ILLUMINATING SETS OF CONSTANT WIDTH

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Abstract. The problem of illuminating the boundary of sets having constant width is considered and a bound for the number of directions needed is given. As a corollary, an estimate for Borsuk's partition problem is inferred. Also, the illumination number of sufficiently symmetric strictly convex bodies is determined.

§1. *Introduction.* Let $\mathbf{x}$ be a point on the boundary ∂K of a convex body* K in Euclidean space, $\mathbf{R}^n$. A direction $\mathbf{u} \in S^{n-1}$ is said to *illuminate* K at $\mathbf{x}$ if the line $\{x + t\mathbf{u} \mid t \in \mathbf{R}\}$ "enters" $\mathbf{K}$ in $\mathbf{x}$. More precisely, $\mathbf{u}$ illuminates K at $\mathbf{x}$ if $\mathbf{x} + t\mathbf{u}$ is an interior point of K, for some positive t. Instead of saying "$\mathbf{u}$ illuminates K at $\mathbf{x}$" we will just say "$\mathbf{u}$ illuminates $\mathbf{x}$". This will cause no confusion, because the convex body K should be clear from the context.

Directions $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in S^{n-1}$ are said to illuminate K if every point on the boundary of K is illuminated by at least one of these directions. We denote by $I(K)$ the minimal number of directions sufficient to illuminate K and call it the *illumination number* of K. This concept of illumination was introduced by V. G. Boltjansky in [**2**]. There he proved that for convex bodies K, $I(K)$ is equal to $H(K)$, the number of smaller, positively homothetic copies of K required to cover K (see [**3**]).

The maximum values of $I(K)$ for convex bodies K in $\mathbf{R}^n$, are unknown when $n > 2$. By the above result of Boltjansky, Hadwiger's conjecture, about the covering of a set by homothetic copies of it, is equivalent to

$$I(\text{convex body in } \mathbf{R}^n) \leqslant I(n\text{-dimensional parallelogram}) = 2^n.$$

See [**3**] for a discussion of this conjecture.

A set of *constant width* d is a convex body such that the distance between any two distinct parallel supporting hyperplanes of it is d (see [**5**, pp. 122–131], [**4**]). In this note we prove

THEOREM 1. *If W is a set of constant width in $\mathbf{R}^n$ then*

$$I(W) < 5n\sqrt{n}(4 + \log n)\left(\frac{3}{2}\right)^{n/2}.$$

The research exposed in this note was done while I was at the Hebrew University in Jerusalem, as a student of Professor Gil Kalai. I am grateful to Kalai for his interest, encouragement and advice, and for valuable improvements of the original manuscript.

* A convex body is a compact convex set that has interior points.

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Using this theorem, and the equivalence $I(W) = H(W)$, we get

COROLLARY 2. *Every set of constant width* $W \subset \mathbf{R}^n$ *can be covered by less than* $5n\sqrt{n}(4+\log n)(3/2)^{n/2}$ *homothetic copies of itself, having some homothety coefficient* α, $0<\alpha<1$.

Since every bounded set of positive diameter is contained in a set of constant width having the same diameter (see [**5**, p. 126]), we have the following estimate for Borsuk's partition problem.

COROLLARY 3. *Every set of diameter* d $(0<d<\infty)$ *in* $\mathbf{R}^n$ *can be covered by less than* $5n\sqrt{n}(4+\log n)(3/2)^{n/2}$ *sets having smaller diameters.*

To the best of our knowledge, this is, asymptotically, the best bound known for Borsuk's partition problem (see [**7**] for references and results concerning this problem). The approach to Borsuk's problem through the the illumination problem is suggested in [**3**] and [**8**, p. 420].

Theorem 1 is proved using a probabilistic argument: The probability that a "small" region in ∂W will be illuminated by a random, uniformly distributed, direction is estimated from below. A straightforward computation then shows that if "enough" directions are chosen randomly, uniformly, and independently, the probability that they will completely illuminate W is nonzero. An essential part of the proof relies on finding lower bounds for volumes of spherical sets of a certain type. In [**11**] analogous results for sets in $\mathbf{R}^n$ give lower bounds for the volumes of sets having constant width.

The only known result in the direction opposite to Theorem 1 is that $I(K) \geq n+1$ for every convex body $K \subset \mathbf{R}^n$ ([**3**]). It is not known if there is a set of constant width $W \subset \mathbf{R}^n$, that cannot be illuminated by $n+1$ directions.

Rogers [**10**] has shown that every $K \subset \mathbf{R}^n$, having diameter $0<d<\infty$ and invariant under the group of congruences that leave invariant an n-dimensional regular simplex, can be partitioned into $n+1$ subsets, each with diameter $<d$. Inspired by this, we prove in Section 4 that if K is a strictly convex body† invariant under a group of orthogonal transformations that is generated by reflections through hyperplanes and acts irreducibly‡ on $\mathbf{R}^n$, then $I(K)=n+1$. This can give an alternate proof of Rogers' result.

§2. First we will introduce some notation and give a condition for the illumination of a single boundary point. Throughout this note, K will denote an arbitrary convex body, and W will denote a set of constant width in $\mathbf{R}^n$. For a set $A \subset S^{n-1}$ we define

$$A^+ = \{\mathbf{u} \in S^{n-1} \mid \mathbf{u} \,.\, \mathbf{v} > 0 \text{ for all } \mathbf{v} \in A\}.$$

† A strictly convex body is a convex body whose boundary contains no line segment.

‡ "G acts irreducibly on $\mathbf{R}^n$" means that there is no subspace of $\mathbf{R}^n$, other than 0 and $\mathbf{R}^n$, which is invariant under all elements of G.

When $\mathbf{x}$ is a boundary point of a convex body K, we use $N_K(\mathbf{x})$ to denote the set of inward normal unit vectors of K at $\mathbf{x}$:

$$N_K(\mathbf{x}) = \{\mathbf{u} \in S^{n-1} \mid \mathbf{u} \,.\, \mathbf{p} \geq \mathbf{u} \,.\, \mathbf{x} \text{ for all } \mathbf{p} \in K\}.$$

Notice that $N_K(\mathbf{x})$ is nonempty for $\mathbf{x} \in \partial K$.

LEMMA 4. *Let K be a convex body in $\mathbf{R}^n$, let $\mathbf{x}$ be a boundary point of K, and let $\mathbf{u} \in S^{n-1}$. Then $\mathbf{x}$ is illuminated by the direction $\mathbf{u}$, if, and only if, $\mathbf{u} \in N_K(\mathbf{x})^+$.*

Proof. Suppose $\mathbf{u} \in N_K(\mathbf{x})^+$. If the line $L = \{\mathbf{x} + t\mathbf{u} \mid t \in \mathbf{R}\}$ contains no interior points of K, then it is a supporting line of K. Therefore there is a hyperplane, $H = \{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \,.\, \mathbf{w} = \mathbf{x} \,.\, \mathbf{w}\}$, supporting K and containing L. Thus one of the vectors $\pm\mathbf{w}/\|\mathbf{w}\|$ is in $N_K(\mathbf{x})$. That is a contradiction to $\mathbf{u} \in N_K(\mathbf{x})^+$, because $H \supset L$ implies that $\mathbf{w} \,.\, \mathbf{u} = 0$. Therefore L contains interior points of K. Pick any $\mathbf{v} \in N_K(\mathbf{x})$. K is contained in the half space $\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \,.\, \mathbf{v} \geq \mathbf{x} \,.\, \mathbf{v}\}$. Since $\mathbf{u} \,.\, \mathbf{v} > 0$, this means that the points $\mathbf{x} + t\mathbf{u}$ with $t < 0$ are not in K. Thus the interior points of L correspond to positive values of t, and $\mathbf{u}$ illuminates $\mathbf{x}$.

Now suppose that $\mathbf{u}$ illuminates $\mathbf{x}$. Let t be a positive number such that $\mathbf{p} = \mathbf{x} + t\mathbf{u}$ is an interior point of K. Because $\mathbf{p}$ is an interior point, for every $\mathbf{v} \in N_K(\mathbf{x})$ we have $\mathbf{p} \,.\, \mathbf{v} > \mathbf{x} \,.\, \mathbf{v}$, which implies $\mathbf{u} \,.\, \mathbf{v} > 0$, so $\mathbf{u} \in N_K(\mathbf{x})^+$.

From now on, we work with an arbitrary, but fixed, set of constant width $W \subset \mathbf{R}^n$. The lemma above shows that if E is a subsets of the boundary of W, then one direction can illuminate E, if, and only if,

$$\bigcap_{\mathbf{x} \in E} N_W(\mathbf{x})^+ = \left(\bigcup_{\mathbf{x} \in E} N_W(\mathbf{x})\right)^+$$

is nonempty. The following proposition will help us find subsets of ∂W that are "easily" illuminated. For a subset $A \subset S^{n-1}$ define $U_W(A)$ to be the union of the sets $N_W(\mathbf{x})$, $\mathbf{x} \in \partial W$, that intersect A:

$$U_W(A) = \bigcup_{N_W(\mathbf{x}) \cap A \neq \varnothing} N_W(\mathbf{x}).$$

A direction in $U_W(A)^+$ illuminates every point $\mathbf{x} \in W$ that satisfies

$$N_W(\mathbf{x}) \cap A \neq \varnothing.$$

In order to show that when A is chosen properly these points are "easily" illuminated, we want to prove that $U_W(A)^+$ is "large". Our means of doing so is by estimating the diameter of $U_W(A)$. (We view S^{n-1} with the metric induced by the Euclidean metric in $\mathbf{R}^n$. The diameters of subsets of S^{n-1} refer to this metric.)

PROPOSITION 5. *Let A be a nonempty subset of S^{n-1}. Then*

$$\text{diameter } U_W(A) \leq 1 + \text{diameter } A.$$

Proof. Since $U_W(A)$ does not change when we replace W with a positively homothetic copy of itself, we may, and will, assume that W has constant width

1, and therefore also diameter 1. Let $\mathbf{v}_1, \mathbf{v}_2$ be unit vectors in $U_W(A)$. By the definition of $U_W(A)$, there are points $\mathbf{x}_1, \mathbf{x}_2 \in \partial W$ such that $\mathbf{v}_i \in N_W(\mathbf{x}_i)$ and $N_W(\mathbf{x}_i) \cap A \neq \emptyset$ for $i = 1, 2$. Suppose $\mathbf{u}_i$ is in $N_W(\mathbf{x}_i) \cap A$, $i = 1, 2$. Since $\mathbf{u}_i$ is an inward normal of W at $\mathbf{x}_i$, and since W has constant width 1, the hyperplane $\{\mathbf{p} \in \mathbf{R}^n \mid \mathbf{p} \,.\, \mathbf{u}_i = \mathbf{x}_i \,.\, \mathbf{u}_i + 1\}$ is a supporting hyperplane of W. The only point on this hyperplane whose distiance from $\mathbf{x}_i$ is not greater than 1 is $\mathbf{x}_i + \mathbf{u}_i$. Since diameter $W = 1$, we conclude that $\mathbf{x}_i + \mathbf{u}_i \in \partial W$ for $i = 1, 2$. Therefore

$$1 = (\text{diameter } W)^2 \geq \|(\mathbf{x}_1 + \mathbf{u}_1) - \mathbf{x}_2\|^2 = \|\mathbf{x}_1 - \mathbf{x}_2\|^2 + 2\mathbf{u}_1 \,.\, (\mathbf{x}_1 - \mathbf{x}_2) + 1$$

and

$$1 = (\text{diameter } W)^2 \geq \|(\mathbf{x}_2 + \mathbf{u}_2) - \mathbf{x}_1\|^2 = \|\mathbf{x}_2 - \mathbf{x}_1\|^2 + 2\mathbf{u}_2 \,.\, (\mathbf{x}_2 - \mathbf{x}_1) + 1.$$

Summing these inequalities and rearranging we get

$$(\mathbf{u}_1 - \mathbf{u}_2) \,.\, (\mathbf{x}_2 - \mathbf{x}_1) \geq \|\mathbf{x}_2 - \mathbf{x}_1\|^2.$$

Because $(\mathbf{u}_1 - \mathbf{u}_2) \,.\, (\mathbf{x}_2 - \mathbf{x}_1) \leq \|\mathbf{u}_1 - \mathbf{u}_2\| \, \|\mathbf{x}_2 - \mathbf{x}_1\|$, this implies $\|\mathbf{u}_1 - \mathbf{u}_2\| \geq \|\mathbf{x}_2 - \mathbf{x}_1\|$. So

$$\|\mathbf{x}_2 - \mathbf{x}_1\| \leq \text{diameter } A. \tag{2.1}$$

As with $\mathbf{x}_i + \mathbf{u}_i$, the points $\mathbf{x}_i + \mathbf{v}_i$ lie in ∂W, and therefore

$$1 \geq \|(\mathbf{x}_1 + \mathbf{v}_1) - (\mathbf{x}_2 + \mathbf{v}_2)\| \geq \|\mathbf{v}_1 - \mathbf{v}_2\| - \|\mathbf{x}_2 - \mathbf{x}_1\|.$$

Using (2.1), this implies

$$1 + \text{diameter } A \geq \|\mathbf{v}_1 - \mathbf{v}_2\|.$$

We use μ to denote the standard probability measure on S^{n-1}. Define

$$g(n, d) = \inf\{\mu(A^+) \mid A \subset S^{n-1}, \text{diameter } A \leq d\}.$$

Let $N(n, \varepsilon)$ be the number of sets having diameter ε that is required to cover S^{n-1}. The core of this note is:

PROPOSITION 6. *For* $0 < \varepsilon < \sqrt{2} - 1$ *we have*

$$I(W) \leq 1 + \frac{\log N(n, \varepsilon)}{-\log(1 - g(n, 1 + \varepsilon))}.$$

Proof. It is easily verified that $0 < g(n, 1 + \varepsilon) < 1$ (if $\emptyset \neq A \subset S^{n-1}$ then A^+ is contained in a hemisphere, so that $\mu(A^+) \leq \frac{1}{2}$. If also diameter $A = d < \sqrt{2}$ then A^+ contains a spherical cap of radius $\sqrt{2} - d$ around any point of A), so that the right-hand side is well defined. Let M be a natural number satisfying

$$M > \frac{\log N(n, \varepsilon)}{-\log(1 - g(n, 1 + \varepsilon))}.$$

It is sufficient to show that M directions can illuminate W. Set $N = N(n, \varepsilon)$, and let $A_1, \ldots, A_N$ be a covering of S^{n-1} with sets of diameter ε. By Proposition 5, we have diameter $U_W(A_i) \leq 1 + \varepsilon$, and therefore

$$\mu(U_W(A_i)^+) \geq g(n, 1 + \varepsilon), \qquad i = 1, 2, \ldots, N.$$

Pick M directions $\mathbf{u}_1, \ldots, \mathbf{u}_M$ at random, uniformly and independently distributed on S^{n-1}. Take any i, j, $1 \leq i \leq N$, $1 \leq j \leq M$. The probability that $\mathbf{u}_j$ will be in $U_W(A_i)^+$ is $\mu(U_W(A_i)^+)$, which is at least $g(n, 1+\varepsilon)$. Therefore the probability that $U_W(A_i)^+$ will contain none of the points $\mathbf{u}_1, \ldots, \mathbf{u}_M$ is at most $(1-g(n, 1+\varepsilon))^M$. Thus the probability p that at least one $U_W(A_l)^+$, $1 \leq l \leq N$ will contain no points of $\mathbf{u}_1, \ldots, \mathbf{u}_M$ satisfies

$$p \leq \sum_{l=1}^{N} (1-g(n, 1+\varepsilon))^M < N(1-g(n, 1+\varepsilon))^{\log N / -\log(1-g(n,1+\varepsilon))} = 1.$$

This shows that one can choose M directions, so that each set $U_W(A_l)^+$, $l = 1, \ldots, N$, contains at least one of them. Let $\mathbf{v}_1, \ldots, \mathbf{v}_M$ be such directions, and let $\mathbf{x}$ be a point of ∂W. We claim that one of these directions illuminates $\mathbf{x}$. Since $N_W(\mathbf{x})$ is nonempty, and the sets $A_1, \ldots, A_N$ cover S^{n-1}, one of them, say A_i, intersects $N_W(\mathbf{x})$. By the definition of $U_W(A_i)$, we have $N_W(\mathbf{x}) \subset U_W(A_i)$. So that

$$N_W(\mathbf{x})^+ \supset U_W(A_i)^+.$$

$U_W(A_i)^+$ contains at least one of $\mathbf{v}_1, \ldots, \mathbf{v}_M$, say $\mathbf{v}_k$. We have

$$\mathbf{v}_k \in U_W(A_i)^+ \subset N_W(\mathbf{x})^+$$

and therefore, by Lemma 4, $\mathbf{v}_k$ illuminates $\mathbf{x}$. This shows that the directions $\mathbf{v}_1, \ldots, \mathbf{v}_M$ illuminate W.

In order to deduce Theorem 1 from Proposition 6, we only have to estimate $g(n, 1+\varepsilon)$ from below, and $N(n, \varepsilon)$ from above. The former will be done in the next section, and the latter is dealt with by the following well known fact.

LEMMA 7. $N(n, \varepsilon) \leq (1+4/\varepsilon)^n$.

Proof. Let E be a maximal subset of S^{n-1} having the property that $\|\mathbf{u}-\mathbf{v}\| > \frac{1}{2}\varepsilon$ for $\mathbf{u} \neq \mathbf{v}$ in E. The maximality of E shows that the balls with radius $\frac{1}{2}\varepsilon$ and centers in E cover S^{n-1}, therefore

$$|E| \geq N(n, \varepsilon).$$

All the balls $B(\mathbf{u}, \varepsilon/4)$, $\mathbf{u} \in E$, are disjoint and are contained in the ball $B(0, 1+\varepsilon/4)$. Comparing volumes gives

$$|E|(\varepsilon/4)^n \leq (1+\varepsilon/4)^n,$$

or

$$|E| \leq \left(\frac{4}{\varepsilon}+1\right)^n.$$

Remark. Better estimates for $N(n, \varepsilon)$ are known (see [**9**]), but do not seem to contribute any significant improvement to Theorem 1.

§3. In this section we give a lower bound for $g(n, d)$, and prove Theorem 1.

PROPOSITION 8. *Let $d>0$ and let A be a nonempty subset of S^{n-1} having diameter $\leqslant d$. Suppose $\mathbf{u}\in S^{n-1}$, $a>0$ and A is contained in the half-space $\{\mathbf{p}\in\mathbf{R}^n \mid \mathbf{p}\,.\,\mathbf{u}\geqslant a\}$, then*

$$A^+\cup TA^+\supset D_0(\mathbf{u}, \arctan(2a/d)),$$

where $T:\mathbf{R}^n\to\mathbf{R}^n$ is the reflection through the line determined by $\mathbf{u}$, $-\mathbf{u}$:

$$T\mathbf{p}=2(\mathbf{p}\,.\,\mathbf{u})\mathbf{u}-\mathbf{p},$$

and $D_0(\mathbf{u}, \psi)$ is the open spherical cap consisting of all unit vectors having an angle with $\mathbf{u}$, which is smaller than ψ.

Proof. Suppose $\mathbf{x}$ is a point in S^{n-1} but not in $A^+\cup TA^+$, and let θ be the angular distance between $\mathbf{x}$ and $\mathbf{u}$, $0\leqslant\theta\leqslant\pi$. Write

$$\mathbf{x}=(\cos\theta)\mathbf{u}+(\sin\theta)\mathbf{v}, \tag{3.1}$$

where $\mathbf{v}$ is a unit vector orthogonal to $\mathbf{u}$ (we ignore the trivial case $n=1$). Since $\mathbf{x}\notin A^+$, there is a point $\mathbf{y}\in A$ with

$$0\geqslant\mathbf{y}\,.\,\mathbf{x}=\mathbf{y}\,.\,\mathbf{u}\cos\theta+\mathbf{y}\,.\,\mathbf{v}\sin\theta. \tag{3.2}$$

Since $T^{-1}=T$ and $\mathbf{x}\notin TA^+$ we have $T\mathbf{x}\notin A^+$. Thus there is a point $\mathbf{z}\in A$ with

$$0\geqslant\mathbf{z}\,.\,T\mathbf{x}=\mathbf{z}\,.\,\mathbf{u}\cos\theta-\mathbf{z}\,.\,\mathbf{v}\sin\theta. \tag{3.3}$$

Summing (3.2) and (3.3), and using $\|\mathbf{y}-\mathbf{z}\|\leqslant d$, $\sin\theta\geqslant 0$, we have

$$0\geqslant(\mathbf{y}\,.\,\mathbf{u}+\mathbf{z}\,.\,\mathbf{u})\cos\theta+(\mathbf{y}-\mathbf{z})\,.\,\mathbf{v}\sin\theta\geqslant(\mathbf{y}\,.\,\mathbf{u}+\mathbf{z}\,.\,\mathbf{u})\cos\theta-d\sin\theta. \tag{3.4}$$

Temporarily suppose that $\theta<\frac{1}{2}\pi$. Then $\cos\theta>0$, so by (3.4)

$$\tan\theta\geqslant\frac{\mathbf{y}\,.\,\mathbf{u}+\mathbf{z}\,.\,\mathbf{u}}{d}\geqslant\frac{2a}{d}.$$

The last inequality is justified by the hypothesis that $A\subset\{\mathbf{p}\in\mathbf{R}^n \mid \mathbf{p}\,.\,\mathbf{u}\geqslant a\}$. Whether or not $\theta<\frac{1}{2}\pi$, we have $\theta\geqslant\arctan(2a/d)$. This shows that $A^+\cup TA^+\supset D_0(\mathbf{u}, \arctan(2a/d))$.

PROPOSITION 9.

$$g(n,d)\geqslant\frac{1}{\sqrt{8\pi n}}\left(\frac{3}{2}+\frac{(2n+1)d^2-(2n+2)}{4n+4-2d^2n}\right)^{-(n-1)/2} \quad \textit{for } 0<d\leqslant\sqrt{2}.$$

Proof. Let $0<d\leqslant\sqrt{2}$ and let A be a nonempty subset of S^{n-1} having diameter $\leqslant d$. By Jung's theorem [**5**, p. 111], there is, in $\mathbf{R}^n$, a ball with radius $d\sqrt{n/(2n+2)}$ containing A. Let $\mathbf{q}$ be the center of this ball. Write $\mathbf{q}=t\mathbf{u}$ with $t\geqslant 0$ and $\mathbf{u}\in S^{n-1}$. For every $\mathbf{x}\in A$ we have

$$d^2\frac{n}{2n+2}\geqslant\|\mathbf{x}-\mathbf{q}\|^2=\|\mathbf{x}-t\mathbf{u}\|^2=1-2t\mathbf{x}\,.\,\mathbf{u}+t^2. \tag{3.5}$$

Since $1-2t\mathbf{x}\,.\,\mathbf{u}+t^2=1-(\mathbf{x}\,.\,\mathbf{u})^2+(\mathbf{x}\,.\,\mathbf{u}-t)^2\geqslant 1-(\mathbf{x}\,.\,\mathbf{u})^2$, inequality (3.5) implies

$$d^2 \frac{n}{2n+2} \geqslant 1-(\mathbf{x} \,.\, \mathbf{u})^2. \tag{3.6}$$

From (3.5), $t \geqslant 0$ and $d \leqslant \sqrt{2}$ it can be seen that $\mathbf{x} \,.\, \mathbf{u} \geqslant 0$, so, using (3.6), we obtain

$$\mathbf{x} \,.\, \mathbf{u} \geqslant \sqrt{1-d^2 \frac{n}{2n+2}}.$$

Set

$$a=\sqrt{1-d^2 \frac{n}{2n+2}}. \tag{3.7}$$

The above argument shows that A is contained in the half-space

$$\{\mathbf{p} \in \mathbf{R}^n \,|\, \mathbf{p} \,.\, \mathbf{u} \geqslant a\}.$$

Proposition 8 can be applied, yielding

$$A^+ \cup TA^+ \supset D_0(\mathbf{u}, \arctan(2a/d)).$$

Now, since T is an orthogonal transformation, we have $\mu(A^+)=\mu(TA^+)$, and

$$\begin{aligned}
\mu(A^+) &= \tfrac{1}{2}(\mu(A^+)+\mu(TA^+)) \geqslant \tfrac{1}{2}\mu(A^+ \cup TA^+) \\
&\geqslant \tfrac{1}{2}\mu\left(D_0\left(\mathbf{u}, \arctan \frac{2a}{d}\right)\right)=\frac{1}{2} \frac{\operatorname{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{\operatorname{Vol}_{n-1} S^{n-1}} \\
&= \frac{\operatorname{Vol}_{n-1} D_0(\mathbf{u}, \arctan 2a/d)}{2n\Omega_n}.
\end{aligned} \tag{3.8}$$

Here Ω_n denotes the volume of the n-dimensional unit ball, and Vol_{n-1} is the $(n-1)$-dimensional volume.

Let D' be the orthogonal projection of $D_0(\mathbf{u}, \arctan 2a/d)$ to the hyperplane $\{\mathbf{p} \in \mathbf{R}^n \,|\, \mathbf{p} \,.\, \mathbf{u}=0\}$. Obviously we have

$$\operatorname{Vol}_{n-1} D_0\left(\mathbf{u}, \arctan \frac{2a}{d}\right) \geqslant \operatorname{Vol}_{n-1} D'. \tag{3.9}$$

D' is an $(n-1)$-dimensional ball having radius

$$\sin\left(\arctan \frac{2a}{d}\right)=\left(1+\frac{d^2}{4a^2}\right)^{-1/2},$$

so

$$\operatorname{Vol}_{n-1} D'=\Omega_{n-1}\left(1+\frac{d^2}{4a^2}\right)^{-(n-1)/2}. \tag{3.10}$$

Using (3.8), (3.9), (3.10), we get

$$\mu(A^+) \geqslant \frac{\Omega_{n-1}}{2n\Omega_n}\left(1+\frac{d^2}{4a^2}\right)^{-(n-1)/2}. \tag{3.11}$$

Now

$$\frac{\Omega_{n-1}}{\Omega_n}=\frac{\pi^{(n-1)/2}/\Gamma((1+n)/2)}{\pi^{n/2}/\Gamma(1+n/2)}=\frac{\Gamma(1+n/2)}{\sqrt{\pi}\Gamma((1+n)/2)} \tag{3.12}$$

where Γ is the Gamma function. Since $\log\Gamma$ is convex ([1, p. 12]), we have

$$\Gamma(1+n/2)\Gamma(n/2)\geqslant\Gamma((1+n)/2)^2,$$

and therefore

$$\frac{\Gamma(1+n/2)}{\Gamma((1+n)/2)}\geqslant\frac{\Gamma(1+n/2)}{\Gamma((1+n)/2)}\sqrt{\frac{\Gamma((1+n)/2)^2}{\Gamma(1+n/2)\Gamma(n/2)}}$$
$$=\sqrt{\frac{\Gamma(1+n/2)}{\Gamma(n/2)}}=\sqrt{\frac{n}{2}}. \tag{3.13}$$

Using (3.11), (3.12), (3.13), we get

$$\mu(A^+)\geqslant\frac{1}{2n}\frac{1}{\sqrt{\pi}}\sqrt{\frac{n}{2}}\left(1+\frac{d^2}{4a^2}\right)^{-(n-1)/2}.$$

And after substituting the value of a,

$$\mu(A^+)\geqslant\frac{1}{\sqrt{8\pi n}}\left(1+\frac{d^2}{4-2d^2n/(n+1)}\right)^{-(n-1)/2}$$
$$=\frac{1}{\sqrt{8\pi n}}\left(\frac{3}{2}+\frac{(2n+1)d^2-2n-2}{4n+4-2d^2n}\right)^{-(n-1)/2}.$$

Now proving Theorem 1 is just a matter of putting the pieces together.

Proof of Theorem 1. Since $t<-\log(1-t)$ for $0<t<1$, Proposition 6 implies

$$I(W)<1+\frac{\log N(n,\varepsilon)}{g(n,1+\varepsilon)},\qquad 0<\varepsilon<\sqrt{2}-1.$$

Choose

$$\varepsilon=\sqrt{1+\frac{1}{2n+1}}-1.$$

From Lemma 7 and Proposition 9 we get

$$I(W)<1+\frac{\log N(n,\varepsilon)}{g(n,1+\varepsilon)}\leqslant 1+\frac{\log\left(1+\frac{4}{\varepsilon}\right)^n}{g\left(n,\sqrt{\frac{2n+2}{2n+1}}\right)}$$
$$\leqslant 1+\sqrt{8\pi n}\left(\frac{3}{2}\right)^{(n-1)/2} n\log\left(1+\frac{4}{\varepsilon}\right)$$
$$=1+4n\sqrt{\pi n/3}\left(\frac{3}{2}\right)^{n/2}\log\left(1+\frac{4}{\varepsilon}\right).$$

Since one easily sees that $\varepsilon > 1/(4n+3)$, we have

$$I(W) < 1 + 4n\sqrt{\pi n/3}\log(13+16n)\left(\frac{3}{2}\right)^{n/2} \leqslant 5n\sqrt{n}(4+\log n)\left(\frac{3}{2}\right)^{n/2}$$

Remarks. 1. In **[11]** we give a lower bound for the volumes of sets of constant width in $\mathbf{R}^n$, using results analogous to Propositions 8 and 9.

2. The factor $5n\sqrt{n}(4+\log n)$ in Theorem 1 should not be taken seriously. It can be improved with some more careful estimates. However, any improvement of the exponential factor, $(3/2)^{n/2}$, would be interesting. A possible way to do this may be to try to get a better lower bound for $g(n, d)$. An advance in this direction may lead to better estimates for the minimal volume of a set having constant width 1 in $\mathbf{R}^n$.

§4.

THEOREM 10. *Let $K \subset \mathbf{R}^n$ be a strictly convex body, invariant under a group of orthogonal transformations G that is generated by reflections through hyperplanes and acts irreducibly on $\mathbf{R}^n$. Then $I(K) = n+1$.*

We preface the proof with a few definitions and a lemma. A unit vector $\mathbf{r}$ is called a *root* of G if the orthogonal reflection through the subspace orthogonal to $\mathbf{r}$ is an element of G. We denote this reflection by $S_\mathbf{r}$:

$$S_\mathbf{r}\mathbf{x} = \mathbf{x} - 2(\mathbf{x}\,.\,\mathbf{r})\mathbf{r}, \qquad \mathbf{x} \in \mathbf{R}^n.$$

If $\mathbf{v}, \mathbf{r}$ are roots of G then $S_\mathbf{r}\mathbf{v}$ is also a root of G, because $S_{S_\mathbf{r}\mathbf{v}} = S_\mathbf{r}S_\mathbf{v}S_\mathbf{r}$. In particular $-\mathbf{r} = S_\mathbf{r}\mathbf{r}$ is a root.

LEMMA 11. *Let G be as in Theorem 10. If $n > 1$ then there are $n+1$ roots of G, $\mathbf{r}_0, \ldots, \mathbf{r}_n$, such that every nonzero $\mathbf{x} \in \mathbf{R}^n$ has a negative inner product with at least one of them.*

At least when G is finite this follows from known results (see **[6]**).

Proof of the Lemma. We will say that a set of vectors $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is *almost independent* if every proper subset of it is linearly independent but $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is linearly dependent. It is easily checked that $\{\mathbf{v}_0, \ldots, \mathbf{v}_m\}$ is almost independent, if, and only if, $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are linearly independent and $\mathbf{v}_0$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$ with *nonzero* coefficients.

Because of the hypotheses on G and because $n \geqslant 2$, G has at least one root, $\mathbf{r}$. $\{\mathbf{r}, -\mathbf{r}\}$ is an almost independent set. Suppose $\{\mathbf{r}_0, \ldots, \mathbf{r}_m\}$ is the largest almost independent set of roots of G. Let U be the subspace generated by $\mathbf{r}_0, \ldots, \mathbf{r}_m$. We claim that $U = \mathbf{R}^n$ and therefore $n = m$. Let $\mathbf{r}$ be a root of G and suppose that U is not invariant under $S_\mathbf{r}$. This implies $\mathbf{r} \notin U$ and $S_\mathbf{r}\mathbf{r}_i \neq \mathbf{r}_i$ for some $i = 0, 1, \ldots, m$. Assume, without loss of generality, that $S_\mathbf{r}\mathbf{r}_0 \neq \mathbf{r}_0$. $\mathbf{r}_0$ is a linear combination of $\mathbf{r}_1, \ldots, \mathbf{r}_m$ with nonzero coefficients. Since $S_\mathbf{r}\mathbf{r}_0 - \mathbf{r}_0$ is a nonzero multiple of $\mathbf{r}$, $S_\mathbf{r}\mathbf{r}_0$ is a linear combination of $\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_m$ with nonzero coefficients. Because $\mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_m$ are linearly independent, this means that the set $\{S_{\mathbf{r}}\mathbf{r}_0, \mathbf{r}, \mathbf{r}_1, \ldots, \mathbf{r}_m\}$ is almost independent, contradicting the choice of $\{\mathbf{r}_0, \ldots, \mathbf{r}_m\}$.

This forces us to conclude that U is invariant under the reflections generating G, and therefore under every transformation in G. G acts irreducibly on $\mathbf{R}^n$ and $U \neq 0$. This implies $U = \mathbf{R}^n$ and $n = m$.

Because $\{\mathbf{r}_0, \ldots, \mathbf{r}_n\}$ is almost independent, it is possible to write

$$0 = \sum_{i=0}^{n} a_i \mathbf{r}_i$$

with all coefficients nonzero. We replace some of the roots $\mathbf{r}_i$ by their negatives, to have all the coefficients a_i positive. If $\mathbf{x}$ is any nonzero vector, we must have $\mathbf{x} \cdot \mathbf{r}_i \neq 0$ for some $i = 0, 1, \ldots, n$, because the $\mathbf{r}_i$ span $\mathbf{R}^n$. Since $0 = \mathbf{x} \cdot 0 = \sum_{i=0}^{n} a_i(\mathbf{x} \cdot \mathbf{r}_i)$ and $a_i > 0$, we must have $\mathbf{x} \cdot \mathbf{r}_i < 0$ for some i.

Proof of Theorem 10. As mentioned in Section 1, $I(K) \geqslant n+1$ holds for every convex body in $\mathbf{R}^n$, thus we only need to show that $I(K) \leqslant n+1$. The Theorem is obviously true when $n = 1$, so we assume $n > 1$.

Let $\mathbf{r}_0, \ldots, \mathbf{r}_n$ be the roots of G guaranteed by the lemma and let $\mathbf{x} \in \partial K$. First observe that the origin is necessarily an interior point of K so $\mathbf{x} \neq 0$. For some $\mathbf{r}_i$, $\mathbf{x} \cdot \mathbf{r}_i < 0$. We claim that this $\mathbf{r}_i$ illuminates $\mathbf{x}$. $\mathbf{x} - 2(\mathbf{x} \cdot \mathbf{r}_i)\mathbf{r}_i = S_{\mathbf{r}_i}\mathbf{x} \in K$, since $S_{\mathbf{r}_i} \in G$. For some positive t, $\mathbf{x} + t\mathbf{r}_i$ is an interior point of K, because $\mathbf{x}$, $\mathbf{x} - 2(\mathbf{x} \cdot \mathbf{r}_i)\mathbf{r}_i \in K$, K is strictly convex and $-2(\mathbf{x} \cdot \mathbf{r}_i) > 0$. This verifies the claim and we see that $\mathbf{r}_0, \ldots, \mathbf{r}_n$ illuminate K.

Remark. *A* set of constant width is strictly convex; therefore Theorem 10 applies to (sufficiently symmetric) sets of constant width.

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