Chapter 9
LINEAR INTEGRAL EQUATIONS IN ONE VARIABLE

The general form of an integral equation for the unknown function of one variable $\psi(x)$ is

$$A(x)\psi(x) = \psi_0(x) + \lambda \int_{a(x)}^{b(x)} J[x, y, \psi(y)]dy \tag{9.1a}$$

When $J[x, y, \psi(y)]$ is of the form $K(x, y)\psi(y)$, eq. 9.1a is the linear integral equation

$$A(x)\psi(x) = \psi_0(x) + \lambda \int_{a(x)}^{b(x)} K(x, y)\psi(y)dy \tag{9.1b}$$

where $A(x)$, $\psi_0(x)$, $a(x)$, $b(x)$, and $K(x, y)$ are known functions and λ is a specified constant. $\psi_0(x)$ is referred to as the *inhomogeneous function* and $K(x,y)$ is the *kernel of the equation*.

If both limits a and b are constants, the integral equation is called a *Fredholm integral equation*. If a is constant and $b(x) = x$, the integral equation is the *Volterra equation*.

Except when $K(x, y)$ is one of a few special cases, there are no techniques for solving eqs. 9.1 in a closed form. As such, for many integral equations, numerical and other approximation methods are essential for estimating $\psi(x)$.

9.1 Fredholm Equations of the First Kind with Non-singular Kernel

If $A(x) = 0$ for all $x \in [a, b]$, the *Fredholm equation of the first kind* is of the form

$$\psi_0(x) = \lambda \int_a^b K(x, y)\psi(y)dy \tag{9.2}$$

where $K(x, y)$ is analytic at all x and y in $[a, b]$.

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Solution by Quadrature Sum

One commonly used approach for solving Fredholm equations is to approximate the integral by a quadrature sum

$$\int_a^b K(x,y)\psi(y)dy \simeq \sum_{m=1}^{N} w_m K(x,y_m)\psi(y_m) \tag{9.3}$$

With this, eq. 9.2 becomes

$$\psi_0(x) = \lambda \sum_{m=1}^{N} w_m K(x,y_m)\psi(y_m) \tag{9.4}$$

Setting x to each abscissa point in the set $\{y_k\}$, and using the notation

$$K(y_k, y_m) \equiv K_{km} \tag{9.5}$$

we obtain a set of coupled equations, the matrix form of which is

$$\begin{pmatrix} \psi_0(y_1) \\ \bullet \\ \bullet \\ \psi_0(y_N) \end{pmatrix} = \lambda \begin{pmatrix} w_1K_{11} & \bullet\bullet & w_NK_{1N} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1K_{N1} & \bullet\bullet & w_NK_{NN} \end{pmatrix} \begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_N) \end{pmatrix} \tag{9.6a}$$

It is straightforward to obtain the solution as

$$\begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_N) \end{pmatrix} = \frac{1}{\lambda} \begin{pmatrix} w_1K_{11} & \bullet\bullet & w_NK_{1N} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1K_{N1} & \bullet\bullet & w_NK_{NN} \end{pmatrix}^{-1} \begin{pmatrix} \psi_0(y_1) \\ \bullet \\ \bullet \\ \psi_0(y_N) \end{pmatrix} \tag{9.6b}$$

If one of the abscissa points (e.g., y_n) is such that $K_{nm} = 0$, then all the elements in the nth row of the coefficient matrix are zero. Thus, the determinant of the matrix of eq. 9.6a is

$$\begin{vmatrix} w_1K_{11} & & & \bullet\bullet & & & w_NK_{1N} \\ \bullet & & & & & & \bullet \\ \bullet & & & & & & \bullet \\ w_1K_{(n-1)1} & & & \bullet\bullet & & & w_NK_{(n-1)N} \\ 0 & 0 & 0 & \bullet\bullet & 0 & 0 & 0 \\ w_1K_{(n+1)1} & & & \bullet\bullet & & & w_NK_{(n+1)N} \\ \bullet & & & & & & \bullet \\ \bullet & & & & & & \bullet \\ w_1K_{N1} & & & \bullet\bullet & & & w_NK_{NN} \end{vmatrix} = 0 \tag{9.7}$$

Therefore, the kernel matrix is singular. Thus, one must choose a quadrature rule such that K_{nm} is non-zero for all abscissae of the quadrature rule.

Example 9.1: Solution to a Fredholm equation of the first kind by quadratures

Since the solution to the Fredholm equation of the first kind

$$\int_{-1}^{1} xe^{y(1-x)}\psi(y)dy = e^{x} - e^{-x} \tag{9.8a}$$

is

$$\psi(x) = e^{-x} \tag{9.8b}$$

Using a quadrature rule to approximate the integral

$$\int_{0}^{1} xe^{y(1-x)}\psi(y)dy \simeq \sum_{k=1}^{N} w_k x e^{x_k(1-x)}\psi(x_k) \tag{9.9}$$

we see that for $x = 0$, the kernel $xe^{y(1-x)}$ is zero. Thus, if we use a Gauss–Legendre quadrature rule (the most appropriate for an integral from –1 to 1), we must choose an even order rule, so that $x = 0$ is not one of the abscissa.

To obtain a sense of the accuracy obtained using an even order Gauss–Legendre rule, we approximate the integral as in eq. 9.9 by a six-point quadrature. We obtain

$$\begin{pmatrix} \psi(0.93247) \\ \psi(0.66121) \\ \psi(0.23862) \\ \psi(-0.23862) \\ \psi(-0.66121) \\ \psi(-0.93247) \end{pmatrix} = \begin{pmatrix} -4.00895 \\ 2.49535 \\ -7.24554 \\ -11.67704 \\ 9.36377 \\ -25.87992 \end{pmatrix} \tag{9.10a}$$

The exact values at these points are given by

$$\begin{pmatrix} \psi_{exact}(0.93247) \\ \psi_{exact}(0.66121) \\ \psi_{exact}(0.23862) \\ \psi_{exact}(-0.23862) \\ \psi_{exact}(-0.66121) \\ \psi_{exact}(-0.93247) \end{pmatrix} = \begin{pmatrix} 0.39358 \\ 0.51623 \\ 0.78771 \\ 1.26950 \\ 1.93713 \\ 2.54078 \end{pmatrix} \tag{9.10b}$$

Clearly, the results presented in eq. 9.10a do not represent e^{-x}. They are highly inaccurate and oscillate wildly from point to point. In addition, the results are not improved by using a larger quadrature set to approximate to the integral. □

This example illustrates the well known fact that the Fredholm integral of the first kind is an ill-conditioned problem (see, for example, Baker, C., et. al., 1964).

There are several methods presented in the literature for smoothing these results. For example, the *Galerkin* approach involves choosing the value of x at which the solution is to be obtained and approximating $\psi(y)$ by a sum over known *basis functions* $\phi_k(y)$, which are chosen by the user. Then, with

$$\psi(y) \simeq \sum_{k=1}^{N} a_k \phi_k(y) \tag{9.11a}$$

the *error parameter*

$$\varepsilon\left(\vec{a}\right) \equiv \left|\psi_0(x) - \sum_{k=1}^{N} a_k \int_a^b K(x,y)\phi_k(y)dy\right|^2 \tag{9.11b}$$

is minimized (for example, by the method of least squares, minimizing with respect to the coefficients a_k). Excellent treatments of such approaches are given by Twomey, S., 1963, Baker, C., et. al., 1964, Hanson, R., 1971, and Hansen, P., 1992.

Series approximation

If $K(x, y)$ and $\psi_0(x)$ are analytic at all points in $[a, b]$, then $\psi(x)$ is also analytic everywhere in $[a, b]$. If $x_0 \, \varepsilon \, [a, b]$, we can $\psi(x)$ as

$$\psi(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \tag{9.12a}$$

that is valid at all x in $[a, b]$. A truncated series

$$\psi(x) = \sum_{k=0}^{N} c_k (x - x_0)^k \tag{9.12b}$$

yields an approximate solution to eq. 9.2.

Substituting eq. 9.12b into eq. 9.2, we obtain

$$\psi_0(x) \simeq \lambda \sum_{k=0}^{N} c_k \int_a^b K(x,y)(y - x_0)^k dy \equiv \lambda \sum_{k=0}^{N} c_k I_k(x) \tag{9.13}$$

Because the integrand is a known function, the integral $I_k(x)$ can be evaluated in a closed form or can be accurately approximated by some quadrature rule. Therefore, $I_k(x)$ is known at every x.

To determine the coefficients c_k,. $\psi_0(x)$ and each $I_k(x)$ are expanded in their Taylor series and we equate the coefficients of corresponding powers of $(x–x_0)$ up to $(x–x_0)^N$.

Using an approach that is suggested by the Galerkin method, a more convenient way to develop the series method is to write $K(x, y)$, $\psi_0(x)$, and $\psi(x)$ as sums over orthogonal polynomials, where the polynomials used depend on the limits of the integral.

Referring to ch. 4, if x and y vary over $[0, \infty]$ and if $\psi_0(x)$ and $K(x, y)$ can be written as

$$K(x,y) \equiv e^{-y}L(x,y) \tag{9.14a}$$

and

$$\psi_0(x) = e^{-x}\Omega_0(x) \tag{9.14b}$$

such that

$$\lim_{y\to\infty} e^{-y}L(x,y) = 0 \tag{9.15a}$$

and

$$\lim_{x\to\infty} e^{-x}\Omega_0(x) = 0 \tag{9.15b}$$

then one might expand these functions as a series involving Laguerre polynomials. Likewise, if $-\infty \leq x$ and $y \leq \infty$, and if $K(x,y)$ and $\psi_0(x)$ can be written as

$$K(x,y) \equiv e^{-y^2}L(x,y) \tag{9.16a}$$

and

$$\psi_0(x) = e^{-x^2}\Omega_0(x) \tag{9.16b}$$

with

$$\lim_{y\to\infty} e^{-y^2}L(x,y) = 0 \tag{9.17a}$$

and

$$\lim_{x\to\infty} e^{-x^2}\Omega(x) = 0 \tag{9.17b}$$

one might write the series for these functions in terms of the Hermite polynomials. If x and y vary over finite limits a and b, one can transform this interval to [–1, 1] and write the series for $\psi_0(x)$ and $K(x, y)$ in terms of Legendre polynomials.

It was shown in ch. 4, eqs. 4.9 through 4.16, which for any limits of integration (finite or infinite) any integral can be converted to one integrated over [–1, 1]. It was also shown that if the exponential factors for $x \varepsilon [0, \infty]$ or $x \varepsilon [-\infty,\infty]$ are not explicitly part of the integrand, one usually achieves accurate results by transforming integrals over an infinite domain to [–1, 1] and using a Gauss–Legendre quadrature. Thus, for a kernel and/or inhomogeneous term that does not contain the exponential explicitly, one should consider transforming the integral to [–1, 1] and expanding the functions over Legendre polynomials. This is an example of what Baker calls *expansion methods* (Baker, C. T. H., 1977, pp. 205–214).

Legendre polynomials form an infinite set of mutually orthogonal polynomial functions over the interval [–1, 1] such that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{(2n+1)}\delta_{nm} \tag{\textbf{\textit{4.100b}}}$$

As such, we transform the domains of ψ, ψ_0, and K to $x, y \varepsilon [-1, 1]$ then expand the unknown function, the inhomogeneous function and the kernel as

$$\psi(x) = \sum_{k=0}^{\infty} c_k P_k(x) \tag{9.18a}$$

$$\psi_0(x) = \sum_{k=0}^{\infty} \phi_k P_k(x) \tag{9.18b}$$

and

$$K(x, y) = \sum_{\substack{k=0 \\ m=0}}^{\infty} \mu_{km} P_k(x) P_m(y) \tag{9.18c}$$

from which we determine the coefficients

$$\phi_k = \frac{(2k+1)}{2}\int_{-1}^{1} \psi_0(x)P_k(x)dx \tag{9.19a}$$

and

$$\mu_{km} = \frac{(2k+1)(2m+1)}{4}\int_{-1}^{1}\int_{-1}^{1} K(x, y)P_k(x)P_m(y)dxdy \tag{9.19b}$$

Then, the Fredholm equation of the first kind becomes

$$\sum_{k=0}^{\infty} \phi_k P_k(x) = \sum_{\substack{k=0 \\ m=0 \\ n=0}}^{\infty} \mu_{km} c_n P_k(x) \int_{-1}^{1} P_m(y) P_n(y) dy = \sum_{\substack{k=0 \\ m=0 \\ n=0}}^{\infty} \mu_{km} c_n P_k(x) \frac{2}{(2m+1)} \delta_{mn}$$

$$= \sum_{\substack{k=0 \\ m=0}}^{\infty} \mu_{km} c_m P_k(x) \frac{2}{(2m+1)} \tag{9.20}$$

Equating the coefficients of the various Legendre polynomials, we obtain

$$\phi_k = \sum_{m=0}^{\infty} \frac{2}{(2m+1)} \mu_{km} c_m \tag{9.21a}$$

Approximating the infinite series by a finite sum, we obtain a finite set of linear equations

$$\phi_k = \sum_{m=0}^{N} \frac{2}{(2m+1)} \mu_{km} c_m \tag{9.21b}$$

In this way, the only integrals we may have to approximate by quadratures are those of eqs. 9.19, which are integrals of known functions. It has been shown in ch. 4 that when the integrand is analytic over the range of integration, quadrature rules usually yield accurate results for such integrals. And approximating an infinite series by a finite sum is equivalent to approximating a function by a truncated Taylor sum. Such an approximation of an analytic function has been shown to be quite accurate. Thus, as will be seen, this method yields stable and accurate results.

We define the $N \times N$ matrix A with elements

$$a_{km} = \frac{2}{(2m+1)} \mu_{km} \tag{9.22}$$

and the column vectors

$$F \equiv \begin{pmatrix} \phi_1 \\ \bullet \\ \bullet \\ \phi_N \end{pmatrix} \tag{9.23a}$$

and

$$C \equiv \begin{pmatrix} c_1 \\ \bullet \\ \bullet \\ c_N \end{pmatrix} \tag{9.23b}$$

to write eq. 9.21b as a matrix equation which is solved straightforwardly by matrix inversion.

Once one has obtained values of the coefficients c_k, one can determine the approximation to $\psi(x)$ at any x in the domain of ψ.

Example 9.2: Solution to a Fredholm equation of the first kind by series expansion

We again consider

$$\int_{-1}^{1} x e^{y(1-x)} \psi(y) dy = e^{x} - e^{-x} \tag{\textbf{\textit{9.8a}}}$$

the solution to which is

$$\psi(x) = e^{-x} \tag{\textbf{\textit{9.8b}}}$$

from which

$$\begin{pmatrix} \psi_{exact}(1.0) \\ \psi_{exact}(0.5) \\ \psi_{exact}(0.0) \\ \psi_{exact}(-0.5) \\ \psi_{exact}(-1.0) \end{pmatrix} = \begin{pmatrix} 0.36788 \\ 0.60653 \\ 1.00000 \\ 1.64872 \\ 2.71828 \end{pmatrix} \tag{9.24}$$

Taking five terms in the sums of eqs. 9.18, the approximations to $\psi(x)$, $\psi_0(x)$, and $K(x, y)$ are

$$\psi(x) \simeq \sum_{k=0}^{4} c_k P_k(x) \tag{9.25a}$$

$$\psi_0(x) \simeq \sum_{k=0}^{4} \phi_k P_k(x) \tag{9.25b}$$

and

$$K(x, y) = \sum_{\substack{k=0 \\ m=0}}^{4} \mu_{km} P_k(x) P_m(y) \tag{9.25c}$$

We note that $\psi_0(x)$ given in eq. 9.8a is an odd function. Referring to eq. 9.19a, since the integral is evaluated over symmetric limits, ϕ_0, ϕ_2, and ϕ_4 are zero since their integrands are odd functions. The non-zero coefficients are

$$\phi_1 = \frac{1}{2}\int_{-1}^{1}(e^x - e^{-x})P_1(x)dx = \frac{1}{2}\int_{-1}^{1}x(e^x - e^{-x})dx = 6e^{-1} \tag{9.26a}$$

and

$$\begin{aligned}\phi_3 &= \frac{7}{2}\int_{-1}^{1}(e^x - e^{-x})P_3(x)dx = \frac{7}{4}\int_{-1}^{1}(e^x - e^{-x})\left(5x^3 - 3x\right)dx \\ &= 7\left(37e^{-1} - 5e^1\right)\end{aligned} \tag{9.26b}$$

Rather than write down all 25 coefficients μ_{mk}, we present two as examples:

$$\begin{aligned}\mu_{21} &= \frac{15}{4}\int_{-1}^{1}xP_2(x)\int_{-1}^{1}e^{y(1-x)}P_1(y)dydx\frac{15}{8}\int_{-1}^{1}x\left(3x^2 - 1\right)\int_{-1}^{1}e^{y(1-x)}ydydx \\ &= \frac{15}{4}\int_{-1}^{1}x(3x^2 - 1)\frac{\left[(2-x)e^{-(1-x)} - xe^{(1-x)}\right]}{(1-x)^2}dx\end{aligned} \tag{9.27a}$$

and

$$\begin{aligned}\mu_{43} = &\frac{63}{4}\int_{-1}^{1}xP_4(x)\int_{-1}^{1}e^{y(1-x)}P_3(y)dydx = \frac{63}{64}\int_{-1}^{1}x(35x^4 - 30x^2 + 3)\times \\ &\left[e^{(1-x)}\left(\frac{1}{(1-x)} - \frac{6}{(1-x)^2} + \frac{15}{(1-x)^3} - \frac{15}{(1-x)^4}\right) + \right. \\ &\left. e^{-(1-z)}\left(\frac{1}{(1-x)} - \frac{6}{(1-x)^2} + \frac{15}{(1-x)^3} - \frac{15}{(1-x)^4}\right)\right]dydx\end{aligned} \tag{9.27b}$$

Such integrals cannot be evaluated in closed form.

It is straightforward to show that these integrands are not singular at $x = 1$. Therefore, they are well approximated by quadrature sums. In this example, we approximate them using a 20-point Gauss–Legendre quadrature rule.

For this five polynomial series approximation, we obtain

$$\Psi = \begin{pmatrix}\psi(1.0)\\ \psi(0.5)\\ \psi(0.0)\\ \psi(-0.5)\\ \psi(-1.0)\end{pmatrix} = \begin{pmatrix}0.36251\\ 0.60773\\ 0.99665\\ 1.65340\\ 2.70208\end{pmatrix} \tag{9.28}$$

The accuracy of these results is indicated by their percent differences from the exact values given in eq. 9.24. These differences are

$$\begin{pmatrix} \Delta(1.0) \\ \Delta(0.5) \\ \Delta(0.0) \\ \Delta(-0.5) \\ \Delta(-1.0) \end{pmatrix} = \begin{pmatrix} 1.5\% \\ 0.2\% \\ 0.3\% \\ 0.3\% \\ 0.6\% \end{pmatrix} \tag{9.29}$$

which indicates that this five term series approximation yields a fairly accurate approximation to the solution to a Fredholm equation of the first kind. □

9.2 Fredholm Equations of the Second Kind with Non-singular Kernel

Referring to eq. 9.1, when $A(x)$ is non-zero for all $x \in [a, b]$, we can divide the Fredholm equation by $A(x)$, rename the inhomogeneous function and kernel as

$$\frac{\psi_0(x)}{A(x)} \to \psi_0(x) \tag{9.30}$$

and

$$\frac{K(x,y)}{A(x)} \to K(x,y) \tag{9.31}$$

to obtain the *Fredholm equation of the second kind*

$$\psi(x) = \psi_0(x) + \lambda \int_a^b K(x,y)\psi(y)dy \tag{9.32}$$

Solution by quadrature sum

We approximate the integral of eq. 9.32 by a quadrature sum to obtain

$$\psi(x) \simeq \psi_0(x) + \lambda \sum_{m=1}^{N} w_m K(x, y_m)\psi(y_m) \tag{9.33}$$

With at least one value of $\psi_0(y_k)$ non-zero, we set x to each of the quadrature points in the set $\{y_m\}$, to obtain

$$\begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_N) \end{pmatrix} = \begin{pmatrix} \psi_0(y_1) \\ \bullet \\ \bullet \\ \psi_0(y_N) \end{pmatrix} + \lambda \begin{pmatrix} w_1 K_{11} & \bullet\bullet & w_N K_{1N} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1 K_{N1} & \bullet\bullet & w_N K_{NN} \end{pmatrix} \begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_N) \end{pmatrix} \tag{9.34}$$

which is solved straightforwardly by matrix inversion.

An approximation to $\psi(x)$ at any x can then be obtained by substituting these values of $\psi(y_k)$ into the sum in eq. 9.33.

Example 9.3: Solution to a Fredholm equation of the second kind by quadratures

The solution to

$$\psi(x) = e^x - \int_{-1}^{1} x e^{y(1-x)} \psi(y) dy \tag{9.35}$$

is

$$\psi(x) = e^{-x} \tag{9.8b}$$

Using a four-point Gauss–Legendre quadrature rule, the integral equation is approximated by

$$\psi(x) \simeq e^x - \sum_{m=1}^{4} w_m x e^{y_m(1-x)} \psi(y_m) \tag{9.36a}$$

Setting x to each y_m, this becomes

$$\psi(y_k) \simeq e^{y_k} - \sum_{m=1}^{4} w_m y_k e^{y_m(1-y_k)} \psi(y_m) \tag{9.36b}$$

We define the 4×4 kernel matrix with elements

$$a_{km} = w_m y_k e^{y_m(1-y_k)} \tag{9.37}$$

Then, eq. 9.36b can be expressed as

$$\begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_4) \end{pmatrix} = \begin{pmatrix} e^{y_1} \\ \bullet \\ \bullet \\ e^{y_4} \end{pmatrix} - \begin{pmatrix} w_1 y_1 e^{y_1(1-y_1)} & \bullet\bullet & w_4 y_1 e^{y_4(1-y_1)} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1 y_4 e^{y_1(1-y_4)} & \bullet\bullet & w_4 y_4 e^{y_4(1-y_4)} \end{pmatrix} \begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_4) \end{pmatrix} \tag{9.38}$$

the solution to which is given by

$$\begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_4) \end{pmatrix} = \left[1 + \begin{pmatrix} w_1 y_1 e^{y_1(1-y_1)} & \bullet\bullet & w_4 y_1 e^{y_4(1-y_1)} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1 y_4 e^{y_1(1-y_4)} & \bullet\bullet & w_4 y_4 e^{y_4(1-y_4)} \end{pmatrix} \right]^{-1} \begin{pmatrix} e^{y_1} \\ \bullet \\ \bullet \\ e^{y_4} \end{pmatrix} \tag{9.39}$$

From this, we obtain

$$\Psi = \begin{pmatrix} \psi(0.86114) \\ \psi(0.33998) \\ \psi(-0.33998) \\ \psi(-0.86114) \end{pmatrix} = \begin{pmatrix} 0.42268 \\ 0.71178 \\ 1.40492 \\ 2.36585 \end{pmatrix} \tag{9.40a}$$

Comparing this to

$$\Psi_{exact} = \begin{pmatrix} \psi_{exact}(0.86114) \\ \psi_{exact}(0.33998) \\ \psi_{exact}(-0.33998) \\ \psi_{exact}(-0.86114) \end{pmatrix} = \begin{pmatrix} 0.42268 \\ 0.71178 \\ 1.40492 \\ 2.36586 \end{pmatrix} \tag{9.40b}$$

we see that the method of quadratures yields an extremely accurate result. The largest of the 4% differences from the exact values is 1.3×10^{-5}.

To obtain values of $\psi(x)$ at any value of x, we substitute the values given in eq. 9.40a into the sum of eq. 9.36a. From this we find

$$\begin{pmatrix} \psi(1.0) \\ \psi(0.5) \\ \psi(0.0) \\ \psi(-0.5) \\ \psi(-1.0) \end{pmatrix} = \begin{pmatrix} 0.36788 \\ 0.60653 \\ 0.00000 \\ 1.64872 \\ 2.71828 \end{pmatrix} \tag{9.41}$$

which is identical to the exact values of $\psi(x)$ at $x = \{1.0, 0.5, 0.0, -0.5, -1.0\}$ to five decimals. The largest percent difference from the exact values is 7.4×10^{-5}. □

Approximating ψ(y) by spline interpolation

Another approach is to approximate $\psi(y)$ by a constant in the Fredholm integral. To do so, we divide $[a, b]$ into small segments, writing the integral as

$$\int_a^b K(x,y)\psi(y)dy = \sum_{m=0}^{N-1} \int_{x_m}^{x_{m+1}} K(x,y)\psi(y)dy \tag{9.42}$$

where

$$x_0 = a \tag{9.43a}$$

and

$$x_N = b \tag{9.43b}$$

We note that the points x_m are chosen by the user and are not necessarily abscissae of a quadrature rule.

Although it is not necessary to do so, for the sake of simplicity, we take these small intervals to be evenly spaced by defining

$$x_{m+1} - x_m \equiv \Delta x \tag{9.44}$$

with Δx independent of m. By taking Δx to be small enough, we can approximate $\psi(y)$ over each segment $x_m \le y \le x_{m+1}$ by the constant

$$\psi(y) \simeq \alpha\psi(x_m) + \beta\psi(x_{m+1}) \tag{9.45}$$

where α and β are chosen by the user. This is a cardinal spline interpolation over $[x_m, x_{m+1}]$. Unless there is some reason to choose otherwise, a reasonable choice would be to take

$$\alpha = \beta = \frac{1}{2} \tag{9.46}$$

With these values for α and β, the Fredholm equation of the second kind becomes

$$\begin{aligned}\psi(x) &\simeq \psi_0(x) + \frac{\lambda}{2}\sum_{m=0}^{N-1} [\psi(x_m) + \psi(x_{m+1})] \int_{x_m}^{x_{m+1}} K(x,y)dy \\ &\equiv \psi_0(x) + \frac{\lambda}{2}\sum_{m=0}^{N-1} [\psi(x_m) + \psi(x_{m+1})] I_m(x)\end{aligned} \tag{9.47}$$

where

$$I_m(x) \equiv \int_{x_m}^{x_{m+1}} K(x,y)dy \tag{9.48}$$

Then, with $0 \le k \le N–1$, we set $x = \{x_0, \ldots, x_{N–1}\}$ to obtain

$$\begin{aligned}\psi(x_k) &\simeq \psi_0(x_k) + \frac{\lambda}{2}\sum_{m=0}^{N-1}[\psi(x_m) + \psi(x_{m+1})]I_m(x_k)\\ &= \psi_0(x_k) + \frac{\lambda}{2}\{I_0(x_k)\psi(x_0) + [I_0(x_k) + I_1(x_k)]\psi(x_1) +\\ &\cdots + [I_{N-2}(x_k) + I_{N-1}(x_k)]\psi(x_{N-1}) + I_{N-1}(x_k)\psi(x_N)\}\end{aligned} \tag{9.49}$$

which yields a set of simultaneous linear equations for the set $\{\psi(x_k)\}$.

Example 9.4: Solution to a Fredholm equation of the second kind by a cardinal spline interpolation

The integral equation

$$\psi(x) = 1 + \int_0^1 xe^{-y(1-x)}\psi(y)dy \tag{9.50a}$$

has solution

$$\psi(x) = e^x \tag{9.50b}$$

For simplicity of illustration, we obtain the solution at $x = \{0, 1/3, 2/3, 1\}$ taking $\Delta x = 1/3$. Then, with $\lambda = \psi_0 = 1$ and with $x_0 = 0$, $x_N = x_3 = 1$, and

$$I_0(x) = \int_0^{1/3} xe^{-y(1-x)}dy = \begin{cases} \dfrac{x\left[1 - e^{-(1-x)/3}\right]}{(1-x)} & x \neq 1 \\ \dfrac{x}{3} & x = 1 \end{cases} \tag{9.51a}$$

$$I_1(x) = \int_{1/3}^{2/3} xe^{-y(1-x)}dy = \begin{cases} \dfrac{x\left[e^{-(1-x)/3} - e^{-2(1-x)/3}\right]}{(1-x)} & x \neq 1 \\ \dfrac{x}{3} & x = 1 \end{cases} \tag{9.51b}$$

and

$$I_2(x) = \int_{2/3}^{1} xe^{-y(1-x)}dy = \begin{cases} \dfrac{x\left[e^{-2(1-x)/3} - e^{-(1-x)}\right]}{(1-x)} & x \neq 1 \\ \dfrac{x}{3} & x = 1 \end{cases} \tag{9.51c}$$

eq. 9.49, for this example, is

$$\left[1-\frac{1}{2}I_0(0)\right]\psi(0)-\frac{1}{2}[I_0(0)+I_1(0)]\psi(\frac{1}{3})-\frac{1}{2}[I_1(0)+I_2(0)]\psi(\frac{2}{3})-\frac{1}{2}I_2(0)\psi(1)=1 \tag{9.52a}$$

$$-\frac{1}{2}I_0(\frac{1}{3})\psi(0)+\left[1-\frac{1}{2}I_0(\frac{1}{3})-\frac{1}{2}I_1(\frac{1}{3})\right]\psi(\frac{1}{3})-\frac{1}{2}\left[I_1(\frac{1}{3})+I_2(\frac{1}{3})\right]\psi(\frac{2}{3})-\frac{1}{2}I_2(\frac{1}{3})\psi(1)=1 \tag{9.52b}$$

$$-\frac{1}{2}I_0(\frac{2}{3})\psi(0)+\frac{1}{2}\left[I_0(\frac{2}{3})+I_1(\frac{2}{3})\right]\psi(\frac{1}{3})+\left[1-\frac{1}{2}I_1(\frac{2}{3})-\frac{1}{2}I_2(\frac{2}{3})\right]\psi(\frac{2}{3})-\frac{1}{2}I_2(\frac{2}{3})\psi(1)=1 \tag{9.52c}$$

$$-\frac{1}{2}I_0(1)\psi(0)-\frac{1}{2}[I_0(1)+I_1(1)]\psi(\frac{1}{3})-\frac{1}{2}[I_1(1)+I_2(1)]\psi(\frac{2}{3})+\left[1-\frac{1}{2}I_2(1)\right]\psi(1)=1 \tag{9.52d}$$

We obtain

$$\begin{pmatrix}\psi(0)\\ \psi(\frac{1}{3})\\ \psi(\frac{2}{3})\\ \psi(1)\end{pmatrix}=\begin{pmatrix}1.00000\\ 1.40497\\ 1.96740\\ 2.74895\end{pmatrix} \tag{9.53a}$$

Comparing these results to

$$\begin{pmatrix}\psi_{exact}(0)\\ \psi_{exact}(\frac{1}{3})\\ \psi_{exact}(\frac{2}{3})\\ \psi_{exact}(1)\end{pmatrix}=\begin{pmatrix}1.00000\\ 1.39561\\ 1.94773\\ 2.71828\end{pmatrix} \tag{9.53b}$$

we see that the method yields reasonably accurate results. □

We note that this approach is not applicable to Fredholm equations of the first kind. Writing the Fredholm integral as

$$\int_a^b K(x,y)\psi(y)dy=\sum_{k=0}^{N-1}\int_{x_k}^{x_{k+1}}K(x,y)\psi(y)dy \tag{\textbf{\textit{9.42}}}$$

and approximating $\psi(y)$ as

$$\psi(y)\simeq\frac{1}{2}[\psi(x_k)+\psi(x_{k+1})]\quad x_k\le y\le x_{k+1} \tag{9.54}$$

the Fredholm integral of the first kind becomes

$$\psi_0(x) \simeq \frac{\lambda}{2}\sum_{m=0}^{N-1}[\psi(x_k) + \psi(x_{k+1})]I_k(x) \tag{9.55a}$$

Setting x to each of the x_k, we obtain the set of equations

$$\psi_0(x_m) \simeq \frac{\lambda}{2}\sum_{m=0}^{N-1}[\psi(x_k) + \psi(x_{k+1})]I_k(x_m) \tag{9.55b}$$

Expressing these equations in matrix form, we obtain the solution

$$\Psi = M^{-1}\Psi_0 \tag{9.56}$$

where the matrix M is given by

$$M = \frac{\lambda}{2}\begin{pmatrix} I_0(x_1) & [I_0(x_1)+I_1(x_1)] & \bullet\bullet & [I_{N-2}(x_1)+I_{N-1}(x_1)] & I_{N-1}(x_1) \\ \bullet & & & & \\ \bullet & & & & \\ I_0(x_k) & [I_0(x_k)+I_1(x_k)] & \bullet\bullet & [I_{N-2}(x_k)+I_{N-1}(x_k)] & I_{N-1}(x_k) \\ \bullet & & & & \\ \bullet & & & & \\ I_0(x_N) & [I_0(x_N)+I_1(x_N)] & \bullet\bullet & [I_{N-2}(x_N)+I_{N-1}(x_N)] & I_{N-1}(x_N) \end{pmatrix} \tag{9.57}$$

Using the Gauss–Jordan elimination method for finding M^{-1} (see ch. 5), we can cast M into a form such that two columns are identical. Thus, M is singular. We demonstrate this by example.

Example 9.5: Matrix for a Fredholm equation of the first kind is singular

The 4×4 matrix M is given by

$$M = \frac{\lambda}{2}\begin{pmatrix} I_0(x_1) & [I_0(x_1)+I_1(x_1)] & [I_1(x_1)+I_2(x_1)] & I_2(x_1) \\ I_0(x_2) & [I_0(x_2)+I_1(x_2)] & [I_1(x_2)+I_2(x_2)] & I_2(x_2) \\ I_0(x_3) & [I_0(x_3)+I_1(x_3)] & [I_1(x_3)+I_2(x_3)] & I_2(x_3) \\ I_0(x_4) & [I_0(x_4)+I_1(x_4)] & [I_1(x_4)+I_2(x_4)] & I_2(x_4) \end{pmatrix} \tag{9.58}$$

Performing the Gauss–Jordan operations on the columns of M,

$$col_2 \rightarrow col_2 - col_1 \tag{9.59a}$$

and

$$col_3 \rightarrow col_3 - col_4 \tag{9.59b}$$

we obtain

$$M \to \frac{\lambda}{2}\begin{pmatrix} I_0(x_1) & I_1(x_1) & I_1(x_1) & I_2(x_1) \\ I_0(x_2) & I_1(x_2) & I_1(x_2) & I_2(x_2) \\ I_0(x_3) & I_1(x_3) & I_1(x_3) & I_2(x_3) \\ I_0(x_4) & I_1(x_4) & I_1(x_4) & I_2(x_4) \end{pmatrix} \tag{9.60}$$

Since the second and third columns of this matrix are identical, M is singular. □

Interpolation of the kernel

Another approach to solving the Fredholm equation of the second kind is to interpolate the kernel over the interval of integration.

We approximate

$$K(x,y) \simeq \sum_{m=1}^{N} K(x,y_m)v_m(y) \tag{9.61}$$

with

$$v_k(x) = \frac{[q(x)-q(x_1)]\ldots[q(x)-q(x_{k-1})][q(x)-q(x_{k+1})]\ldots[q(x)-q(x_N)]}{[q(x_k)-q(x_1)]\ldots[q(x_k)-q(x_{k-1})][q(x_k)-q(x_{k+1})]\ldots[q(x_k)-q(x_N)]} \tag{1.18}$$

where $q(x)$ is some function appropriate to the kernel being represented. Then, the Fredholm equation of the second kind becomes

$$\psi(x) \simeq \psi_0(x) + \lambda\sum_{m=1}^{N} K(x,y_m)\int_a^b v_m(y)\psi(y)dy \tag{9.62}$$

We define

$$\beta_m \equiv \int_a^b v_m(y)\psi(y)dy \tag{9.63}$$

so that eq. 9.62 is written as

$$\psi(x) = \psi_0(x) + \lambda\sum_{m=1}^{N} K(x,y_m)\beta_m \tag{9.64}$$

Substituting this into eq. 9.63, we have

$$\beta_k = \int_a^b v_k(y)\left[\psi_0(y) + \lambda \sum_{m=1}^{N} K(y, y_m)\beta_m\right] dy \tag{9.65}$$

With

$$\alpha_k \equiv \int_a^b v_k(y)\psi_0(y)dy \tag{9.66a}$$

and

$$\Gamma_{km} \equiv \int_a^b v_k(y)K(y, y_m)dy \tag{9.66b}$$

eq. 9.65 can be written as the set of linear equations

$$\beta_k = \alpha_k + \lambda \sum_{m=1}^{N} \Gamma_{km}\beta_m \tag{9.67}$$

The solution for the set $\{\beta_k\}$ is found by standard methods. Then, $\psi(x)$ at any x is given by eq. 9.64.

Example 9.6: Solution to a Fredholm equation of the second kind by interpolation of the kernel

We again consider

$$\psi(x) = e^x - \int_{-1}^{1} xe^{y(1-x)}\psi(y)dy \tag{9.35}$$

which has solution

$$\psi(x) = e^{-x} \tag{9.8}$$

Since the kernel is an exponential function with a non-negative exponent for x ε [–1, 1], we choose

$$q(y) = e^y \tag{9.68}$$

in the interpolating function $v_k(y)$ in eq. 1.18. We then interpolate the kernel over the nine points {–1, –0.75, . . ., 0.75, 1} to obtain

$$\alpha_k = \int_{-1}^{1} e^y v_k(y)dy = \int_{-1}^{1} e^y \frac{\ldots[e^y - e^{x_{k-1}}][e^y - e^{x_{k+1}}]\ldots}{\ldots[e^{x_k} - e^{x_{k-1}}][e^{x_k} - e^{x_{k+1}}]\ldots} dy \tag{9.69a}$$

and

$$\Gamma_{km} = \int_{-1}^{1} y e^{x_m(1-y)} v_k(y) dy = \int_{-1}^{1} y e^{x_m(1-y)} \frac{\ldots[e^y - e^{x_{k-1}}][e^y - e^{x_{k+1}}]\ldots}{\ldots[e^{x_k} - e^{x_{k-1}}][e^{x_k} - e^{x_{k+1}}]\ldots} dy \tag{9.69b}$$

Although these integrals for α_k and Γ_{km} can be evaluated in closed form, to save ourselves effort, we evaluate them numerically using a 20-point Gauss–Legendre rule.

Solving for the set $\{\beta_k\}$ and substituting these values into eq. 9.64, we obtain

$$\begin{pmatrix} \psi(-1.00) \\ \psi(-0.75) \\ \psi(-0.25) \\ \psi(0.25) \\ \psi(0.75) \\ \psi(1.00) \end{pmatrix} = \begin{pmatrix} 2.71800 \\ 2.11678 \\ 1.28396 \\ 0.77881 \\ 0.47217 \\ 0.36809 \end{pmatrix} \tag{9.70a}$$

which is an accurate approximation to

$$\begin{pmatrix} \psi_{exact}(-1.00) \\ \psi_{exact}(-0.75) \\ \psi_{exact}(-0.25) \\ \psi_{exact}(0.25) \\ \psi_{exact}(0.75) \\ \psi_{exact}(1.00) \end{pmatrix} = \begin{pmatrix} 2.71828 \\ 2.11700 \\ 1.28402 \\ 0.77880 \\ 0.47237 \\ 0.36788 \end{pmatrix} \tag{9.70b}\square$$

Expansion in orthogonal polynomials

As discussed for Fredholm equations of the first kind, we can solve equations of the second kind by expanding $\psi(x)$, $\psi_0(x)$, and $K(x,y)$ in series involving orthogonal polynomials. If, for example, we transform the domain of these functions to [–1, 1], then expand these functions in terms of Legendre polynomials, we have

$$\psi(x) = \sum_{k=0}^{\infty} c_k P_k(x) \tag{\textbf{\textit{9.18a}}}$$

$$\psi_0(x) = \sum_{k=0}^{\infty} \phi_k P_k(x) \tag{9.18b}$$

and

$$K(x,y) = \sum_{\substack{k=0\\m=0}}^{\infty} \mu_{km} P_k(x) P_m(y) \tag{9.18c}$$

Using the orthogonality condition for Legendre polynomials eq. 4.100b, the Fredholm equation of the second kind can be expressed as

$$\sum_{k=0}^{\infty} c_k P_k(x) = \sum_{k=0}^{\infty} \phi_k P_k(x) + \sum_{\substack{k=0\\m=0}}^{\infty} \mu_{km} c_m P_k(x) \frac{2}{(2m+1)} \tag{9.71}$$

Thus, the set of linear equations for the coefficients set $\{c_k\}$ are

$$c_k = \phi_k + \sum_{m=0}^{\infty} \frac{2}{(2m+1)} \mu_{km} c_m \tag{9.72a}$$

which we approximate by the finite sum

$$c_k = \phi_k + \sum_{m=0}^{N} \frac{2}{(2m+1)} \mu_{km} c_m \tag{9.72b}$$

Solving for the set $\{c_k\}$, we obtain the approximate solution

$$\psi(x) \simeq \sum_{k=0}^{N} c_k P_k(x) \tag{9.73}$$

Example 9.7: Solution to a Fredholm equation of the second kind by series expansion

We again consider

$$\psi(x) = e^x - \int_{-1}^{1} x e^{y(1-x)} \psi(y) dy \tag{9.35}$$

which has solution

$$\psi(x) = e^{-x} \tag{\textbf{\textit{9.8b}}}$$

With

$$\psi(x) \simeq \sum_{k=0}^{4} c_k P_k(x) \tag{9.74a}$$

$$e^x \simeq \sum_{k=0}^{4} \phi_k P_k(x) \tag{9.74b}$$

and

$$xe^{y(1-x)} \simeq \sum_{\substack{k=0 \\ m=0}}^{4} \mu_{km} P_k(x) P_m(y) \tag{9.74c}$$

we solve the matrix equation

$$\begin{pmatrix} c_0 \\ \bullet \\ \bullet \\ c_4 \end{pmatrix} = \begin{pmatrix} e^{x_0} \\ \bullet \\ \bullet \\ e^{x_4} \end{pmatrix} - \begin{pmatrix} 2\mu_{00} & \bullet\bullet & \frac{2}{9}\mu_{04} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ 2\mu_{40} & \bullet\bullet & \frac{2}{9}\mu_{44} \end{pmatrix} \begin{pmatrix} c_0 \\ \bullet \\ \bullet \\ c_4 \end{pmatrix} \tag{9.75}$$

for the coefficients c_k. Substituting these into eq. 9.74a, we can approximate $\psi(x)$ at any x ε [–1,1]. We find

$$\begin{pmatrix} \psi(1.00) \\ \psi(0.50) \\ \psi(0.00) \\ \psi(-0.50) \\ \psi(-1.00) \end{pmatrix} = \begin{pmatrix} 0.34486 \\ 0.61254 \\ 0.99804 \\ 1.64621 \\ 2.70190 \end{pmatrix} \tag{9.76a}$$

which compares reasonably well with

$$\begin{pmatrix} \psi_{exact}(1.00) \\ \psi_{exact}(0.50) \\ \psi_{exact}(0.00) \\ \psi_{exact}(-0.50) \\ \psi_{exact}(-1.00) \end{pmatrix} = \begin{pmatrix} 0.36788 \\ 0.60653 \\ 1.00000 \\ 1.64872 \\ 2.71828 \end{pmatrix} \tag{9.76b}$$

except at $x = 1.00$, where the largest error is 6.3%. □

Neumann series

The *Neumann series in* λ for the inhomogeneous Fredholm equation is obtained by replacing $\psi(y)$ in the integral of eq. 9.32 by

$$\psi_0(y) + \lambda \int_a^b K(y,z)\psi(z)dz$$

to obtain

$$\begin{aligned}&\psi(x) = \psi_0(x) + \lambda \int_a^b K(x,y)\left[\psi_0(y) + \lambda \int_a^b K(y,z)\psi(z)dz\right]dy = \\ &\psi_0(x) + \lambda \int_a^b K(x,y)\psi_0(y)dy + \lambda^2 \int_a^b \int_a^b K(x,y)K(y,z)\psi(z)dydz \qquad (9.77)\end{aligned}$$

Repeating this process ad infinitum, we obtain the Neumann series in λ,

$$\begin{aligned}&\psi(x) = \\ &\psi_0(x) + \lambda \int_a^b K(x,y)\psi_0(y)dy + \lambda^2 \int_a^b \int_a^b K(x,y)K(y,z)\psi_0(z)dydz \\ &+ \lambda^3 \int_a^b \int_a^b \int_a^b K(x,y)K(y,z)K(z,w)\psi_0(w)dydzdw + ... \\ &\equiv \psi_0(x) + \sum_{m=1}^{\infty} \lambda^m I_m(x) \qquad (9.78)\end{aligned}$$

where

$$I_m(x) \equiv \underbrace{\int_a^b \int_a^b \bullet \bullet \int_a^b K(x,y)K(y,z) \bullet \bullet K(t,u)\psi_0(u)dydz \bullet \bullet dtdu}_{m\ integrals} \qquad (9.79)$$

If the value of λ is such that the infinite series of eq. 9.78 converges (see, for example, the *Cauchy ratio test* for convergence, Cohen, H., 1992, pp. 128–129), we can approximate the Neumann series by truncating it to obtain

$$\psi(x) \simeq \psi_0(x) + \sum_{m=1}^{N} \lambda^m I_m(x) \qquad (9.80)$$

As discussed in Appendix 1, such a series in λ also allows us to create a Pade Approximant, a diagonal form of which, should be more accurate than the truncated Neumann series.

Example 9.8: Neumann series for an inhomogeneous Fredholm equation of the second kind

It is straightforward to show that the solution to

$$\psi(x) = e^{x/5} - \frac{1}{5}\int_{-1}^{1} x e^{y(1-x)/5}\psi(y)dy \tag{9.81}$$

is

$$\psi(x) = e^{-x/5} \tag{9.82}$$

Writing eq. 9.81 as

$$\psi(x) = e^{x/5} - \lambda\int_{-1}^{1} x e^{y(1-x)/5}\psi(y)dy \tag{9.83}$$

the three term Neumann sum is

$$\psi(x) \simeq e^{x/5} - \lambda\int_{-1}^{1} x e^{y(1-x)/5} e^{y/5} dy + \lambda^2 \int_{-1}^{1}\int_{-1}^{1} x e^{y(1-x)/5} y e^{z(1-y)/5} e^{z/5} dydz \tag{9.84}$$

With

$$I_1(x) \equiv \int_{-1}^{1} x e^{y(1-x)/5} e^{y} dy = \frac{5x}{(2-x)}\left[e^{(2-x)/5} - e^{-(2-x)/5}\right] \tag{9.85a}$$

and

$$\begin{aligned} I_2(x) &\equiv \int_{-1}^{1}\int_{-1}^{1} x e^{y(1-x)/5} y e^{z(1-y)/5} e^{z/5} dzdy \\ &= \int_{-1}^{1} \frac{xy}{(2-y)} e^{y(1-x)/5}\left[e^{(2-y)/5} - e^{-(2-y)/5}\right]dy \end{aligned} \tag{9.85b}$$

(which, at a given x, we approximate by a quadrature sum), we obtain an approximation to $\psi(x)$ at any $x \in [-1, 1]$. With $\lambda = 1/5$, we obtain

$$\begin{aligned} \psi(x) \simeq{}& e^{x/5} - \frac{x}{(2-x)}\left[e^{(2-x)/5} - e^{-(2-x)/5}\right] \\ &+ \frac{1}{25}\int_{-1}^{1} \frac{xy}{(2-y)} e^{y(1-x)/5}\left[e^{(2-y)/5} - e^{-(2-y)/5}\right]dy \end{aligned} \tag{9.86}$$

To approximate the integrals $I_m(x)$ of eq. 9.79, we use a 20-point Gauss–Legendre quadrature rule. Setting $x = \{-0.75, -0.25, 0.25, 0.75\}$, we find

$$\begin{pmatrix} \psi(-0.75) \\ \psi(-0.25) \\ \psi(0.25) \\ \psi(0.75) \end{pmatrix} = \begin{pmatrix} 1.13651 \\ 1.04559 \\ 0.95420 \\ 0.86155 \end{pmatrix} \tag{9.87a}$$

which is a reasonable approximation to

$$\begin{pmatrix} \psi_{exact}(-0.75) \\ \psi_{exact}(-0.25) \\ \psi_{exact}(0.25) \\ \psi_{exact}(0.75) \end{pmatrix} = \begin{pmatrix} 1.16183 \\ 1.05127 \\ 0.95123 \\ 0.86071 \end{pmatrix} \tag{9.87b}$$

The [2, 2] Pade Approximant is found by requiring that

$$\psi^{[2,2]}(x) = \frac{p_0(x) + \lambda p_1(x)}{1 + \lambda q_1(x)} \tag{9.88}$$

be identical to

$$\psi_2(x) = \psi_0(x) + \lambda I_1(x) + \lambda^2 I_2(x) \tag{9.89}$$

This results in

$$\psi^{[2,2]}(x) = \frac{\psi_0(x) + \lambda\left[I_1(x) - \psi_0(x)\frac{I_2(x)}{I_1(x)}\right]}{1 - \lambda\frac{I_2(x)}{I_1(x)}} \tag{9.90}$$

From this, we obtain

$$\begin{pmatrix} \psi(-0.75) \\ \psi(-0.25) \\ \psi(0.25) \\ \psi(0.75) \end{pmatrix} = \begin{pmatrix} 1.14092 \\ 1.04632 \\ 0.95397 \\ 0.86152 \end{pmatrix} \tag{9.91}$$

which, as expected, is slightly more accurate than the Neumann series. □

9.3 Eigensolutions of Fredholm Equations of the Second Kind with Non-singular Kernel

When $\psi_0(x) = 0$ for all $x \varepsilon [a, b]$, the Fredholm equation of the second kind becomes the *homogeneous Fredholm equation*

$$\psi(x) = \lambda \int_a^b K(x, y)\psi(y)dy \tag{9.92}$$

which only has solutions for specific values of λ. The *eigenvalue of the kernel* is $1/\lambda$ and $\psi_\lambda(x)$, the solution to eq. 9.92 for that specific value of λ, is the corresponding *eigenfunction of the kernel.*

Solution by quadratures

Using a N-point quadrature rule to approximate the integral, eq. 9.92 becomes

$$\psi(x) \simeq \lambda \sum_{m=1}^{N} w_m K(x, y_m)\psi(y_m) \tag{9.93}$$

Setting x in this expression to each quadrature point in the set $\{y_k\}$, eq. 9.93 can be written in matrix form as

$$\left[1 - \lambda \begin{pmatrix} w_1 K_{11} & \bullet\bullet & w_N K_{1N} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ w_1 K_{N1} & \bullet\bullet & w_N K_{NN} \end{pmatrix}\right] \begin{pmatrix} \psi(y_1) \\ \bullet \\ \bullet \\ \psi(y_N) \end{pmatrix} = 0 \tag{9.94}$$

The eigenvalues of the kernel are obtained from

$$\begin{vmatrix} (1 - \lambda w_1 K_{11}) & \bullet\bullet & -w_N K_{1N} \\ \bullet & & \bullet \\ \bullet & & \bullet \\ -w_1 K_{N1} & \bullet\bullet & (1 - \lambda w_N K_{NN}) \end{vmatrix} = 0 \tag{9.95}$$

An approximation to the corresponding eigenfunctions are obtained by solving $(N - 1)$ of the equations of eq. 9.94 for $(N - 1)$ values $\psi(y_k)$ in terms of one of the ψ quantities. For example, we can solve for each $\psi(y_k)$ for $k \geq 2$ in terms of $\psi(y_1)$. Thus, the value of each ratio $\psi(y_k)/\psi(y_1)$ is known and eq. 9.93 can be expressed as

$$\psi_\lambda(x) =$$
$$\lambda[w_1K(x,y_1)\psi_\lambda(y_1) + w_2K(x,y_2)\psi_\lambda(y_2) + ... + w_NK(x,y_N)\psi_\lambda(y_N)]$$
$$= \lambda\psi_\lambda(y_1)\left[w_1K(x,y_1) + \frac{\psi_\lambda(y_2)}{\psi_\lambda(y_1)}w_2K(x,y_2) + ... + \frac{\psi_\lambda(y_N)}{\psi_\lambda(y_1)}w_NK(x,y_N)\right] \quad (9.96)$$

The undetermined coefficient $\psi_\lambda(y_1)$ is obtained from a normalization condition defined by the user.

Example 9.9: Eigensolution by quadratures

Let us consider

$$\psi(x) = \lambda\int_0^1 e^{xy}\psi(y)dy \quad (9.97)$$

We transform the Gauss–Legendre data to points over the interval [0, 1] (as described in ch. 4). These are

$$y_k^{[0,1]} = \frac{1 + y_k^{[-1,1]}}{2} \quad (9.98a)$$

and

$$w_k^{[0,1]} = \frac{w_k^{[-1,1]}}{2} \quad (9.98b)$$

Approximating the integral in eq. 9.97 by a three-point quadrature rule we obtain

$$\psi(x) \simeq \lambda\sum_{m=1}^{3} w_k e^{xy_k}\psi(y_k) \quad (9.99)$$

Setting x to each of the points $\{y_1, y_2, y_3\}$, we have

$$\begin{pmatrix} 1-\lambda w_1e^{y_1y_1} & -\lambda w_2e^{y_1y_2} & -\lambda w_3e^{y_1y_3} \\ -\lambda w_1e^{y_2y_1} & 1-\lambda w_2e^{y_2y_2} & -\lambda w_3e^{y_2y_3} \\ -\lambda w_1e^{y_3y_1} & -\lambda w_2e^{y_3y_2} & 1-\lambda w_3e^{y_3y_3} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix} = 0 \quad (9.100a)$$

The values of λ, found from

$$\begin{vmatrix} 1-\lambda w_1e^{y_1y_1} & -\lambda w_2e^{y_1y_2} & -\lambda w_3e^{y_1y_3} \\ -\lambda w_1e^{y_2y_1} & 1-\lambda w_2e^{y_2y_2} & -\lambda w_3e^{y_2y_3} \\ -\lambda w_1e^{y_3y_1} & -\lambda w_2e^{y_3y_2} & 1-\lambda w_3e^{y_3y_3} \end{vmatrix} = 0 \quad (9.100b)$$

are given by

$$\{\lambda_1, \lambda_2, \lambda_3\} = \{0.73908, 9.43870, 290.88072\} \tag{9.101}$$

Referring to eq. 9.96, the corresponding eigenfunctions are

$$\psi_{\lambda_1}(x) = \psi_1^{\lambda_1}\left[0.20530e^{0.88730x} + 0.25994e^{0.50000x} + 0.13009e^{0.11270x}\right] \tag{9.102a}$$

$$\psi_{\lambda_2}(x) = \psi_1^{\lambda_2}\left[2.62186e^{0.88730x} - 1.05829e^{0.50000x} - 2.81603e^{0.11270x}\right] \tag{9.102b}$$

$$\psi_{\lambda_3}(x) = \psi_1^{\lambda_3}\left[80.80020e^{0.88730x} - 199.95011e^{0.50000x} + 122.19328e^{0.11270x}\right] \tag{9.102c}$$

As noted in example 3.4 of Baker, C. T. H., 1977, p. 177, it has been shown that λ_1, the smallest value of λ for this kernel, is bounded by

$$0.73888 < \lambda < 0.73926 \tag{9.103}$$

Our result is consistent with this. □

Eigensolution of a degenerate kernel

A kernel that can be written as

$$K(x, y) = \sum_{k=1}^{N} A_k(x)B_k(y) \tag{9.104}$$

is called a *degenerate kernel*. The eigensolutions for such a kernel can be determined exactly.

With eq. 9.104, the homogeneous Fredholm equation becomes

$$\psi(x) = \lambda \sum_{k=1}^{N} A_k(x) \int_a^b B_k(y)\psi(y)dy \equiv \lambda \sum_{k=1}^{N} A_k(x)\beta_k \tag{9.105}$$

Substituting this expression for $\psi(y)$ into

$$\beta_k \equiv \int_a^b B_k(y)\psi(y)dy \tag{9.106}$$

eq. 9.105 becomes

$$\beta_k = \lambda \sum_{m=1}^{N} \beta_m \int_a^b B_k(y) A_m(y) dy \equiv \lambda \sum_{m=1}^{N} \mu_{km} \beta_m \tag{9.107}$$

This set of equations only has solution for specific values λ (the inverses of which are the eigenvalues of the kernel). Referring to eq. 9.105, the elements of the set $\{\beta_k{}^{\lambda}\}$ yield the corresponding eigenfunction given by

$$\psi_\lambda(x) = \lambda \sum_{k=1}^{N} \beta_k^{\lambda} A_k(x) \tag{9.108}$$

Example 9.10: Eigensolution to a Fredholm equation of the second kind with a degenerate kernel

Writing

$$\psi(x) = \lambda \int_0^1 (x+y)\psi(y)dy = \lambda\left[x\int_0^1 \psi(y)dy + \int_0^1 y\psi(y)dy\right] \equiv \lambda[\beta_1 x + \beta_2] \tag{9.109}$$

we define

$$\beta_1 \equiv \int_0^1 \psi(y)dy \tag{9.110a}$$

and

$$\beta_2 \equiv \int_0^1 y\psi(y)dy \tag{9.110b}$$

Substituting $\psi(y)$ given in eq. 9.109 into eqs. 9.110, we obtain

$$\beta_1 = \lambda \int_0^1 [\beta_1 y + \beta_2]dy = \lambda\left[\frac{\beta_1}{2} + \beta_2\right] \tag{9.111a}$$

and

$$\beta_2 = \lambda \int_0^1 y[\beta_1 y + \beta_2]dy = \lambda\left[\frac{\beta_1}{3} + \frac{\beta_2}{2}\right] \tag{9.111b}$$

from which

$$\lambda_{\pm} = 2\left(1 \pm \frac{1}{\sqrt{3}}\right) \tag{9.112}$$

With

$$\beta_2^{\pm} = \left(\frac{1}{\lambda_{\pm}} - \frac{1}{2}\right)\beta_1^{\pm} \tag{9.113}$$

the eigenfunctions are given by eq. 9.109 to be

$$\psi_{\pm} = \beta_1^{\pm}\lambda_{\pm}\left[x + \frac{1}{\lambda_{\pm}} - \frac{1}{2}\right] \tag{9.114}$$ □

Approximate eigensolutions by interpolating the kernel

When a nondegenerate kernel has an infinite number of eigensolutions to eq. 9.92, any finite approximation method will yield a finite subset of such solutions.

Referring to the discussion given in example 9.6, we interpolate the kernel over a set of points $\{y_k\}$ selected by the user. We write

$$K(x,y) \simeq \sum_{k=1}^{N} K(x,y_k)v_k(y) \tag{\textbf{\textit{9.61}}}$$

where

$$v_k(y) = \frac{[q(y) - q(y_1)]...[q(y) - q(y_{k-1})][q(y) - q(y_{k+1})]...[q(y) - q(y_N)]}{[q(y_k) - q(y_1)]...[q(y_k) - q(y_{k-1})][q(y_k) - q(y_{k+1})]...[q(y_k) - q(y_N)]} \tag{\textbf{\textit{1.18}}}$$

In this way, the kernel is approximated by a degenerate kernel, and the homogeneous Fredholm equation becomes

$$\psi(x) = \lambda \sum_{m=1}^{N} K(x,x_m)\int_a^b v_m(y)\psi(y)dy \tag{9.115}$$

As before, we define

$$\beta_k \equiv \int_a^b v_k(y)\psi(y)dy \tag{\textbf{\textit{9.63}}}$$

with which eq. 9.115 can be written as

$$\psi(x) = \lambda \sum_{k=1}^{N} K(x, x_k)\beta_k \tag{9.116}$$

This is an approximation to the eigenfunction corresponding to λ.

Using the expression of eq. 9.116 for $\psi(y)$, eq. 9.63 becomes

$$\beta_k = \lambda \sum_{m=1}^{N} \left[\int_a^b v_k(y) K(y, x_m) dy\right] \beta_m \tag{9.117}$$

With

$$\Gamma_{km} \equiv \int_a^b v_k(y) K(y, x_m) dy \tag{\textbf{\textit{9.66b}}}$$

eq. 9.117 is the eigenvalue equation

$$\beta_k = \lambda \sum_{m=1}^{N} \Gamma_{km} \beta_m \tag{9.118}$$

The eigenvalues and the values of the β_k are determined from eq. 9.118 by standard methods (see, for example, matrix methods of Jacobi, Givens and Householder introduced in ch. 5). The corresponding eigenfunctions are then given by eq. 9.116.

The accuracy of the interpolation of eq. 9.61, which can be determined independently of the method used to find the eigenpairs, will be a measure of the accuracy of the results.

Example 9.11: Eigensolution by interpolation of the kernel

We again consider

$$\psi(x) = \lambda \int_0^1 e^{xy} \psi(y) dy \tag{\textbf{\textit{9.97}}}$$

Since the kernel is an exponential function of y with a non-negative exponent for $x \,\varepsilon\, [0, 1]$, we choose

$$q(y) = e^y \tag{\textbf{\textit{9.68}}}$$

as the interpolation function in $v_k(y)$. For the purpose of illustration, we interpolate the kernel over the three points $\{y_1, y_2, y_3\} = \{0, .5, 1\}$. As such, we will determine three eigensolutions of the kernel.

The level of accuracy of the interpolation of the kernel is represented in Table 9.1.

y	Interpolated	Exact
0.1	1.03884	1.04081
0.2	1.08043	1.08329
0.3	1.12479	1.12750
0.4	1.17183	1.17351
0.5	1.22140	1.22140
0.6	1.27324	1.27125
0.7	1.32693	1.32313
0.8	1.38187	1.37713
0.9	1.43722	1.43333

Table 9.1 Interpolated values of e^{xy} at $x = 0.4$

In generating this table for $x = 0.4$, the largest percent difference between the interpolated and exact values of the kernel is 0.34%. Thus, the accuracy of the eigensolutions is expected to be at this level.

With k, r, and s taking on the values 1, 2, and 3, and with $r \neq s \neq k$, we have

$$\Gamma_{km} = \int_0^1 e^{yx_m} v_k(y)dy = \int_0^1 e^{yx_m} \frac{[e^y - e^{x_r}][e^y - e^{x_s}]}{[e^{x_k} - e^{x_r}][e^{x_k} - e^{x_s}]} dy$$

$$= \frac{\left[\frac{\left(e^{(2+x_m)}-1\right)}{(2+x_m)} - \left(e^{x_r} + e^{x_s}\right)\frac{\left(e^{(1+x_m)}-1\right)}{(1+x_m)} + e^{x_r+x_s}\frac{(e^{x_m}-1)}{x_m}\right]}{[e^{x_k} - e^{x_r}][e^{x_k} - e^{x_s}]} \tag{9.119}$$

where

$$\frac{(e^{x_m} - 1)}{x_m} = 1 \tag{9.120}$$

for $x_m = 0$.

From

$$|1 - \lambda\Gamma| = \left|1 - \lambda\begin{pmatrix} 0.15473 & 0.13579 & 0.10060 \\ 0.68639 & 0.90912 & 1.21862 \\ 0.15888 & 0.25253 & 0.39906 \end{pmatrix}\right| = 0 \tag{9.121}$$

we obtain

$$(\lambda_1, \lambda_2, \lambda_3) = (0.73874,\ 9.46937,\ 273.84781) \tag{9.122}$$

Referring to eq. 9.116, the corresponding eigenfunctions are

$$\begin{aligned}\psi_1(x) &= \beta_1\lambda_1\left[K(x,0) + \frac{\beta_2}{\beta_1}K(x,0.5) + \frac{\beta_3}{\beta_1}K(x,1)\right] \\ &= \beta_1\left[0.73874 + 5.37741e^{0.5x} + 1.54551e^{x}\right]\end{aligned} \tag{9.123a}$$

$$\psi_2(x) = \beta_1 \lambda_2 \left[K(x, 0) + \frac{\beta_2}{\beta_1} K(x, 0.5) + \frac{\beta_3}{\beta_1} K(x, 1) \right]$$
$$= \beta_1 \left[9.46937 + 1.02738 e^{0.5x} - 6.01102 e^{x} \right] \tag{9.123b}$$

and

$$\psi_3(x) = \beta_1 \lambda_3 \left[K(x, 0) + \frac{\beta_2}{\beta_1} K(x, 0.5) + \frac{\beta_3}{\beta_1} K(x, 1) \right]$$
$$= \beta_1 \left[273.84781 - 423.54533 e^{0.5x} + 160.46100 e^{x} \right] \tag{9.123c}$$

where β_1 is determined by a user-defined normalization condition.

Again, we see that λ_1, the smallest value of λ, is consistent with

$$0.73888 < \lambda < 0.73926 \tag{9.103}$$

as noted in example 3.4 of Baker , C. T. H., 1977, p. 177. □

9.4 Volterra Equations with Non-singular Kernel

The general form of a linear Volterra integral equation of the first kind is

$$\phi_0(x) = \lambda \int_a^x L(x, y) \psi(y) dy \tag{9.124}$$

and the Volterra equation of the second kind is

$$\psi(x) = \psi_0(x) + \lambda \int_a^x K(x, y) \psi(y) dy \tag{9.125}$$

where $K(x, y)$ is analytic at all x and at all $y \leq x$.

Since the integral is zero when $x = a$, we note from eq. 9.124 that in order for the equation of the first kind to have a solution, $\phi_0(x)$ must satisfy

$$\phi_0(a) = 0 \tag{9.126a}$$

The solution to the Volterra equation of the second kind satisfies

$$\psi(a) = \psi_0(a) \tag{9.126b}$$

Converting the Volterra equation of the first kind into the Volterra equation of the second kind

Differentiating the Volterra of the first kind, we have

$$\phi_0'(x) = \lambda \int_a^x \frac{\partial L(x,y)}{\partial x} \psi(y) dy + L(x,x)\psi(x) \tag{9.127a}$$

If $L(x,x) = 0$ for every x, this becomes a Volterra equation of the first kind. If $L(x, x) \neq 0$ for all x, this can be written as

$$\psi(x) = \frac{\phi_0'(x)}{L(x,x)} - \lambda \int_a^x \frac{1}{L(x,x)} \frac{\partial L(x,y)}{\partial x} \psi(y) dy \tag{9.127b}$$

Defining

$$\frac{\phi_0'(x)}{L(x,x)} \equiv \psi_0(x) \tag{9.128a}$$

and

$$-\frac{1}{L(x,x)} \frac{\partial L(x,y)}{\partial x} \equiv K(x,y) \tag{9.128b}$$

eq. 9.127b becomes

$$\psi(x) = \psi_0(x) + \lambda \int_a^x K(x,y)\psi(y) dy \tag{\textbf{\textit{9.125}}}$$

which is a Volterra equation of the second kind.

Thus, we can solve a Volterra equation of the first kind by solving the equivalent Volterra equation of the second kind.

Example 9.12: Converting a Volterra equation of the first kind to a Volterra equation of the second kind

The solution to

$$xe^x = \int_0^x e^{(x-y)} \psi(y) dy \tag{9.129}$$

is

$$\psi(x) = e^x \tag{\textbf{\textit{9.50b}}}$$

With

$$\phi_0'(x) = (1 + x)e^x \tag{9.130}$$

and

$$L(x, x) = e^0 = 1 \tag{9.131}$$

eq. 9.129 becomes

$$\psi(x) = (1 + x)e^x - \int_0^x e^{(x-y)}\psi(y)dy \tag{9.132}$$

It is a trivial exercise to show that

$$\psi(x) = e^x \tag*{(\textbf{\textit{9.50b}})}$$

is the solution to eq. 9.132. □

Taylor series approximation for the Volterra equation of the second kind

The Taylor series for $\psi(x)$ expanded around a is

$$\psi(x) = \psi(a) + \psi'(a)(x-a) + \frac{1}{2!}\psi''(a)(x-a)^2 + \frac{1}{3!}\psi'''(a)(x-a)^3 + \ldots \tag{9.133}$$

The first term in the series is

$$\psi(a) = \psi_0(a) \tag*{(\textbf{\textit{9.126b}})}$$

and from the derivative of $\psi(x)$ given in eq. 9.125, we have

$$\psi'(x) = \psi_0'(x) + \lambda z \int_a^x \frac{\partial K(x,y)}{\partial x}\psi(y)dy + \lambda K(x,x)\psi(x) \tag{9.134a}$$

Thus

$$\psi'(a) = \psi_0'(a) + \lambda K(a,a)\psi(a) = \psi_0'(a) + \lambda K(a,a)\psi_0'(a) \tag{9.134b}$$

The derivative of $\psi'(x)$ given in eq. 9.134a yields

$$\psi''(x) = \psi_0''(x) + \lambda \int_a^x \frac{\partial^2 K(x,y)}{\partial x^2} \psi(y) dy + \\ \lambda \frac{\partial K(x,y)}{\partial x}\bigg|_{y=x} \psi(x) + \lambda \frac{\partial K(x,y)}{\partial y}\bigg|_{y=x} \psi(x) + \lambda K(x,x) \psi'(x) \tag{9.135a}$$

from which

$$\psi''(a) = \psi_0''(a) + \lambda \frac{\partial K(x,y)}{\partial x}\bigg|_{\substack{y=a \\ x=a}} \psi_0(a) + \lambda \frac{\partial K(x,y)}{\partial y}\bigg|_{\substack{y=a \\ x=a}} \psi_0(a) \\ + \lambda K(a,a) \psi'(a)$$

where $\psi'(a)$ is given in eq. 9.134b.

Example 9.13: Solution to a Volterra equation by Taylor series

The solution to

$$\psi(x) = 1 + x + \int_0^x e^{(x-y)} \psi(y) dy \tag{9.136}$$

is

$$\psi(x) = \frac{1}{4} + \frac{x}{2} + \frac{3}{4} e^{2x} \tag{9.137}$$

With

$$\psi_0(0) = 1 \tag{9.138a}$$

and

$$K(0,0) = 1 \tag{9.138b}$$

eq. 9.134b yields

$$\psi'(0) = 2 \tag{9.139a}$$

and from eq. 9.135b

$$\psi''(a) = 3 \tag{9.139b}$$

Therefore,

$$\psi(x) \simeq \psi_0(0) + \psi'(0)x + \frac{1}{2!}\psi''(0)x^2 = 1 + 2x + \frac{3}{2}x^2 \tag{9.140}$$

We compare this to the MacLaurin expansion of the solution given in eq. 9.137,

$$\psi_{exact}(x) = \frac{1}{4} + \frac{1}{2}x + \frac{3}{4}\left(1 + 2x + \frac{4x^2}{2!} + \ldots\right) = 1 + 2x + \frac{3}{2}x^2 + \ldots \tag{9.141}$$

Thus, the Taylor series approximation is identical to the series representation of the known solution up to x^2. □

Approximating ψ(y) by a spline interpolation

As we did for the Fredholm equation, we develop an approach in which we approximate $\psi(y)$ by a constant in the Volterra integral. We begin by denoting the limits of the integral $[a, x]$ as $[x_0, x_N]$. To approximate the solution to the Volterra equation at x_N, we divide $[x_0, x_N]$ into small segments, writing the integral in the Volterra equation as

$$\int_{x_0}^{x_N} K(x_N, y)\psi(y)dy = \sum_{m=0}^{N-1} \int_{x_m}^{x_{m+1}} K(x_N, y)\psi(y)dy \tag{9.142}$$

where, as before, for convenience, we take these intervals to be evenly spaced of width Δx.

By taking Δx to be small enough, we can approximate $\psi(y)$ over each segment $x_k \leq y \leq x_{k+1}$ by the constant

$$\psi(y) \simeq \alpha\psi(x_m) + \beta\psi(x_{m+1}) \tag{9.45}$$

where α and β, chosen by the user, are here taken to be 1/2 as note in eq. 9.46. Then eq. 9.142 becomes

$$\int_{x_0}^{x_N} K(x_N, y)\psi(y)dy \simeq \frac{1}{2}\sum_{m=0}^{N-1} [\psi(x_m) + \psi(x_{m+1})] \int_{x_m}^{x_{m+1}} K(x_N, y)dy \tag{9.143}$$

Defining

$$I_m(x_N) \equiv \int_{x_m}^{x_{m+1}} K(x_N, y)dy \tag{9.144}$$

the Volterra equation of the second kind becomes

$$\psi(x_N) \simeq \psi_0(x_N) + \frac{\lambda}{2}\sum_{m=0}^{N-1} [\psi(x_m) + \psi(x_{m+1})]I_m(x_N) \tag{9.145}$$

Referring to eq. 9.126b,

$$\psi(x_0) = \psi(a) = \psi_0(a) \tag{9.146}$$

Then, eq. 9.145 becomes

$$\left(1 - \frac{\lambda}{2} I_{N-1}(x_N)\right)\psi(x_N) \simeq \psi_0(x_N) + \frac{\lambda}{2}\psi_0(a)I_0(x_N) +$$
$$\frac{\lambda}{2}\{\psi(x_1)I_0(x_N) + [\psi(x_1) + \psi(x_2)]I_1(x_N) + \ldots$$
$$+[\psi(x_{N-2}) + \psi(x_{N-1})]I_{N-2}(x_N) + \psi(x_{N-1})I_{N-1}(x_N)$$

To determine the set $\{\psi(x_k)\}$, we begin with $x = x_1$ in the Volterra equation to obtain

$$\psi(x_1) = \psi_0(x_1) + \lambda \int_{x_0}^{x_1} K(x_1, y)\psi(y)dy \simeq \psi_0(x_1) + \frac{\lambda}{2}[\psi_0(x_0) + \psi(x_1)]I_0(x_1) \tag{9.148}$$

from which

$$\left(1 - \frac{\lambda}{2} I_0(x_1)\right)\psi(x_1) \simeq \psi_0(x_1)\left(1 + \frac{\lambda}{2} I_0(x_1)\right) \tag{9.149}$$

Then, with $\psi(x_1)$ given by eq. 9.149, we set $x = x_2$ in the Volterra equation to obtain

$$\psi(x_2) = \psi_0(x_2) + \lambda \int_{x_0}^{x_1} K(x_2, y)\psi(y)dy + \lambda \int_{x_1}^{x_2} K(x_2, y)\psi(y)dy$$
$$\simeq \psi_0(x_2) + \frac{\lambda}{2}[\psi(x_0) + \psi(x_1)]I_0(x_2) + \frac{\lambda}{2}[\psi(x_1) + \psi(x_2)]I_1(x_2) \tag{9.150}$$

Then, $\psi(x_2)$ is found from

$$\left(1 - \frac{\lambda}{2} I_1(x_2)\right)\psi(x_2) \simeq \psi_0(x_2) + \frac{\lambda}{2}[\psi(x_0) + \psi(x_1)]I_0(x_2) + \frac{\lambda}{2} I_1(x_2)\psi(x_1) \tag{9.151}$$

and so on.

Example 9.14: Solution to a Volterra equation of the second kind

As noted earlier,

$$\psi(x) = 1 + x + \int_0^x e^{(x-y)}\psi(y)dy \tag{\textbf{\textit{9.136}}}$$

has solution

$$\psi(x) = \frac{1}{4} + \frac{x}{2} + \frac{3}{4}e^{2x} \tag{9.137}$$

For simplicity of illustration, we obtain the solution at $x = 0.9$ taking $\Delta x = 0.3$. Then, with $x_0 = 0$ and $x_N = x_3 = 0.9$, and

$$\psi(x_0) = \psi_0(0) = 1 \tag{9.152}$$

we obtain $\psi(x_3)$ from eq. 9.147 in terms of $\psi(x_1)$ and $\psi(x_2)$. These quantities are determined from eqs. 9.149 and 9.151.

Approximating $\psi(y)$ by

$$\psi(y) \simeq \frac{1}{2}[\psi(x_m) + \psi(x_{m+1})] \quad x_m \leq y \leq x_{m+1} \tag{9.153}$$

eq. 9.145 becomes

$$\begin{aligned}\psi(x_3) &\simeq \psi_0(x_3) + \frac{1}{2}[\psi(x_0) + \psi(x_1)]I_0(x_3)\\ &+ \frac{1}{2}[\psi(x_1) + \psi(x_2)]I_1(x_3) + \frac{1}{2}[\psi(x_2) + \psi(x_3)]I_2(x_3)\end{aligned} \tag{9.154a}$$

from which

$$\begin{aligned}&\left[1 - \frac{1}{2}\, I_2(x_3)\right]\psi(x_3) = \psi_0(x_3) +\\ &\frac{1}{2}\,\psi_0(x_0)I_0(x_3) + \frac{1}{2}\,[I_0(x_3) + I_1(x_3)]\psi(x_1) + \frac{1}{2}\,[I_1(x_3) + I_2(x_3)]\psi(x_2)\end{aligned} \tag{9.154b}$$

From eq. 9.149, $\psi(x_1)$ is given by

$$\left[1 - \frac{1}{2}\, I_0(x_1)\right]\psi(x_1) \simeq \psi_0(x_1) + \frac{1}{2}\,\psi_0(x_0)I_0(x_1) \tag{9.155}$$

and with this value of $\psi(x_1)$, eq. 9.151 yields

$$\left[1 - \frac{1}{2}\, I_1(x_2)\right]\psi(x_2) \simeq \psi_0(x_2) + \frac{1}{2}\,\psi_0(x_0)I_0(x_2) + \frac{1}{2}\,[I_0(x_2) + I_1(x_2)]\psi(x_1) \tag{9.156}$$

With

$$I_k(x_m) = \int_{x_k}^{x_{K+1}} K(x_m, y)dy = \int_{x_k}^{x_{k+1}} e^{(x_m - y)}dy = e^{x_m}(e^{-x_k} - e^{-x_{k+1}}) \qquad (9.157)$$

we obtain

$$\begin{pmatrix} \psi(0.3) \\ \psi(0.6) \\ \psi(0.9) \end{pmatrix} = \begin{pmatrix} 1.78764 \\ 3.11604 \\ 5.44382 \end{pmatrix} \qquad (9.158a)$$

Comparing these to

$$\begin{pmatrix} \psi_{exact}(0.3) \\ \psi_{exact}(0.6) \\ \psi_{exact}(0.9) \end{pmatrix} = \begin{pmatrix} 1.76659 \\ 3.04009 \\ 5.23724 \end{pmatrix} \qquad (9.158b)$$

we see that this method yields a reasonably accurate approximate solution. □

9.5 Fredholm Equations with Weakly Singular Kernel

A weakly singular kernel satisfies

$$\lim_{y \to x} K(x, y) = \infty \qquad (9.159a)$$

and

$$\lim_{y \to x} (x - y)K(x, y) = 0 \qquad (9.159b)$$

Equations of this type, which arises in applied problems, are of the form

$$\psi(x) = \psi_0(x) + \lambda \int_a^b \frac{L(x, y)}{|x - y|^p} \psi(y)dy \qquad 0 < p < 1 \qquad (9.160a)$$

and

$$\psi(x) = \psi_0(x) + \lambda \int_a^b L(x, y)\ell n|x - y|\psi(y)dy \qquad (9.160b)$$

where $L(x, y)$ is analytic at all x and y in $[a, b]$.

An example of such an equation is the Kirkwood–Riseman formula, from which one determines the viscosity and the translational diffusion constants of macromolecules:

$$\psi(x) = \psi_0(x) - \lambda \int_{-1}^{1} \frac{1}{\sqrt{|x-y|}} \psi(y) dy \tag{9.161}$$

(see Kirkwood, J. G., and Riseman, J., 1948).

Lagrange interpolation methods

To approximate the solution to either of eq. 9.160 by Lagrange-like interpolation, we write

$$\psi(y) \simeq \sum_{m=1}^{M} \psi(y_m) v_m(y) \tag{9.162}$$

where

$$v_m(y) = \frac{(q(y)-q(y_1))\dots(q(y)-q(y_{m-1}))(q(y)-q(y_{m+1}))\dots(q(y)-q(y_M))}{(q(y_m)-q(y_1))\dots(q(y_m)-q(y_{m-1}))(q(y_m)-q(y_{m+1}))\dots(q(y_m)-q(y_M))} \tag{1.18}$$

With this approximation of $\psi(y)$, eqs. 9.160 become

$$\psi(x) = \psi_0(x) + \lambda \sum_{m=1}^{M} \psi(x_m) \int_a^b \frac{L(x,y)}{|x-y|^p} v_m(y) dy \quad 0<p<1 \tag{9.163a}$$

and

$$\psi(x) = \psi_0(x) + \lambda \sum_{m=1}^{M} \psi(x_m) \int_a^b L(x,y) \ell n|x-y| v_m(y) dy \tag{9.163b}$$

Setting x to each y_k, we obtain

$$\psi(y_k) = \psi_0(y_k) + \lambda \sum_{m=1}^{M} \psi(y_m) \int_a^b \frac{L(y_k,y)}{|y_k-y|^p} v_m(y) dy \tag{9.163c}$$

and

$$\psi(y_k) = \psi_0(y_k) + \lambda \sum_{m=1}^{M} \psi(y_m) \int_a^b L(y_k, y)\ell n|y_k - y| v_m(y) dy \tag{9.163d}$$

Depending on $L(x, y)$, the integrals in eqs. 9.163c and 9.163d are then evaluated in closed form or are approximated numerically by quadrature sums.

Referring to ch. 4, eqs. 4.152–4.155, when approximating these integrals by quadrature sums, the singularity structure of the integrand should be "smoothed out" by writing

$$\int_a^b \frac{L(y_k, y) v_m(y)}{|y_k - y|^p} dy = \int_a^b \frac{[L(y_k, y) v_m(y) - L(y_k, y_k) v_m(y_k)]}{|y_k - y|^p} dy + L(y_k, y_k) v_m(y_k) \int_a^b \frac{1}{|y_k - y|^p} dy \tag{9.164a}$$

and

$$\int_a^b L(y_k, y) v_m(y) \ell n|y_k - y| dy = \int_a^b [L(y_k, y) v_m(y) - L(y_k, y_k) v_m(y_k)] \ell n|y_k - y| dy + L(y_k, y_k) v_m(y_k) \int_a^b \ell n|y_k - y| dy \tag{9.164b}$$

With

$$v_m(y_k) = \delta_{km} \tag{\textbf{\textit{1.19}}}$$

eq. 9.164 become

$$\int_a^b \frac{L(y_k, y) v_m(y)}{|y_k - y|^p} dy = \int_a^b \frac{[L(y_k, y) v_m(y) - L(y_k, y_k) \delta_{km}]}{|y_k - y|^p} dy + L(y_k, y_k) \delta_{km} \int_a^b \frac{1}{|y_k - y|^p} dy \tag{9.165a}$$

and

$$\int_a^b L(y_k, y)v_m(y)\ell n|y_k - y|dy = \int_a^b [L(y_k, y)v_m(y) - L(y_k, y_k)\delta_{km}]\ell n|y_k - y|dy \\ + L(y_k, y_k)\delta_{km}\int_a^b \ell n|y_k - y|dy \tag{9.165b}$$

The first integrals on the right sides of eqs. 9.165 are approximated by quadrature sums. If, for a small parameter ε, a quadrature point y_n satisfies

$$|y_n - y_k| < \varepsilon \tag{9.166}$$

that term in the quadrature sum is taken to be zero.

The second integrals on the right sides of eqs. 9.165 can be evaluated in closed form as follows:

$$\int_a^b \frac{1}{|x-y|^p}dy = \int_a^x \frac{1}{(x-y)^p}dy + \int_x^b \frac{1}{(y-x)^p}dy = \frac{(x-a)^{1-p} + (b-x)^{1-p}}{1-p} \tag{9.167a}$$

and

$$\int_a^b \ell n|x-y|dy = \int_a^x \ell n(x-y)dy + \int_x^b \ell n(y-x)dy = \\ (x-a)\ell n(x-a) + (b-x)\ell n(b-x) - (b-a) \tag{9.167b}$$

Unless one has a sense of the behavior of $\psi(y)$, and so has a sense of the interpolating function $q(y)$, it is reasonable to use polynomial interpolation by taking $q(y) = y$ and interpolating with

$$v_m(y) \equiv \frac{(y-y_1)\ldots(y-y_{m-1})(y-y_{m+1})\ldots(y-y_N)}{(y_m-y_1)\ldots(y_m-y_{m-1})(y_m-y_{m+1})\ldots(y_m-y_N)} \tag{1.7}$$

Example 9.15: Solution to a Fredholm equation with a weakly singular kernel using Lagrange interpolation

It is straightforward to show that the solution to

$$\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}\int_{-1}^1 \frac{e^{x-y}}{\sqrt{|x-y|}}\psi(y)dy \tag{9.168}$$

is

$$\psi(x) = e^x \tag{9.50b}$$

Approximating ψ(y) by

$$\psi(y) = \sum_{m=1}^{M} \psi(y_m) v_m(y) \tag{9.162}$$

eq. 9.168 becomes

$$\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}e^x \sum_{m=1}^{M} \psi(y_m) \int_{-1}^{1} \frac{e^{-y} v_m(y)}{\sqrt{|x-y|}} dy \tag{9.169a}$$

To illustrate, we take the somewhat crude interpolation over five points y = {–1.0, –0.5, 0.0, 0.5, 1.0}. Then, eq. 9.169a becomes

$$\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}e^x \sum_{m=1}^{5} \psi(y_m) \int_{-1}^{1} \frac{e^{-y} v_m(y)}{\sqrt{|x-y|}} dy \tag{9.169b}$$

With $v_m(y)$ the polynomial given in eq. 1.7, we then set x to each point in the set $\{y_m\}$ = {–1.0, –0.5, 0.0, 0.5, 1.0} and evaluate

$$\begin{aligned}\int_{-1}^{1} \frac{e^{-y} v_m(y)}{\sqrt{|x_k-y|}} dy &= \int_{-1}^{1} \frac{\left[e^{-y} v_m(y) - e^{-x_k}\delta_{km}\right]}{\sqrt{|x_k-y|}} dy + e^{-x_k}\delta_{km} \int_{-1}^{1} \frac{1}{\sqrt{|x_k-y|}} dy \\ &\simeq \sum_{n=1}^{N} w_n \frac{\left[e^{-y} v_m(y_n) - e^{-x_k}\delta_{km}\right]}{\sqrt{|x_k-y_n|}} + 2e^{-x_k}\delta_{km}\left[\sqrt{1+x_k} + \sqrt{1-x_k}\right]\end{aligned} \tag{9.170}$$

We obtain

$$\begin{pmatrix} \psi(-1.0) \\ \psi(-0.5) \\ \psi(0.0) \\ \psi(0.5) \\ \psi(1.0) \end{pmatrix} = \begin{pmatrix} 0.36751 \\ 0.60603 \\ 1.00058 \\ 1.65099 \\ 2.72063 \end{pmatrix} \tag{9.171a}$$

which compares well with the exact result e^x given by

$$\begin{pmatrix} \psi_{exact}(-1.0) \\ \psi_{exact}(-0.5) \\ \psi_{exact}(0.0) \\ \psi_{exact}(0.5) \\ \psi_{exact}(1.0) \end{pmatrix} = \begin{pmatrix} 0.36788 \\ 0.60653 \\ 1.00000 \\ 1.64872 \\ 2.71828 \end{pmatrix} \tag{9.171b}$$

The largest error in these results is 0.13%.

The results given in eq. 9.171a can then be substituted into eq. 9.169b to yield values for $\psi(x)$ at any x. □

As the reader will show in Problem 14, a second approach using Lagrange (or Lagrange-like) interpolation is to write

$$\int_a^b \frac{L(x,y)\psi(y)}{|x-y|^p} dy \simeq \sum_{m=1}^{M} L(x,y_m)\psi(y_m) \int_a^b \frac{v_m(y)}{|x-y|^p} dy \tag{9.172a}$$

and

$$\int_a^b L(x,y)\psi(y)\ell n|x-y|dy \simeq \sum_{m=1}^{M} L(x,y_m)\psi(y_m) \int_a^b v_m(y)\ell n|x-y|dy \tag{9.172b}$$

If the interpolation functions $v_m(y)$ are polynomials of order M, they can be written as sums of powers of y with $\alpha_M = 1$. That is,

$$\int_a^b \frac{v_m(y)}{|x_k-y|^p} dy = \sum_{r=0}^{M} \alpha_r \int_a^b \frac{y^r}{|x_k-y|^p} dy \tag{9.173a}$$

and

$$\int_a^b v_m(y)\ell n|x_k-y|dy = \sum_{r=0}^{M} \alpha_r \int_a^b y^r \ell n|x_k-y|dy \tag{9.173b}$$

can be evaluated in closed form.

Writing

$$\int_a^b \frac{y^r}{|x-y|^p} dy = \int_a^x \frac{y^r}{(x-y)^p} dy + \int_x^b \frac{y^r}{(y-x)^p} dy \tag{9.174}$$

we make the substitutions

$$(x-y)^p = z^q \qquad y \in [a,x] \tag{9.175a}$$

and

$$(y-x)^p = z^q \quad y \in [x,b] \tag{9.175b}$$

to obtain

$$\int_a^b \frac{y^r}{|x-y|^p}dy = \frac{q}{p}\left[\int_0^{(x-a)^{p/q}}\left(x-z^{1/p}\right)^r z^{q/p-q-1}dz + \int_0^{(b-x)^{p/q}}\left(x+z^{1/p}\right)^r z^{q/p-q-1}dz\right] \tag{9.176}$$

Another approach using polynomial interpolation is to write

$$\int_a^b \frac{L(x,y)}{|x-y|^p}\psi(y)dy = \int_a^b \frac{[L(x,y)-L(x,x)]}{|x-y|^p}\psi(y)dy + L(x,x)\int_a^b \frac{1}{|x-y|^p}\psi(y)dy \tag{9.177}$$

Since the integrand of the first integral is finite (zero) at $y = x$, the first integral is well approximated by the N-point quadrature sum

$$\int_a^b \frac{[L(x,y)-L(x,x)]}{|x-y|^p}\psi(y)dy \simeq \sum_{k=1}^{N} w_k \frac{[L(x,y_k)-L(x,x)]}{|x-y_k|^p}\psi(y_k) \tag{9.178}$$

To evaluate the second integral, we interpolate $\psi(y)$ over a subset (which can be the entire set) of the quadrature abscissae $\{y_k\}$. That is, we interpolate over the points $\{Y_m\}$ ε $\{y_k\}$. Then, the second integral is approximated by

$$\int_a^b \frac{1}{|x-y|^p}\psi(y)dy \simeq \sum_{m=1}^{M}\psi(Y_m)\int_a^b \frac{v_m(y)}{|x-y|^p}dy \tag{9.179}$$

with $M \leq N$. Writing

$$v_m(y) = \frac{y^M + c_1 y^{M-1} + \ldots}{(Y_m-Y_1)\ldots(Y_m-Y_{m-1})(Y_m-Y_{m+1})\ldots(Y_m-Y_M)} \tag{9.180}$$

the integrals in eq. 9.179 can be evaluated in closed form as described above.

The approximation to the integral equation then becomes

$$\psi(x) \simeq \psi_0(x) + \lambda\sum_{k=1}^{N} w_k \frac{[L(x,y_k)-L(x,x)]}{|x-y|^p}\psi(y_k) + \lambda L(x,x)\sum_{m=1}^{M}\psi(Y_m)\int_a^b \frac{v_m(y)}{|x-y|^p}dy \tag{9.181}$$

With $\{\psi(Y_m)\}$ ε $\{\psi(y_m)\}$, this can be solved by standard methods.

In practice, we find that it is best to choose a quadrature rule that contains abscissae $\{Y_m\}$ that are approximately equally spaced over $[a, b]$.

Example 9.16: Solution to a Fredholm equation with a weakly singular kernel using Lagrange interpolation

We again consider

$$\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}\int_{-1}^{1} \frac{e^{x-y}}{\sqrt{|x-y|}}\psi(y)dy \qquad \textbf{\textit{(9.168)}}$$

which has solution

$$\psi(x) = e^x \qquad \textbf{\textit{(9.50b)}}$$

The approximation of eq. 9.181 for this equation is

$$\psi(x) \simeq e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}e^x\sum_{k=1}^{N} w_k \frac{\left[e^{-y_k} - e^{-x}\right]}{\sqrt{|x-y|}}\psi(y_k) + \frac{1}{2}e^x\sum_{m=1}^{M}\psi(Y_m)\int_{-1}^{1}\frac{v_m(y)}{\sqrt{|x-y|}}dy \qquad (9.182)$$

The quadrature set used to evaluate the first integral is selected based on the points over which $\psi(y)$ is interpolated in the second integral. If the interval [–1,1] is divided into four segments of approximately equal widths, the five points should be approximately {–1.0, –0.5, 0.0, 0.5, 1.0}. Since 0.0 is one of these points, we look for an odd order quadrature rule for which 0.0 is one of the abscissae and that has abscissae close ±1.0 and ±0.5.

Referring to Stroud, A.H., and Secrest, D., 1966, p. 101, the 17 point Gauss–Legendre quadrature abscissae and weights are shown in Table 9.2.

$N = 17$	
x	w
0.00000	0.17945
±0.17848	0.17656
±0.35123	0.16800
±0.51269	0.15405
±0.65767	0.13514
±0.78151	0.11188
±0.88024	0.06504
±0.95068	0.05546
±0.99057	0.02415

Table 9.2 Seventeen point Gauss–Legendre quadrature data

We use this quadrature rule to evaluate the first integral, and the points

$$\{Y_m\}_{17} = \{-0.99058, -0.51260, 0.00000, 0.51260, 0.99058\} \tag{9.183}$$

from this quadrature set to interpolate $\psi(y)$ in the second integral. Then eq. 9.182 becomes

$$\psi(x) \simeq e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \\ \frac{1}{2}e^x \sum_{k=1}^{17} w_k \frac{\left[e^{-y_k} - e^{-x}\right]}{\sqrt{|x-y|}}\psi(y_k) + \frac{1}{2}e^x \sum_{m=1}^{5} \psi(Y_m) \int_{-1}^{1} \frac{v_m(y)}{\sqrt{|x-y|}}dy \tag{9.184}$$

where

$$v_m(y) = \frac{(y-Y_1)\ldots(y-Y_{m-1})(y-Y_{m+1})\ldots(y-Y_M)}{(Y_m-Y_1)\ldots(Y_m-Y_{m-1})(Y_m-Y_{m+1})\ldots(Y_m-Y_M)} \tag{9.185}$$

Results at a sample of the 17 abscissae are

$$\begin{pmatrix} \psi(0.99058) \\ \psi(0.78151) \\ \psi(0.51269) \\ \psi(0.17848) \\ \psi(0.00000) \\ \psi(-0.51269) \\ \psi(-0.88024) \\ \psi(-0.99058) \end{pmatrix} = \begin{pmatrix} 2.69145 \\ 2.18359 \\ 1.66886 \\ 1.19523 \\ 1.00031 \\ 0.59916 \\ 0.41431 \\ 0.37118 \end{pmatrix} \tag{9.186a}$$

which is a reasonably accurate approximation to

$$\begin{pmatrix} \psi_{exact}(0.99058) \\ \psi_{exact}(0.78151) \\ \psi_{exact}(0.51269) \\ \psi_{exact}(0.17848) \\ \psi_{exact}(0.00000) \\ \psi_{exact}(-0.51269) \\ \psi_{exact}(-0.88024) \\ \psi_{exact}(-0.99058) \end{pmatrix} = \begin{pmatrix} 2.69278 \\ 2.18478 \\ 1.66978 \\ 1.19540 \\ 1.00000 \\ 0.59888 \\ 0.41468 \\ 0.37136 \end{pmatrix} \tag{9.186b}$$

The largest error in these results is 0.09% with an average error of 0.05%. This indicates that this approach is a bit more accurate than the method described in example 9.15. □

Spline Interpolation Methods

To apply spline interpolation methods, we subdivide [a, b] into several segments, writing eqs. 9.160 as

$$\psi(x) = \psi_0(x) + \lambda \sum_{m=1}^{M} \int_{z_m}^{z_{m+1}} \frac{L(x, y)}{|x - y|^p} \psi(y) dy \tag{9.187a}$$

and

$$\psi(x) = \psi_0(x) + \lambda \sum_{m=1}^{M} \int_{z_m}^{z_{m+1}} L(x, y) \ell n|x - y| \psi(y) dy \tag{9.187b}$$

where

$$z_1 \equiv a \tag{9.188a}$$

and

$$z_{M+1} \equiv b \tag{9.188b}$$

As before, the segments can be equally spaced, or defined by some other method such as taking a subset of the abscissae of a Gaussian quadrature rule. Then, over each segment, $\psi(y)$ is approximated by a constant. Unlike the approximation of eq. 9.45, here we approximate $\psi(y)$ by

$$\psi(y) \simeq \psi(\alpha z_m + \beta z_{m+1}) \equiv \psi(y_m) \quad z_m \leq y_m \leq z_{m+1} \tag{9.189}$$

where, unless there is a reason to do otherwise, we take y_m to be the midpoint of the mth segment. Then, setting x to each y_k, eqs. 9.187 become

$$\psi(y_k) \simeq \psi_0(y_k) + \lambda \sum_{m=1}^{M} \psi(y_m) \int_{z_m}^{z_{m+1}} \frac{L(y_k, y)}{|y_k - y|^p} dy \tag{9.190a}$$

and

$$\psi(y_k) \simeq \psi_0(y_k) + \lambda \sum_{m=1}^{M} \psi(y_m) \int_{z_m}^{z_{m+1}} L(y_k, y) \ell n|y_k - y| dy \tag{9.190b}$$

which are solved by standard techniques.

Example 9.17: Solution of a Fredholm equation with a weakly singular kernel using spline interpolation

We again consider

$$\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}\int_{-1}^{1}\frac{e^{x-y}}{\sqrt{|x-y|}}\psi(y)dy \qquad \textbf{(9.168)}$$

which has solution

$$\psi(x) = e^x \qquad \textbf{(9.50b)}$$

We define N segments by the points $\{z_1, \ldots, z_{N+1}\}$ from which we determine the points at which we will determine $\psi(y)$ at

$$y_k \equiv \frac{1}{2}(z_k + z_{k+1}) \qquad (9.191)$$

Then eq. 9.168 is approximated by

$$\psi(y_k) = e^{y_k}\left[1 - \sqrt{(1+y_k)} - \sqrt{(1-y_k)}\right] + \frac{1}{2}\sum_{m=1}^{N}\psi(y_m)\int_{z_m}^{z_{m+1}}\frac{e^{y_k-y}}{\sqrt{|y_k-y|}}dy \qquad (9.192)$$

As noted above, we could approximate each of the integrals by "smoothing out" the singularity at $y = y_k$ in the integrands by writing

$$\int_{z_m}^{z_{m+1}}\frac{e^{y_k-y}}{\sqrt{|y_k-y|}}dy = \int_{z_m}^{z_{m+1}}\frac{e^{y_k-y}-1}{\sqrt{|y_k-y|}}dy + \int_{z_m}^{z_{m+1}}\frac{1}{\sqrt{|y_k-y|}}dy \simeq$$
$$\sum_{n=1}^{N} w_n\frac{e^{y_k-y_n}-1}{\sqrt{|y_k-y_n|}} + \int_{z_m}^{z_{m+1}}\frac{1}{\sqrt{|y_k-y|}}dy \qquad (9.193)$$

and easily evaluating the second integral for each of the cases $y_k \geq z_{m+1}$, $z_{m+1} \geq y_k \geq z_{m+1}$, and $y_k \leq z_m$.

To illustrate another approach, we have evaluated the integrals for these three cases by expanding the exponential in a Taylor series to approximate

$$\int_{z_m}^{z_{m+1}}\frac{e^{y_k-y}}{\sqrt{|y_k-y|}}dy \simeq \sum_{n=0}^{4}\frac{1}{n!}\int_{z_m}^{z_{m+1}}\frac{(y_k-y)^n}{\sqrt{|y_k-y|}}dy \qquad (9.194)$$

A sample of the results we obtain are given below.

For $N = 10$ segments, we find

$$\begin{pmatrix} \psi(-.9) \\ \psi(-.5) \\ \psi(.1) \\ \psi(.9) \end{pmatrix} = \begin{pmatrix} 0.51386 \\ 0.67232 \\ 1.00968 \\ 2.08540 \end{pmatrix} \tag{9.195a}$$

which is a poor approximation to

$$\begin{pmatrix} \psi_{exact}(-.9) \\ \psi_{exact}(-.5) \\ \psi_{exact}(.1) \\ \psi_{exact}(.9) \end{pmatrix} = \begin{pmatrix} 0.40657 \\ 0.60653 \\ 1.10517 \\ 2.45960 \end{pmatrix} \tag{9.195b}$$

These results differ from the exact values by an average of 13.2% with a maximum error of 26.4%.

For $N = 20$ segments, we obtain

$$\begin{pmatrix} \psi(-.95) \\ \psi(-.55) \\ \psi(.05) \\ \psi(.45) \\ \psi(.85) \end{pmatrix} = \begin{pmatrix} 0.49036 \\ 0.64835 \\ 0.97569 \\ 1.33916 \\ 2.24827 \end{pmatrix} \tag{9.196a}$$

the exact values of which are

$$\begin{pmatrix} \psi_{exact}(-.95) \\ \psi_{exact}(-.55) \\ \psi_{exact}(.05) \\ \psi_{exact}(.45) \\ \psi_{exact}(.85) \end{pmatrix} = \begin{pmatrix} 0.38674 \\ 0.57695 \\ 1.05127 \\ 1.56831 \\ 2.58571 \end{pmatrix} \tag{9.196b}$$

The average and maximum errors of these results are 12.6% and 26.8%, respectively.

These results indicate that this cardinal spline interpolation method converges very slowly. Thus, the results of the Lagrange interpolation method are much more accurate for interpolation over a small set of points. □

Schlitt's method

A method, proposed by Schlitt, D.W., (1968), involves "smoothing out" the singularity by subtracting and adding the unknown function $\psi(x)$ to obtain

$$\psi(x) = \psi_0(x) + \lambda \int_a^b \frac{L(x,y)}{|x-y|^p}[\psi(y) - \psi(x)]dy + \lambda\psi(x)\int_a^b \frac{L(x,y)}{|x-y|^p}dy \tag{9.197}$$

With

$$J(x) \equiv \int_a^b \frac{L(x,y)}{|x-y|^p}dy \tag{9.198}$$

eq. 9.197 becomes

$$\psi(x)[1 - \lambda J(x)] = \psi_0(x) + \lambda \int_a^b \frac{L(x,y)}{|x-y|^p}[\psi(y) - \psi(x)]dy \tag{9.199}$$

If $\psi(x)$ is analytic at all $y \,\varepsilon\, [a, b]$, $\psi(y) - \psi(x)$ can be expanded in a Taylor series, all terms of which contain $(x - y)^n$ with $n \geq 1$. Therefore, the integrand of the integral on the right hand side of eq. 9.199 can be approximated by a quadrature sum. With x set to each point in the abscissae set, we obtain

$$\psi(y_k)[1 - \lambda J(y_k)] = \psi_0(y_k) + \lambda \sum_{\substack{m=1 \\ m \neq k}}^{N} \frac{L(y_k, y_m)}{|y_k - y_m|^p}[\psi(y_m) - \psi(y_k)] \tag{9.200a}$$

where, because $p < 1$, all terms in the series for $[\psi(y) - \psi(x)]/|x - y|^p$ have positive powers of $(x - y)$. Therefore, the $m = k$ term in the sum is zero.

This results in the set of equations

$$\psi(y_k)\left[1 - \lambda J(y_k) + \lambda \sum_{\substack{m=1 \\ m \neq k}}^{N} \frac{L(y_k, y_m)}{|y_k - y_m|^p}\right] - \lambda \sum_{\substack{m=1 \\ m \neq k}}^{N} \frac{L(y_k, y_m)}{|y_k - y_m|^p}\psi(y_m) = \psi_0(y_k) \tag{9.200b}$$

the solution to which is obtained by standard methods. An example of this is left as an exercise for the reader (see Problem 16).

9.6 Fredholm Equations with Kernels Containing a Pole Singularity

Integral equations in which the kernel has a pole (or *Cauchy*) singularity are of the form

$$\psi(x) = \psi_0(x) + \lambda \int_a^b \frac{L(x,y)}{(y-z)_P} \psi(y)dy \quad a \le z \le b \tag{9.201}$$

where the subscript P indicates that the integral is a principal value integral. z can either be a constant or a function of x.

When the singularity is at a constant value of y, it is called a *fixed pole*. An example of such an equation is the *Lippmann–Schwinger* equation for the scattering wave function of a non-relativistic particle (Lippmann, B., and Schwinger, J., 1950). With E a constant related to the energy of the incident particle, the form of this equation is

$$\psi(x) = \psi_0(x) - \frac{2}{\pi} \int_0^\infty \frac{L(x,y)}{(y-E)_P} \psi(y)dy \tag{9.202a}$$

An integral equation with a Cauchy kernel that is developed from quantum field theory is the *Omnes equation* (Omnes, R., 1958)

$$\psi(x) = \psi_0(x) + \frac{1}{\pi} \int_1^\infty \frac{L(y)}{(y-x)_P} \psi(y)dy \tag{9.202b}$$

With x a variable, a pole singularity at $y = x$ is a *variable* or *movable pole*.

Solution to an equation with a fixed pole

Referring to the Lippmann–Schwinger equation, we note that when we approximate integrals by quadrature sums, a semi-infinite interval $[a, \infty]$ suggests that we might use Laguerre quadratures. However, as noted in ch. 4, unless the integrand of an integral explicitly contains e^{-y}, using a Gauss–Laguerre quadrature is less reliable than transforming the range of integration to [–1, 1] and using a Gauss–Legendre quadrature. As such, with $z =$ constant, we consider eq. 9.201 in the form

$$\psi(x) = \psi_0(x) + \lambda \int_{-1}^1 \frac{L(x,y)}{(y-z)_P} \psi(y)dy \quad -1 \le z \le 1 \tag{9.203}$$

To accurately approximate the solution to this integral equation, we write eq. 9.203 as

$$\psi(x) = \psi_0(x) + \lambda \int_{-1}^{1} \frac{[L(x,y) - L(x,z)]}{(y-z)} \psi(y)dy + \lambda L(x,z) \int_{-1}^{1} \frac{1}{(y-z)_P} \psi(y)dy \tag{9.204}$$

Since the integrand of the first integral is no longer singular, the integral is well approximated by a Gauss–Legendre quadrature sum

$$\int_{-1}^{1} \frac{[L(x,y) - L(x,z)]}{(y-z)} \psi(y)dy \simeq \sum_{m=1}^{N} w_m \frac{[L(x,y_m) - L(x,z)]}{(y_m - z)} \psi(y_m) \tag{9.205}$$

To approximate the second integral, we express $\psi(y)$ by an interpolating polynomial

$$\psi(y) \simeq \sum_{m=1}^{M} v_m(y)\psi(Y_m) \tag{\textbf{\textit{9.162}}}$$

where $M \leq N$ and the set $\{Y_m\}$ over which the interpolation is constructed is a subset (possibly the entire set) of $\{y_m\}$, the quadrature abscissae.

We note that $\{Y_m\}$ being a subset of the abscissae of the N-point quadrature rule are some of the zeros of the Legendre polynomial $P_N(y)$. Thus, referring to eq. 1.15, we can express the interpolating function as

$$v_m(y) = \frac{P_N(y)}{(y - Y_m)P'_N(Y_m)} \tag{9.206}$$

and the second integral of eq. 9.204 becomes

$$\begin{aligned} \int_{-1}^{1} \frac{1}{(y-z)_P} \psi(y)dy &\simeq \sum_{m=1}^{M} \frac{\psi(Y_m)}{P'_N(Y_m)} \int_{-1}^{1} \frac{P_N(y)}{(y-z)_P(y-Y_m)} dy \\ &= -\sum_{m=1}^{M} \frac{\psi(Y_m)}{P'_N(Y_m)} \frac{1}{(Y_m - z)} \int_{-1}^{1} \left[\frac{P_N(y)}{(Y_m - y)} - \frac{P_N(y)}{(z-y)_P} \right] dy \end{aligned} \tag{9.207}$$

The advantage of this form of the second integral is that each term can be expressed in terms of *the Legendre function of the second kind*, Q_N, many properties of which are well established (see, for example, Cohen, H., 1992, pp. 299–306 and pp. 370–371). The *Neumann representation* of Q_N is given by

$$Q_N(z) = \frac{1}{2}\int_{-1}^{1} \frac{P_N(y)}{(z-y)} dy \tag{9.208}$$

Then, eq. 9.207 can be written as

$$\int_{-1}^{1} \frac{1}{(y-z)_P} \psi(y) dy \simeq -2\sum_{m=1}^{M} \frac{\psi(Y_m)}{P'_N(Y_m)} \frac{[Q_N(Y_m) - Q_N(z)]}{(Y_m - z)} \tag{9.209}$$

Combining the results given in eqs. 9.204 and 9.208, the approximated integral equation is

$$\begin{aligned}\psi(x) = \psi_0(x) + \lambda \sum_{m=1}^{N} w_m \frac{[L(x,y_m) - L(x,z)]}{(y_m - z)} \psi(y_m) \\ - 2\lambda L(x,z) \sum_{m=1}^{M} \frac{\psi(Y_m)}{P'_N(Y_m)} \frac{[Q_N(Y_m) - Q_N(z)]}{(Y_m - z)}\end{aligned} \tag{9.210}$$

which is then solved by standard methods.

If z has a value close to one of the quadrature abscissae y_m, such that for some small ε

$$|y_m - z| < \varepsilon \tag{9.211}$$

that term in the first sum is replaced by

$$\frac{[L(x,y_m) - L(x,z)]}{(y_m - z)} \to \left.\frac{\partial L(x,y)}{\partial y}\right|_{y=z} \tag{9.212}$$

If y_m, is also one of the interpolation points Y_m, that term in the second sum is replaced by

$$\frac{[Q_N(Y_m) - Q_N(z)]}{(Y_m - z)} \to Q'_N(z) \tag{9.213}$$

From the author’s experience, it is found that the most accurate method of approximating the $P_N(y)$, $Q(y)$, $P'_N(y)$, and $Q'_N(y)$ is from the recurrence relations satisfied by these functions. The Legendre polynomials satisfy

$$(\ell+2)P_{\ell+2}(y) - y(2\ell+3)P_{\ell+1}(y) + (\ell+1)P_\ell(y) = 0 \tag{9.214a}$$

for $\ell \geq 0$. Taking one derivative of this expression, we obtain

$$(\ell+2)P'_{\ell+2}(y) - y(2\ell+3)P'_{\ell+1}(y) + (\ell+1)P'_{\ell}(y) = (2\ell+3)P_{\ell+1}(y) \quad (9.214b)$$

By substituting the expression of eq. 9.214a into the Neumann expression for $Q_N(z)$ we obtain

$$(\ell+2)\int_{-1}^{1}\frac{P_{\ell+2}(y)}{(z-y)_P}dy - (2\ell+3)\int_{-1}^{1}\frac{yP_{\ell+1}(y)}{(z-y)_P}dy + (\ell+1)\int_{-1}^{1}\frac{P_{\ell}(y)}{(z-y)_P}dy = 0 \quad (9.215a)$$

Referring to eq. 9.210, this can be written as

$$2(\ell+2)Q_{\ell+2}(z) - (2\ell+3)\int_{-1}^{1}\frac{(y-z)P_{\ell+1}(y)}{(z-y)_P}dy$$
$$-(2\ell+3)z\int_{-1}^{1}P_{\ell+1}(y)dy + 2(\ell+1)Q_{\ell}(z) = 0 \quad (9.215b)$$

The Legendre polynomials satisfy the *orthonormalization condition*

$$\int_{-1}^{1}P_{\ell}(y)P_m(y)dy = \frac{2}{(2m+1)}\delta_{\ell m} \quad (9.216)$$

Therefore, with

$$P_0(z) = 1 \quad (9.217)$$

and $\ell \geq 0$, we obtain

$$\int_{-1}^{1}\frac{(y-z)P_{\ell+1}(y)}{(z-y)_P}dy = -\int_{-1}^{1}P_0(y)P_{\ell+1}(y)dy = 0 \quad (9.218)$$

Thus, from eq. 9.215b, the Legendre function of the second kind satisfies

$$(\ell+2)Q_{\ell+2}(y) - y(2\ell+3)Q_{\ell+1}(y) + (\ell+1)Q_{\ell}(y) = 0 \quad (9.219a)$$

and from the first derivative of this expression,

$$(\ell+2)Q'_{\ell+2}(y) - y(2\ell+3)Q'_{\ell+1}(y) + (\ell+1)Q'_{\ell}(y) = (2\ell+3)Q_{\ell+1}(y) \quad (9.219b)$$

Example 9.18: Solution to a Fredholm equation with a fixed pole singularity in the kernel using interpolation

(a) The solution to the Lippmann–Schwinger type of equation

$$\psi(x) = 2e^x + \frac{1}{\ell n(3)} \int_{-1}^{1} \frac{e^{(x-y)}}{(y - \frac{1}{2})_P} \psi(y) dy \tag{9.220}$$

is

$$\psi(x) = e^x \tag{\textbf{\textit{9.50b}}}$$

We write eq. 9.220 as

$$\psi(x)=2e^x+\frac{1}{\ell n(3)}\int_{-1}^{1} \frac{\left[e^{(x-y)}-e^{(x-\frac{1}{2})}\right]}{(y-\frac{1}{2})}\psi(y)dy+\frac{e^{(x-\frac{1}{2})}}{\ell n(3)}\int_{-1}^{1}\frac{1}{(y-\frac{1}{2})_P}\psi(y)dy \tag{9.221}$$

Using the 17-point Gauss–Legendre quadrature rule with abscissae $\{y_m\}$ to approximate the first integral, we obtain

$$\int_{-1}^{1} \frac{\left[e^{(x-y)} - e^{(x-\frac{1}{2})}\right]}{(y - \frac{1}{2})} \psi(y) dy \simeq \sum_{m=1}^{17} w_m \frac{\left[e^{(x-y_m)} - e^{(x-\frac{1}{2})}\right]}{(y_m - \frac{1}{2})} \psi(y_m) \tag{9.222}$$

Approximating $\psi(y)$ in the second integral by a polynomial over$\{Y_m\}$, an M-point subset of $\{y_m\}$, we have

$$\begin{aligned}\int_{-1}^{1} \frac{1}{\left(y - \frac{1}{2}\right)_P} \psi(y) dy &\simeq -2 \sum_{m=1}^{M} \frac{\psi(Y_m)}{P'_{17}(Y_m)} \frac{\left[Q_{17}(Y_m) - Q_{17}\left(\frac{1}{2}\right)\right]}{\left(Y_m - \frac{1}{2}\right)} \\ &= -2 \sum_{m=1}^{17} C_m \frac{\psi(Y_m)}{P'_{17}(Y_m)} \frac{\left[Q_{17}(Y_m) - Q_{17}\left(\frac{1}{2}\right)\right]}{\left(Y_m - \frac{1}{2}\right)}\end{aligned} \tag{9.223}$$

where C_m is 0 if y_m is not in the subset $\{Y_m\}$ and is 1 if y_m is in $\{Y_m\}$. Setting x to each quadrature point in the set $\{y_k\}$, eq. 9.221 is approximated by

$$\psi(y_k) = 2e^{y_k} + \frac{1}{\ell n(3)}\left[\sum_{m=1}^{17} w_m \frac{\left[e^{(y_k - y_m)} - e^{(y_k - \frac{1}{2})}\right]}{\left(y_m - \frac{1}{2}\right)}\right.$$
$$\left. - \sum_{m=1}^{17} C_m \frac{\psi(y_m)}{P'_{17}(y_m)} \frac{\left[Q_{17}(y_m) - Q\left(\frac{1}{2}\right)\right]}{\left(y_m - \frac{1}{2}\right)}\right] \tag{9.224}$$

With $\{Y_m\} = \{y_m\}$, so that all $C_m = 1$, we solve eq. 9.224 by matrix inversion. We obtain results that are essentially the exact values of $\psi(x)$. Each of the computed values of $\psi(x)$ differ from that value of e^x by a difference of $1.3 \times 10^{-13}\%$.We do find that the accuracy of the results are sensitive to the points over which $\psi(y)$ is interpolated. For example, if $\{Y_m\} = \{y_1, y_3, \ldots, y_{15}, y_{17}\}$ (the odd index points of $\{y_m\}$), a sample of typical results are

$$\psi(y_k) \simeq 2e^{y_k} + \frac{1}{\ell n(3)}\left[\sum_{m=1}^{17} w_m \frac{\left[e^{(y_k - y_m)} - e^{(y_k - \frac{1}{2})}\right]}{(y_m - \frac{1}{2})}\psi(y_m)\right.$$
$$\left. -2e^{(y_k - \frac{1}{2})}\sum_{m=1}^{5} \frac{1}{P'_{17}(Y_m)} \frac{\left[Q_{17}(Y_m) - Q_{17}(\frac{1}{2})\right]}{(Y_m - \frac{1}{2})}\psi(Y_m)\right]$$

$$= 2e^{y_k} + \frac{1}{\ell n(3)}\sum_{m=1}^{17}\left[w_m \frac{\left[e^{(y_k - y_m)} - e^{(y_k - \frac{1}{2})}\right]}{(y_m - \frac{1}{2})}\psi(y_m)\right.$$
$$\left. - 2e^{(y_k - \frac{1}{2})} C_m \frac{1}{P'_{17}(y_m)} \frac{\left[Q_{17}(y_m) - Q_{17}(\frac{1}{2})\right]}{(y_m - \frac{1}{2})}\psi(y_m)\right] \tag{9.225}$$

Solving this by standard methods, we obtain

$$\begin{pmatrix} \psi(0.99058) \\ \psi(0.51269) \\ \psi(0.00000) \\ \psi(-0.65767) \\ \psi(-0.95068) \end{pmatrix} = \begin{pmatrix} 2.08635 \\ 1.29373 \\ 0.77479 \\ 0.40139 \\ 0.29944 \end{pmatrix} \tag{9.226a}$$

which differs from

$$\begin{pmatrix} \psi_{exact}(0.99058) \\ \psi_{exact}(0.51269) \\ \psi_{exact}(0.00000) \\ \psi_{exact}(-0.65767) \\ \psi_{exact}(-0.95068) \end{pmatrix} = \begin{pmatrix} 2.69278 \\ 1.66978 \\ 1.00000 \\ 0.51806 \\ 0.38648 \end{pmatrix} \tag{9.226b}$$

by approximately 22.5%.This example suggests that accurate results are only obtained by interpolating over the entire quadrature set. □

Solution to an equation with a variable pole

We consider an integral equation with a kernel that has a variable pole by setting $z = x$ in eq. 9.201 to obtain

$$\psi(x) = \psi_0(x) + \lambda \int_{-1}^{1} \frac{L(x,y)}{(y-x)_P} \psi(y)dy - 1 \leq x \leq 1 \tag{9.227a}$$

As we did with the integral equation with a kernel with a fixed pole, we write eq. 9.227a as

$$\psi(x) = \psi_0(x) + \lambda \int_{-1}^{1} \frac{[L(x,y) - L(x,x)]}{(y-x)} \psi(y)dy + \lambda L(x,x) \int_{-1}^{1} \frac{1}{(y-x)_P} \psi(y)dy \tag{9.227b}$$

then approximate the first integral by the quadrature sum

$$\int_{-1}^{1} \frac{[L(x,y) - L(x,x)]}{(y-x)} \psi(y)dy \simeq \sum_{m=1}^{N} w_m \frac{[L(x,y_m) - L(x,x)]}{(y_m - x)} \psi(y_m) \tag{9.228}$$

For the second integral, we refer to the results of example 9.18 and interpolate $\psi(y)$ over the entire set of quadrature abscissae $\{y_m\}$ as

$$\psi(y) \simeq \sum_{m=1}^{N} v_m(y)\psi(y_m) \qquad \textbf{\textit{(9.162)}}$$

We again construct the interpolating functions in terms of Legendre polynomials as

$$v_m(y) = \frac{P_N(y)}{(y - y_m)P'_N(y_m)} \qquad \textbf{\textit{(9.206)}}$$

Then,

$$\begin{aligned}
\int_{-1}^{1} \frac{1}{(y-x)_P}\psi(y)dy &\simeq \sum_{m=1}^{N} \frac{\psi(y_m)}{P'_N(y_m)} \int_{-1}^{1} \frac{P_N(y)}{(y-x)_P(y-y_m)}dy \\
= -\sum_{m=1}^{N} \frac{\psi(y_m)}{P'_N(y_m)} \frac{1}{(y_m - x)} &\int_{-1}^{1} \left[\frac{P_N(y)}{(y_m - y)} - \frac{P_N(y)}{(x-y)_P}\right]dy \\
= -2\sum_{m=1}^{M} \frac{\psi(y_m)}{P'_N(y_m)} &\frac{[Q_N(y_m) - Q_N(x)]}{(y_m - x)}
\end{aligned} \qquad (9.229)$$

Combining the results given in eqs. 9.228 and 9.229, the approximated integral equation is

$$\begin{aligned}
\psi(x) \simeq \psi_0(x) + \lambda \sum_{m=1}^{N} w_m \frac{[L(x, y_m) - L(x,x)]}{(y_m - x)}\psi(y_m) \\
- 2\lambda L(x,x) \sum_{m=1}^{N} \frac{\psi(y_m)}{P'_N(y_m)} \frac{[Q_N(y_m) - Q_N(x)]}{(y_m - x)}
\end{aligned} \qquad (9.230)$$

By setting x to each y_k of the quadrature abscissae, we obtain a set of N linear equations for $\psi(y_k)$ which is solved by standard methods. Again, the values of the Legendre functions are found from the recurrence relations of eqs. 9.214 and 9.219. The $m = k$ terms in the two sums must be replaced by

$$\frac{[L(y_k, y_m) - L(y_k, y_k)]}{(y_m - y_k)} \to \left.\frac{\partial L(y_k, y)}{\partial y}\right|_{y=y_k} \qquad (9.231a)$$

and

$$\frac{[Q_N(y_m) - Q_N(y_k)]}{(y_m - y_k)} \to Q'_N(y_k) \qquad (9.231b)$$

Example 9.19: Solution to a Fredholm equation with a variable pole singularity in the kernel using interpolation

The solution to the Omnes type of equation

$$\psi(x) = e^x\left[1 + \ell n\left(\frac{1+x}{1-x}\right)\right] + \int_{-1}^{1} \frac{e^{(x-y)}}{(y-x)_P}\psi(y)dy \tag{9.232}$$

is

$$\psi(x) = e^x \tag*{(\textbf{\textit{9.50b}})}$$

We write eq. 9.232 as

$$\psi(x) = e^x\left[1 + \ell n\left(\frac{1+x}{1-x}\right)\right] + \int_{-1}^{1} \frac{\left[e^{(x-y)} - 1\right]}{(y-x)}\psi(y)dy + \int_{-1}^{1} \frac{1}{(y-x)_P}\psi(y)dy \tag{9.233a}$$

Using the 17-point Gauss–Legendre quadrature rule again, this integral equation is approximated by

$$\begin{aligned}\psi(y_k) \simeq\ & e^{y_k}\left[1 + \ell n\left(\frac{1+y_k}{1-y_k}\right)\right] \\ & + \sum_{m=1}^{17}\left[w_m \frac{\left[e^{(y_k-y_m)} - 1\right]}{(y_m - y_k)} - 2\frac{1}{P'_{17}(y_m)}\frac{[Q_{17}(y_m) - Q_{17}(y_k)]}{(y_m - y_k)}\right]\psi(y_m)\end{aligned} \tag{9.233b}$$

We again obtain highly accurate results. The average difference of the 17 values from the corresponding exact values is $5.5 \times 10^{-13}\%$ and the individual difference of each value of $\psi(y_k)$ is of this order of magnitude. □

Problems

1. The homogeneous Fredholm integral equation of the first kind $\dfrac{1 - e^{(1+x)}}{(1+x)} = \displaystyle\int_0^1 e^{-xy}\psi(y)dy$ has solution $\psi(x) = e^{-x}$
   Transform the range of integration to [–1, 1] and expand $\psi_0(x)$, $\psi(x)$, and $K(x,y)$ in series over Legendre polynomials. From this, find the approximate solution to the above equation by approximating $\psi(x) \simeq \sum_{m=0}^{2} \beta_m P_m(x)$ at $x = \{0.20, 0.60, 1.00\}$. Compare this approximate solution to the exact values at these points

2. The Fredholm integral equation of the first kind $\sqrt{\dfrac{\pi}{1+x^2}} = \int_{-\infty}^{\infty} e^{-x^2y^2}\psi(y)dy$ has solution $\psi(x) = e^{-x^2}$

   (a) Expand each of $\psi_0(x)$, $\psi(x)$, and $K(x,y)$ in three term series over approximations Hermite polynomials. From this, find the approximate solution for $\psi(x)$ at $x = \{2, 10, 50\}$.

   (b) Transform the range of integration to [–1, 1] and expand $\psi_0(x)$, $\psi(x)$, and $K(x,y)$ in series over Legendre polynomials. From this, find the approximate solution to the above equation for a three term series approximation to $\psi(x)$ at $x = \{2, 10, 50\}$. At these points, determine the values of $1 - x^2 + \dfrac{x^4}{2!}$ the first three terms in the MacLaurin series for the solution. Compare these values to the results of parts (a) and (b).

3. The solution to the integral equation $\psi(x) = 1 - \dfrac{\sin(\pi x)}{\pi} + \int_0^1 x\cos(\pi xy)\psi(y)dy$ is $\psi(x) = 1$
   Find an approximate solution to this equation using

   (a) a 3-point Gauss–Legendre quadrature rule
   (b) a three term Neumann series in λ, then setting $\lambda = 1$
   (c) a 3-point interpolation of $x\cos(\pi xy)$ of the form $x\cos(\pi xy) \simeq \sum_{k=1}^{3} x\cos(\pi xy_k)v_k(y)$ over the points $y = \{0.0,\ 0.5,\ 1.0\}$. With $q(y) = \cos(\pi y)$ interpolate using $v_k(y) = \dfrac{(q(y) - q(y_r))(q(y) - q(y_s))}{(q(y_k) - q(y_r))(q(y_k) - q(y_s))}$ with k, r, and s taking the values 1, 2, and 3, and with $k \neq r \neq s$.

4. The solution to the integral equation $\psi(x) = 1 - \dfrac{2\sin(\pi x)}{\pi} + \int_{-1}^{1} x\cos(\pi xy)\psi(y)dy$ is $\psi(x) = 1$
   Find an approximate solution to this equation by expanding the functions ψ and ψ_0 and the kernel $K(x, y)$ in series over the three lowest order Legendre polynomials as in example 9.5.

5. Find the exact solution to $\psi(x) = x + \int_0^1 (3x + y)\psi(y)dy$ using the fact that the kernel is degenerate.

6. Find the two exact eigensolutions to $\psi(x) = \lambda \int_0^1 \sin[\pi(x + y)]\psi(y)dy$

7. Find approximations to the eigensolutions of the homogeneous Fredholm equation of the second kind given in Problem 6 by

   (a) approximating the integral by a 3-point Gauss–Legendre quadrature rule.
   (b) Interpolating the kernel over the points $y = \{0.0, 0.5, 1.0\}$ as described in example 9.9, using $q(y) = \sin(\pi y)$

8. Find the Volterra integral equation of the second kind $\psi(x) = \psi_0(x) + \int_0^x e^{(x-y)}\psi(y)dy$ that has the same solution as the Volterra equation of the first kind $\int_0^x e^{(x-y)}\psi(y)dy = e^x(e^x - 1)$

9. Find the first three non-zero terms in the Taylor series for $\psi(x)$ which satisfies $\psi(x) = x^2 + \int_0^x x^2y^2\psi(y)dy$ which has solution $\psi_{exact}(x) = xe^{\frac{1}{5}x^5}$
10. Use the method of approximating $\psi(y)$ by the constant $\alpha\psi(x_k) + \beta\psi(x_{k+1})$ for the Volterra integral equation $\psi(x) = 1 + x + \int_0^x e^{(x-y)}\psi(y)dy$ the solution to which is $\psi(x) = \frac{1}{4} + \frac{x}{2} + \frac{3}{4}e^{2x}$
    Find the approximate solution to this equation at $x = 0.15$ for
    (a) $\alpha = 0, \beta = 1$
    (b) $\alpha = \beta = 1/2$
    (c) $\alpha = 1/3, \beta = 2/3$
    (d) $\alpha = 2/3, \beta = 1/3$
11. (a) Show that the solution to the Volterra equation $\psi(x) = x^2 - \int_1^x \frac{x}{y}\psi(y)dy$ is $\psi(x) = x$
    (b) By approximating $\psi(y)$ by the constant $[\psi(x_k) + \psi(x_{k+1})]/2$, estimate this solution at $x = 1.15$ with $\Delta x = 0.05$.
12. Approximate the solution to $\psi(x) = \left[x - \sqrt{1+x}(8x^2 - 4x + 3) - \sqrt{1-x}\right.$ $\left.(8x^2 + 4x + 3)\right] + \frac{15}{2}\int_{-1}^{1} \frac{xy}{\sqrt{|x-y|}}\psi(y)dy$ by approximating $\psi(x)$ by a polynomial Lagrange interpolation. Compare your result to the analytic solution $\psi(x) = x$.
13. One can show that the solution to $\psi(x) = e^x[3 - (1+x)\ell n(1+x) - (1+x)$ $\ell n(1+x)] + \int_{-1}^{1} e^{x-y}\ell n|x-y|\psi(y)dy$ is $\psi(x) = e^x$
    Using the Lagrange polynomial interpolation of $\psi(y)$ over the three points {–1, 0, 1}, obtain an approximate solution to the above equation. Use these approximate values to estimate $\psi(-0.5)$ and $\psi(0.5)$.
14. By approximating $e^{-y}\psi(y)$ by a Lagrange polynomial interpolation over the three points {–1, 0, 1}, obtain an approximate solution to $\psi(x) = e^x[3 - (1+x)\ell n(1+x) - (1+x)\ell n(1+x)] + \int_{-1}^{1} e^{x-y}\ell n|x-y|\psi(y)dy$
    Compare your results to the exact values $\psi(x) = e^x$ at these three points.
15. Approximate the solution to $\psi(x) = \left[x - \sqrt{1+x}\left(8x^2 - 4x + 3\right) - \sqrt{1-x}\right.$ $\left.\left(8x^2 + 4x + 3\right)\right] + \frac{15}{2}\int_{-1}^{1} \frac{xy}{\sqrt{|x-y|}}\psi(y)dy$ by writing this equation as $\psi(x) =$ $\left[x - \sqrt{1+x}(8x^2 - 4x + 3) - \sqrt{1-x}(8x^2 + 4x + 3)\right] + \frac{15}{2}\int_{-1}^{1} \frac{(xy-x^2)}{\sqrt{|x-y|}}\psi(y)dy +$ $\frac{15}{2}x^2\int_{-1}^{1} \frac{1}{\sqrt{|x-y|}}\psi(y)dy$ then estimating the value of the first integral by a Gauss–Legendre quadrature rule, and interpolating $\psi(y)$ in the second integral by a polynomial Lagrange interpolation over four points that are a subset of

the quadrature abscissae and are close to {–1, –1/3, 1/3, 1}. Compare your approximate results to the analytic solution $\psi(x) = x$.

16. Use the Schlitt method described above to approximate the solution to $\psi(x) = e^x\left[1 - \sqrt{(1+x)} - \sqrt{(1-x)}\right] + \frac{1}{2}\int_{-1}^{1}\frac{e^{x-y}}{\sqrt{|x-y|}}\psi(y)dy$

    Compare your results to the analytic solution $\psi(x) = e^x$.

17. The integral equation $\psi(x) = \dfrac{\ell n(3)}{4}x + \displaystyle\int_{-1}^{1}\frac{xy}{(y-\frac{1}{2})_P}\psi(y)dy$ has solution $\psi(x) = x$

    Approximate the solution to this integral equation by interpolating $\psi(y)$ as $\psi(y) = \sum\limits_{m=1}^{N}\psi(y_m)v_m(y)$ with $v_m(y) = \frac{P_N(y)}{P'_N(y_m)(y-y_m)}$ where $P_N(y)$ is the Legendre polynomial of order N. Take $N = 5$.

18. The integral equation $\psi(x) = x\left[1 - 2x + x^2\ell n\left(\dfrac{1+x}{1-x}\right)\right] + \displaystyle\int_{-1}^{1}\frac{xy}{(y-x)_P}\psi(y)dy$ has solution $\psi(x) = x$

    Approximate the solution to this integral equation by interpolating $\psi(y)$ as $\psi(y) = \sum\limits_{m=1}^{5}\psi(y_m)v_m(y)$ with $v_m(y) = \dfrac{P_5(y)}{P'_5(y_m)(y-y_m)}$ where $P_N(y)$ is the Legendre polynomial of order N.

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