Chapter 2
Discontinuous System Theory

As in Luo (2011), the passability of a flow to the separation boundary of two different dynamical systems is presented. The accessible and inaccessible subdomains are introduced first for a theory of discontinuous dynamic systems. On the accessible domains, the corresponding dynamic systems are introduced. The flow orientation and singular sets of the separation boundary are discussed. The passability and tangency (grazing) of a flow to the separation boundary between two adjacent accessible domains are presented, and the necessary and sufficient conditions for such passability and tangency of the flow to the boundary are presented. The product of the normal components of vector fields to the boundary is presented, and the corresponding conditions for the flow passability to the boundary are discussed.

2.1 Domain Accessibility

For any discontinuous dynamical system, there are many vector fields defined on different domains in phase space, and such distinct vector fields between two vector fields in two adjacent domains cause flows at the boundary of the domains to be nonsmooth or discontinuous. To investigate the dynamics of discontinuous dynamical systems, consider a discontinuous dynamical system on a universal domain $\mho \subset R^n$, and the passability of a flow from one domain to its adjacent domains is discussed first. Thus, subdomains Ω_α $(\alpha \in I,\ I = \{1, 2, \cdots, N\})$ of the universal domain $\mho$ are introduced and the vector fields on the subdomains may be defined differently. If there is a vector field on a subdomain, then this subdomain is said to be an accessible domain. Otherwise, such a domain is said to be an inaccessible domain. Thus, the domain accessibility can provide a design possibility for discontinuous dynamical systems. The corresponding definitions of the domain accessibility are given as follows.

Definition 2.1. A subdomain in the universal domain $\mho$ in a discontinuous dynamical system is termed an *accessible* subdomain, if at least a specific, continuous vector field can be defined on such a subdomain.

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Definition 2.2. A subdomain in a universal domain $\mho$ in discontinuous dynamical systems is termed an *inaccessible* subdomain, if no any vector fields can be defined on such a subdomain.

Since the accessible and inaccessible subdomains exist in discontinuous dynamical systems, the universal domain $\mho$ is classified into connectable and separable domains. The connectable domain is defined as follows.

Definition 2.3. A domain $\mho$ in phase space is termed a *connectable domain* if all the accessible subdomains of the universal domain can be connected without any inaccessible subdomain.

Similarly, a definition of the separable domain is given as follows.

Definition 2.4. A domain is termed a *separable domain*, if the accessible subdomains in the universal domain are separated by inaccessible domains.

Since any discontinuous dynamical system possesses different vector fields defined on each accessible subdomain, the corresponding dynamical behaviors in those accessible subdomains Ω_α are distinguishing. The different behaviors in distinct subdomains cause flow complexity in the domain $\mho$ of discontinuous dynamical systems. The boundary between two adjacent, accessible subdomains is a bridge of dynamical behaviors in two domains for flow continuity. Any connectable domain is bounded by the universal boundary $S \subseteq R^{n-1}$, and each subdomain is bounded by the subdomain boundary surface $S_{\alpha\beta} \subset R^{n-1}$ $(\alpha, \beta \in I)$ with or without the partial universal boundary. For instance, consider an n-dimensional connectable domain in phase space, as shown in Fig. 2.1a through an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an n_2-dimensional, subvector $\mathbf{x}_{n_2}$ $(n_1 + n_2 = n)$. The shaded area Ω_α is a specific subdomain, and the other subdomains are white. The dark, solid curve represents the original boundary of the domain $\mho$. For the separable domain, there is at least an inaccessible subdomain to separate the accessible subdomains. The union of inaccessible subdomains is also called the "inaccessible sea." The inaccessible sea is the complement of the accessible subdomains to the universal (original) domain $\mho$. That is determined by $\Omega_0 = \mho \backslash \cup_{\alpha \in I} \Omega_\alpha$. The accessible subdomains in the domain $\mho$ are also called the "islands." For illustration of such a definition, a separable domain is shown in Fig. 2.1b. The thick curve is the boundary of the universal domain, and the gray area is the inaccessible sea. The white regions are the accessible domains (or islands). The hatched region represents a specific accessible subdomain (island).

From one accessible island to another, the transport laws are needed for motion continuity, which is not discussed in this book. For information about this topic, the reader can refer to Luo (2011).

2.2 Discontinuous Dynamical Systems

Consider a dynamic system consisting of N subdynamic systems in a universal domain $\mho \subset R^n$. The universal domain is divided into N accessible subdomains Ω_α $(\alpha \in I)$ and the union of inaccessible domain Ω_0. The union of all accessible

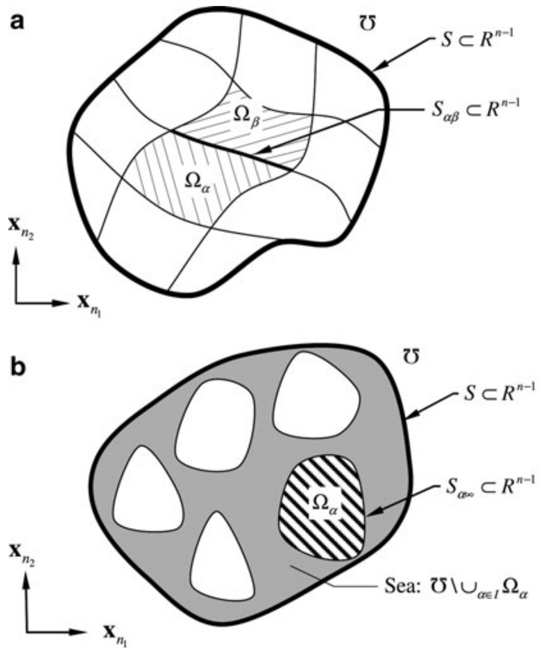

Fig. 2.1 Phase space: (**a**) connectable and (**b**) separable domains ($n_1 + n_2 = n$)

subdomains $\cup_{\alpha \in I} \Omega_\alpha$ and $\mho = \cup_{\alpha \in I} \Omega_\alpha \cup \Omega_0$ is the universal domain, as shown in Fig. 2.1 by an n_1-dimensional, subvector $\mathbf{x}_{n_1}$ and an n_2-dimensional, subvector $\mathbf{x}_{n_2}$ ($n_1 + n_2 = n$). For the connectable domain in Fig. 2.1a, $\Omega_0 = \varnothing$. In Fig. 2.1b, the union of the inaccessible subdomains is the sea, and $\Omega_0 = \mho \backslash \cup_{\alpha \in I} \Omega_\alpha$ is the complement of the union of the accessible subdomain. On the α^{th} open subdomain Ω_α, there is a C^{r_α}-continuous system ($r_\alpha \geq 1$) in the form of

$$\dot{\mathbf{x}}^{(\alpha)} \equiv \mathbf{F}^{(\alpha)}(\mathbf{x}^{(\alpha)}, t, \mathbf{p}_\alpha) \in R^n, \quad \mathbf{x}^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_n^{(\alpha)})^{\mathrm{T}} \in \Omega_\alpha. \tag{2.1}$$

The time is denoted by t and $\dot{\mathbf{x}} = \mathrm{d}\mathbf{x}/\mathrm{d}t$. In an accessible subdomain Ω_α, the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ with parameter vector $\mathbf{p}_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, \ldots, p_\alpha^{(l)})^{\mathrm{T}} \in R^l$ is C^{r_α}-continuous ($r_\alpha \geq 1$) in $\mathbf{x} \in \Omega_\alpha$ and for all time t; and the continuous flow in (2.1) $\mathbf{x}^{(\alpha)}(t) = \Phi^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t, \mathbf{p}_\alpha)$ with $\mathbf{x}^{(\alpha)}(t_0) = \Phi^{(\alpha)}(\mathbf{x}^{(\alpha)}(t_0), t_0, \mathbf{p}_\alpha)$ is C^{r+1}-continuous for time t.

For discontinuous dynamical systems, the following assumptions will be adopted herein.

H2.1

The flow switching between two adjacent subsystems is time-continuous.

H2.2

For an unbounded, accessible subdomain Ω_α, there is a bounded domain $D_\alpha \subset \Omega_\alpha$ and the corresponding vector field and its flow are bounded, i.e.,

$$||\mathbf{F}^{(\alpha)}|| \leq K_1(\text{const}) \quad \text{and} \quad ||\Phi^{(\alpha)}|| \leq K_2(\text{const}) \quad \text{on} \quad D_\alpha \quad \text{for } t \in [0, \infty). \tag{2.2}$$

H2.3
For a bounded, accessible subdomain Ω_α, there is a bounded domain $D_\alpha \subset \Omega_\alpha$ and the corresponding vector field is bounded, but the flow may be unbounded, i.e.,

$$||\mathbf{F}^{(\alpha)}|| \le K_1(\text{const}) \text{ and } ||\Phi^{(\alpha)}|| \le \infty \text{ on } D_\alpha \text{ for } t \in [0,\infty). \tag{2.3}$$

2.3 Flow Passability to Boundary

Since dynamical systems on different accessible subdomains are distinguishing, the relation between flows in the two subdomains should be developed herein for flow continuity. For a subdomain Ω_α, there are k_α-adjacent subdomains with k_α-pieces of boundaries ($k_\alpha \le N-1$). Consider a boundary of any two adjacent subdomains, formed by the intersection of the two closed subdomains (i.e., $\partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j$, $i,j \in I, j \ne i$), as shown in Fig. 2.2.

Definition 2.5. The boundary in n-dimensional phase space is defined as

$$\begin{aligned} S_{ij} &\equiv \partial\Omega_{ij} = \bar{\Omega}_i \cap \bar{\Omega}_j \\ &= \left\{ \mathbf{x} | \varphi_{ij}(\mathbf{x},t,\lambda) = 0, \varphi_{ij} \text{ is } C^r \text{ - continuous } (r \ge 1) \right\} \subset R^{n-1}. \end{aligned} \tag{2.4}$$

Definition 2.6. The two subdomains Ω_i and Ω_j are *disjoint* if the boundary $\partial\Omega_{ij}$ is an empty set (i.e., $\partial\Omega_{ij} = \varnothing$).

From the definition, $\partial\Omega_{ij} = \partial\Omega_{ji}$. The flow on the boundary $\partial\Omega_{ij}$ can be determined by

$$\dot{\mathbf{x}}^{(0)} = \mathbf{F}^{(0)}(\mathbf{x}^{(0)},t) \text{ with } \varphi_{ij}(\mathbf{x}^{(0)},t,\lambda) = 0, \tag{2.5}$$

where $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, \ldots, x_n^{(0)})^{\mathrm{T}}$. With specific initial conditions, one always obtains different flows on $\varphi_{ij}(\mathbf{x}^{(0)},t,\lambda) = \varphi_{ij}(\mathbf{x}_0^{(0)},t_0,\lambda) = 0$.

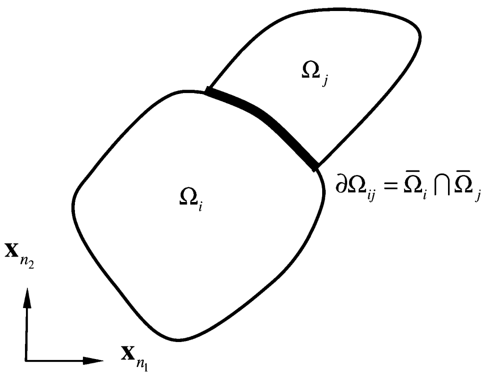

Fig. 2.2 Subdomains Ω_α and Ω_β, the corresponding boundary $\partial\Omega_{\alpha\beta}$

Definition 2.7. If the intersection of three or more subdomains,

$$\Gamma_{\alpha_1\alpha_2\ldots\alpha_k} \equiv \cap_{\alpha=\alpha_1}^{\alpha_k} \bar{\Omega}_\alpha \subset R^r \quad (r = 0, 1, \ldots, n-2), \tag{2.6}$$

where $\alpha_k \in I$ and $k \geq 3$ is nonempty, the subdomain intersection is termed the *singular* set.

For $r = 0$, the singular sets are singular points, which are also termed *the corner points or vertex*. In other words, any corner point is the intersection of n-linearly independent, $(n-1)$-dimensional boundary surfaces in an n-dimensional state space. For $r = 1$, the singular sets will be curves, which are termed the one-dimensional singular edges to the $(n-1)$-dimensional boundary. Similarly, any one-dimensional singular edge is the intersection of $(n-1)$-linearly independent, $(n-1)$-dimensional boundary surfaces in an n-dimensional state space. For $r \in \{2, 3, \ldots, n-2\}$, the singular sets are the r-dimensional singular surfaces to the $(n-1)$-dimensional discontinuous boundary. In Fig. 2.3, the singular set for three closed domains $\{\bar{\Omega}_i, \bar{\Omega}_j, \bar{\Omega}_k\}$ $(i, j, k \in I)$ is sketched. The circular symbols represent intersection sets. The largest solid circular symbol stands for the singular set Γ_{ijk}. The corresponding discontinuous boundaries relative to the singular set are labeled by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$, and $\partial\Omega_{ik}$. The singular set possesses the hyperbolic or parabolic behavior depending on the properties of the separation boundary, which can be referred to Luo (2005, 2006, 2011). The flow on the singular sets can be similarly defined as in (2.5), by a dynamical system with the corresponding boundary constraints. The detailed discussion is given later.

Definition 2.8. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, then a resultant flow of two flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ is called as a *semipassable flow* from domain Ω_i to Ω_j at point $(\mathbf{x}_m, t_m)$ to boundary

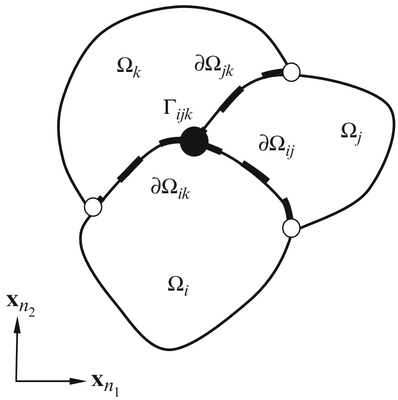

Fig. 2.3 A singular set for the intersection of three domains $\{\bar{\Omega}_i, \bar{\Omega}_j, \bar{\Omega}_k\}$ $(i, j, k \in I)$. The *circular symbols* represent intersection sets. The *large solid circular symbol* stands for the singular set Γ_{ijk}. The corresponding discontinuous boundaries are marked by $\partial\Omega_{ij}$, $\partial\Omega_{jk}$, and $\partial\Omega_{ik}$

$\partial\Omega_{ij}$ if the two flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ in the neighborhood of $\partial\Omega_{ij}$ possess the following properties

$$\begin{aligned}
&\text{either} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
&\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i,
\end{aligned} \tag{2.7}$$

where the normal vector of the boundary $\partial\Omega_{ij}$ is

$$\mathbf{n}_{\partial\Omega_{ij}} = \nabla\varphi_{ij}|_{\mathbf{x}=\mathbf{x}_m} = \left(\frac{\partial\varphi_{ij}}{\partial x_1}, \frac{\partial\varphi_{ij}}{\partial x_2}, \ldots, \frac{\partial\varphi_{ij}}{\partial x_n}\right)^{\mathrm{T}} |_{\mathbf{x}=\mathbf{x}_m}. \tag{2.8}$$

The notations $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ and $t_{m\pm} = t_m \pm 0$ are used. $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ represents that the normal vector of boundary at $(\mathbf{x}_m, t_m)$ points to domain Ω_j. In addition, a boundary $\partial\Omega_{ij}$ to semipassable flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ from domain Ω_i to domain Ω_j is called the *semipassable* boundary (expressed by $\overrightarrow{\partial\Omega}_{ij}$). For a geometrical explanation of the semipassable flow to the boundary, consider a flow $\mathbf{x}^{(i)}(t)$ of discontinuous dynamical system in (2.1) passing through boundary $\partial\Omega_{ij}$ from domain Ω_i to domain Ω_j. At time t_m, the flow $\mathbf{x}^{(i)}(t)$ arrives to the boundary $\partial\Omega_{ij}$, and there is a small neighborhood $(t_{m-\varepsilon}, t_{m+\varepsilon})$ of time t_m, which is arbitrarily selected. Before the flow $\mathbf{x}^{(i)}(t)$ reaches to the boundary $\partial\Omega_{ij}$, a point $\mathbf{x}^{(i)}(t_{m-\varepsilon})$ lies in domain Ω_i. As $\varepsilon \to 0$, the time increments $\Delta t \equiv \varepsilon \to 0$. A point $\mathbf{x}_m$ on the boundary is the limit of $\mathbf{x}^{(i)}(t_{m-\varepsilon})$ as $\varepsilon \to 0$, and the point $\mathbf{x}_m$ must satisfy the boundary constraint of $\varphi_{ij}(\mathbf{x}, t) = 0$. After the flow $\mathbf{x}^{(i)}(t)$ passes through the boundary at point $\mathbf{x}_m$, the flow $\mathbf{x}^{(i)}(t)$ will switch to the flow $\mathbf{x}^{(j)}(t)$ on the side of domain Ω_j. $\mathbf{x}^{(j)}(t_{m+\varepsilon})$ is a point in the neighborhood of boundary, and a point $\mathbf{x}_m$ on the boundary is also the limit of $\mathbf{x}^{(j)}(t_{m+\varepsilon})$ as $\varepsilon \to 0$. The coming and leaving flow vectors are $\mathbf{x}^{(i)}(t_m) - \mathbf{x}^{(i)}(t_{m-\varepsilon})$ and $\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_m)$, respectively. Whether the flow passes through the boundary or not is dependent on the properties of both coming and leaving flows in the neighborhood of boundary. The processes of a flow passing through the boundary of $\partial\Omega_{ij}$ from domain Ω_i to Ω_j are shown in Fig. 2.4 for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ and $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$, respectively. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of the boundary $\partial\Omega_{ij}$, determined by $\varphi_{ij}(\mathbf{x}, t) = 0$. When a coming flow $\mathbf{x}^{(i)}(t)$ in domain Ω_i arrives to the semipassable boundary $\overrightarrow{\partial\Omega}_{ij}$, the flow of $\mathbf{x}^{(i)}(t)$ can also be tangential to or bouncing on (or switching back from) the semipassable boundary $\partial\Omega_{ij}$. However, once a leaving flow $\mathbf{x}^{(j)}(t)$ in domain Ω_j leaves the semipassable boundary $\overrightarrow{\partial\Omega}_{ij}$, the leaving flow cannot pass through the boundary $\overrightarrow{\partial\Omega}_{ij}$, but the leaving flow $\mathbf{x}^{(j)}(t)$ can tangentially leave the semipassable boundary. Thus, tangential (or grazing) flows to the boundary are very important, which is discussed later in this chapter. In the following discussion, no any control and transport laws will be inserted on the boundary. The direction of $\mathbf{t}_{\partial\Omega_{ij}} \times \mathbf{n}_{\partial\Omega_{ij}}$ is the positive direction by the right-hand rule.

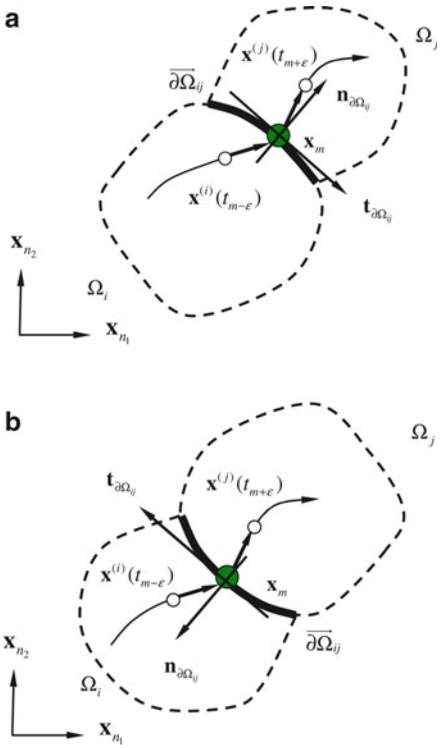

Fig. 2.4 A flow passing through the semipassable boundary $\overrightarrow{\partial\Omega}_{ij}$ from domain Ω_i to Ω_j: (**a**) $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ and (**b**) $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$. $\mathbf{x}^{(i)}(t_{m-\varepsilon})$, $\mathbf{x}^{(j)}(t_{m+\varepsilon})$, and $\mathbf{x}_m$ are three points in Ω_i and Ω_j and on the boundary $\partial\Omega_{ij}$, respectively. Two vectors $\mathbf{n}_{\partial\Omega_{ij}}$ and $\mathbf{t}_{\partial\Omega_{ij}}$ are the normal and tangential vectors of $\partial\Omega_{ij}$

Theorem 2.1. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i,j)$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$. *Suppose* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, *then two flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r_i}_{[t_{m-\varepsilon},t_m)}$*- and* $C^{r_j}_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2, \alpha = i,j)$ *for time t, respectively.* $\|\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}\| < \infty$ $(\alpha = i,j)$. *The resultant flow of* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *to the boundary* $\partial\Omega_{ij}$ *is semipassable from domain* Ω_i *to* Ω_j *if and only if*

$$
\left.\begin{array}{ll}
\text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) > 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
\text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) < 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.
\end{array}\right\} \tag{2.9}
$$

Proof. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m$ and $\mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, then the two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^r_{[t_{m-\varepsilon},t_m)}$- and $C^r_{(t_m,t_{m+\varepsilon}]}$-continuous $(r \geq 2)$ for time t, respectively. $\|\ddot{\mathbf{x}}^{(\alpha)}(t)\| < \infty$ $(\alpha \in \{i,j\})$ for $0 < \varepsilon << 1$. Consider $a \in [t_{m-\varepsilon}, t_{m-})$ and $b \in (t_{m-}, t_{m+\varepsilon}]$. Application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ with $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ ($\alpha \in \{i,j\}$) to $\mathbf{x}^{(\alpha)}(a)$ and $\mathbf{x}^{(\alpha)}(b)$ gives

$$\begin{aligned}
\mathbf{x}^{(i)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) = \mathbf{x}^{(i)}(a) + \dot{\mathbf{x}}^{(i)}(a)(t_{m-} - \varepsilon - a) + o(t_{m-} - \varepsilon - a),\\
\mathbf{x}^{(j)}(t_{m+\varepsilon}) &\equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) = \mathbf{x}^{(j)}(b) + \dot{\mathbf{x}}^{(j)}(t_{m+} + \varepsilon - b) + o(t_{m+} + \varepsilon - b).
\end{aligned}$$

Let $a \to t_{m-}$ and $b \to t_{m+}$, the limits of the foregoing equations lead to

$$\left.\begin{aligned}
\mathbf{x}^{(i)}(t_{m-\varepsilon}) &\equiv \mathbf{x}^{(i)}(t_{m-} - \varepsilon) = \mathbf{x}^{(i)}(t_{m-}) - \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon + o(\varepsilon),\\
\mathbf{x}^{(j)}(t_{m+\varepsilon}) &\equiv \mathbf{x}^{(j)}(t_{m+} + \varepsilon) = \mathbf{x}^{(j)}(t_{m+}) + \dot{\mathbf{x}}^{(j)}(t_{m+})\varepsilon + o(\varepsilon).
\end{aligned}\right\}$$

Because of $0<\varepsilon<<1$, the ε^2 and higher-order terms of the foregoing equations can be ignored. Therefore, with the first equation of (2.9), the following relations exist,

$$\left.\begin{aligned}
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-})\varepsilon > 0,\\
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] &= \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+})\varepsilon > 0.
\end{aligned}\right\}$$

From Definition 2.8, the flow at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ is semipassable from domain Ω_i to Ω_j under the condition in the first inequality equations of (2.9). In a similar manner, the flow at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$ is semipassable under conditions in the second inequality equation in (2.9), and vice versa. ■

Theorem 2.2. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$, then two vector fields of $\mathbf{F}^{(i)}(\mathbf{x}, t, \mathbf{p}_i)$ and $\mathbf{F}^{(j)}(\mathbf{x}, t, \mathbf{p}_j)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$, $\alpha = i,j$) for time t, respectively. $\|\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}\| < \infty$ ($\alpha = i,j$). The resultant flow of two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ is semipassable from domain Ω_i to Ω_j if and only if*

$$\begin{aligned}
&\text{either} \quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \text{ and}\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0\end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j\\
&\text{or} \quad \left.\begin{aligned}&\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \text{ and}\\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0\end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i,
\end{aligned} \tag{2.10}$$

where $\mathbf{F}^{(i)}(t_{m-}) \equiv \mathbf{F}^{(i)}(\mathbf{x}_m, t_{m-}, \mathbf{p}_i)$ and $\mathbf{F}^{(j)}(t_{m+}) \equiv \mathbf{F}^{(j)}(\mathbf{x}_m, t_{m+}, \mathbf{p}_j)$.

Proof. For a point $\mathbf{x}_m \in \partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. With (2.1), the first inequality equation of (2.10) gives

$$\left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0. \end{aligned}\right\}$$

From Theorem 2.1 and Definition 2.8, the resultant flow at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$ is semipassable. In a similar fashion, the resultant flow of two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i$ is semipassable under conditions in the second inequality equations of (2.10). ■

Definition 2.9. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$, then two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are called *nonpassable flows of the first kind* at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ (or termed *sink flows* at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$) if flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in vicinity of $\partial\Omega_{ij}$ possess the following properties

$$\begin{aligned} &\text{either} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.11}$$

Definition 2.10. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there is a time interval $(t_m, t_{m+\varepsilon}]$. Suppose $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$, then two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are called *nonpassable flows of the second kind* at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ (or termed *source flows* at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$) if the flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ in neighborhood of $\partial\Omega_{ij}$ possess the following properties

$$\begin{aligned} &\text{either} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] < 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] > 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.12}$$

The boundary $\partial\Omega_{ij}$ for two *sink* flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ is called a *nonpassable boundary of the first kind*, donated by $\widetilde{\partial\Omega}_{ij}$ (or termed *a sink boundary* between Ω_i and Ω_j). The boundary $\partial\Omega_{ij}$ for two *source* flows $\mathbf{x}^{(i)}(t)$ and

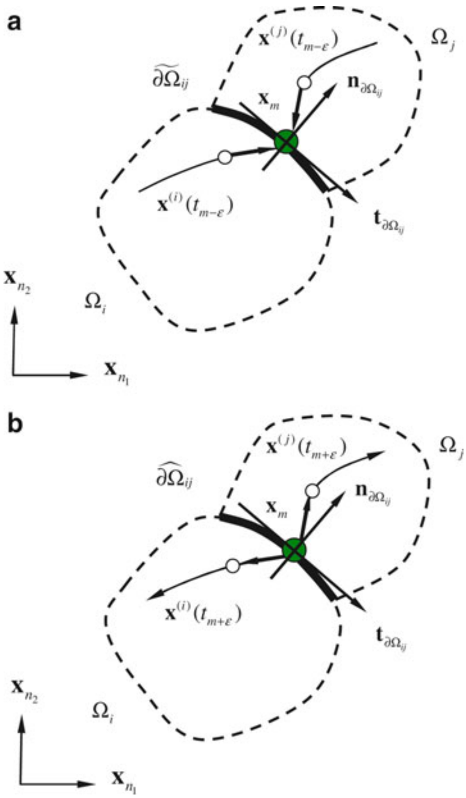

Fig. 2.5 Nonpassable flow to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$: (**a**) sink flow to $\widetilde{\partial\Omega}_{ij}$ (or the nonpassable flow of the first kind), (**b**) source flow to $\widehat{\partial\Omega}_{ij}$ (or the nonpassable flow of the second kind). $\mathbf{x}_m \equiv (\mathbf{x}_{n_1}(t_m), \mathbf{x}_{n_2}(t_m))^{\mathrm{T}}$, $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv (\mathbf{x}_{n_1}^{(\alpha)}(t_{m\pm\varepsilon}), \mathbf{x}_{n_2}^{(\alpha)}(t_{m\pm\varepsilon}))^{\mathrm{T}}$, and $\alpha = \{i, j\}$ where $t_{m\pm\varepsilon} = t_m \pm \varepsilon$ for an arbitrary small $\varepsilon > 0$

$\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ is called a *nonpassable boundary of the second kind*, denoted by $\widehat{\partial\Omega}_{ij}$ (or termed *a source boundary* between Ω_i and Ω_j). The sink and source flows to the boundary $\partial\Omega_{ij}$ between Ω_i and Ω_j are illustrated in Fig. 2.5a, b. The flows in the neighborhood of boundary $\partial\Omega_{ij}$ are depicted. When a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha = i, j$) in domain Ω_α arrives to the nonpassable boundary of the first kind $\widetilde{\partial\Omega}_{ij}$, the flow can be either tangential to or sliding on the nonpassable boundary $\widetilde{\partial\Omega}_{ij}$. For the nonpassable boundary of the second kind $\widehat{\partial\Omega}_{ij}$, a flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha = i, j$) in the domain Ω_α can be either tangential to or bouncing on the nonpassable boundary $\widehat{\partial\Omega}_{ij}$. In this chapter, only the flows tangential to the nonpassable boundary are discussed.

Theorem 2.3. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there is a time interval $[t_{m-\varepsilon}, t_m)$. Suppose $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$, then the flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$-continuous for time t* and $\|\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}\| < \infty$ ($r_\alpha \geq 2$). *Two flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ to boundary $\partial\Omega_{ij}$ are nonpassable flows of the first kind (or sink flows) if and only if*

$$
\begin{array}{ll}
\text{either} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) > 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m-}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
\text{or} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) < 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m-}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.
\end{array}
\tag{2.13}
$$

Proof. Following the proof procedure of Theorem 2.1, Theorem 2.3 can be proved. ■

Theorem 2.4. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there is a time interval* $[t_{m-\varepsilon}, t_m)$*.* $\mathbf{x}^{(\alpha)}(t_{m-}) = \mathbf{x}_m$*. The vector field* $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$*-continuous and* $||\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}|| < \infty$ $(r_\alpha \geq 1)$*. Two flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *to boundary* $\partial\Omega_{ij}$ *are nonpassable flows of the first kind (or sink flows) if and only if*

$$
\begin{array}{ll}
\text{either} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \text{ and} \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m-}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
\text{or} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \text{ and} \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \mathbf{F}^{(j)}(t_{m-}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i,
\end{array}
\tag{2.14}
$$

where $\mathbf{F}^{(\alpha)}(t_{m-}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m-}, \mathbf{p}_\alpha)$ $(\alpha \in \{i, j\})$*.*

Proof. Following the proof procedure of Theorem 2.2, Theorem 2.4 can be easily proved. ■

Theorem 2.5. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there is a time interval* $(t_m, t_{m+\varepsilon}]$*. Suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$*, then* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$*-continuous for time* t *with* $||\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}|| < \infty$ $(r_\alpha \geq 2)$*. Two flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *to boundary* $\partial\Omega_{ij}$ *are nonpassable flows of the second kind (or source flows) if and only if*

$$
\begin{array}{ll}
\text{either} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m+}) < 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
\text{or} & \left.\begin{array}{l} \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m+}) > 0 \\ \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i.
\end{array}
\tag{2.15}
$$

Proof. Following the procedure of the proof of Theorem 2.1, Theorem 2.5 can be proved. ■

Theorem 2.6. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there is a time interval* $(t_m, t_{m+\varepsilon}]$*. Suppose* $\mathbf{x}^{(\alpha)}(t_{m+}) = \mathbf{x}_m$*, then the vector field* $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$*-continuous and* $\|\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}\| < \infty$ $(r_\alpha \geq 1)$*. Two flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *to boundary* $\partial\Omega_{ij}$ *are nonpassable flows of the second kind (or source flows) if and only if*

$$
\begin{aligned}
&\text{either} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\
&\text{or} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i,
\end{aligned}
\tag{2.16}
$$

where $\mathbf{F}^{(\alpha)}(t_{m+}) \triangleq \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m+}, \mathbf{p}_\alpha)$ $(\alpha = i, j)$.

Proof. Following the proof procedure of Theorem 2.2, Theorem 2.6 can be easily proved. ■

2.4 Tangential Flows to Boundary

In this section, the flow local singularity and tangential flow are discussed. The corresponding necessary and sufficient conditions are presented.

Definition 2.11. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$, then a flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$- and/or $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2)$. A point $(\mathbf{x}_m, t_m)$ on boundary $\partial\Omega_{ij}$ is critical to flow $\mathbf{x}^{(\alpha)}(t)$ if

$$
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = 0 \text{ and /or } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) = 0. \tag{2.17}
$$

Theorem 2.7. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*). Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i, j\})$*, then a flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$*- and/or* $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2)$*. The vector field* $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ *is* $C^{r_\alpha-1}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r_\alpha-1}_{(t_m, t_{m+\varepsilon}]}$*-continuous for time t, respectively.* $\|\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}\| < \infty$*. A point* $(\mathbf{x}_m, t_m)$ *on the boundary* $\partial\Omega_{ij}$ *is critical to flow* $\mathbf{x}^{(\alpha)}(t)$ *if and only if*

$$
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0 \text{ and/or } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+}) = 0, \tag{2.18}
$$

where $\mathbf{F}^{(\alpha)}(t_{m\pm}) = \mathbf{F}^{(\alpha)}(\mathbf{x}, t_{m\pm}, \mathbf{p}_\alpha)$.

Proof. Using (2.1) and Definition 2.11, Theorem 2.7 can be proved. ■

The tangential vector of the coming and leaving flows $\mathbf{x}^{(\alpha)}(t_{m\pm})$ to the boundary $\partial\Omega_{ij}$ in domain Ω_α ($\alpha \in \{i,j\}$) is normal to the normal vector of the boundary, so the coming flow is tangential to the boundary.

Definition 2.12. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$), then a flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon},t_m)}$- and $C^{r_\alpha}_{(t_m,t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$) for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ if the following conditions hold.

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0. \tag{2.19}$$

$$\text{either} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] < 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta, \tag{2.20}$$

$$\text{or} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \tag{2.21}$$

where $\alpha, \beta \in \{i,j\}$ *but* $\beta \neq \alpha$.

The normal vector $\mathbf{n}_{\partial\Omega_{ij}}$ is normal to the tangential plane. Without any switching laws, equation (2.19) gives

$$\dot{\mathbf{x}}^{(\alpha)}(t_{m-}) = \dot{\mathbf{x}}^{(\alpha)}(t_{m+}) \quad \text{but} \quad \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) \neq \dot{\mathbf{x}}^{(0)}(t_m). \tag{2.22}$$

The above equation implies that the flow $\mathbf{x}^{(\alpha)}$ on the boundary $\partial\Omega_{ij}$ is at least C^1-continuous. To demonstrate the above definition, consider a flow in domain Ω_i tangential to the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$, as shown in Fig. 2.6. The gray-filled symbols represent two points ($\mathbf{x}^{(i)}_{m\pm\varepsilon} = \mathbf{x}^{(i)}(t_m \pm \varepsilon)$) on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a large circular symbol. This tangential flow is also termed *a grazing flow*.

Theorem 2.8. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *between two adjacent domains* Ω_α ($\alpha = i, j$). *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$). *Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$), *then a flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_m)}$- *and* $C^{r_\alpha}_{(t_m,t_{m+\varepsilon}]}$-*continuous* ($r_\alpha \geq 2$) *for time t.* $||\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}|| < \infty$. *The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* Ω_α *is tangential to boundary* $\partial\Omega_{ij}$ *at point* $(\mathbf{x}_m, t_m)$ *if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.23}$$

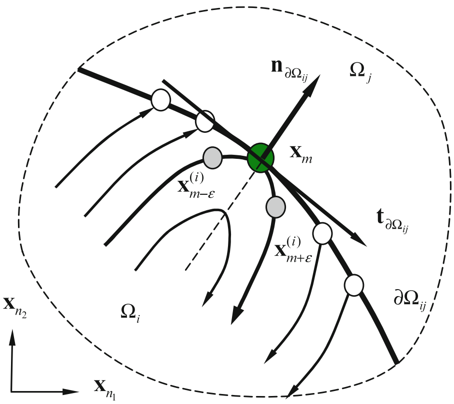

Fig. 2.6 A flow in domain Ω_i tangential to boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j$. The *gray-filled symbols* represent two points ($\mathbf{x}^{(i)}_{m-\varepsilon}$ and $\mathbf{x}^{(i)}_{m+\varepsilon}$) on the flow before and after the tangency. The tangential point $\mathbf{x}_m$ on the boundary $\partial\Omega_{ij}$ is depicted by a *large circular symbol*

$$\begin{aligned}
&\text{either} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) < 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\
&\text{or} \quad \left.\begin{aligned} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha,
\end{aligned} \tag{2.24}$$

where $\alpha, \beta \in \{i, j\}$ *but* $\beta \neq \alpha$.

Proof. Equation (2.23) is identical to the first condition in (2.18). Consider

$$\begin{aligned}
\mathbf{x}^{(\alpha)}(t_{m\pm}) &\equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon \mp \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm} \pm \varepsilon) + o(\varepsilon), \\
&= \mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \mp \varepsilon\dot{\mathbf{x}}^{(\alpha)}(t_{m\pm\varepsilon}) + o(\varepsilon).
\end{aligned}$$

For $0 < \varepsilon << 1$, the higher-order terms in the above equation can be ignored. Therefore,

$$\left.\begin{aligned}
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}), \\
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \varepsilon\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}).
\end{aligned}\right\}$$

From (2.24), the first case is

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) > 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) < 0$$

from which (2.20) holds for $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ ($\beta \neq \alpha$). However, the second case is

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-\varepsilon}) < 0 \text{ and } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+\varepsilon}) > 0,$$

from which (2.21) holds for $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha$. Therefore, from Definition 2.12, the flow $\mathbf{x}^{(\alpha)}(t)$ for time $t \in [t_{m-\varepsilon}, t_{m+\varepsilon}]$ in Ω_α is tangential to the boundary $\partial\Omega_{ij}$. ■

Notice that the aforementioned theorem can be used for surface boundary.

Theorem 2.9. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$), then a vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$) for time t. $\|\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}\| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.25}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{array}\right. \tag{2.26}$$

where $\alpha, \beta \in \{i, j\}$ but $\beta \neq \alpha$.

Proof. Using (2.1) and Theorem 2.8, Theorem 2.9 can be proved. ■

For simplicity, consider $(n-1)$-dimensional *planes* in state space as the separation boundary in discontinuous dynamical systems, and the corresponding tangency to the $(n-1)$-dimensional boundary planes is discussed as follows. Because the normal vector $\mathbf{n}_{\partial\Omega_{ij}}$ for the $(n-1)$-dimensional plane boundaries does not change with location, the corresponding conditions for a flow to tangential to such plane boundaries can help one understand the concept of a flow tangential to the general separation boundary in discontinuous dynamical systems. The $(n-1)$-dimensional surfaces as general separation boundaries are discussed in the next chapter.

Theorem 2.10. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at t_m on the $(n-1)$-dimensional plane boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$), then a flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 3$) for time t and $\|\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}\| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the $(n-1)$-dimensional plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.27}$$

$$\left.\begin{array}{lll} \text{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \text{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{array}\right\} \tag{2.28}$$

where $\alpha, \beta \in \{i, j\}$ but $\beta \neq \alpha$.

Proof. Equation (2.27) is identical to (2.19), thus the first condition in (2.19) is satisfied. From Definition 2.12, consider the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$ $(\beta \neq \alpha)$ first. Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$ and a flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon},t_m)}$- and $C^{r_\alpha}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 3)$ for time t, then for $a \in [t_{m-\varepsilon}, t_m)$ and $a \in (t_m, t_{m+\varepsilon}]$, the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ up to the third-order term is given as follows

$$\begin{aligned}\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} - \varepsilon) = \mathbf{x}^{(\alpha)}(a) + \dot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a) \\ + \ddot{\mathbf{x}}^{(\alpha)}(a)(t_{m\pm} \pm \varepsilon - a)^2 + o((t_{m\pm} \pm \varepsilon - a)^2).\end{aligned}$$

As $a \to t_{m\pm}$, the limit of the foregoing equation leads to

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_m \pm \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm}) \pm \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon + \ddot{\mathbf{x}}^{(\alpha)}(t_{m\pm})\varepsilon^2 + o(\varepsilon^2).$$

The ignorance of the ε^3 and higher-order terms, deformation of the above equation, and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon - \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon + \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2.$$

With (2.27), one can obtain

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2.$$

For the plane boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$, using the first inequality equation of (2.28), the foregoing two equations lead to

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2 > 0,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2 < 0.$$

From Definition 2.12, the first inequality equation of (2.28) is obtained. Similarly, using the second inequality of (2.28), one can obtain

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m-})\varepsilon^2 < 0,$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}^{(\alpha)}(t_{m+})\varepsilon^2 > 0.$$

for the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha$. Therefore, under (2.28), the flow $\mathbf{x}^{(\alpha)}(t)$ in domain Ω_α is tangential to the plane boundary $\partial\Omega_{ij}$, vice versa. ■

Theorem 2.11. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m on the $(n-1)$-dimensional plane boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i,j\}$), then the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mu_\alpha)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2$) for time t and $\|\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}\| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.29}$$

$$\left.\begin{array}{lll} \text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(\alpha)}(t_{m\pm}) < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \text{or } \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(\alpha)}(t_{m\pm}) > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{array}\right\} \tag{2.30}$$

where $\alpha, \beta \in \{i,j\}$ but $\alpha \neq \beta$, and the total differentiation ($p, q \in \{1, 2, \ldots, n\}$)

$$D\mathbf{F}^{(\alpha)}(t_{m\pm}) = \left\{ \left[\frac{\partial F_p^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)}{\partial x_q}\right]_{n\times n} \mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)}{\partial t} \right\}\Big|_{(\mathbf{x}_m, t_{m\pm})}. \tag{2.31}$$

Proof. Using (2.1) and (2.29), the first condition in (2.19) is satisfied. The derivative of (2.1) with respect to time t gives

$$\ddot{\mathbf{x}} \equiv D\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha) = \left[\frac{\partial F_p^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)}{\partial x_q}\right]_{n\times n} \dot{\mathbf{x}} + \frac{\partial}{\partial t}\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha).$$

For $t = t_{m\pm}$ and $\mathbf{x} = \mathbf{x}_m$, the left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ to the above equation gives

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \ddot{\mathbf{x}}(t_{m\pm}) = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \left\{ \left[\frac{\partial F_p^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)}{\partial x_q}\right]_{n\times n} \mathbf{F}^{(\alpha)}(t_{m\pm}) + \frac{\partial \mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)}{\partial t} \right\}\Big|_{(\mathbf{x}_m, t_{m\pm})},$$

where $\mathbf{F}^{(\alpha)}(\mathbf{x}_m, t_{m\pm}, \mathbf{p}_\alpha) \triangleq \mathbf{F}^{(\alpha)}(t_{m\pm})$. Using (2.30), the above equation leads to (2.28). From Theorem 2.10, the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$, vice versa. ■

Definition 2.13. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m on the $(n-1)$-dimensional *plane* boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$

$(\alpha \in \{i,j\})$, then a flow $\mathbf{x}^{(\alpha)}(t)$ is $C^{r_\alpha}_{[t_{m-\varepsilon},t_m)}$- and $C^{r_\alpha}_{(t_m,t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2l_\alpha)$ for time t. The flow $\mathbf{x}^{(\alpha)}(t)$ in $\boldsymbol{\Omega}_\alpha$ is tangential to the plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ with the $(2l_\alpha - 1)$th-order if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{k_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha}}\Big|_{t=t_{m\pm}} = 0 \text{ for } k_\alpha = 1, 2, \ldots, 2l_\alpha - 1, \tag{2.32}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m\pm}} \neq 0, \tag{2.33}$$

$$\text{either} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \boldsymbol{\Omega}_\beta \tag{2.34}$$

$$\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \boldsymbol{\Omega}_\alpha, \tag{2.35}$$

where $\alpha, \beta \in \{i,j\}$ but $\beta \neq \alpha$.

Theorem 2.12. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ *at time* t_m *on the* $(n-1)$*-dimensional plane boundary* $\partial\Omega_{ij}$ *between two adjacent domains* $\boldsymbol{\Omega}_\alpha$ $(\alpha = i,j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*). Suppose* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$*, then a flow* $\mathbf{x}^{(\alpha)}(t)$ *is* $C^{r_\alpha}_{[t_{m-\varepsilon},t_m)}$*- and* $C^{r_\alpha}_{(t_m,t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2l_\alpha + 1)$ *for time* t*.* $\|\mathrm{d}^{r_\alpha}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha}\| < \infty$*. The flow* $\mathbf{x}^{(\alpha)}(t)$ *in* $\boldsymbol{\Omega}_\alpha$ *is tangential to the plane boundary* $\partial\Omega_{ij}$ *at point* $(\mathbf{x}_m, t_m)$ *with the* $(2l_\alpha - 1)$th*-order if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{k_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha}}\Big|_{t=t_{m\pm}} = 0 \quad \text{for } (k_\alpha = 1, 2, \ldots 2l_\alpha - 1), \tag{2.36}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m\pm}} \neq 0, \tag{2.37}$$

$$\left.\begin{array}{ll} \text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dfrac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m\pm}} < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \boldsymbol{\Omega}_\beta \\ \text{or} \quad\;\; \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dfrac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m\pm}} > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \boldsymbol{\Omega}_\alpha, \end{array}\right\} \tag{2.38}$$

where $\beta \in \{i,j\}$ *but* $\beta \neq \alpha$*.*

Proof. For (2.36) and (2.37), the first two conditions in Definition 2.13 are satisfied. Consider the boundary $\partial\Omega_{ij}$ with $\mathbf{n}_{\partial\Omega_{ij}} \to \boldsymbol{\Omega}_\beta$ $(\beta \neq \alpha)$ first. Choose

$a \in [t_{m-\varepsilon}, t_m)$ or $a \in (t_m, t_{m-\varepsilon}]$, and application of the Taylor series expansion of $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ to $\mathbf{x}^{(\alpha)}(a)$ and up to the $(2l_\alpha)$th-order term gives

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(a) + \sum_{k_\alpha=1}^{2l_\alpha-1} \frac{\mathrm{d}^{k_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha}}\Big|_{t=a}(t_{m\pm} \pm \varepsilon - a)^{k_\alpha}$$
$$+ \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=a}(t_{m\pm} \pm \varepsilon - a)^{2l_\alpha} + o((t_{m\pm} \pm \varepsilon - a)^{2l_\alpha}).$$

As $a \to t_{m\pm}$, the foregoing equation becomes

$$\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon}) \equiv \mathbf{x}^{(\alpha)}(t_{m\pm} \pm \varepsilon) = \mathbf{x}^{(\alpha)}(t_{m\pm}) + \sum_{k_\alpha=1}^{2l_\alpha-1} \frac{\mathrm{d}^{k_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha}}\Big|_{t=t_{m\pm}}(\pm\varepsilon)^{k_\alpha}$$
$$+ \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m\pm}}\varepsilon^{2l_\alpha} + o(\pm\varepsilon^{2l_\alpha}).$$

With (2.36) and (2.37), the deformation of the above equation and left multiplication of $\mathbf{n}_{\partial\Omega_{ij}}$ produces

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m-}) - \mathbf{x}^{(\alpha)}(t_{m-\varepsilon})] = -\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m-}}\varepsilon^{2l_\alpha},$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(\alpha)}(t_{m+\varepsilon}) - \mathbf{x}^{(\alpha)}(t_{m+})] = \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \frac{\mathrm{d}^{2l_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{2l_\alpha}}\Big|_{t=t_{m+}}\varepsilon^{2l_\alpha}.$$

Under (2.38), the condition in (2.34) is satisfied, and vice versa. Therefore, the flow $\mathbf{x}^{(\alpha)}(t)$ in domain Ω_α is tangential to $\partial\Omega_{ij}$ with the $(2l_\alpha - 1)$th-order for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta$. Similarly, under the condition in (2.38), the flow $\mathbf{x}^{(\alpha)}(t)$ in domain Ω_α is tangential to boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ with the $(2l_\alpha - 1)$th-order for $\mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha$. Hence, the theorem is proved. ■

Theorem 2.13. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in \partial\Omega_{ij}$ at time t_m on the $(n-1)$-dimensional plane boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$). Suppose $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ ($\alpha \in \{i, j\}$), then the vector field $\mathbf{F}^{(\alpha)}(\mathbf{x}, t, \mathbf{p}_\alpha)$ is $C^{r_\alpha}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_\alpha}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2l_\alpha$) for time t. $||\mathrm{d}^{r_\alpha+1}\mathbf{x}^{(\alpha)}/\mathrm{d}t^{r_\alpha+1}|| < \infty$. The flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m, t_m)$ with the* $(2l_\alpha - 1)$th-*order if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{k_\alpha-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for } k_\alpha = 1, 2, \ldots, 2l_\alpha - 1, \tag{2.39}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2l_\alpha-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) \neq 0, \tag{2.40}$$

$$\begin{aligned}&\text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2l_\alpha-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) < 0 \quad \text{for } \partial\Omega_{ij} \to \Omega_\beta \\ &\text{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D^{2l_\alpha-1}\mathbf{F}^{(\alpha)}(t_{m\pm}) > 0 \quad \text{for } \partial\Omega_{ij} \to \Omega_\alpha,\end{aligned} \tag{2.41}$$

where the total differentiation

$$D^{k_\alpha-1}\mathbf{F}^{(\alpha)}(t_m) = D^{k-2}\left\{\left[\frac{\partial F_p^{(\alpha)}(\mathbf{x},t,\mathbf{p}_\alpha)}{\partial x_q}\right]_{n\times n}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}^{(\alpha)}(\mathbf{x},t,\mathbf{p}_\alpha)}{\partial t}\right\}\Big|_{(\mathbf{x}_m,t_m)}, \tag{2.42}$$

with $p,q \in \{1,2,\ldots,n\}$, $k_\alpha \in \{2,3,\ldots 2l_\alpha\}$, *and* $\beta \in \{i,j\}$ *but* $\alpha \neq \beta$.

Proof. The k_α-order derivative of (2.1) with respect to time gives

$$\begin{aligned}\frac{\mathrm{d}^{k_\alpha}\mathbf{x}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha}}\Big|_{(\mathbf{x}_m,t_m)} &= \frac{\mathrm{d}^{k_\alpha-1}\dot{\mathbf{x}}^{(\alpha)}(t)}{\mathrm{d}t^{k_\alpha-1}}\Big|_{(\mathbf{x}_m,t_m)} = \frac{\mathrm{d}^{k_\alpha-1}\mathbf{F}^{(\alpha)}(\mathbf{x},t,\mathbf{p}_\alpha)}{\mathrm{d}t^{k_\alpha-1}}\Big|_{(\mathbf{x}_m,t_m)} \equiv D^{k_\alpha-1}\mathbf{F}^{(\alpha)}(t_m) \\ &= D^{k-2}\left\{\left[\frac{\partial F_p^{(\alpha)}(\mathbf{x},t,\mathbf{p}_\alpha)}{\partial x_q}\right]_{n\times n}\dot{\mathbf{x}} + \frac{\partial \mathbf{F}^{(\alpha)}(\mathbf{x},t,\mathbf{p}_\alpha)}{\partial t}\right\}\Big|_{(\mathbf{x}_m,t_m)}.\end{aligned}$$

Using the foregoing equation to the conditions in (2.39)–(2.42), the flow $\mathbf{x}^{(\alpha)}(t)$ in Ω_α is tangential to the plane boundary $\partial\Omega_{ij}$ at point $(\mathbf{x}_m,t_m)$ with the $(2l_\alpha-1)$th order from Theorem 2.12. Therefore, this theorem is proved. ■

2.5 Switching Bifurcations of Passable Flows

In this section, the switching bifurcation between the passable and nonpassable flows to the boundary is discussed. In addition, the switching bifurcation between the sink and source flows on the boundary is also discussed. The switching bifurcations are defined first, and then the sufficient and necessary conditions for such switching bifurcations are developed. The L-functions of flows are introduced to develop criteria for the switching bifurcations from sufficient and necessary conditions.

Definition 2.14. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1},\mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon},t_m)$ and $(t_m,t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon},t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 1,\ \alpha = i,j)$ for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m,t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of a flow* from the semipassable flow to the nonpassable flow of the first kind (or called *the sliding bifurcation from* $\overrightarrow{\partial\Omega}_{ij}$ *to* $\widetilde{\partial\Omega}_{ij}$) if

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) \neq 0, \tag{2.43}$$

$$\text{either} \quad \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j \tag{2.44}$$

$$\text{or} \quad \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i. \tag{2.45}$$

Definition 2.15. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$-continuous $(r_\alpha \geq 1,\ \alpha = i, j)$ for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation of a flow from the passable flow to the nonpassable flow of the second kind* (or called *the source bifurcation from* $\overrightarrow{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$) if

$$\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m\pm}) \neq 0, \tag{2.46}$$

$$\text{either} \quad \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] < 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j \tag{2.47}$$

$$\text{or} \quad \left.\begin{aligned} &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] > 0 \\ &\mathbf{n}_{\partial\Omega_{ij}}^{\mathrm{T}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i. \tag{2.48}$$

From the two definitions, the switching bifurcations of a flow from the semi-passable boundary to the nonpassable boundaries of the first and second kinds are

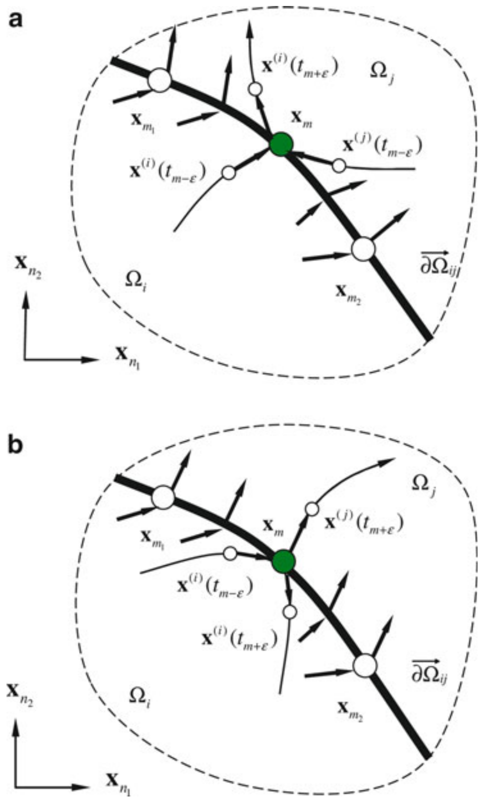

Fig. 2.7 (**a**) The sliding bifurcation and (**b**) the source bifurcation on the semipassable boundary $\overrightarrow{\partial\Omega}_{ij}$. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$ ($\alpha \in \{i,j\}$) and $\mathbf{x}_m$ lie in the corresponding domains Ω_α and on the boundary $\partial\Omega_{ij}$, respectively

presented in Fig. 2.7. The source (or *sink*) bifurcation of a flow to the boundary requires the tangential bifurcation of the coming (or *leaving*) flow to the boundary. Similarly, the switching bifurcation of a passable flow from $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ is defined as follows.

Definition 2.16. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$) and $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$ ($\alpha = i,j$) are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$) for time t. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ is termed *the switching bifurcation* of a flow from $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for} \quad \alpha = i,j \tag{2.49}$$

and

$$\text{either} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] < 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_j \tag{2.50}$$

$$\text{or} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] > 0 \text{ and} \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\Omega_{ij}} \to \Omega_i. \tag{2.51}$$

The above definitions give the three possible switching bifurcations of the semipassable flow to the boundary $\overrightarrow{\partial\Omega}_{ij}$. The corresponding theorems can be stated for necessary and sufficient conditions. The proofs can be completed as in Theorems 2.8–2.10.

Theorem 2.14. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon},t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 1$, $\alpha = i,j$) for time t, respectively. The sliding bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$ occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0, \tag{2.52}$$

$$\left.\begin{array}{lll} \text{either } & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or } & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{array}\right\} \tag{2.53}$$

$$\left.\begin{array}{l} \text{either} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.54}$$

Proof. Following the proof procedures in Theorems 2.8 and 2.9, the above theorem can be easily proved. ■

Theorem 2.15. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m on the $(n-1)$-dimensional plane* *boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2$, $\alpha = i,j$) for time t, respectively. The sliding bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$ occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0, \tag{2.55}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.56}$$

Proof. Following the proof procedures of Theorems 2.10 and 2.11, the above theorem can be easily proved. ∎

Theorem 2.16. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α ($\alpha = i,j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$-continuous ($r_\alpha \geq 1$, $\alpha = i,j$) for time t, respectively. The source bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0, \tag{2.57}$$

$$\left.\begin{array}{lll} \text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} \quad\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{array}\right\} \tag{2.58}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.59}$$

Proof. Following the proof procedures of Theorems 2.8 and 2.9, the above theorem can be easily proved. ∎

Theorem 2.17. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m on the $(n-1)$-dimensional plane* *boundary* $\partial\Omega_{ij}$ *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. *The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*- and* $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$*-continuous* $(r_\alpha \geq 2, \alpha = i, j)$ *for time* t*, respectively. The source bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *from* $\overrightarrow{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$ *occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0, \tag{2.60}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \\ \mathbf{n}^{T}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \\ \mathbf{n}^{T}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.61}$$

Proof. Following the proof procedures of Theorems 2.10 and 2.11, the above theorem can be easily proved. ■

Theorem 2.18. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. *The flows* $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ *are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 1)$ *for time* t. *The switching bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *from* $\overrightarrow{\partial\Omega}_{ij}$ *to* $\overleftarrow{\partial\Omega}_{ij}$ *occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\mp}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0, \tag{2.62}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{array}\right\} \tag{2.63}$$

with $\alpha, \beta = i, j$ *but* $\beta \neq \alpha$.

Proof. Following the proof procedures of Theorems 2.8 and 2.9, the above theorem can be easily proved. ■

Theorem 2.19. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *at time* t_m *on the* $(n-1)$*-dimensional plane boundary* $\partial\Omega_{ij}$ *between two adjacent domains* Ω_α $(\alpha = i, j)$. *For an arbitrarily small*

$\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i,j)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 1)$ for time t. The switching bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\overleftarrow{\partial\Omega}_{ij}$ occurs if and only if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\mp}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0, \tag{2.64}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\mp}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\mp}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array}\right\} \tag{2.65}$$

Proof. Following the proof procedures of Theorems 2.10 and 2.11, the above theorem can be easily proved. ■

Definition 2.17. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \partial\Omega_{ij}$ for time t_m and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$ $(\alpha \in \{i,j\})$. The $L_{\alpha\beta}$-functions of flows to the boundary $\partial\Omega_{ij}$ is defined as

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \mathbf{p}_\beta) = [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m\mp})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(t_{m\pm})], \tag{2.66}$$

where $\beta \in \{i,j\}$ but $\beta \neq \alpha$.

From the foregoing definition, the passable flows and nonpassable flows (including sink and source flows) at the boundary $\partial\Omega_{\alpha\beta}$, respectively, require the L-function satisfies

$$\left.\begin{array}{l} L_{\alpha\beta}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \mathbf{p}_\beta) > 0 \text{ on } \overrightarrow{\partial\Omega}_{\alpha\beta}, \\ L_{\alpha\beta}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \mathbf{p}_\beta) < 0 \text{ on } \overline{\partial\Omega}_{\alpha\beta} = \widetilde{\partial\Omega}_{\alpha\beta} \cup \widehat{\partial\Omega}_{\alpha\beta}. \end{array}\right\} \tag{2.67}$$

The switching bifurcation of a flow at point $(\mathbf{x}_m, t_m)$ on the boundary $\partial\Omega_{\alpha\beta}$ requires

$$L_{\alpha\beta}(\mathbf{x}_m, t_m, \mathbf{p}_\alpha, \mathbf{p}_\beta) = 0. \tag{2.68}$$

If the $L_{\alpha\beta}$-function of a flow is defined on one side of the neighborhood of the boundary $\partial\Omega_{\alpha\beta}$, one can obtain

$$L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) = [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon})] \times [\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon})]. \tag{2.69}$$

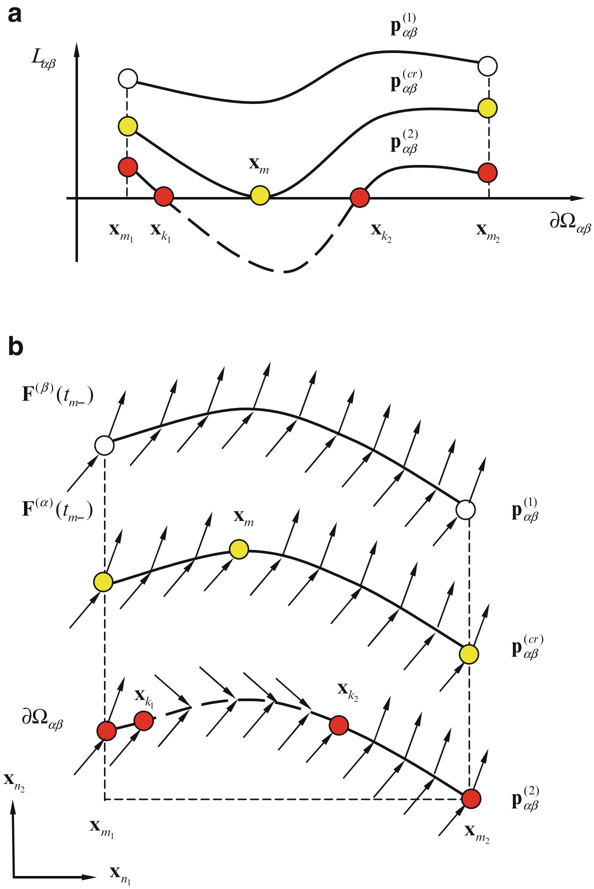

Fig. 2.8 (**a**) The $L_{\alpha\beta}$-functions of flows and (**b**) the vector fields between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the boundary $\partial\Omega_{\alpha\beta}$. The point $\mathbf{x}_m$ for $\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})}$ is the critical point for the switching bifurcation. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the onset and vanishing of the nonpassable flow for parameter on the boundary $\overrightarrow{\partial\Omega}_{\alpha\beta}$. The *dashed* and *solid curves* represent $L_{\alpha\beta}<0$ and $L_{\alpha\beta}\geq 0$, respectively

If $L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) < 0$ and $\mathbf{n}_{\partial\Omega_{\alpha\beta}}^{\mathrm{T}} \cdot \mathbf{F}^{(\alpha)}(t_{m-}) = 0$, the flow $\mathbf{x}^{(\alpha)}(t)$ at $(\mathbf{x}_m, t_m)$ is tangential to the boundary $\partial\Omega_{\alpha\beta}$.

Consider the $L_{\alpha\beta}$-function varying with the parameter vector $\mathbf{p}_{ij} \in \{\mu_\alpha\}_{\alpha\in\{i,j\}}$ for the switching flow from $\overrightarrow{\partial\Omega}_{\alpha\beta}$ to $\widetilde{\partial\Omega}_{\alpha\beta}$. The $L_{\alpha\beta}$-functions of flows at different locations of the boundary are distinct. The $L_{\alpha\beta}$-functions of flows between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the boundary $\partial\Omega_{\alpha\beta}$ are sketched in Fig. 2.8 for a parameter vector $\mathbf{p}_{\alpha\beta}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$. For a specific value $\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$, there is a point $\mathbf{x}_m$ on the boundary for the bifurcation of a flow switching from $\overrightarrow{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the onset and vanishing points of the sink flow for system parameter vector $\mathbf{p}_{\alpha\beta}$ on the boundary $\partial\Omega_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta} < 0$ and $L_{\alpha\beta} \geq 0$, respectively. For parameter vector $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)} \to \mathbf{p}_{\alpha\beta}^{(cr)}$, the $L_{\alpha\beta}$-function for a flow $\mathbf{x} \in (\mathbf{x}_{m_1}, \mathbf{x}_{m_2})$ on the boundary is positive (i.e., $L_{\alpha\beta} > 0$). Thus, the boundary $\partial\Omega_{\alpha\beta}$ is semipassable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})} \to \mathbf{p}_{\alpha\beta}^{(2)}$, there are two ranges of $L_{\alpha\beta} > 0$ for $\mathbf{x} \in [\mathbf{x}_{m_1}, \mathbf{x}_{k_1}) \cup (\mathbf{x}_{k_2}, \mathbf{x}_{m_2}]$ and a range of $L_{\alpha\beta} < 0$ for $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$. From (2.67), the flow at the portion of $\mathbf{x} \in$

$(\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ on the boundary $\partial\Omega_{\alpha\beta}$ is nonpassable. The flow at the portion of boundary with $L_{\alpha\beta} > 0$ is semipassable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)} \to \mathbf{p}_{\alpha\beta}^{(2)}$, the point $(\mathbf{x}_m, \mathbf{p}_{\alpha\beta}^{(\mathrm{cr})})$ on the boundary $\partial\Omega_{\alpha\beta}$ is the onset point of the nonpassable flow. However, for $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(2)} \to \mathbf{p}_{\alpha\beta}^{(1)}$, such a point is the vanishing point of the nonpassable flow. For three critical points $(\mathbf{x}_m, \mathbf{x}_{k_1}, \mathbf{x}_{k_2})$, the $L_{\alpha\beta}$-function of flows is zero (i.e., $L_{\alpha\beta} = 0$). For $L_{\alpha\beta}$ in Fig. 2.8a, the corresponding vector fields varying with the system parameter on the boundary $\partial\Omega_{\alpha\beta}$ are illustrated in Fig. 2.8b. $\mathbf{F}^{(\alpha)}(t_{m-})$ and $\mathbf{F}^{(\beta)}(t_{m\pm})$ are the limits of the vector fields in domains Ω_α and Ω_β to the boundary $\partial\Omega_{\alpha\beta}$, respectively. This nonpassable flow on the boundary $\partial\Omega_{\alpha\beta}$ with $L_{\alpha\beta} < 0$ is a sink flow. The critical points $(\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ have the same properties as point $\mathbf{x}_m$ for $\mathbf{p}_{\alpha\beta}^{(cr)}$. Namely, $L_{\alpha\beta}(\mathbf{x}_m) = 0$, $L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}) < 0$, or $L_{\beta\beta}(\mathbf{x}_{m\pm\varepsilon}) > 0$.

If the two critical points have the different properties, the sliding flow between two different critical points is discussed later. The $L_{\alpha\beta}$-functions of flows are $L_{\alpha\beta}(\mathbf{x}_{k_1}) = 0$ and $L_{\alpha\alpha}(\mathbf{x}_{k_1\pm\varepsilon}) < 0$ for point $\mathbf{x}_{k_1}$ but $L_{\alpha\beta}(\mathbf{x}_{k_2}) = 0$ and $L_{\beta\beta}(\mathbf{x}_{k_2\pm\varepsilon}) < 0$ for point $\mathbf{x}_{k_2}$. From the $L_{\alpha\beta}$-function of flows, Theorems 2.14, 2.16, and 2.18 can be restated.

Theorem 2.20. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m+\varepsilon}, t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2$, $\alpha = i,j$) for time t, respectively. The sliding bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.70}$$

$$\mathbf{n}^{\mathbf{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \text{and} \quad \alpha = i,j. \tag{2.71}$$

Proof. Applying the L_{ij}-functions of flows in Definition 2.17 to Theorem 2.14, the foregoing theorem can be easily proved. ■

Theorem 2.21. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$-continuous ($r_\alpha \geq 2$, $\alpha = i,j$) for time t, respectively. The source bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\overrightarrow{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.72}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{ii}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_i) < 0. \tag{2.73}$$

Proof. Applying the L-function of flows in Definition 2.17 to Theorem 2.16, this theorem can be easily proved. ■

Theorem 2.22. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *at time* t_m *between two adjacent domains* Ω_α *(*$\alpha = i, j$*). For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(\alpha)}(t)$ *(*$\alpha = i, j$*) are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous (*$r_\alpha \geq 2$*) for time* t*. The switching bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *from* $\overrightarrow{\partial\Omega}_{ij}$ *to* $\overleftarrow{\partial\Omega}_{ij}$ *occurs if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.74}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) < 0 \quad (\alpha = i, j). \tag{2.75}$$

Proof. Applying the L-function of flows in Definition 2.17 to Theorem 2.18, the theorem can be easily proved. ■

Remark. For the $(n-1)$-dimensional *plane* boundary $\partial\Omega_{ij}$, the second conditions in (2.71), (2.73), and (2.75) in Theorems 2.20–2.22 can be replaced by (2.56), (2.61), and (2.65) in Theorems 2.15, 2.17, and 2.19, respectively.

For the passable flow at $\mathbf{x}(t_m) \equiv \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$, consider the time interval $[t_{m_1}, t_{m_2}]$ for $[\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ on the boundary, and the L-functions of flows (i.e., L_{ij}) for $t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ is also positive, i.e., $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) > 0$. To determine the switching bifurcation, the local minimum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is introduced. Because $\mathbf{x}_m$ is a vector function of time t_m, the two total derivatives of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ are introduced, i.e.,

$$\begin{aligned} DL_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = \nabla L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j).\mathbf{F}^{(0)}_{ij}(\mathbf{x}_m, t_m) \\ + \frac{\partial L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)}{\partial t_m}, \end{aligned} \tag{2.76}$$

$$D^k L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = D^{k-1}(DL_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)) \quad \text{for} \quad k = 1, 2, \ldots. \tag{2.77}$$

Thus, the local minimum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is determined by

$$D^k L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0 \quad (k = 1, 2, \ldots, 2l-1), \tag{2.78}$$

$$D^{2l} L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) > 0. \tag{2.79}$$

Definition 2.18. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and

$(t_m, t_{m+\varepsilon}])$. $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2l,\ \alpha = i,j)$ for time t. *The local minimum set* of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is defined by

$$
{}_{\min}L_{ij}(t_m) = \left\{ L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) \middle| \begin{array}{l} \forall t_m \in [t_{m_1}, t_{m_2}], \exists \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ \text{so that } D^k L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0 \\ \text{for } k = 1, 2, \cdots 2l-1 \text{ and} \\ D^{2l} L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) > 0. \end{array} \right\} \tag{2.80}
$$

From the local minimum set of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$, the corresponding global minimum can be determined as follows.

Definition 2.19. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i,j)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2l,\ \alpha = i,j)$ for time t, respectively. The global minimum set of $L_{ij}(\mathbf{x}_m, t_m, \mu_i, \mu_j)$ is defined by

$$
{}_{G\min}L_{ij}(t_m) = \min_{t_m \in [t_{m_1}, t_{m_2}]} \left\{ \begin{array}{l} {}_{\min}L_{ij}(t_m), L_{ij}(\mathbf{x}_{m_1}, t_{m_1}, \mathbf{p}_i, \mathbf{p}_j), \\ L_{ij}(\mathbf{x}_{m_2}, t_{m_2}, \mathbf{p}_i, \mathbf{p}_j). \end{array} \right\} \tag{2.81}
$$

From the foregoing definition, Theorems 2.20–2.22 can be expressed through the global minimum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$. So the following corollaries can be achieved, which give the conditions for onsets of switching bifurcations.

Corollary 2.1. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for time* t_m *between two adjacent domains* Ω_α $(\alpha = i,j)$. *For an arbitrarily small* $\varepsilon > 0$, *there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. *The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r}_{[t_{m+\varepsilon}, t_m)}$- *and* $C^{r}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-*continuous* $(r_\alpha \geq 2l,\ \alpha = i,j)$ *for time t, respectively. The necessary and sufficient conditions for the sliding bifurcation onset of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *on the boundary* $\overrightarrow{\partial\Omega}_{ij}$ *are*

$$
{}_{G\min}L_{ij}(t_m) = 0, \tag{2.82}
$$

$$
\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \text{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_j) < 0. \tag{2.83}
$$

Proof. $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ replaced by its global minimum in Theorem 2.20 gives this corollary. This corollary is proved. ■

Corollary 2.2. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ *for time* t_m *between two adjacent domains* Ω_α

$(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$-continuous $(r_\alpha \geq 2l,\ \alpha = i,j)$ for time t, respectively. The necessary and sufficient conditions for the source bifurcation onset of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ are

$$_{G\,\min}L_{ij}(t_m) = 0, \tag{2.84}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{ii}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_i) < 0. \tag{2.85}$$

Proof. $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ replaced by its global minimum in Theorem 2.21 gives this corollary. ■

Corollary 2.3. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overrightarrow{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α $(\alpha = i,j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\mp}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 2l)$ for time t. The necessary and sufficient conditions for the switching bifurcation onset of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\overrightarrow{\partial\Omega}_{ij}$ are*

$$_{G\,\min}L_{ij}(t_m) = 0, \tag{2.86}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(t)}(t_{m\pm}) = 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) < 0 \quad \text{for} \quad \alpha = i,j. \tag{2.87}$$

Proof. $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ replaced by its global minimum in Theorem 2.22 gives this corollary. ■

2.6 Switching Bifurcations of Nonpassable Flows

The onset and vanishing of the sliding and source flows on the boundary were discussed. The fragmentations of the sliding and source flows on the boundary are of great interest in this section. This kind of bifurcation is still a switching bifurcation. The definitions for such fragmentation bifurcations of flows on the nonpassable boundary are similar to the switching bifurcations of flows from the semipassable boundary to nonpassable boundary. The necessary and sufficient conditions for the fragmentation bifurcation from a nonpassable flow to a passable flow on the boundary are quite similar to the sliding and source bifurcations from a passable flow to a nonpassable flow.

Definition 2.20. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m+\varepsilon}, t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 1,\ \alpha = i, j)$ for time t, respectively. The tangential bifurcation of the flow $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ is termed *the switching bifurcation* of a nonpassable flow *of the first kind* from $\widetilde{\partial\Omega}_{ij}$ to $\overrightarrow{\partial\Omega}_{ij}$ (or simply called *the sliding fragmentation bifurcation*) if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m\mp}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m-}) \neq 0, \tag{2.88}$$

$$\text{either} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{2.89}$$

$$\text{or} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{2.90}$$

Definition 2.21. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{[t_{m+\varepsilon}, t_m)}$-continuous $(r_\alpha \geq 1,\ \alpha = i, j)$ for time t, respectively. The tangential bifurcation of flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ is termed *the switching bifurcation* of a nonpassable flow *of the second kind* from $\widehat{\partial\Omega}_{ij}$ to $\overrightarrow{\partial\Omega}_{ij}$ (or simply called *the source fragmentation bifurcation*) if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(j)}(t_{m+}) \neq 0, \tag{2.91}$$

$$\text{either} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{2.92}$$

$$\text{or} \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+}) > 0] \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{aligned}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{2.93}$$

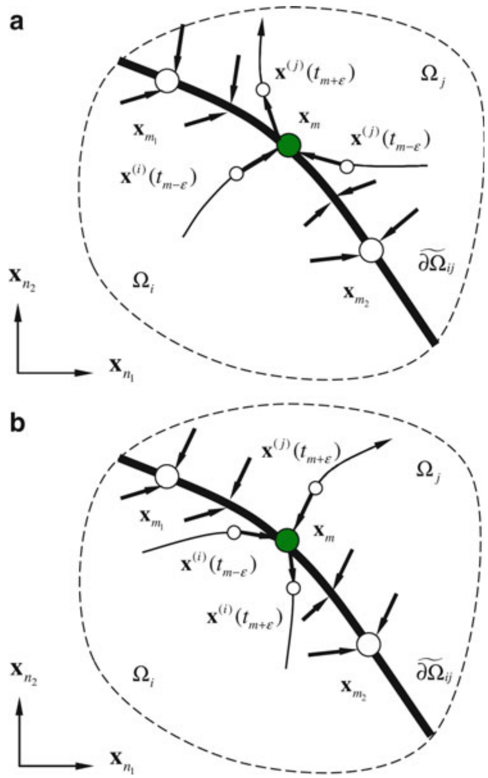

Fig. 2.9 The sliding fragmentation bifurcation to the sink boundary $\widetilde{\partial\Omega}_{ij}$ in domain: (**a**) Ω_j and (**b**) Ω_i. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$, $\mathbf{x}^{(\beta)}(t_{m-\varepsilon})$, and $\mathbf{x}_m$ lie in the corresponding domains and on the boundary $\partial\Omega_{ij}$, respectively. $\alpha, \beta \in \{i, j\}$ but $\alpha \neq \beta$ and $n_1 + n_2 = n$

For the fragmentation bifurcation of the nonpassable flow on the boundary, the vector fields near the sink and source boundaries are sketched in Figs. 2.9 and 2.10, respectively. The switching from the sink or source flow to the semipassable flow has two possibilities. Therefore, the conditions in Definitions 2.20 and 2.21 have been changed accordingly. Before the fragmentation bifurcation of the nonpassable flow occurs on the boundary, the flow $\mathbf{x}^{(\alpha)}(t)$ ($\alpha \in \{i, j\}$) exists for $t \in [t_{m-\varepsilon}, t_{m-})$ or $t \in (t_{m+}, t_{m+\varepsilon}]$ on the sink or source boundary. Only the sliding flow exists on such a boundary. After the fragmentation bifurcation occurs, the sliding flow on the boundary will split into at least two portions of the sliding and semipassable motions. This phenomenon is called *the fragmentation of the sliding flow on the boundary*, which can help one easily understand the sliding dynamics on the boundary. In addition, for the nonpassable boundary, if flows on both sides of the nonpassable boundary possess the local singularity at the boundary, the nonpassable flow of the first kind switches into the nonpassable flow of the second kind, and vice versa. The local singularity of such switchability is similar to the switching between the two semipassable flows on the boundary, and the corresponding definition of the switching bifurcation is given as follows.

Definition 2.22. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) at time t_m between two adjacent domains

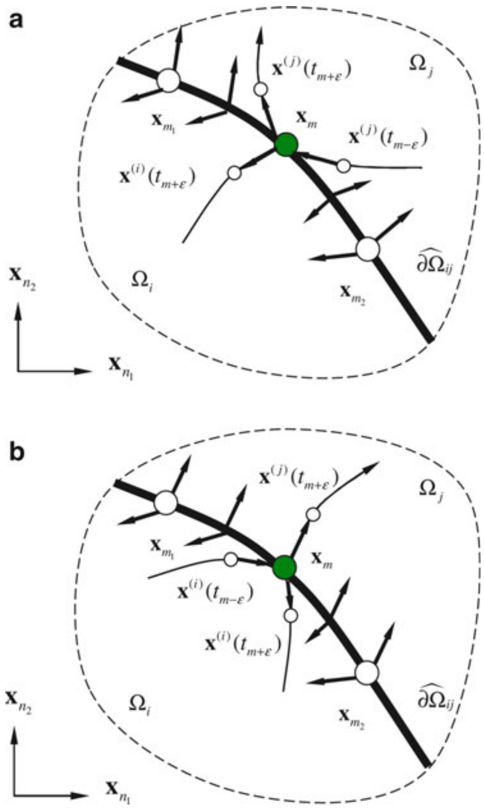

Fig. 2.10 The source fragmentation bifurcation to the source boundary $\widehat{\partial\Omega}_{ij}$ in domain: (**a**) Ω_j and (**b**) Ω_i. Four points $\mathbf{x}^{(\alpha)}(t_{m\pm\varepsilon})$, $\mathbf{x}^{(\beta)}(t_{m+\varepsilon})$, and $\mathbf{x}_m$ lie in the corresponding domains and on the boundary $\partial\Omega_{ij}$, respectively. $\alpha, \beta \in \{i, j\}$ but $\alpha \neq \beta$ and $n_1 + n_2 = n$

Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$. The flows $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ are $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 1)$ for time t. The tangential bifurcation of the flow $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) is termed the switching bifurcation of a nonpassable flow from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) if

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \dot{\mathbf{x}}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for} \quad \alpha = i, j, \tag{2.94}$$

$$\text{either} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] > 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] < 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] < 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] > 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \tag{2.95}$$

$$\text{or} \quad \left.\begin{aligned} &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m-}) - \mathbf{x}^{(i)}(t_{m-\varepsilon})] < 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(i)}(t_{m+\varepsilon}) - \mathbf{x}^{(i)}(t_{m+})] > 0 \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m-}) - \mathbf{x}^{(j)}(t_{m-\varepsilon})] > 0 \text{ and} \\ &\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot [\mathbf{x}^{(j)}(t_{m+\varepsilon}) - \mathbf{x}^{(j)}(t_{m+})] < 0 \end{aligned}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \tag{2.96}$$

Theorem 2.23. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 1,\ \alpha = i, j)$ *for time t, respectively. The sliding fragmentation bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *from* $\widetilde{\partial\Omega}_{ij}$ *to* $\overrightarrow{\partial\Omega}_{ij}$ *occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0, \tag{2.97}$$

$$\left.\begin{array}{lll} \text{either} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 & \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{array}\right\} \tag{2.98}$$

$$\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array} \tag{2.99}$$

Proof. Following the proof procedures of Theorems 2.8 and 2.9, the above theorem can be easily proved. ■

Theorem 2.24. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *at time* t_m *on the* $(n-1)$*-dimensional plane boundary* $\partial\Omega_{ij}$ *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(i)}(t)$ *and* $\mathbf{x}^{(j)}(t)$ *are* $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$*- and* $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2,\ \alpha = i, j)$ *for time t, respectively. The sliding fragmentation bifurcation of the flow* $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ *at point* $(\mathbf{x}_m, t_m)$ *from* $\widetilde{\partial\Omega}_{ij}$ *to* $\overrightarrow{\partial\Omega}_{ij}$ *occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0, \tag{2.100}$$

$$\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) < 0 \end{array}\right\} \text{ for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array} \tag{2.101}$$

Proof. Following the proof procedures of Theorems 2.10 and 2.11, the above theorem can be easily proved. ■

Theorem 2.25. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ for time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous $(r_\alpha \geq 1, \alpha = i, j)$ for time t, respectively. The source fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0, \tag{2.102}$$

$$\begin{aligned} &\text{either } \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \qquad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i, \end{aligned} \tag{2.103}$$

$$\begin{aligned} &\text{either} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.104}$$

Proof. Following the proof procedures of Theorems 2.8 and 2.9, the above theorem can be easily proved. ■

Theorem 2.26. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ at time t_m on the $(n-1)$-dimensional plane boundary $\partial\Omega_{ij}$ between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$ and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$ continuous $(r_\alpha \geq 1, \alpha = i, j)$ for time t, respectively. The source fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ from $\widehat{\partial\Omega}_{ij}$ to $\overrightarrow{\partial\Omega}_{ij}$ occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m\pm}) = 0 \quad \text{and} \quad \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0, \tag{2.105}$$

$$\begin{aligned} &\text{either} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j \\ &\text{or} \quad \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) < 0 \end{array}\right\} \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{aligned} \tag{2.106}$$

Proof. Following the proof procedures in Theorems 2.10 and 2.11, the above theorem can be easily proved. ∎

Theorem 2.27. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$*) at time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ *are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 1)$ *for time t. The switching bifurcation of the flow at point* $(\mathbf{x}_m, t_m)$ *from* $\widetilde{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$ *to* $\widetilde{\partial\Omega}_{ij}$*) occurs if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0, \tag{2.107}$$

$$\left.\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\beta \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m-\varepsilon}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m+\varepsilon}) > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_\alpha, \end{array}\right. \tag{2.108}$$

with $\alpha, \beta = i, j$ *but* $\alpha \neq \beta$.

Proof. Following the proof procedures of Theorems 2.8 and 2.9, the above theorem can be easily proved. ∎

Theorem 2.28. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$*) at time* t_m *on the* $(n-1)$*-dimensional plane boundary* $\partial\Omega_{ij}$ *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(\alpha)}(t_{m\pm}) = \mathbf{x}_m$*. The flows* $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ *are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2)$ *for time t. The switching bifurcation of the flow at point* $(\mathbf{x}_m, t_m)$ *from* $\widetilde{\partial\Omega}_{ij}$ *to* $\widehat{\partial\Omega}_{ij}$ *(or* $\widehat{\partial\Omega}_{ij}$ *to* $\widetilde{\partial\Omega}_{ij}$*) exists if and only if*

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{for} \quad \alpha = i, j, \tag{2.109}$$

$$\begin{array}{ll} \text{either} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) < 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) > 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_j, \\ \text{or} & \left.\begin{array}{l} \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(i)}(t_{m\pm}) > 0 \\ \mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot D\mathbf{F}^{(j)}(t_{m\pm}) < 0 \end{array}\right\} \quad \text{for } \mathbf{n}_{\partial\Omega_{ij}} \to \Omega_i. \end{array} \tag{2.110}$$

Proof. Following the proof procedures of Theorems 2.10 and 2.11, the above theorem can be easily proved. ∎

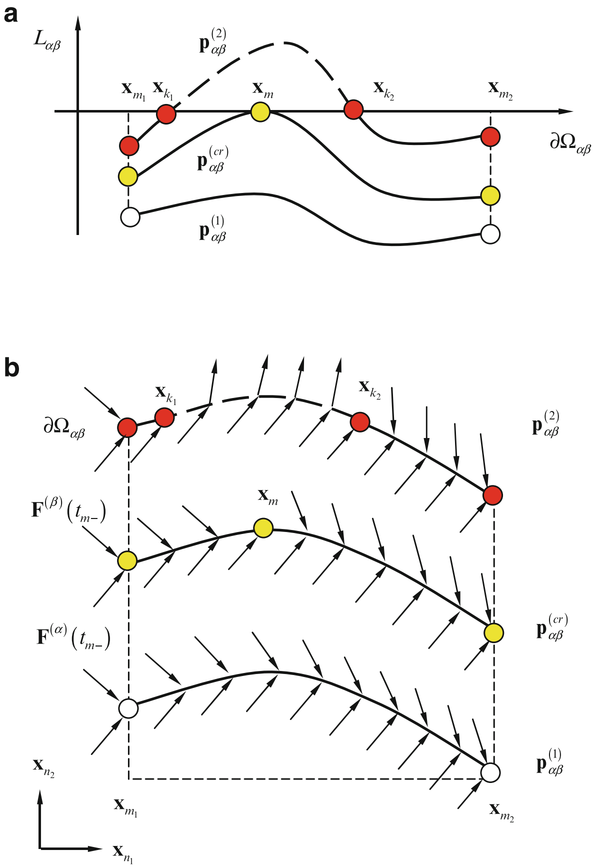

Fig. 2.11 (**a**) The L-function of flows ($L_{\alpha\beta}$) and (**b**) the vector fields between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the boundary $\widetilde{\partial\Omega}_{\alpha\beta}$. The point $\mathbf{x}_m$ for $\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})}$ is the critical point for the switching bifurcation. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are the starting and vanishing of the passable flow on the boundary $\widetilde{\partial\Omega}_{\alpha\beta}$. The *dashed* and *solid curves* represent $L_{\alpha\beta}>0$ and $L_{\alpha\beta}\leqslant 0$, respectively ($n_1+n_2=n$)

Similarly, the $L_{\alpha\beta}$-functions of flows varying with $\mathbf{p}_{ij}\in\{\mu_\alpha\}_{\alpha\in\{i,j\}}$ is used to discuss the switching of the nonpassable flow from $\overline{\partial\Omega}_{\alpha\beta}$ to $\overrightarrow{\partial\Omega}_{\alpha\beta}$. The $L_{\alpha\beta}$-functions of a nonpassable flow to the boundary is $L_{\alpha\beta}<0$ with varying boundary location. The $L_{\alpha\beta}$-function for flows between two points $\mathbf{x}_{m_1}$ and $\mathbf{x}_{m_2}$ on the sink boundary $\widetilde{\partial\Omega}_{\alpha\beta}$ are sketched in Fig. 2.11 for $\mathbf{p}_{\alpha\beta}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$. The $L_{\alpha\beta}$-function is sketched in Fig. 2.11a, and the corresponding vector fields varying with system parameters on the boundary $\partial\Omega_{\alpha\beta}$ are illustrated in Fig. 2.11b. $\mathbf{F}^{(\alpha)}(t_{m-})$ and $\mathbf{F}^{(\beta)}(t_{m\pm})$ are limits of the vector fields to the boundary $\partial\Omega_{\alpha\beta}$ in domains Ω_α and Ω_β, respectively. The boundary relative to the nonpassable flows with $L_{\alpha\beta}<0$ is a sink boundary. There is a specific value $\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})}$ between $\mathbf{p}_{\alpha\beta}^{(1)}$ and $\mathbf{p}_{\alpha\beta}^{(2)}$. For this specific value, a point $\mathbf{x}_m$ on the sink boundary can be found for the sliding fragmentation bifurcation on the boundary. Two points $\mathbf{x}_{k_1}$ and $\mathbf{x}_{k_2}$ are onset and vanishing points of the passable flow on the boundary $\partial\Omega_{\alpha\beta}$ for $\mathbf{p}_{\alpha\beta}$. The dashed and solid curves represent $L_{\alpha\beta}>0$ and $L_{\alpha\beta}\leq 0$, respectively. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(1)}\to\mathbf{p}_{\alpha\beta}^{(\mathrm{cr})}$, the $L_{\alpha\beta}$-functions of flows for $\mathbf{x}\in(\mathbf{x}_{m_1},\mathbf{x}_{m_2})$ on the boundary is negative (i.e., $L_{\alpha\beta}<0$). Therefore, the boundary $\partial\Omega_{\alpha\beta}$ is nonpassable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}_{\alpha\beta}^{(cr)}\to\mathbf{p}_{\alpha\beta}^{(2)}$,

$L_{\alpha\beta} < 0$ are for $\mathbf{x} \in [\mathbf{x}_{m_1}, \mathbf{x}_{k_1}) \cup (\mathbf{x}_{k_2}, \mathbf{x}_{m_2}]$, and $L_{\alpha\beta} > 0$ are for $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$. From (2.67), the flow for the portion of $\mathbf{x} \in (\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ boundary with $L_{\alpha\beta} > 0$ is semipassable. For $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}^{(1)}_{\alpha\beta} \to \mathbf{p}^{(2)}_{\alpha\beta}$, the point $(\mathbf{x}_m, \mathbf{p}^{(\mathrm{cr})}_{\alpha\beta})$ on the boundary $\partial\Omega_{\alpha\beta}$ is the onset point of the semipassable flow on the boundary. The sliding flow on the boundary will be fragmentized. However, for $\mathbf{p}_{\alpha\beta}$ varying from $\mathbf{p}^{(2)}_{\alpha\beta} \to \mathbf{p}^{(1)}_{\alpha\beta}$, the sliding fragmentation disappears at such a point. At three critical points $(\mathbf{x}_m, \mathbf{x}_{k_1}, \mathbf{x}_{k_2})$, $L_{\alpha\beta} = 0$. The flow at the critical points $(\mathbf{x}_{k_1}, \mathbf{x}_{k_2})$ has the same properties as at the critical point $\mathbf{x}_m$. If the two critical points have the different properties, the sliding flow between the two different critical points is discussed later. From the $L_{\alpha\beta}$-function of flows, the criteria for the sliding fragmentation bifurcation can be given as similar to Theorems 2.22, 2.25, and 2.27. Thus, the corresponding bifurcation conditions are stated herein.

Theorem 2.29. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2$, $\alpha = i, j$) for time t, respectively. The sliding fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ exists if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.111}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \text{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_j) < 0. \tag{2.112}$$

Proof. Applying the L-function of flows in Definition 2.17 to Theorem 2.22, the foregoing theorem can be easily proved. ■

Theorem 2.30. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2$, $\alpha = i, j$) for time t, respectively. The source fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.113}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(j)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_j) < 0. \tag{2.114}$$

Proof. Applying the L_{ij} and L_{jj}-functions of flows in Definition 2.17 to Theorem 2.25, the foregoing theorem can be easily proved. ■

Theorem 2.31. *For a discontinuous dynamical system in (2.1), there is a point* $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overset{\smile}{\partial\Omega}_{ij}$ *(or* $\overset{\frown}{\partial\Omega}_{ij}$*) for time* t_m *between two adjacent domains* Ω_α $(\alpha = i, j)$*. For an arbitrarily small* $\varepsilon > 0$*, there are two time intervals (i.e.,* $[t_{m-\varepsilon}, t_m)$ *and* $(t_m, t_{m+\varepsilon}]$*), and* $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$*. The flows* $\mathbf{x}^{(\alpha)}(t)$ $(\alpha = i, j)$ *are* $C^{r_\alpha}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$*-continuous* $(r_\alpha \geq 2)$ *for time t. The switching bifurcation of the flow at point* $(\mathbf{x}_m, t_m)$ *from* $\overset{\smile}{\partial\Omega}_{ij}$ *to* $\overset{\frown}{\partial\Omega}_{ij}$ *(or* $\overset{\frown}{\partial\Omega}_{ij}$ *to* $\overset{\smile}{\partial\Omega}_{ij}$*) occurs if and only if*

$$L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0, \tag{2.115}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) < 0 \quad \text{for} \quad \alpha = i, j. \tag{2.116}$$

Proof. Applying the L_{ij} and $L_{\alpha\alpha}$-functions of flows in Definition 2.17 to Theorem 2.27, the foregoing theorem can be easily proved. ■

For the nonpassable flow at $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\overset{\frown}{\partial\Omega}_{ij}$), consider a time interval $[t_{m_1}, t_{m_2}]$ for $[\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$ on the boundary, for $t_m \in [t_{m_1}, t_{m_2}]$ and $\mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}]$, $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) < 0$. To determine the switching bifurcation, the local maximum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ can be determined. With (2.78) and (2.79), the local maximum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is determined by

$$D^k L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0 \quad (k = 0, 1, 2, \ldots, 2l - 1), \tag{2.117}$$

$$D^{2l} L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) < 0. \tag{2.118}$$

Definition 2.23. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overset{\smile}{\partial\Omega}_{ij}$ (or $\overset{\frown}{\partial\Omega}_{ij}$) at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous $(r \geq 2l)$ for time t. The local maximum set of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is defined by

$${}_{\max}L_{ij}(t_m) = \left\{ L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) \left| \begin{array}{l} \forall t_m \in [t_{m_1}, t_{m_2}], \exists \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}], \\ \text{so that } D^k L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) = 0 \\ \text{for } k = \{1, 2, \ldots 2l - 1\} \text{ and} \\ D^{2l} L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j) < 0. \end{array} \right. \right\} \tag{2.119}$$

From the local maximum set of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$, the corresponding global maximum can be determined as follows.

Definition 2.24. For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \overset{\smile}{\partial\Omega}_{ij}$ (or $\overset{\frown}{\partial\Omega}_{ij}$) at time t_m between two adjacent domains Ω_α $(\alpha = i, j)$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\mp})$. The global maximum set of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ is defined by

$$_{G\max}L_{ij}(t_m) = \max_{t_m \in [t_{m_1}, t_{m_2}]} \left\{ \begin{array}{l} _{\max}L_{ij}(t_m), L_{ij}(\mathbf{x}_{m_1}, t_{m_1}, \mathbf{p}_i, \mathbf{p}_j), \\ L_{ij}(\mathbf{x}_{m_2}, t_{m_2}, \mathbf{p}_i, \mathbf{p}_j). \end{array} \right\} \tag{2.120}$$

From the foregoing definition, Theorems 2.22, 2.25, and 2.27 can be expressed through the global minimum of $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$. So the following corollaries can be achieved, which give the condition of sliding fragmentation bifurcation.

Corollary 2.4. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m-}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_m)}$- and $C^{r_j}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2l$, $\alpha = i, j$) for time t, respectively. The sliding fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widetilde{\partial\Omega}_{ij}$ occurs if and only if*

$$_{G\max}L_{ij}(t_m) = 0, \tag{2.121}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m-}) \neq 0 \quad \text{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_j) < 0. \tag{2.122}$$

Proof. In Theorem 2.29, replacing $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ by its global maximum value $_{G\max}L_{ij}(t_m)$ gives the above corollary. ■

Corollary 2.5. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widehat{\partial\Omega}_{ij}$ at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m+})$. The flows $\mathbf{x}^{(i)}(t)$ and $\mathbf{x}^{(j)}(t)$ are $C^{r_i}_{[t_{m-\varepsilon}, t_{m+\varepsilon}]}$- and $C^{r_j}_{(t_m, t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2l$, $\alpha = i, j$) for time t, respectively. The source fragmentation bifurcation of the flow $\mathbf{x}^{(i)}(t) \cup \mathbf{x}^{(j)}(t)$ at point $(\mathbf{x}_m, t_m)$ on the boundary $\widehat{\partial\Omega}_{ij}$ occurs if and only if*

$$_{G\max}L_{ij}(t_m) = 0, \tag{2.123}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}} \cdot \mathbf{F}^{(i)}(t_{m+}) \neq 0 \quad \text{and} \quad L_{jj}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_j) < 0. \tag{2.124}$$

Proof. In Theorem 2.30, replacing $L_{ij}(\mathbf{x}_m, t_m, \mathbf{p}_i, \mathbf{p}_j)$ by the global maximum $_{G\max}L_{ij}(t_m)$ gives the above corollary. ■

Corollary 2.6. *For a discontinuous dynamical system in (2.1), there is a point $\mathbf{x}(t_m) = \mathbf{x}_m \in [\mathbf{x}_{m_1}, \mathbf{x}_{m_2}] \subset \widetilde{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$) at time t_m between two adjacent domains Ω_α ($\alpha = i, j$). For an arbitrarily small $\varepsilon > 0$, there are two time intervals (i.e., $[t_{m-\varepsilon}, t_m)$ and $(t_m, t_{m+\varepsilon}]$), and $\mathbf{x}^{(i)}(t_{m\pm}) = \mathbf{x}_m = \mathbf{x}^{(j)}(t_{m\pm})$. The flows $\mathbf{x}^{(\alpha)}(t)$*

($\alpha = i,j$) are $C^{r_\alpha}_{[t_{m-\varepsilon},t_{m+\varepsilon}]}$-continuous ($r_\alpha \geq 2l$) for time t. The switching bifurcation of the flow at point $(\mathbf{x}_m, t_m)$ from $\widetilde{\partial\Omega}_{ij}$ to $\widehat{\partial\Omega}_{ij}$ (or $\widehat{\partial\Omega}_{ij}$ to $\widetilde{\partial\Omega}_{ij}$) occurs if and only if

$$ {}_{G\max}L_{ij}(t_m) = 0, \tag{2.125}$$

$$\mathbf{n}^{\mathrm{T}}_{\partial\Omega_{ij}}.\mathbf{F}^{(\alpha)}(t_{m\pm}) = 0 \quad \text{and} \quad L_{\alpha\alpha}(\mathbf{x}_{m\pm\varepsilon}, t_{m\pm\varepsilon}, \mathbf{p}_\alpha) < 0 \quad \text{for} \quad \alpha = i,j. \tag{2.126}$$

Proof. In Theorem 2.31, replacing $L_{ij}(\mathbf{x}_m, t_m, \boldsymbol{\mu}_i, \boldsymbol{\mu}_j)$ through its global maximum ${}_{G\max}L_{ij}(t_m)$ gives the above corollary. ■

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