Chapter 2
Generalized Monotone Maps and Complementarity Problems

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Abstract In this chapter, we present some classes of generalized monotone maps and their relationship with the corresponding concepts of generalized convexity. We present results of generalized monotone maps that are used in the analysis and solution of variational inequality and complementarity problems. In addition, we obtain various characterizations and establish a connection between affine pseudomonotone mapping, affine quasimonotone mapping, positive-subdefinite matrices, generalized positive-subdefinite matrices, and the linear complementarity problem. These characterizations are useful for extending the applicability of Lemke's algorithm for solving the linear complementarity problem.

2.1 Introduction

Generalized monotonicity plays an important role in solving mathematical programming, complementarity problems, and variational inequalities. Generalized monotone maps are of fundamental importance and arise in economic applications. Different types of generalized monotonicity are related to various kinds of generalized convexity of the underlying function. It is well known [25] that a differentiable function is convex if and only if its gradient is a monotone map; see also [16]. In [27], the notion of a monotone map is generalized to that of a pseudomonotone map and a differentiable pseudoconvex function is characterized by the pseudomonotonicity of the gradient. A similar relationship exists between strictly pseudoconvex and quasiconvex functions as well as strongly convex and strongly pseudoconvex and the corresponding monotonicity property [29] of their gradient. Generalized monotone maps provide first-order characterizations of

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generalized convex functions. During the last few decades extensive research has been devoted to generalized convexity in view of finding solutions of nonconvex optimization problems. A large number of articles have appeared on this subject where several existence results and algorithmic implications are studied. The importance of various kinds of generalized monotonicity concepts both for theory and for solution methods of variational inequalities and complementarity problems are well known. John [23, 24] presents uses of generalized concavity and generalized monotonicity in consumer theory and general equilibrium theory. In [3], application of pseudomonotone maps to economics is discussed and it is shown that the concept of pseudomonotonicity is strongly related to a notion of rationality of consumer behaviour. In this chapter, we discuss various characterizations and the role of generalized monotone maps that are used in the analysis and solution of complementarity problems.

Given a nonempty subset K of R^n and a mapping $\mathscr{F} : R^n \to R^n$, the variational inequality problem $\mathrm{VI}(K,\mathscr{F})$ is to find a vector $x^* \in K$ such that

$$(x-x^*)^t \mathscr{F}(x^*) \geq 0 \qquad \forall\, x \in K. \tag{2.1}$$

One typically assumes that the set K is closed and convex. The set K is polyhedral in many applications. When $K = R^n_+$, the nonnegative orthant of R^n, the above problem reduces to the nonlinear complementarity problem NCP$(\mathscr{F})$ which is stated as follows.

Find a vector x^* such that

$$x^* \in R^n_+, \qquad \mathscr{F}(x^*) \in R^n_+, \qquad x^{*t}\mathscr{F}(x^*) = 0. \tag{2.2}$$

For a given matrix $A \in R^{n\times n}$ and a vector $q \in R^n$ when $\mathscr{F}(x)$ is an affine function (i.e., $\mathscr{F}(x) = Ax + q$) then the problem NCP$(\mathscr{F})$ reduces to the linear complementarity problem LCP(q,A). Complementarity problems are treated as a part of mathematical programming and equilibrium problems. The complementarity problem has gained importance because it is a unified study of several optimization problems and game problems. This subject has a wide range of applications encompassing fields such as economics, control theory, engineering, game theory, and optimization. For a comprehensive survey of theory, algorithms, and applications on finite-dimensional variational inequalities and nonlinear complementarity problems, we refer the reader to the article by Harker and Pang [22].

Given a convex cone K in R^n and a mapping $\mathscr{F} : R^n \to R^n$, the generalized complementarity problem GCP$(K,\mathscr{F})$ is to find a vector $x^* \in K$ such that

$$\mathscr{F}(x^*) \in K^* \quad \text{and} \quad x^{*t}\mathscr{F}(x^*) = 0, \tag{2.3}$$

where K^* is the dual cone of K; that is $K^* = \{y \in R^n \,:\, y^t x \geq 0,\ \forall\, x \in K\}$. The feasible set of GCP$(K,\mathscr{F})$ is defined as

$$\mathrm{FEA}(K,\mathscr{F}) = \{x \in K \,:\, \mathscr{F}(x) \in K^*\}.$$

The problem GCP$(K,\mathscr{F})$ is said to be feasible if FEA$(K,\mathscr{F})$ is nonempty.

Geometrically, the problem GCP$(K,\mathscr{F})$ finds a vector $x^* \in K$ with the property that its image under the mapping $\mathscr{F}$ lies in the dual cone of K which is orthogonal to x^*. The nonnegative orthant is self-dual (i.e., $(R^n_+)^* = R^n_+$), therefore it is easy to see that GCP$(R^n_+,\mathscr{F})$ reduces to NCP$(\mathscr{F})$ as given by (2.2). Karamardian [26] obtained a relationship of the solution set between the generalized complementarity problem and variational inequality and proved that GCP$(K,\mathscr{F})$ and VI$(K,\mathscr{F})$ have the same solution set. See also [7, p. 31] and [22].

Proposition 2.1 ([26]). *Let K be a convex cone. Then $x^* \in K$ solves the problem* VI$(K,\mathscr{F})$ *if and only if x^* solves GCP$(K,\mathscr{F})$.*

Even though every generalized complementarity problem is a variational inequality problem, the converse is not true in general. The most basic result on the existence of a solution to the variational inequality problem VI$(K,\mathscr{F})$ requires the set K to be compact and convex and the mapping $\mathscr{F}$ to be continuous. See [22] and the references cited therein. The basic existence result is presented below.

Theorem 2.1. *Let K be a nonempty, compact, and convex subset of R^n and let the map $\mathscr{F} : K \to R^n$ be continuous. Then there exists a solution to the problem VI$(K,\mathscr{F})$.*

2.2 Preliminaries

We consider matrices and vectors with real entries. Let R^n_+ denote the nonnegative orthant in R^n and $R^{n\times n}$ denote the set of all $n\times n$ real matrices. For any matrix $A \in R^{m\times n}$, a_{ij} denotes its ith row and jth column entry. For any matrix $A \in R^{m\times n}$, let $A_{i\cdot}$ denote its ith row and $A_{\cdot j}$ denote its jth column. For any set $\alpha \subseteq \{1,2,\ldots,n\}$, $\bar{\alpha}$ denotes its complement in $\{1,2,\ldots,n\}$. If A is a matrix of order n, $\alpha \subseteq \{1,2,\ldots,n\}$ and $\beta \subseteq \{1,2,\ldots,n\}$, then $A_{\alpha\beta}$ denotes the submatrix of A consisting of only the rows and columns of A whose indices are in α and β, respectively. Any vector $x \in R^n$ is a column vector unless otherwise specified and x^t denotes the row transpose of x. For any matrix $A \in R^{n\times n}$, A^t denotes its transpose. We say that a vector $y \in R^n$ is *unisigned* if either $y \in R^n_+$ or $-y \in R^n_+$. Given a symmetric matrix $S \in R^{n\times n}$, its *inertia* is the triple $(\nu_+(S), \nu_-(S), \nu_0(S))$ where $\nu_+(S), \nu_-(S)$, and $\nu_0(S)$ denote the number of positive, negative, and zero eigenvalues of S, respectively. Given $x \in R^n$, x^+ and x^- are the vectors of R^n defined by $x_i^+ := \max\{x_i, 0\}$ and $x_i^- := \max\{-x_i, 0\}$ $\forall\, i$. Clearly, $x = x^+ - x^-$. A cone is said to be pointed if $K \cap (-K) = \{0\}$. A cone is said to be solid if its interior is nonempty. Given $\Omega \subseteq R^n$, we denote the interior of Ω by $\mathrm{int}(\Omega)$.

Let $\Omega \subseteq R^n$ be a convex set and $f : \Omega \to R$. Different kinds of generalized convexity were established in the literature by retaining some of the properties of convex functions and a large number of articles have appeared on this subject. Each type of generalized monotone map is related to a generalized convex function. We recall the definitions of generalized convex functions and review some of the characterizations from the literature [2, 33] which are needed for further discussions.

Definition 2.1. f is said to be

(i) *Convex* on Ω if for all $x,y \in \Omega$, and $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$
(ii) *Strictly convex* on Ω if for all $x,y \in \Omega$, $x \neq y$ and $0 < \lambda < 1$, $f(\lambda x + (1-\lambda)y) < \lambda f(x) + (1-\lambda)f(y)$
(iii) *Quasiconvex* on Ω if for all $x,y \in \Omega$ and $0 \leq \lambda \leq 1$, $f(\lambda x + (1-\lambda)y) \leq \max\{f(x), f(y)\}$
(iv) *Strictly quasiconvex* on Ω if for all $x,y \in \Omega$ and $0 < \lambda < 1$, $f(x) < f(y) \Rightarrow f(\lambda x + (1-\lambda)y) < f(y)$

We assume differentiability of the function f on the open convex set $\Omega \subseteq R^n$ for providing the definition of following generalized convex functions.

Definition 2.2. f is said to be

(i) *A pseudoconvex* function if for all $x,y \in \Omega$, $(y-x)^t \nabla f(x) \geq 0 \Rightarrow f(y) \geq f(x)$.
(ii) *Strictly pseudoconvex* if for all $x,y \in \Omega$, $x \neq y$, $(y-x)^t \nabla f(x) \geq 0 \Rightarrow f(y) > f(x)$.

Theorem 2.2 ([2, 3.5.11 Theorem, p. 143]). *Let $\Omega \subseteq R^n$ be a nonempty open convex set and $f : \Omega \to R$ be a differentiable pseudoconvex function. Then f is both strictly quasiconvex and quasiconvex.*

Theorem 2.3 ([33, p. 134, 146], [2, p. 137]). *Let $\Omega \subseteq R^n$ be a nonempty open convex set and f be a differentiable function defined on Ω. Then f is quasiconvex if and only if for all $x,y \in \Omega$, either one of the following statements holds true.*

$$(y-x)^t \nabla f(x) > 0 \Rightarrow f(y) > f(x). \tag{2.4}$$

$$f(y) \leq f(x) \Rightarrow (y-x)^t \nabla f(x) \leq 0. \tag{2.5}$$

2.3 Different Types of Generalized Monotone Maps

Various kinds of generalized monotonicity concepts have been introduced in the literature. A large number of publications have appeared which deal with the concepts and characterizations of generalized monotonicity for different subclasses of maps. In this chapter, we present a brief review of basic generalized monotonicity concepts which are needed for presentation of the results that deal with variational inequalities and in particular complementarity problems. We now recall the following definitions from [27, 29].

Definition 2.3. Let $\Omega \subset R^n$ and $\mathscr{F} : \Omega \to R^n$. $\mathscr{F}$ is said to be

(i) *Monotone* on Ω if $x,y \in \Omega \Rightarrow (y-x)^t(\mathscr{F}(y) - \mathscr{F}(x)) \geq 0$
(ii) *Strictly monotone* on Ω if $x,y \in \Omega$, $x \neq y \Rightarrow (y-x)^t(\mathscr{F}(y) - \mathscr{F}(x)) > 0$
(iii) *Pseudomonotone* on Ω if $x,y \in \Omega$, $(y-x)^t \mathscr{F}(x) \geq 0 \Rightarrow (y-x)^t \mathscr{F}(y) \geq 0$

(iv) *Strictly pseudomonotone* on Ω if $x,y \in \Omega,\ x \neq y,\ (y-x)^t \mathscr{F}(x) \geq 0 \Rightarrow (y-x)^t \mathscr{F}(y) > 0$

(v) *Quasimonotone* on Ω if $x,y \in \Omega$, $(y-x)^t \mathscr{F}(x) > 0 \Rightarrow (y-x)^t \mathscr{F}(y) \geq 0$

The following statement follows from the above definitions.

$$\text{Monotonicity} \Rightarrow \text{pseudomonotonicity} \Rightarrow \text{quasimonotonicity}.$$

For further details on different kinds of generalized monotonicity and their relationship, see [21, 29]. The following lemma is useful.

Lemma 2.1 ([27, Lemma 3.1. p. 449]). *Let $\Omega \subset R^n$ and $\mathscr{F} : \Omega \to R^n$ be pseudomonotone on Ω. Then for every $x,y \in \Omega$, we have*

$$(y-x)^t \mathscr{F}(x) > 0 \Rightarrow (y-x)^t \mathscr{F}(y) > 0.$$

The following theorem establishes the equivalence of convexity of a function and monotonicity of its gradient.

Theorem 2.4 ([29]). *Let $\Omega \subset R^n$ and $f : \Omega \to R$ be a differentiable function on an open convex set Ω of R^n. Then f is convex (strictly convex) if and only if $\mathscr{F} = \nabla f$ is monotone (strictly monotone) on Ω.*

The following theorem generalizes a well-known result of a convex mathematical program (the solution set of a convex mathematical program is convex) to a variational inequality problem with a pseudomonotone type of map. See [22] and references cited therein.

Theorem 2.5. *Let K be a nonempty, closed, and convex subset of R^n and the map $\mathscr{F} : K \to R^n$ be continuous and pseudomonotone from $K \to R^n$. Then x^* solves the problem $VI(K,\mathscr{F})$ if and only if $x^* \in K$ and*

$$(y-x)^t \mathscr{F}(y) \geq 0 \ \forall\, y \in K.$$

In particular, the solution set of $\mathrm{VI}(K,\mathscr{F})$ *is convex if it is nonempty.*

Even though a variational inequality problem can have more than one solution, if we assume $\mathscr{F}$ to be strictly monotone on K then $\mathrm{VI}(K,\mathscr{F})$ can have at most one solution. We now state the following existence theorem for the generalized complementarity problem $\mathrm{GCP}(K,\mathscr{F})$.

Theorem 2.6. *Let K be a solid, pointed, closed, convex cone in R^n. If $\mathscr{F}$ is continuous and strictly monotone with respect to K and if the $GCP(K,\mathscr{F})$ is feasible, then the $GCP(K,\mathscr{F})$ has a unique solution.*

Karamardian and Schaible [29] studied different generalizations of monotone maps where different generalizations of monotonicity correspond to some kind of generalized convexity of the function f. For the gradient map $\mathscr{F} = \nabla f$, the following result is observed in [27, 29]. We present the proof for pseudomonotone maps along the same lines of Karamardian [27]. For quasimonotone maps, the proof technique is similar and we refer the reader to the article of Karamardian and Schaible [29].

Theorem 2.7. *Let $\Omega \subset R^n$ be an open convex set and $f : \Omega \to R$ be differentiable on Ω. Then f is pseudoconvex if and only if $\mathscr{F} = \nabla f$ is pseudomonotone on Ω.*

Proof. Suppose that f is pseudoconvex. Let $x, y \in \Omega$ such that $(y-x)^t \nabla f(x) \geq 0$. From the definition of pseudoconvexity, it follows that $f(y) \geq f(x)$. By Theorem 2.2, pseudoconvexity implies quasiconvexity. Consequently, $f(y) \geq f(x) \Rightarrow (y-x)^t \nabla f(y) \geq 0$. Therefore ∇f is pseudomonotone on Ω.

To prove the converse suppose that ∇f is pseudomonotone on Ω. Let $x, y \in \Omega$, $x \neq y$ such that

$$(y-x)^t \nabla f(x) \geq 0. \tag{2.6}$$

To show f is pseudoconvex, we need to show that $f(y) \geq f(x)$.

Assume to the contrary that

$$f(y) < f(x). \tag{2.7}$$

From the mean value theorem, we have

$$f(y) - f(x) = (y-x)^t \nabla f(\bar{x}), \tag{2.8}$$

where

$$\bar{x} = \bar{\lambda} x + (1 - \bar{\lambda}) y \tag{2.9}$$

for some $0 < \bar{\lambda} < 1$. From (2.7), (2.8), and (2.9), we get

$$(x - \bar{x})^t \nabla f(\bar{x}) > 0. \tag{2.10}$$

From (2.10) and Lemma 2.1, we get

$$(x - \bar{x})^t \nabla f(x) > 0.$$

From (2.9), this implies

$$(x - y)^t \nabla f(x) > 0.$$

However, this contradicts (2.6). This completes the proof. ■

Theorem 2.8. *Let $\Omega \subset R^n$ be an open convex set and $f : \Omega \to R$ be differentiable on Ω. Then f is quasiconvex if and only if $\mathscr{F} = \nabla f$ is quasimonotone on Ω.*

Proof. Suppose that f is quasiconvex. Let $x, y \in \Omega$ such that $(y-x)^t \nabla f(x) > 0$. From (2.4), it follows that $f(y) > f(x)$. Now by (2.5), $f(x) < f(y) \Rightarrow (x-y)^t \nabla f(y) \leq 0 \Rightarrow (y-x)^t \nabla f(y) \geq 0$. Therefore ∇f is quasimonotone on Ω.

For the converse part of theorem, the argument is similar to the earlier one. ■

Karamardian, Schaible, and Crouzeix [30] obtained first-order necessary and sufficient conditions for a map to be pseudomonotone or quasimonotone. Let $\Omega \subset R^n$ be an open convex set and $\mathscr{F} : \Omega \to R^n$ be differentiable with Jacobian matrix $J_{\mathscr{F}}(x)$ evaluated at $x \in \Omega$. Let the projection of $\mathscr{F}$ on v be defined by $\psi : I_{x,v} \to R$, $v \in R^n$ where

$$\psi_{x,v}(t) = v^t \mathscr{F}(x+tv), \qquad I_{x,v} = \{t \mid x+tv \in \Omega\}. \tag{2.11}$$

Theorem 2.9 ([30, Theorem 4.1, p. 404]). *Let $\mathscr{F} : \Omega \to R^n$ be differentiable on the open convex set $\Omega \subset R^n$. Then*

(i) $\mathscr{F}$ is quasimonotone on Ω if and only if

$$v^t \mathscr{F}(x) = 0 \Rightarrow v^t J_{\mathscr{F}}(x) v \geq 0; \tag{2.12}$$

$$v^t \mathscr{F}(x) = v^t J_{\mathscr{F}}(x) v = 0, \qquad \hat{t} < 0, v^t \mathscr{F}(x+\hat{t}v) > 0 \Rightarrow \exists\, \tilde{t} > 0, \qquad \tilde{t} \in I_{x,v} \tag{2.13}$$

$$\text{such that } v^t \mathscr{F}(x+tv) \geq 0 \;\forall\, 0 \leq t \leq \tilde{t}.$$

(ii) $\mathscr{F}$ is pseudomonotone on Ω if and only if

$$\begin{gathered} v^t \mathscr{F}(x) = 0 \Rightarrow v^t J_{\mathscr{F}}(x) v \geq 0; \\ v^t \mathscr{F}(x) = v^t J_{\mathscr{F}}(x) v = 0 \;\Rightarrow\; \exists\, \tilde{t} > 0, \qquad \tilde{t} \in I_{x,v} \\ \text{such that } v^t \mathscr{F}(x+tv) \geq 0 \;\forall\, 0 \leq t \leq \tilde{t}. \end{gathered} \tag{2.14}$$

Karamardian, Schaible, and Crouzeix [30] also obtained somewhat different sufficient conditions for a map to be pseudomonotone and strictly pseudomonotone. For the proof we refer the reader to [30].

Theorem 2.10 ([30, Theorem 4.2, p. 406]). *Let $\mathscr{F} : \Omega \to R^n$ be differentiable on the open convex set $\Omega \subset R^n$. Then*

(i) $\mathscr{F}$ is pseudomonotone on Ω if for every $x \in \Omega$ and $v \in R^n$, we have

$$\begin{gathered} v^t \mathscr{F}(x) = 0 \Rightarrow v^t J_{\mathscr{F}}(x) v \geq 0; \\ v^t \mathscr{F}(x) = v^t J_{\mathscr{F}}(x) v = 0 \Rightarrow \exists\, \varepsilon > 0 \quad \text{such that } v^t J_{\mathscr{F}}(x+tv) v \geq 0 \\ \forall\, t \in I_{x,v}, \qquad |t| \leq \varepsilon. \end{gathered} \tag{2.15}$$

(ii) $\mathscr{F}$ is strictly pseudomonotone on Ω if for every $x \in \Omega$ and $v \in R^n$

$$v^t \mathscr{F}(x) = 0 \Rightarrow v^t J_{\mathscr{F}}(x) v \geq 0.$$

For the proof, we refer the reader to [30]. Karamardian, Schaible, and Crouzeix [30] also presented an example from [1] to show that the sufficient condition (2.15) in Theorem 2.10 is not a necessary condition.

2.4 Generalized Monotonicity of Affine Maps

Generalized monotone affine maps have been considered in [9–11, 18, 19, 35, 37, 39]. The special case of the map $\mathscr{F} : \Omega \to R^n$ is an affine map

$$\mathscr{F}(x) = Ax + q,$$

where $A \in R^{n\times n}$ and $q \in R^n$. Special cases of Theorem 2.9 for an affine map $\mathscr{F} : \Omega \to R^n$ are given below.

Theorem 2.11 ([30, Theorem 5.1, p. 408]). *Let $\Omega \subset R^n$ be open and convex. Then for an affine map $\mathscr{F} : \Omega \to R^n$ of the form $\mathscr{F}(x) = Ax + q$ is quasimonotone if and only if it is pseudomonotone on Ω.*

Proof. Note that pseudomonotonicity $\Rightarrow$ quasimonotonicity. Therefore we need to show that quasimonotonicity imples pseudomonotonicity. The Jacobian matrix $J_{\mathscr{F}}(x) = A$ is independent of x. Note that by part (i) of Theorem 2.9 the condition (2.12)

$$v^t\mathscr{F}(x) = 0 \Rightarrow v^tJ_{\mathscr{F}}(x)v \geq 0$$

holds. This reduces to $v^t(Ax + q) = 0 \Rightarrow v^tAv \geq 0$.

The function $\psi : I_{x,v} \to R,\ v \in R^n$, which is a linear function of t, is given by

$$\psi(t) = (v^tAv)t + v^t(Ax + q).$$

It is easy to see that the condition (2.14) is always satisfied because

$$v^t(Ax + q) = v^tAv = 0 \Rightarrow v^t[A(x + tv) + q] = 0,\ \forall\, t.$$

Now by part (ii) of Theorem 2.9, it follows that $\mathscr{F}$ is pseudomonotone. ■

Theorem 2.11 is not true if Ω is not open. Karamardian, Schaible, and Crouzeix [30] provide an example. For a continuous map $\mathscr{F}$ (not necessarily affine), Crouzeix and Schaible [11] observe the following result which uses the nonemptiness of $int(\Omega)$.

Theorem 2.12. *Assume that $\mathscr{F} : \Omega \to R^n$ is continuous on Ω and quasimonotone on $int(\Omega)$. Then it is also quasimonotone on Ω.*

Theorem 2.13 ([30, Theorem 5.2, p. 409]). *Let $\Omega \subset R^n$ be open and convex. Then for an affine map $\mathscr{F} : \Omega \to R^n$ where $\mathscr{F}(x) = Ax + q$ is pseudomonotone if and only if for every $x \in \Omega$ and $v \in R^n$ we have*

$$v^t(Ax + q) = 0 \Rightarrow v^tAv \geq 0.$$

Proof. It is easy to see that the condition (2.14) is always satisfied inasmuch as

$$v^t(Ax + q) = v^tAv = 0 \Rightarrow v^t[A(x + tv) + q] = 0,\ \forall\, t.$$

From Theorem 2.9, the result follows. ■

Karamardian, Schaible, and Crouzeix [30] observe that the sufficiency part of the above theorem remains valid for an arbitrary convex set Ω (not necessarily open).

2.5 Generalized Monotone Affine Maps on R^n_+ and Positive-Subdefinite Matrices

Generalized monotone affine maps arise in linear complementarity problems. In [9], Crouzeix et al. obtained new characterizations of generalized monotone affine maps on R^n_+ using positive subdefinite matrices. A is called a *positive-subdefinite matrix* if for all $x \in R^n$, $x^tAx < 0$ implies A^tx is unisigned. The class of positive-subdefinite matrices (PSBD) is a generalization of the class of positive-semidefinite (PSD) matrices. The study of pseudoconvex and quasiconvex quadratic forms leads to this new class of matrices, and it is useful in the study of quadratic programming problem. The class of symmetric positive-subdefinite matrices was introduced by Martos [34] in connection with a characterization of a pseudoconvex function. Martos did an interesting study of these matrices. Cottle and Ferland [5] followed the path set by Martos in [34] and among other things, obtained converses for some of Martos's results. Rao [40] obtained a characterization of merely positive-subdefinite matrices which enabled the easy recognition of quasiconvex and pseudoconvex quadratic forms. He also studied this class with respect to generalized inverse (g-inverse). Martos was considering the Hessians of quadratic functions, therefore he was concerned only about symmetric matrices. Later Crouzeix et al. [9] and Mohan, Neogy, and Das [35] studied nonsymmetric PSBD matrices in the context of generalized monotonicity and the linear complementarity problem. In this section characterizations of generalized monotone affine maps on R^n_+ using positive-subdefinite matrices, the properties of PSBD matrices, and their applications to linear complementarity problem are presented. It is not surprising that many properties of PSD matrices are lost through the generalization. It is useful to review some matrix classes and their properties which form the basis for further discussions.

The *linear complementarity problem* is a fundamental problem that arises in optimization, game theory, economics, and engineering. It can be stated as follows.

Given a matrix $A \in R^{n\times n}$ and a vector $q \in R^n$, the linear complementarity problem is to find a vector $x \in R^n$ such that

$$x \geq 0, \qquad Ax + q \geq 0, \tag{2.16}$$

$$x^t(Ax+q) = 0. \tag{2.17}$$

This problem is denoted as LCP(q,A). The LCP is normally identified as a problem of mathematical programming and provides a unifying framework for several optimization problems such as linear programming, linear fractional programming, convex quadratic programming, and bimatrix game problems. More specifically, the LCP models the optimality conditions of these problems. The early motivation for studying the linear complementarity problem was that the KKT optimality conditions for linear and quadratic programs reduce to an LCP. The algorithm presented by Lemke and Howson [32] to compute an equilibrium pair of strategies to a bimatrix game, later extended by Lemke [5] (known as Lemke's algorithm) to solve an LCP(q,A), contributed significantly to the development of linear complementarity theory. In fact, the study of the LCP really came into prominence only when Lemke and Howson [32] and Lemke [5] showed that the problem of computing a Nash equilibrium point of a bimatrix game can be posed as an LCP following the publication by Cottle [4]. However, Lemke's algorithm does not solve every instance of the linear complementarity problem, and in some instances of the problem may terminate inconclusively without either computing a solution to it or showing that no solution to it exists. Extending the applicability of Lemke's algorithm to more matrix classes has been considered by many researchers including Eaves [12, 13], Garcia [17], Karamardian [28], and Todd [41]. For recent books on the linear complementarity problem and its applications, see Cottle, Pang, and Stone [7], Murty [36], and Facchinei and Pang [15]. Matrix classes play an important role in studying the theory and algorithms of LCP. The study of special properties of the data matrix A has historically been an important part of LCP research. A variety of classes of matrices is introduced in the context of the linear complementarity problem. Many of the matrix classes encountered in the context of the LCP are commonly found in several applications. Some of these matrix classes are of interest because they characterize certain properties of the LCP and they offer certain nice features from the viewpoint of algorithms. Several algorithms have been designed for the solution of the linear complementarity problem. Many of these methods are matrix class dependent. They work only for LCPs with some special classes of matrices and can give no information otherwise.

It is well known that the positive-semidefiniteness of a matrix A is equivalent to the monotonicity of the affine mapping $\mathscr{F}(x) = Ax + q$, where $A \in R^{n\times n}$ and $q \in R^n$. The class of PSD matrices is a subclass of positive-subdefinite matrices. Let A be a given $n \times n$ matrix, not necessarily symmetric.

Definition 2.4. We say that a real square matrix A of order n is *positive subdefinite* (*PSBD*) if for all $x \in R^n$

$$x^t Ax < 0 \text{ implies either } A^t x \leq 0 \text{ or } A^t x \geq 0.$$

A is said to be *merely positive-subdefinite* (*MPSBD*) if A is a PSBD matrix but not positive-semidefinite (PSD).

Definition 2.5. A is said to be a $P\,(P_0)$*-matrix* if all its principal minors are positive (nonnegative).

A subclass of P_0 occurs in Markov chain analysis and in the study of global univalence in economic theory [38].

Definition 2.6. A is called *copositive* (C_0) (*strictly copositive* (C)) if $x^t Ax \geq 0 \;\forall\, x \geq 0$ $(x^t Ax > 0 \;\forall\, 0 \neq x \geq 0)$. $A \in R^{n\times n}$ is said to be *conegative* if $x^t Ax \leq 0 \;\forall\, x \geq 0$.

A is said to be *copositive-plus* (C_0^+) if $A \in C_0$ and the following implication holds.

$$[x^t Ax = 0, x \geq 0] \Rightarrow (A + A^t)x = 0.$$

We say that $A \in R^{n \times n}$ is *copositive-star* (C_0^*) if $A \in C_0$ and the following implication holds.

$$[x^t Ax = 0, Ax \geq 0, x \geq 0] \Rightarrow A^t x \leq 0.$$

A is called *copositive (strictly copositive, copositive-plus, PSD, PD)* of order k, $0 \leq k \leq n$, if every principal submatrix of order k is copositive (strictly copositive, copositive-plus, PSD, PD).

Definition 2.7. A is said to be *column sufficient* if for all $x \in R^n$ the following implication holds.

$$x_i(Ax)_i \leq 0 \;\forall\; i \;\Rightarrow\; x_i(Ax)_i = 0 \;\forall\; i.$$

A is said to be *row sufficient* if A^t is column sufficient.

A is *sufficient* if A and A^t are both column sufficient.

A matrix A is *sufficient of order k* if all its $k \times k$ principal submatrices are sufficient.

For details on sufficient matrices, see [6, 8, 43].

Definition 2.8. $A \in R^{n \times n}$ is called a *Q-matrix* (or a *matrix satisfying the Q-property*) if for every $q \in R^n$, LCP(q,A) has a solution.

Given a matrix $A \in R^{n \times n}$ and a vector $q \in R^n$ we define the feasible set $F(q,A) = \{x \geq 0 \mid Ax + q \geq 0\}$ and the solution set of LCP(q,A) by $S(q,A) = \{x \in F(q,A) \mid x^t(Ax+q) = 0\}$. We say that A is a *Q_0-matrix* if $F(q,A) \neq \emptyset$ implies $S(q,A) \neq \emptyset$.

A is said to be a *completely Q (Q_0)-matrix* if all its principal submatrices are Q (Q_0)-matrices.

We recall that given a matrix $A \in R^{n \times n}$ and a vector $q \in R^n$, an affine map $\mathscr{F}(x) = Ax + q$ is said to be *pseudomonotone* on R^n_+ if

$$(y-x)^t(Ax+q) \geq 0, \qquad y \geq 0, \qquad x \geq 0 \Rightarrow (y-x)^t(Ay+q) \geq 0.$$

Given $A \in R^{n \times n}$ and $q \in R^n$, Crouzeix et al. [9] prove the following necessary and sufficient condition for an affine map $\mathscr{F}(x) = Ax + q$ to be pseudomonotone on R^n_+.

Proposition 2.2 ([9]). *An affine map $\mathscr{F}$ is pseudomonotone on R^n_+ if and only if*

$$x \in R^n,\;\; x^t Ax < 0 \Rightarrow \begin{cases} A^t x \geq 0 \text{ and } x^t q \geq 0 & \text{or} \\ A^t x \leq 0,\; x^t q \leq 0 & \text{and } x^t(Ax^- + q) < 0. \end{cases}$$

A necessary and sufficient condition for an affine map $\mathscr{F}(x) = Ax + q$ to be quasimonotone on R^n_+ is given below.

Proposition 2.3 ([9]). *An affine map $\mathscr{F}$ is quasimonotone on R^n_+ if and only if*

$$x \in R^n, \;\; x^t A x < 0 \Rightarrow \begin{cases} A^t x \geq 0 & \text{and } x^t q \geq 0 \text{ or} \\ A^t x \leq 0 & \text{and } x^t q \leq 0. \end{cases}$$

The above proposition shows that A is PSBD when $\mathscr{F}$ is quasimonotone (a fortiori, pseudomonotone) on R^n_+.

Theorem 2.14 ([9]). *Assume that $\mathscr{F}$ is quasimonotone on R^n_+ and $q \neq 0$. Then $\mathscr{F}$ is pseudomonotone on R^n_+.*

Definition 2.9. We say that a matrix $A \in R^{n\times n}$ is *pseudomonotone* if $\mathscr{F}(x) = Ax$ is pseudomonotone on the nonnegative orthant.

Theorem 2.15 ([18, Corollary 4]). *If A is pseudomonotone, then A is a row sufficient matrix.*

A row sufficient matrix belongs to Q_0 [7, p. 159], therefore LCP(q,A) is solvable by Lemke's algorithm where A is a pseudomonotone matrix. Gowda [19] showed with an example that the transpose of a pseudomonotone matrix need not be in Q_0 and hence need not be pseudomonotone. However, if A is pseudomonotone then under certain conditions only A^t is a Q_0-matrix. These conditions are stated below in the following theorem.

Theorem 2.16 ([19]). *Suppose that $A \in R^{n\times n}$ is pseudomonotone. Then under each of the following conditions A^t satisfies the copositive star property and hence belongs to Q_0.*

(i) The diagonals of A consist only of zeros.
(ii) The system $0 \neq d \geq 0, A^t d = 0$ has no solution.
(iii) A is invertible.
(iv) $A \in R_0$.
(v) A is normal (i.e., $AA^t = A^t A$).

Gowda [18] observes the following results. For the proofs of these results we refer the reader to the article [18] by Gowda.

Theorem 2.17. *Suppose that LCP(q,A) is feasible and the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone. Then $A \in C_0 \cap P_0$.*

Gowda [18] provides the following example to show that the above result may not hold if the feasibility condition is dropped. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad q = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Then the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone but $A \notin C_0 \cap P_0$. For details see [18].

The following result is a corollary of the above theorem.

Corollary 2.1 ([18]). *Suppose that A is symmetric. Then the pseudomonotone mapping $\mathscr{F}(x) = Ax + q$ is monotone (i.e., A is positive semidefinite (PSD)) if and only if A is copositive.*

The following theorem in [18] shows that the pseudomonotonicity can be described in terms of a single variable in R^n.

Theorem 2.18 ([18, p. 375]). *For the pair (q,A), let*

(i) $\mathscr{A} := \{x : (A^t x)_i > 0$ *for some i and* $x^t(Ax^- + q) \leq 0\}$.
(ii) $\mathscr{B} := \{x : (A^t x)_i < 0$ *for some i and* $x^t(Ax^- + q) \geq 0\}$.
(iii) $\mathscr{C} := \{x : x^t(Ax^- + q) \geq 0\}$, $\mathscr{D} := \{x : x^t(Ax^+ + q) \geq 0\}$.

The mapping $\mathscr{F}(x) = Ax + q$ is pseudomonotone if and only if

(a) $x^t Ax \geq 0 \; \forall \, x \in \mathscr{A} \cup \mathscr{B}$.
(b) $\mathscr{C} \subseteq \mathscr{D}$.

It is easy to see that if A is PSD then for any q, mapping $\mathscr{F}(x) = Ax + q$ is pseudomonotone. Gowda [18] proves that even the converse is true.

Corollary 2.2 ([18, p. 376]). *$A \in R^{n \times n}$ is PSD if and only if for every q, the mapping $\mathscr{F}(x) = Ax + q$ is pseudomonotone.*

Theorem 2.19 ([18]). *Suppose that $A \in R^{n \times n}$ has no zero column. If the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone and LCP(q,A) is feasible, then A is pseudomonotone.*

Theorem 2.20 ([18]). *Suppose that A is pseudomonotone. Then $A \in P_0 \cap Q_0$ and every feasible LCP(q,A) is solvable (by Lemke's algorithm).*

The following result follows immediately from the above two theorems.

Theorem 2.21 ([18]). *Suppose that $A \in R^{n \times n}$ has no zero column. If LCP(q,A) is feasible and the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone, then $A \in P_0 \cap Q_0$ and every feasible LCP(q',A) is solvable (by Lemke's algorithm).*

Theorem 2.22 ([18]). *Suppose that the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone and LCP(q,A) is feasible. Then LCP(q,A) is solvable (by Lemke's algorithm).*

By presenting the following example Gowda [18] shows that stronger conclusions in the above theorem are not possible. For other values q', feasibility of LCP(q',A) does not necessarily imply solvability. Let

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad q' = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Then the map $\mathscr{F}(x) = Ax + q$ is pseudomonotone. LCP(q,A) and LCP(q',A) are feasible but LCP(q',A) is not solvable. Therefore, $A \notin Q_0$.

The following theorem relates the concept of pseudomonotonicity of a matrix A to the class of PSBD matrices.

Theorem 2.23 ([9, Theorem 3.3]). *$A \in R^{n\times n}$ is pseudomonotone if and only if A is PSBD and copositive with the additional condition in the case where $A = ab^t$ that $b_i = 0 \Rightarrow a_i = 0$.*

In fact, the class of pseudomonotone matrices coincides with the class of matrices which are both PSBD and C_0^*. For more details on pseudomonotone and C_0^* matrices see [20].

In general PSBD matrices need not be P_0 or Q_0. We provide the following example.

Example 2.1. Suppose

$$A = \begin{bmatrix} 0 & -3 \\ -5 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We show that A is PSBD by showing that for all $x \in R^n$,

$$x^t Ax < 0 \quad \text{implies either} \quad A^t x \leq 0 \text{ or } A^t x \geq 0.$$

Then $x^t Ax = -8x_1x_2 < 0$ implies x_1 and x_2 are of same sign. Clearly $A \in$ PSBD because

$$A^t x = \begin{bmatrix} -5x_2 \\ -3x_1 \end{bmatrix}$$

implies either $A^t x \leq 0$ or $A^t x \geq 0$ but $A \notin P_0$.

The following example shows that PSBD matrices need not be Q_0 in general.

Example 2.2. Let

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

But $x^t Ax = x_1^2 + 4x_1x_2 < 0$ implies x_1 and x_2 are of different sign. Clearly $A \in$ PSBD because

$$A^t x = \begin{bmatrix} x_1 + 4x_2 \\ 0 \end{bmatrix}$$

implies either $A^t x \leq 0$ or $A^t x \geq 0$. Taking

$$q = \begin{bmatrix} -1 \\ -8 \end{bmatrix}$$

we note that LCP(q,A) is feasible but the problem has no complementary solution. Therefore A is not a Q_0 matrix.

Theorem 2.24 ([9, Proposition 2.1]). *Let $A = ab^t$ with $a,b \in R^n$, $a,b \neq 0$.*
A is PSBD if and only if one of the following conditions holds.

(i) $\exists$ a $t > 0$ such that $b = ta$.
(ii) For all $t > 0$, $b \neq ta$ and either $b \geq 0$ or $b \leq 0$.

Further suppose that $A \in MPSBD$. Then $A \in C_0$ if and only if either ($a \geq 0$ and $b \geq 0$) or ($a \leq 0$ and $b \leq 0$) and $A \in C_0^*$ if and only if A is copositive and $a_i = 0$ whenever $b_i = 0$.

Gowda [18] conjectured that pseudomonotonicity of a matrix A implies pseudomonotonicity of a matrix A^t. Obviously, Gowda's conjecture is true when A is PSD. Crouzeix et al. [9] show that the conjecture is also true when rank$(A) \geq 2$ but it is not true for matrices of rank1.

Combining [9, Theorem 2.1] and [9, Proposition 2.5], we get the following theorem on PSBD matrices.

Theorem 2.25 ([35]). *Suppose $A \in R^{n\times n}$ is PSBD and rank$(A) \geq 2$. Then A^t is PSBD and at least one of the following conditions holds.*

(i) A is PSD.
(ii) $(A + A^t) \leq 0$.
(iii) A is C_0^*.

Theorem 2.26 ([9, Proposition 2.2]). *Assume that $A \in R^{n\times n}$ is MPSBD and rank$(A) \geq 2$. Then*

(a) $\nu_-(A + A^t) = 1$.
(b) $(A + A^t)x = 0 \;\Leftrightarrow\; Ax = A^t x = 0$.

Because a PSBD matrix is a natural generalization of a PSD matrix, it is of interest to determine which of the properties of a PSD matrix also holds for a PSBD matrix. In particular, we may ask whether

(i) A is PSBD if and only if $(A + A^t)$ is PSBD.
(ii) Any PPT (Principal Pivot Transform)[42] of a PSBD matrix is a PSBD matrix.

Mohan, Neogy, and Das [35] observe that these properties are not carried over to PSBD matrices. However PSBD is a complete class in the sense of [7, 3.9.5].

Theorem 2.27 ([35]). *Suppose $A \in R^{n\times n}$ is a PSBD matrix. Then $A_{\alpha\alpha} \in PSBD$ where $\alpha \subseteq \{1,\ldots,n\}$.*

Theorem 2.28 ([35]). *Suppose $A \in R^{n\times n}$ is a PSBD matrix. Let $D \in R^{n\times n}$ be a positive diagonal matrix. Then $A \in PSBD$ if and only if $DAD^t \in PSBD$.*

Theorem 2.29 ([35]). *PSBD matrices are invariant under principal rearrangement; that is if $A \in R^{n\times n}$ is a PSBD matrix and $P \in R^{n\times n}$ is any permutation matrix, then $PAP^t \in PSBD$.*

Lemma 2.2. *Suppose $A \in R^{n\times n}$ is a PSBD matrix with rank$(A) \geq 2$ and $A + A^t \leq 0$. Then at least one of the following conditions holds.*

(i) A is PSD.
(ii) $A \leq 0$.

Theorem 2.30 ([35]). *Suppose $A \in R^{n\times n}$ is a PSBD matrix with rank$(A) \geq 2$. Then A is a Q_0 matrix.*

Proof. By Theorem 2.25 and Lemma 2.2, it follows that either $A \in$ PSD or $A \leq 0$ or $A \in C_0^*$. Therefore $A \in Q_0$ (see [7]). ∎

Theorem 2.31 ([35]). *Suppose A is a PSBD $\cap\, C_0$ matrix with rank$(A) \geq 2$. Then $A \in R^{n\times n}$ is a sufficient matrix.*

Proof. Note that by Theorem 2.25, A^t is a PSBD $\cap\, C_0$ matrix with rank$(A^t) \geq 2$. Now by Theorem 2.23, A and A^t are pseudomonotone. Hence A and A^t are row sufficient by Theorem 2.15. Therefore, A is sufficient. ∎

The following theorem provides a new sufficient condition to solve LCP(q,A) by Lemke's algorithm.

Theorem 2.32. *Suppose $A \in R^{n\times n}$ can be written as $M+N$ where $M \in$ MPSBD $\cap\, C_0^+$, rank$(M) \geq 2$, and $N \in C_0$. If the system $q+Mx-N^t y \geq 0$, $y \geq 0$ is feasible, then Lemke's algorithm for LCP(q,A) with covering vector $d > 0$ terminates with a solution.*

For the proof of the above result we refer the reader to the article by Mohan, Neogy, and Das [35]. The proof follows along similar lines to the proof of Evers [14].

2.6 Generalized Positive-Subdefinite Matrices

The class of generalized positive-subdefinite (GPSBD) matrices is an interesting matrix class introduced by Crouzeix and Komlósi [10]. This class is a generalization of the class of symmetric positive-subdefinite (PSBD) matrices introduced by Martos [34] and nonsymmetric PSBD matrices studied by Crouzeix et al. [9]. The solution set of a linear complementarity problem ($S(q,A)$) can be linked with the set of KKT-stationary points ($S''(q,A)$) of the corresponding quadratic programming problem. The row-sufficient matrices have been characterized by Cottle, Pang, and Venkateswarn [8] as the class for which the solution set of LCP(q,A) is the same as the solution set of KKT points of the corresponding quadratic program. In [37], Neogy and Das showed that the property ($S''(q,A) \subseteq S(q,A)$) holds for generalized positive-subdefinite matrices under some additional assumptions and identified a large subclass of GPSBD matrices as row-sufficient matrices. This has practical relevance to the study of quadratic programming and interior point algorithms.

Definition 2.10. A matrix $A \in R^{n\times n}$ is called a *generalized positive-subdefinite matrix* (GPSBD) [10] if there exist nonnegative multipliers s_i, t_i with $s_i + t_i = 1$, $i = 1,2,\ldots,n$ such that

$$\forall\, x \in R^n, \qquad x^t Ax < 0 \Rightarrow \begin{cases} \text{either} & -s_i x_i + t_i (A^t x)_i \geq 0 \quad \text{for all } i, \\ \text{or} & -s_i x_i + t_i (A^t x)_i \leq 0 \quad \text{for all } i. \end{cases} \tag{2.18}$$

Let S and T be two nonnegative diagonal matrices with diagonal elements s_i, t_i, where $s_i + t_i = 1$ for $i = 1, \ldots, n$. Note that S and T are independent of x. A matrix $A \in R^{n\times n}$ is said to be GPSBD if there exist two nonnegative diagonal matrices S and T with $S + T = I$ such that

$$\forall\, x \in R^n, \qquad x^t Ax < 0 \Rightarrow \begin{cases} \text{either} & -Sx + TA^t x \geq 0, \\ \text{or} & -Sx + TA^t x \leq 0. \end{cases} \tag{2.19}$$

Note that GPSBD reduces to PSBD if $S = 0$. A is called *nondegenerate* GPSBD if for all $x \in R^n$, $x^t Ax < 0$ implies $-Sx + TA^t x \neq 0$ and unisigned; that is, at least one of the inequalities in (2.18) should hold as a strict inequality. A is said to be a *merely generalized positive-subdefinite* (MGPSBD) *matrix* if A is a GPSBD matrix but not a PSBD matrix. The following example is a nontrivial example of a GPSBD matrix.

Example 2.3. Let

$$A = \begin{bmatrix} 0 & 5 & 0 \\ -4 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note that $\nu_-(A + A^t) = 1$.

Then for any $x = [x_1 \;\; x_2 \;\; x_3]^t$, $x^t Ax = x_1 x_2 < 0$ implies x_1 and x_2 are of opposite sign. Clearly, $A^t x = [-4x_2 \;\; 5x_1 + x_3 \;\; -x_2]^t$ and for $x = [-1 \;\; 1 \;\; 7]^t$, $x^t Ax < 0$ does not imply $A^t x$ is unisigned. Therefore, A is not a PSBD matrix. However, with the choice $s_1 = 0, s_2 = 1$ and $s_3 = 0$, it is easy to check that A is a GPSBD matrix.

Theorem 2.33. *Suppose $A \in MGPSBD \cap C_0$ with $0 < t_i < 1$ for all i. Then A is a row-sufficient matrix.*

Proof. Suppose $x_i(A^t x)_i \leq 0$ for $i = 1, \ldots, n$. Let $I_1 = \{i \;:\; x_i > 0\}$ and $I_2 = \{i \;:\; x_i < 0\}$. We need to consider three cases.

Case I. $I_2 = \emptyset$. Then $x^t Ax = x^t A^t x = \sum_i x_i (A^t x)_i \leq 0$. Because $A \in C_0$, $[x_i(A^t x)_i] = 0$, $\forall\, i$.

Case II. $I_1 = \emptyset$. Then $(-x)^t A^t (-x) = x^t A^t x = \sum_i x_i (A^t x)_i \leq 0$. Because $A \in C_0$, $[x_i(A^t x)_i] = 0$, $\forall\, i$.

Case III. Suppose there exists a vector x such that $x_i(A^t x)_i \leq 0$ for $i = 1, 2, \ldots, n$ and $x_k(A^t x)_k < 0$ for at least one $k \in \{1, 2, \ldots, n\}$. Let $I_1 \neq \emptyset$ and $I_2 \neq \emptyset$. $x^t Ax = x^t A^t x = \sum_i [x_i(A^t x)_i] < 0$. This implies $-s_i x_i + t_i (A^t x)_i \geq 0$, $\forall\, i$ or $-s_i x_i + t_i (A^t x)_i \leq 0$, $\forall\, i$.

Without loss of generality, assume $-s_i x_i + t_i (A^t x)_i \geq 0$, $\forall\, i$. Then for all $i \in I_1$, $-s_i x_i^2 + t_i x_i (A^t x)_i \geq 0$. This implies $[x_i(A^t x)_i] \geq (s_i/t_i) x_i^2 > 0$, $\forall\, i \in I_1$. Therefore, $\sum_{i\in I_1} [x_i(A^t x)_i] > 0$. Because $x_i(A^t x)_i \leq 0$ for $i = 1, \ldots, n$, this leads to a contradiction.

Therefore, $[x_i(A^t x)_i] = 0$, $\forall\, i$. So A is row sufficient. ∎

Neogy and Das [37] provide an example to show that the assumption in the above theorem $0 < t_i < 1 \; \forall \; i$ cannot be relaxed.

The following theorem extends the result of Evers [14] and the result obtained in Theorem 2.32 in an earlier section for solving LCP(q,A) by Lemke's algorithm when A satisfies certain conditions stated in the following theorem.

Theorem 2.34. *Suppose* $A \in R^{n\times n}$ *can be written as* $M+N$ *where* $M \in MGPSBD \cap C_0^+$*, is nondegenerate with* $0 < t_i < 1$*,* $\forall \; i$*, and* $N \in C_0$*. If the system* $q+Mx-N^t y \geq 0,\; y \geq 0$ *is feasible, then Lemke's algorithm for LCP*(q,A) *with covering vector* $d > 0$ *terminates with a solution.*

The following example demonstrates that the class MGPSBD $\cap \; C_0^+$ is nonempty.

Example 2.4 ([37]). Consider the copositive-plus matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & 0 & 1 \end{bmatrix}.$$

Take $x = [-1 \;\; -1 \;\; 1]^t$. It is easy to check that A is not MPSBD. However, with choice $s_i = \frac{1}{2} \; \forall i$, A is a MGPSBD matrix.

The following result is a consequence of the characterization of row-sufficient matrices observed by Cottle, Pang, and Venkateswarn [8].

Lemma 2.3. *Suppose* $A \in MGPSBD \cap C_0$ *with* $0 < t_i < 1$ *for all i. For each vector* $q \in R^n$*, if* (x^*, u^*) *is a Karush–Kuhn–Tucker pair of the quadratic program* $QP(q,A): \; [\min x^t(Ax+q);\; x \geq 0,\; Ax+q \geq 0]$*, then* x^* *solves* $LCP(q,A): \; [x \geq 0,\; Ax+q \geq 0,\; x^t(Ax+q) = 0]$.

Proof. From Theorem 2.33 and [8, Theorem 4, p. 238], the result follows. ■

Remark 2.1. From Lemma 2.3, it follows that the solution set of a linear complementarity problem $(S(q,A))$ is related to the set of KKT-stationary points $(S''(q,A))$ of the corresponding quadratic programming problem and the statement $S''(q,A) \subseteq S(q,A)$ holds for MGPSBD matrices with some additional assumptions as stated in Theorem 2.33. For details see Neogy and Das [37].

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