Preface

All three authors of the present book have long-standing experience in teaching graduate courses in multivariate analysis (MVA). These experiences have taught us that aside from distribution theory, projections and the singular value decomposition (SVD) are the two most important concepts for understanding the basic mechanism of MVA. The former underlies the least squares (LS) estimation in regression analysis, which is essentially a projection of one subspace onto another, and the latter underlies principal component analysis (PCA), which seeks to find a subspace that captures the largest variability in the original space. Other techniques may be considered some combination of the two.

This book is about projections and SVD. A thorough discussion of generalized inverse (g-inverse) matrices is also given because it is closely related to the former. The book provides systematic and in-depth accounts of these concepts from a unified viewpoint of linear transformations in finite dimensional vector spaces. More specifically, it shows that projection matrices (projectors) and g-inverse matrices can be defined in various ways so that a vector space is decomposed into a direct-sum of (disjoint) subspaces. This book gives analogous decompositions of matrices and discusses their possible applications.

This book consists of six chapters. Chapter 1 overviews the basic linear algebra necessary to read this book. Chapter 2 introduces projection matrices. The projection matrices discussed in this book are general oblique projectors, whereas the more commonly used orthogonal projectors are special cases of these. However, many of the properties that hold for orthogonal projectors also hold for oblique projectors by imposing only modest additional conditions. This is shown in Chapter 3.

Chapter 3 first defines, for an n by m matrix $\boldsymbol{A}$, a linear transformation $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ that maps an element $\boldsymbol{x}$ in the m-dimensional Euclidean space E^m onto an element $\boldsymbol{y}$ in the n-dimensional Euclidean space E^n. Let $\mathrm{Sp}(\boldsymbol{A}) = \{\boldsymbol{y}|\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}\}$ (the range or column space of $\boldsymbol{A}$) and $\mathrm{Ker}(\boldsymbol{A}) = \{\boldsymbol{x}|\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}\}$ (the null space of $\boldsymbol{A}$). Then, there exist an infinite number of the subspaces V and W that satisfy

$$E^n = \mathrm{Sp}(\boldsymbol{A}) \oplus W \text{ and } E^m = V \oplus \mathrm{Ker}(\boldsymbol{A}), \tag{1}$$

where $\oplus$ indicates a direct-sum of two subspaces. Here, the correspondence between V and $\mathrm{Sp}(\boldsymbol{A})$ is one-to-one (the dimensionalities of the two subspaces coincide), and an inverse linear transformation from $\mathrm{Sp}(\boldsymbol{A})$ to V can be uniquely defined. Generalized inverse matrices are simply matrix representations of the inverse transformation with the domain extended to E^n. However, there are infinitely many ways in which the generalization can be made, and thus there are infinitely many corresponding generalized inverses $\boldsymbol{A}^-$ of $\boldsymbol{A}$. Among them, an inverse transformation in which $W = \mathrm{Sp}(\boldsymbol{A})^\perp$ (the ortho-complement subspace of $\mathrm{Sp}(\boldsymbol{A})$) and $V = \mathrm{Ker}(\boldsymbol{A})^\perp = \mathrm{Sp}(\boldsymbol{A}')$ (the ortho-complement subspace of $\mathrm{Ker}(\boldsymbol{A})$), which transforms any vector in W to the zero vector in $\mathrm{Ker}(\boldsymbol{A})$, corresponds to the Moore-Penrose inverse. Chapter 3 also shows a variety of g-inverses that can be formed depending on the choice of V and W, and which portion of $\mathrm{Ker}(\boldsymbol{A})$ vectors in W are mapped into.

Chapter 4 discusses generalized forms of oblique projectors and g-inverse matrices, and gives their explicit representations when V is expressed in terms of matrices.

Chapter 5 decomposes $\mathrm{Sp}(\boldsymbol{A})$ and $\mathrm{Sp}(\boldsymbol{A}') = \mathrm{Ker}(\boldsymbol{A})^\perp$ into sums of mutually orthogonal subspaces, namely

$$\mathrm{Sp}(\boldsymbol{A}) = E_1 \dot{\oplus} E_2 \dot{\oplus} \cdots \dot{\oplus} E_r$$

and

$$\mathrm{Sp}(\boldsymbol{A}') = F_1 \dot{\oplus} F_2 \dot{\oplus} \cdots \dot{\oplus} F_r,$$

where $\dot{\oplus}$ indicates an orthogonal direct-sum. It will be shown that E_j can be mapped into F_j by $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$ and that F_j can be mapped into E_j by $\boldsymbol{x} = \boldsymbol{A}'\boldsymbol{y}$. The singular value decomposition (SVD) is simply the matrix representation of these transformations.

Chapter 6 demonstrates that the concepts given in the preceding chapters play important roles in applied fields such as numerical computation and multivariate analysis.

Some of the topics in this book may already have been treated by existing textbooks in linear algebra, but many others have been developed only recently, and we believe that the book will be useful for many researchers, practitioners, and students in applied mathematics, statistics, engineering, behaviormetrics, and other fields.

This book requires some basic knowledge of linear algebra, a summary of which is provided in Chapter 1. This, together with some determination on the part of the reader, should be sufficient to understand the rest of the book. The book should also serve as a useful reference on projectors, generalized inverses, and SVD.

In writing this book, we have been heavily influenced by Rao and Mitra's (1971) seminal book on generalized inverses. We owe very much to Professor

C. R. Rao for his many outstanding contributions to the theory of g-inverses and projectors. This book is based on the original Japanese version of the book by Yanai and Takeuchi published by Todai-Shuppankai (University of Tokyo Press) in 1983. This new English edition by the three of us expands the original version with new material.

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Haruo Yanai
Kei Takeuchi
Yoshio Takane

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Yanai, H.; Takeuchi, K.; Takane, Y.

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