ which is additive outside a ball, then f is q-asymptotically close to an additive function, where* $0 < p < q < 1$.

Corollary 2.40. *If a function $f : E_1 \to E_2$ is p-asymptotically close to a function $A : E_1 \to E_2$ which is additive outside a ball, then f has an additive q-asymptotical derivative, where* $0 < p < q < 1$.

2.5 Method of Invariant Means

So far we have dealt with generalizations of Theorem 2.3 in connection with the bounds for the Cauchy difference. Now, we will briefly introduce another generalization of the theorem from the point of view of the domain space of the functions involved.

Let $(G, \cdot)$ be a semigroup or group and let $B(G)$ denote the space of all bounded complex-valued functions on G with the norm

$$\|f\| = \sup\{f(x) \mid x \in G\}.$$

A linear functional m on $B(G)$ is called a *right invariant mean* if the following conditions are satisfied:

(i) $m(\overline{f}) = \overline{m(f)}$ *for* $f \in B(G)$,
(ii) $\inf\{f(x) \mid x \in G\} \leq m(f) \leq \sup\{f(x) \mid x \in G\}$ *for all real-valued* $f \in B(G)$,
(iii) $m(f_x) = m(f)$ *for all* $x \in G$ *and* $f \in B(G)$, *where* $f_x(t) = f(tx)$.

If (*iii*) in the above definition is replaced with $m({}_x f) = m(f)$, where ${}_x f(t) = f(xt)$, then m is called a *left invariant mean*.

When a right (left) invariant mean exists on $B(G)$, we call G *right* (*left*) *amenable*. It is known that if G is a semigroup with both right and left invariant means, then there exists a two-sided invariant mean on $B(G)$ and in this case G is called *amenable*. It is also known that if G is a group, then either right or left amenability of G implies that G is amenable (ref. [127]). We remark that the norm of the functional m is one.

G. L. Forti [105] proved the following theorem.

Theorem 2.41 (Forti). *Assume that* $(G, \cdot)$ *is a right* (*left*) *amenable semigroup. If a function* $f : G \to \mathbb{C}$ *satisfies*

$$|f(x \cdot y) - f(x) - f(y)| \leq \delta$$

for some $\delta \geq 0$ *and for all* $x, y \in G$, *then there exists a homomorphism* $H : G \to \mathbb{C}$ *such that*

$$|f(x) - H(x)| \leq \delta$$

for all $x \in G$.

Proof. Let $m : B(G) \to \mathbb{C}$ be a right invariant mean. We use the notation m_x to indicate that the mean is to be applied with respect to the variable x. Define the function $H : G \to \mathbb{C}$ by

$$H(y) = m_x\big(f(x \cdot y) - f(x)\big).$$

Using the right invariance and the linearity of the functional m, we have

$$\begin{aligned} H(y) + H(z) &= m_x\big(f(x \cdot y) - f(x)\big) + m_x\big(f(x \cdot z) - f(x)\big) \\ &= m_x\big(f(x \cdot y \cdot z) - f(x \cdot z) + f(x \cdot z) - f(x)\big) \\ &= m_x\big(f(x \cdot y \cdot z) - f(x)\big) \\ &= H(y \cdot z), \end{aligned}$$

so that H is a homomorphism. We now get

$$\begin{aligned}|f(y) - H(y)| &= \big|f(y) - m_x\big(f(x \cdot y) - f(x)\big)\big| \\ &= \big|m_x\big(f(x \cdot y) - f(x) - f(y)\big)\big| \\ &\leq \|m_x\| \, |f(x \cdot y) - f(x) - f(y)| \\ &\leq \delta\end{aligned}$$

for all $y \in G$. The proof for the case of a left invariant mean is similar. $\square$

It should be remarked that L. Székelyhidi introduced the invariant mean method in [339, 340]. Also, he proved this theorem for the case when $(G, \cdot)$ is a right (left) amenable group (see [342]).

The method of invariant means does not provide a proof of uniqueness of the homomorphism H. However, it should be remarked that J. Rätz [316] proved the uniqueness of the homomorphism H.

Let us now introduce some terminologies. A vector space E is called a *topological vector space* if the set E is a topological space and if the vector space operations (vector addition and scalar multiplication) are continuous in the topology of E. A *local base* of a topological vector space E is a collection $\mathcal{B}$ of neighborhoods of 0 in E such that every neighborhood of 0 contains a member of $\mathcal{B}$. A topological vector space is called *locally convex* if there exists a local base of which members are convex. A topological space E is a *Hausdorff space* if distinct points of E have disjoint neighborhoods. A topological vector space E is called *sequentially complete* if every Cauchy sequence in E converges to a point of E.

L. Székelyhidi [343] presented that if the equation of homomorphism is stable for functions from a semigroup G into $\mathbb{C}$, then stability also holds true for functions $f : G \to E$, where E is a semi-reflexive complex locally convex Hausdorff topological vector space. Z. Gajda [111] significantly generalized this result, i.e., the stability result of Székelyhidi also holds true for functions $f : G \to E$, where E is a complex locally convex Hausdorff topological vector space which is sequentially complete.

Applying the above result of Gajda to Theorem 2.41, we obtain the following corollary (ref. [111]).

Corollary 2.42. *Let $(G, \cdot)$ be a right (left) amenable semigroup, and let E be a complex topological vector space which is locally convex, Hausdorff, and sequentially complete. If a function $f : G \to E$ satisfies*

$$|f(x \cdot y) - f(x) - f(y)| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in G$, then there exists a unique homomorphism $H : G \to E$ such that

$$|f(x) - H(x)| \leq \delta$$

for all $x \in G$.

2.6 Fixed Point Method

In Theorems 2.3. 2.5, and 2.18, the relevant additive functions A are explicitly constructed from the given function f by means of

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x) \quad \text{or} \quad A(x) = \lim_{n\to\infty} 2^n f(2^{-n} x).$$

This method is called a direct method presented by D. H. Hyers [135] for the first time. On the other hand, another approach for proving the stability was introduced in Section 2.5. This approach is called the method of invariant means. In this section, we will deal with a new method, namely, the fixed point method.

For a nonempty set X, we introduce the definition of the generalized metric on X. A function $d : X^2 \to [0,\infty]$ is called a *generalized metric* on X if and only if d satisfies

(M1) $d(x,y) = 0$ *if and only if* $x = y$;
(M2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;
(M3) $d(x,z) \le d(x,y) + d(y,z)$ *for all* $x, y, z \in X$.

We remark that the only difference between the generalized metric and the usual metric is that the range of the former is permitted to include the infinity.

We now introduce one of the fundamental results of the fixed point theory. For the proof, we refer to [240].

Theorem 2.43. *Let* (X,d) *be a generalized complete metric space. Assume that* $\Lambda : X \to X$ *is a strictly contractive operator with the Lipschitz constant* $L < 1$. *If there exists a nonnegative integer* n_0 *such that* $d\big(\Lambda^{n_0+1}x, \Lambda^{n_0}x\big) < \infty$ *for some* $x \in X$, *then the following statements are true:*

(i) The sequence $\{\Lambda^n x\}$ *converges to a fixed point* x^* *of* Λ;
(ii) x^* *is the unique fixed point of* Λ *in* $X^* = \big\{y \in X \mid d(\Lambda^{n_0}x, y) < \infty\big\}$;
(iii) If $y \in X^*$, *then*

$$d(y, x^*) \le \frac{1}{1-L} d(\Lambda y, y).$$

In 2003, V. Radu proved the Hyers–Ulam–Rassias stability of the additive Cauchy equation (2.1) by using the fixed point method (see [279] or [57]).

Theorem 2.44 (Cădariu and Radu). *Let* E_1 *and* E_2 *be a real normed space and a real Banach space, respectively. Let* p *and* θ *be nonnegative constants with* $p \ne 1$. *If a function* $f : E_1 \to E_2$ *satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big) \tag{2.30}$$

for all $x, y \in E_1$, *then there exists a unique additive function* $A : E_1 \to E_2$ *such that*

$$\|f(x) - A(x)\| \le \frac{2\theta}{|2-2^p|}\|x\|^p \tag{2.31}$$

for any $x \in E_1$.

Proof. We define the set

$$X = \{g : E_1 \to E_2 \mid p \cdot g(0) = 0\}$$

and introduce a generalized metric $d_p : X^2 \to [0, \infty]$ by

$$d_p(g, h) = \sup_{x \neq 0} \|g(x) - h(x)\| / \|x\|^p.$$

We know that (X, d_p) is complete.

Now, we define an operator $\Lambda : X \to X$ by

$$(\Lambda g)(x) = (1/q) g(qx),$$

where $q = 2$ if $p < 1$, and $q = 1/2$ if $p > 1$. Then, we have

$$\begin{aligned} \|(\Lambda g)(x) - (\Lambda h)(x)\| / \|x\|^p &= (1/q) \|g(qx) - h(qx)\| / \|x\|^p \\ &= q^{p-1} \|g(qx) - h(qx)\| / \|qx\|^p \\ &\leq q^{p-1} d_p(g, h) \end{aligned}$$

for any $g, h \in X$. Thus, we conclude that

$$d_p(\Lambda g, \Lambda h) \leq q^{p-1} d_p(g, h)$$

for all $g, h \in X$, i.e., Λ is a strictly contractive operator on X with the Lipschitz constant $L = q^{p-1}$.

If we put $y = x$ in (2.30), then we get

$$\|2f(x) - f(2x)\| \leq 2\theta \|x\|^p$$

for each $x \in E_1$, which implies that

$$d_p(f, \Lambda f) \leq \begin{cases} \theta & (\text{for } \ p < 1), \\ 2^{1-p}\theta & (\text{for } \ p > 1). \end{cases} \tag{a}$$

According to Theorem 2.43 (i), there exists a function $A : E_1 \to E_2$ which is a fixed point of Λ, i.e.,

$$A(2x) = 2A(x)$$

for all $x \in E_1$. In addition, A is uniquely determined in the set

$$X^* = \{g \in X \mid d_p(f, g) < \infty\}.$$

Moreover, due to Theorem 2.43 (i), we see that $d_p(\Lambda^n f, A) \to 0$ as $n \to \infty$, i.e.,

$$A(x) = \lim_{n\to\infty} q^{-n} f(q^n x) \tag{b}$$

for any $x \in E_1$. In view of Theorem 2.43 (iii) and (a), we have

$$d_p(f, A) \le \frac{1}{1 - q^{p-1}} d_p(f, \Lambda f) \le \frac{2\theta}{|2 - 2^p|},$$

which implies the validity of the inequality (2.31).

If we replace x and y in (2.30) with $q^n x$ and $q^n y$, respectively, then we obtain

$$\left\| q^{-n} f\big(q^n(x+y)\big) - q^{-n} f(q^n x) - q^{-n} f(q^n y) \right\| \le L^n \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in E_1$. If we let $n \to \infty$ in the preceding inequality and consider (b), then we have

$$A(x+y) = A(x) + A(y)$$

for any $x, y \in E_1$. □

2.7 Composite Functional Congruences

It is also interesting to study the stability problem when the values of the Cauchy difference $f(x+y) - f(x) - f(y)$ are forced to lie near integers, i.e.,

$$f(x+y) - f(x) - f(y) \in \mathbb{Z} + (-\varepsilon, \varepsilon), \tag{2.32}$$

where ε is a small positive number.

R. Ger and P. Šemrl [123] proved that if a function $f : G \to \mathbb{R}$, where G is a cancellative abelian semigroup, satisfies the condition in (2.32) with $0 < \varepsilon < 1/4$, then there exists a function $p : G \to \mathbb{R}$ such that $p(x+y) - p(x) - p(y) \in \mathbb{Z}$ and $|f(x) - p(x)| \le \varepsilon$.

Such a property of functions satisfying (2.32) is called the *composite functional congruence*. It is a generalization of the functional congruence which was first studied by van der Corput [86].

Before stating the results of Ger and Šemrl, we introduce a theorem of M. Hosszú [134]. We may omit the proof because it is beyond the scope of this book.

Theorem 2.45. *Let G_1 and G_2 be a cancellative abelian semigroup and a divisible abelian group, respectively, in which the equation $nx = y$ has a unique solution $x \in G_2$ for each fixed $y \in G_2$ and any $n \in \mathbb{N}$. The most general form of solutions $f : G_1^2 \to G_2$ of the functional equation*

$$f(x+y, z) + f(x, y) = f(x, y+z) + f(y, z)$$

is $f(x,y) = B(x,y) + g(x+y) - g(x) - g(y)$*, where* $B : G_1^2 \to G_2$ *is an arbitrary skew-symmetric biadditive function and where* $g : G_1 \to G_2$ *is an arbitrary function.*

Let $(G, +)$ be a group and let $U, V \subset G$. We can define the addition and the subtraction between sets by

$$U + V = \{x + y \mid x \in U,\ y \in V\}$$

and

$$U - V = \{x - y \mid x \in U,\ y \in V\}.$$

For convenience, let us define $U^+ = U + U$ and $U^- = U - U$. We remark that if $(G, +)$ is an abelian group, then $(U^+)^- = (U^-)^+$.

A set of *generators* of a group G is a subset S of G such that each element of G can be represented (using the group operations) in terms of members of S, where the repetitions of members of S are allowed.

If G is an abelian group with a finite set of generators, then G is a Cartesian product of infinite cyclic groups $F_1, F_2, \ldots, F_m$ and cyclic groups $H_1, H_2, \ldots, H_n$ of finite order. If $n = 0$, G is called *torsion-free*.

Ger and Šemrl [123] proved the following theorem.

Theorem 2.46. *Let* $(G_1, +)$ *and* $(G_2, +)$ *be a cancellative abelian semigroup and a torsion-free divisible abelian group, respectively. Assume that* U *and* V *are nonempty subsets of* G_2 *with* $(U^+)^- \cap (V^+)^- = \{0\}$*. If a function* $f : G_1 \to G_2$ *satisfies*

$$f(x+y) - f(x) - f(y) \in U + V$$

for all $x, y \in G_1$*, it can be represented by*

$$f = u + v,$$

where $u, v : G_1 \to G_2$ *satisfy the relations*

$$u(x+y) - u(x) - u(y) \in U$$

and

$$v(x+y) - v(x) - v(y) \in V$$

for all $x, y \in G_1$*. The functions* u *and* v *are determined uniquely up to an additive function.*

Proof. There are functions $\psi : G_1^2 \to U$ and $\varphi : G_1^2 \to V$ such that

$$d(x,y) = f(x+y) - f(x) - f(y) = \psi(x,y) + \varphi(x,y)$$

for all $x, y \in G_1$. The commutativity of $(G_1, +)$ implies that d is symmetric.

Claim that ψ is symmetric. We have

$$\psi(x, y) - \psi(y, x) = d(x, y) - \varphi(x, y) - d(y, x) + \varphi(y, x) \in V^-.$$

On the other hand,

$$\psi(x, y) - \psi(y, x) \in U^-,$$

and since $0 \in U^- \cap V^-$, we infer that

$$\begin{aligned} \psi(x, y) - \psi(y, x) &\in U^- \cap V^- \\ &\subset \left(U^- + U^-\right) \cap \left(V^- + V^-\right) \\ &= \left(U^+\right)^- \cap \left(V^+\right)^- \\ &= \{0\}. \end{aligned}$$

Claim that ψ satisfies

$$\psi(x, y + z) + \psi(y, z) = \psi(x + y, z) + \psi(x, y) \tag{a}$$

for all $x, y \in G_1$. A straightforward computation yields that d satisfies the same functional equation. Consequently,

$$\begin{aligned} &\psi(x, y + z) + \psi(y, z) - \psi(x + y, z) - \psi(x, y) \\ &\quad = \varphi(x + y, z) + \varphi(x, y) - \varphi(x, y + z) - \varphi(y, z) \\ &\quad \in \left(U^+\right)^- \cap \left(V^+\right)^- \\ &\quad = \{0\}, \end{aligned}$$

which proves (a).

According to Theorem 2.45, there exists a function $u : G_1 \to G_2$ such that

$$\psi(x, y) = u(x + y) - u(x) - u(y) \in U$$

for any $x, y \in G_1$ (since ψ is symmetric, we take $B(x, y) \equiv 0$ in Theorem 2.45). Now, define $v(x) = f(x) - u(x)$ for all $x \in G_1$. Then, we have

$$\varphi(x, y) = d(x, y) - \psi(x, y) = v(x + y) - v(x) - v(y) \in V$$

for all $x, y \in G_1$.

In order to prove the uniqueness, we assume that there are two representations

$$f = u + v = u' + v'$$

with the properties described above. Putting

$$a = u - u' = v' - v,$$

we get

$$\begin{aligned} &a(x+y) - a(x) - a(y) \\ &\quad = \big(u(x+y) - u(x) - u(y)\big) - \big(u'(x+y) - u'(x) - u'(y)\big) \\ &\quad \in U^-. \end{aligned}$$

Similarly,

$$a(x+y) - a(x) - a(y) \in V^-.$$

These two relations, together with $0 \in U^- \cap V^-$, imply the additivity of a. Moreover, we have

$$u' = u - a \quad \text{and} \quad v' = v + a$$

which ends the proof. □

Let E be a vector space and let $V \subset E$. The intersection of all convex sets in E containing V is called the *convex hull* of V and denoted by $\mathrm{Co}V$. Thus, $\mathrm{Co}V$ is the smallest convex set containing V.

Let U be a subset of a topological vector space and let $\overline{U}$ denote the closure of U. Ger and Šemrl [123] also presented the following theorem.

Theorem 2.47 (Ger and Šemrl). *Let $(G, +)$ be a cancellative abelian semigroup, and let E be a Banach space. Assume that nonempty subsets $U, V \subset E$ satisfy $(U^+)^- \cap (V^+)^- = \{0\}$, $0 \in V$, and V is bounded. If a function $f : G \to E$ satisfies*

$$f(x+y) - f(x) - f(y) \in U + V$$

for all $x, y \in G$, then there exists a function $p : G \to E$ such that

$$p(x+y) - p(x) - p(y) \in U$$

for any $x, y \in G$, and

$$p(x) - f(x) \in \overline{\mathrm{Co}V}$$

for any $x \in G$. Moreover, if $U^- \cap 3\big(\overline{\mathrm{Co}V}\big)^- = \{0\}$, then the function p is unique.

Proof. According to Theorem 2.46, there exist functions $u, v : G \to E$ such that $f = u + v$, $u(x+y) - u(x) - u(y) \in U$, and $v(x+y) - v(x) - v(y) \in V$ for $x, y \in G$. It follows from [105, Theorem 4] that there exists an additive function $a : G \to E$ such that

$$v(x) - a(x) \in \overline{\mathrm{Co}(-V)} = -\overline{\mathrm{Co}V}$$

for all $x \in G$. Putting $p = u + a$ and applying $f = u + v$, we get the desired relation

$$p(x) - f(x) \in \overline{\mathrm{Co}V}$$

for any $x \in G$.

Assume that $p_1, p_2 : G \to E$ are two functions such that

$$p_i(x + y) - p_i(x) - p_i(y) \in U$$

for any $x, y \in G$, and

$$p_i(x) - f(x) \in \overline{\mathrm{Co}V}$$

for all $x \in G$ and $i \in \{1, 2\}$. Then, we have

$$r(x) = p_1(x) - p_2(x) = \bigl(p_1(x) - f(x)\bigr) - \bigl(p_2(x) - f(x)\bigr) \in \bigl(\overline{\mathrm{Co}V}\bigr)^-$$

for all $x \in G$. Consequently, using the notation $V_0 = \overline{\mathrm{Co}V}$, we obtain

$$\begin{aligned} r(x + y) - r(x) - r(y) &\in (U - U) \cap \bigl(V_0^- - V_0^- - V_0^-\bigr) \\ &= U^- \cap 3V_0^- \\ &= \{0\}, \end{aligned}$$

since V_0^- is convex and symmetric with respect to zero. Thus, r is additive and bounded. An extended version of Theorem 2.1 implies that $r(x) = p_1(x) - p_2(x) \equiv 0$. This ends the proof. □

Ger and Šemrl also gave the following corollary in the paper [123].

Corollary 2.48. *Let $(G, +)$ be a cancellative abelian semigroup, and let $\varepsilon \in (0, 1/4)$. If a function $f : G \to \mathbb{R}$ satisfies the congruence*

$$f(x + y) - f(x) - f(y) \in \mathbb{Z} + (-\varepsilon, \varepsilon)$$

for all $x, y \in G$, then there exists a function $p : G \to \mathbb{R}$ such that

$$p(x + y) - p(x) - p(y) \in \mathbb{Z}$$

for any $x, y \in G$, and

$$|f(x) - p(x)| \le \varepsilon$$

for all $x \in G$.

Proof. Let $U = \mathbb{Z}$ and $V = (-\varepsilon, \varepsilon)$. Then, we have $(U^+)^- = \mathbb{Z}$ and $(V^+)^- = (-1, 1)$ and hence $(U^+)^- \cap (V^+)^- = \{0\}$. Therefore, the assertion is an immediate consequence of Theorem 2.47. □

2.8 Pexider Equation

In this section, we prove the Hyers–Ulam–Rassias stability of the Pexider equation,

$$f(x+y) = g(x) + h(y).$$

Let G_1 and G_2 be abelian groups. It is well-known that functions $f, g, h : G_1 \to G_2$ satisfy the Pexider equation if and only if there exist an additive function $A : G_1 \to G_2$ and constants $a, b \in G_2$ such that

$$f(x) = A(x) + a + b, \quad g(x) = A(x) + a, \quad h(x) = A(x) + b$$

for all $x \in G_1$.

In 1993, J. Chmieliński and J. Tabor [67] investigated the stability of the Pexider equation (ref. [126]). This paper seems to be the first one concerning the stability problem of the Pexider equation. We will introduce a theorem presented by K.-W. Jun, D.-S. Shin, and B.-D. Kim [155].

Theorem 2.49 (Jun, Shin, and Kim). *Let G and E be an abelian group and a Banach space, respectively. Let $\varphi : G^2 \to [0, \infty)$ be a function satisfying*

$$\Phi(x) = \sum_{i=1}^{\infty} 2^{-i}\left(\varphi\left(2^{i-1}x, 0\right) + \varphi\left(0, 2^{i-1}x\right) + \varphi\left(2^{i-1}x, 2^{i-1}x\right)\right) < \infty$$

and

$$\lim_{n\to\infty} 2^{-n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in G$. If functions $f, g, h : G \to E$ satisfy the inequality

$$\|f(x+y) - g(x) - h(y)\| \le \varphi(x, y) \tag{2.33}$$

for all $x, y \in G$, then there exists a unique additive function $A : G \to E$ such that

$$\begin{aligned} \|f(x) - A(x)\| &\le \|g(0)\| + \|h(0)\| + \Phi(x), \\ \|g(x) - A(x)\| &\le \|g(0)\| + 2\|h(0)\| + \varphi(x, 0) + \Phi(x), \\ \|h(x) - A(x)\| &\le 2\|g(0)\| + \|h(0)\| + \varphi(0, x) + \Phi(x) \end{aligned} \tag{2.34}$$

for all $x \in G$.

Proof. If we put $y = x$ in (2.33), then we have

$$\|f(2x) - g(x) - h(x)\| \le \varphi(x, x) \tag{a}$$

for all $x \in G$. Putting $y = 0$ in (2.33) yields that

$$\|f(x) - g(x) - h(0)\| \le \varphi(x, 0) \tag{b}$$

for any $x \in G$. It follows from (b) that

$$\|g(x) - f(x)\| \le \|h(0)\| + \varphi(x, 0) \tag{c}$$

for each $x \in G$. If we put $x = 0$ in (2.33), then we get

$$\|f(y) - g(0) - h(y)\| \le \varphi(0, y)$$

for $y \in G$. Thus, we obtain

$$\|h(x) - f(x)\| \le \|g(0)\| + \varphi(0, x) \tag{d}$$

for all $x \in G$.

Let us define

$$u(x) = \|g(0)\| + \|h(0)\| + \varphi(0, x) + \varphi(x, 0) + \varphi(x, x).$$

Using the inequalities (a), (c), and (d), we have

$$\begin{aligned}
&\|f(2x) - 2f(x)\| \\
&\quad \le \|f(2x) - g(x) - h(x)\| + \|g(x) - f(x)\| + \|h(x) - f(x)\| \\
&\quad \le \|g(0)\| + \|h(0)\| + \varphi(0, x) + \varphi(x, 0) + \varphi(x, x) \\
&\quad = u(x)
\end{aligned} \tag{e}$$

for all $x \in G$. Replacing x with $2x$ in (e), we get

$$\left\|f\left(2^2 x\right) - 2f(2x)\right\| \le u(2x) \tag{f}$$

for any $x \in G$. It then follows from (e) and (f) that

$$\begin{aligned}
\left\|f\left(2^2 x\right) - 2^2 f(x)\right\| &\le \left\|f\left(2^2 x\right) - 2f(2x)\right\| + 2\|f(2x) - 2f(x)\| \\
&\le u(2x) + 2u(x)
\end{aligned}$$

for all $x \in G$.

Applying an induction argument on n, we will prove that

$$\|f(2^n x) - 2^n f(x)\| \le \sum_{i=1}^{n} 2^{i-1} u(2^{n-i} x) \tag{g}$$

for all $x \in G$ and $n \in \mathbb{N}$. In view of (e), the inequality (g) is true for $n = 1$. Assume that (g) is true for some $n > 0$. Substituting $2x$ for x in (g), we obtain

$$\left\| f\left(2^{n+1}x\right) - 2^n f(2x)\right\| \leq \sum_{i=1}^{n} 2^{i-1} u\left(2^{n+1-i}x\right)$$

for any $x \in G$. Hence, it follows from (e) that

$$\begin{aligned}\left\| f\left(2^{n+1}x\right) - 2^{n+1} f(x)\right\| &\leq \left\| f\left(2^{n+1}x\right) - 2^n f(2x)\right\| + 2^n \| f(2x) - 2f(x)\| \\ &\leq \sum_{i=1}^{n} 2^{i-1} u\left(2^{n+1-i}x\right) + 2^n u(x) \\ &= \sum_{i=1}^{n+1} 2^{i-1} u\left(2^{n+1-i}x\right)\end{aligned}$$

for all $x \in G$, which proves the inequality (g).

By (g), we have

$$\|2^{-n} f(2^n x) - f(x)\| \leq \sum_{i=1}^{n} 2^{i-1-n} u(2^{n-i}x) \qquad (h)$$

for all $x \in G$ and $n \in \mathbb{N}$. Moreover, if $m, n \in \mathbb{N}$ with $m < n$, then it follows from (e) that

$$\begin{aligned}&\|2^{-n} f(2^n x) - 2^{-m} f(2^m x)\| \\ &\quad\leq \sum_{i=m}^{n-1} \left\|2^{-i} f(2^i x) - 2^{-(i+1)} f\left(2^{i+1}x\right)\right\| \\ &\quad\leq \sum_{i=m}^{n-1} 2^{-(i+1)} u(2^i x) \\ &\quad= \sum_{i=m}^{n-1} 2^{-(i+1)} \left(\|g(0)\| + \|h(0)\| + \varphi(0, 2^i x) + \varphi(2^i x, 0) + \varphi(2^i x, 2^i x)\right) \\ &\quad\leq 2^{-m} \left(\|g(0)\| + \|h(0)\|\right) \\ &\qquad + \sum_{i=m}^{\infty} 2^{-(i+1)} \left(\varphi(0, 2^i x) + \varphi(2^i x, 0) + \varphi(2^i x, 2^i x)\right) \\ &\quad\to 0 \quad \text{as} \quad m \to \infty\end{aligned}$$

for all $x \in G$. Hence, $\{2^{-n} f(2^n x)\}$ is a Cauchy sequence for every $x \in G$. Since E is a Banach space, we can define a function $A : G \to E$ by

$$A(x) = \lim_{n\to\infty} 2^{-n} f(2^n x).$$

In view of (2.33), we obtain

$$\|2^{-n} f(2^n x + 2^n y) - 2^{-n} g(2^n x) - 2^{-n} h(2^n y)\| \le 2^{-n}\varphi(2^n x, 2^n y)$$

for all $x \in G$ and $n \in \mathbb{N}$. It follows from (c) that

$$\|2^{-n} g(2^n x) - 2^{-n} f(2^n x)\| \le 2^{-n}\big(\|h(0)\| + \varphi(2^n x, 0)\big) \tag{i}$$

for any $x \in G$ and $n \in \mathbb{N}$. Since

$$\begin{aligned} 2^{-n}\varphi(2^n x, 0) \;&\le\; 2\sum_{i=n}^{\infty} 2^{-(i+1)}\big(\varphi(0, 2^i x) + \varphi(2^i x, 0) + \varphi(2^i x, 2^i x)\big) \\ &\to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

it follows from (i) that

$$\lim_{n\to\infty} 2^{-n} g(2^n x) = \lim_{n\to\infty} 2^{-n} f(2^n x) = A(x) \tag{j}$$

for each $x \in G$. Also, by (d), we have

$$\|2^{-n} h(2^n x) - 2^{-n} f(2^n x)\| \le 2^{-n}\big(\|g(0)\| + \varphi(0, 2^n x)\big) \tag{k}$$

for all $x \in G$ and $n \in \mathbb{N}$. Similarly, it follows from (k) that

$$\lim_{n\to\infty} 2^{-n} h(2^n x) = \lim_{n\to\infty} 2^{-n} f(2^n x) = A(x) \tag{l}$$

for each $x \in G$. Thus, by (2.33), (j), (l), and the commutativity of G, we get

$$\begin{aligned} 0 &= \left\| \lim_{n\to\infty} \big(2^{-n} f(2^n x + 2^n y) - 2^{-n} g(2^n x) - 2^{-n} h(2^n y)\big) \right\| \\ &= \|A(x + y) - A(x) - A(y)\| \end{aligned}$$

for all $x, y \in G$.

Taking the limit in (h) as $n \to \infty$ yields

$$\begin{aligned}
&\|A(x) - f(x)\| \\
&\quad \le \lim_{n\to\infty} \sum_{i=1}^{n} 2^{i-1-n} u(2^{n-i}x) \\
&\quad = \lim_{n\to\infty} \big(1 - 2^{-n}\big)\big(\|g(0)\| + \|h(0)\|\big) \\
&\qquad + \lim_{n\to\infty} \sum_{i=1}^{n} 2^{-i} \big(\varphi\big(0, 2^{i-1}x\big) + \varphi\big(2^{i-1}x, 0\big) + \varphi\big(2^{i-1}x, 2^{i-1}x\big)\big) \\
&\quad = \|g(0)\| + \|h(0)\| + \Phi(x)
\end{aligned}$$

for each $x \in G$, which proves (2.34).

It remains to prove the uniqueness of A. Assume that $A' : G \to E$ is another additive function which satisfies the inequalities in (2.34). Then we have

$$\begin{aligned}
&\|A(x) - A'(x)\| \\
&\quad \le 2^{-n}\|A(2^n x) - f(2^n x)\| + 2^{-n}\|f(2^n x) - A'(2^n x)\| \\
&\quad \le 2^{-n+1}\big(\|g(0)\| + \|h(0)\| + \Phi(2^n x)\big) \\
&\quad = 2^{-(n-1)}\big(\|g(0)\| + \|h(0)\|\big) \\
&\qquad + 2 \sum_{i=n+1}^{\infty} 2^{-i} \big(\varphi\big(0, 2^{i-1}x\big) + \varphi\big(2^{i-1}x, 0\big) + \varphi\big(2^{i-1}x, 2^{i-1}x\big)\big) \\
&\quad \to 0 \quad \text{as} \quad n \to \infty
\end{aligned}$$

for each $x \in G$, which implies that $A(x) = A'(x)$ for all $x \in G$. $\square$

Corollary 2.50. *Let E_1 and E_2 be Banach spaces and let $\theta \ge 0$ and $p \in [0, 1)$ be constants. If functions $f, g, h : E_1 \to E_2$ satisfy the inequality*

$$\|f(x + y) - g(x) - h(y)\| \le \theta\big(\|x\|^p + \|y\|^p\big)$$

for all $x, y \in E_1$, then there exists a unique linear function $A : E_1 \to E_2$ such that

$$\begin{aligned}
\|f(x) - A(x)\| &\le \|g(0)\| + \|h(0)\| + \frac{4\theta}{2 - 2^p}\|x\|^p, \\
\|g(x) - A(x)\| &\le \|g(0)\| + 2\|h(0)\| + \frac{6 - 2^p}{2 - 2^p}\theta\|x\|^p, \\
\|h(x) - A(x)\| &\le 2\|g(0)\| + \|h(0)\| + \frac{6 - 2^p}{2 - 2^p}\theta\|x\|^p
\end{aligned}$$

for any $x \in E_1$.

In 2000, Y.-H. Lee and K.-W. Jun [232] investigated the Hyers–Ulam–Rassias stability of the Pexider equation on the restricted domains (ref. [274]).

2.9 Remarks

T. Aoki [7] appears to be the first to extend the theorem of Hyers (Theorem 2.3) for additive functions. Indeed, Aoki provided a proof of the special case of the theorem of Th. M. Rassias when the given function is additive.

It was Th. M. Rassias [285] who was the first to prove the stability of the linear function in Banach spaces.

D. G. Bourgin [28] stated the following result without proof, which is similar to the theorem of Găvruta (Theorem 2.18).

Let E_1 and E_2 be Banach spaces and let $\varphi : [0,\infty)^2 \to [0,\infty)$ be a monotone nondecreasing function such that

$$\Phi(x) = \sum_{k=1}^{\infty} 2^{-k} \varphi\big(2^k \|x\|, 2^k \|x\|\big) < \infty$$

for all $x \in E_1$. If a function $f : E_1 \to E_2$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \le \varphi\big(\|x\|, \|y\|\big)$$

for all $x, y \in E_1$, then there exists a unique additive function $A : E_1 \to E_2$ such that

$$\|f(x) - A(x)\| \le \Phi(x)$$

for any $x \in E_1$.

G. L. Forti [104] used a similar idea in proving his stability theorem for a class of functional equations of the form

$$f\big(F(x,y)\big) = H\big(f(x), f(y)\big)$$

with f as the unknown function, which includes the additive Cauchy equation as a special case. In the special case of the additive Cauchy equation, it proves that the result of Bourgin with an abelian semigroup $(G,+)$ instead of the Banach space E_1 holds true under an additional condition such as

$$\lim_{k\to\infty} 2^{-k} \varphi\big(2^k \|x\|, 2^k \|y\|\big) = 0$$

for all $x, y \in G$.

Recently, S.-M. Jung and S. Min [193] have proved the Hyers–Ulam–Rassias stability of the functional equations of the type $f(x+y) = H\big(f(x), f(y)\big)$ by using the fixed point method, where H is a bounded linear transformation.

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