Chapter 2
Variational Formulae

Abstract We study extrinsic geometry (properties depending on the second fundamental form) of a codimension-one foliation subject to ($\mathscr{F}$-truncated) variations of metrics along the leaves. In Sect. 2.3.1 we develop formulae for the deformation of geometric quantities as the Riemannian metric varies along the leaves. Then, in Sect. 2.3.2, we study variation properties of the functionals depending on the principal curvatures of the leaves and the $\mathscr{F}$-truncated families of metrics, in particular, for conformal metrics along the leaves. The last section of Chap. 2 contains applications to umbilical foliations and minimization of the total bending of the unit normal vector field.

2.1 Introduction

The problem of minimizing geometric quantities has been very popular for many years: recall, for example, classical isoperimetric inequalities, Fenchel estimates of total curvature of curves (and some generalizations like these in [29]), and Kuiper's work on tight and taut submanifolds (see [15] and the bibliographies therein).

In the context of foliations, one has several results of Langevin (and co-authors) [26–28] and so on, and the authors' work [47]. In all cases mentioned earlier, one considers a fixed Riemannian manifold and looks for geometric objects (curves, hypersurfaces, foliations) minimizing geometric quantities defined usually as integrals of curvatures of different types. On the other hand, there is some interest ([33, 34, 48, 50], and so on) in prescribing geometric quantities of given objects (say, foliations): given a foliated manifold $(M,\mathscr{F})$ and a geometric quantity Q (function, vector or tensor field) one may search for a Riemannian metric g on M for which a given geometric invariant (say, curvature of some sort) coincides with Q.

In Chap. 2, the authors describe a new approach combining the two we just mentioned: given a foliated manifold $(M,\mathscr{F})$ and a geometric quantity Q (say, integral of a curvature-like invariant) we look for Riemannian metrics which minimize Q in the class of $\mathscr{F}$-truncated metrics (i.e., the unit vector field N

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orthogonal to $\mathscr{F}$ is the same for all metrics of the variation family). Certainly (as in some of the cases mentioned before) such Riemannian structures need not exist, but if they do, they usually have interesting geometric properties.

The key objective of Chap. 2 is to study properties of Riemannian structures minimizing a quantity Q (for $\mathscr{F}$-truncated metrics). Here Q is built of the invariants of extrinsic geometry of $\mathscr{F}$, that is of the second fundamental forms of the leaves and their invariants arising from its characteristic polynomial: symmetric functions of the principal curvatures.

We denote by $\mathscr{M} = \mathscr{M}(M,\mathscr{F},N)$ the space of smooth Riemannian structures of finite volume on M with N being a unit normal to $\mathscr{F}$. Elements of $\mathscr{M}$ will be called *$\mathscr{F}$-truncated metrics*. Let $\mathscr{M}_1 \subset \mathscr{M}$ be the subspace of metrics of unit volume. By $\mathscr{F}\mathscr{M}$ (and $\mathscr{F}\mathscr{M}_1$), we denote the foliation on $\mathscr{M}$ (respectively, $\mathscr{M}_1$) by leaves consisting of metrics conformally equivalent along $\mathscr{F}$.

Given $f \in C^2(\mathbb{R}^n)$, we study variational properties of the functional

$$I_f : g \mapsto \int_M f(\overrightarrow{\tau})\,\mathrm{d}\,\mathrm{vol}_g, \quad \overrightarrow{\tau} = (\tau_1,\dots,\tau_n), \quad g \in \mathscr{M} \tag{2.1}$$

and related functionals (total mean curvatures σ_i, power sums τ_i, etc.). In Sect. 3.7, the function f is the trace of a $(1,1)$-tensor.

2.2 Auxiliary Results

2.2.1 Biregular Foliated Coordinates

The coordinate system described in the following lemma, see [15, Sect. 5.1], is here called a *biregular foliated chart*.

Lemma 2.1. *Let $(M,\mathscr{F},N)$ be a differentiable manifold with a codimension-one foliation $\mathscr{F}$ and a vector field N transversal to $\mathscr{F}$. Then for any $q \in M$ there exists a coordinate system $(x_0,x_1,\dots x_n)$ on a neighborhood $U_q \subset M$ (centered at q) such that the leaves on U_q are given by $\{x_0 = c\}$ (hence the coordinate vector fields $\partial_i = \partial_{x_i}$, $i \ge 1$, are tangent to leaves), and N is directed along $\partial_0 = \partial_{x_0}$ (one may assume $N = \partial_0$ at q).*

If $(M,\mathscr{F},g)$ is a foliated Riemannian manifold and N is the unit normal then in biregular foliated coordinates $(x_0,x_1,\dots,x_n)$, g has the form:

$$g = g_{00}\,\mathrm{d}x_0^2 + \textstyle\sum_{i,j>0} g_{ij}\,\mathrm{d}x_i \mathrm{d}x_j. \tag{2.2}$$

As usual, let g^{ij} denote the entries of the matrix inverse to (g_{ij}) and $g_{ij,k}$ be the derivative of g_{ij} in the direction of ∂_k.

Lemma 2.2. *For a metric (2.2) in biregular foliated coordinates (of a codimension-one foliation $\mathscr{F}$) on (M,g), one has*

$$N = \partial_0/\sqrt{g_{00}} \quad \textit{(the unit normal)}$$

$$\Gamma^j_{i0} = (1/2)\sum\nolimits_s g_{is,0}g^{sj}, \quad \Gamma^i_{00} = -(1/2)\sum\nolimits_s g_{00,s}g^{si}, \quad \Gamma^0_{ij} = -g_{ij,0}/(2\,g_{00}),$$

$$b_{ij} = \Gamma^0_{ij}\sqrt{g_{00}} = -\frac{1}{2}\,g_{ij,0}/\sqrt{g_{00}} \quad \textit{(the second fundamental form)},$$

$$A^j_i = -\Gamma^j_{i0}/\sqrt{g_{00}} = \frac{-1}{2\sqrt{g_{00}}}\sum\nolimits_s g_{is,0}\,g^{sj} \quad \textit{(the Weingarten operator)},$$

$$(b_m)_{ij} = A^{s_1}_{s_2}\ldots A^{s_{m-1}}_{s_m}A^{s_m}_i g_{js_1} \quad \textit{(the mth "power" of } b_{ij}),$$

$$\tau_i = \left(\frac{-1}{2\sqrt{g_{00}}}\right)^i \sum\nolimits_{\{r_a\},\{s_b\}} g_{r_1s_2,0}\cdots g_{r_{i-1}s_i,0}\,g^{s_1r_1}\ldots g^{s_ir_i}.$$

In particular,

$$\tau_1 = \frac{-1}{2\sqrt{g_{00}}}\sum_{r,s} g_{rs,0}\,g^{rs}, \quad \tau_2 = \frac{1}{4\,g_{00}}\sum_{r_1,r_2,s_1,s_2} g_{r_1s_2,0}\,g_{r_2s_1,0}\,g^{s_1r_1}\,g^{s_2r_2}, \quad \text{etc.}$$

Proof. This is standard and left to the reader. For convenience, observe that the formula for A follows from that for b and $A^j_i = \sum_s b_{is}g^{sj}$. Notice that $(A^m)^j_i = \sum_{\{s_l\}} A^j_{s_2}A^{s_2}_{s_3}\ldots A^{s_{m-1}}_{s_m}A^{s_m}_i$. Formulae for b_m follow from the above and $(A^m)^s_i g_{sj} = g(A^m e_i, e_j) = (b_m)_{ij}$. Formulae for τ's follow directly from the above and the equality $\tau_i = \mathrm{Tr}\,(A^i)$. $\square$

For example, we apply Lemma 2.2 to the tensor $h(b)$ of (1.13).

Proposition 2.1. *The tensor $h(b)$ of (1.13) with generic functions f_0, f_1 in biregular foliated coordinates has a form:*

$$h(b)_{ac} = f_0(0)\,g_{ac} - \frac{f_{0,\tau_1}(0)}{2\sqrt{g_{00}}}\left(\sum\nolimits_{i,j} g^{ij}\psi_{ij}\right) g_{ac} + f_1(0)\,\psi_{ac} + o([\psi_{ac}]), \tag{2.3}$$

where $\psi_{ij} = g_{ij,0}$. If f_2 is also generic and $f_0(0) = f_{0,\tau_1}(0) = f_1(0) = 0$ then the second-order approximation of $h(b)$ is

$$h(b)_{ac} = \left(\frac{f_{0,\tau_2}(0)}{4\,g_{00}}\sum_{i,r,s_1,s_2} g^{s_1i}g^{s_2r}\psi_{rs_1}\psi_{is_2} + \frac{f_{0,\tau_1\tau_1}(0)}{8\,g_{00}}\left(\sum_{i,j} g^{ij}\psi_{ij}\right)^2\right) g_{ac}$$
$$-\frac{f_{1,\tau_1}(0)}{2\sqrt{g_{00}}}\left(\sum\nolimits_{i,j} g^{ij}\psi_{ij}\right)\psi_{ac} + \frac{f_2(0)}{4\,g_{00}}\sum\nolimits_{i,j} g^{ij}\,\psi_{ai}\psi_{cj} + o([\psi_{ac}]^2). \tag{2.4}$$

Proof. Take biregular foliated coordinates on M (with the origin at $q\in M$) as in Lemma 2.1. By definition (1.13), the $\mathscr{F}$-components of the tensor are

$$h(b)_{ac}=\sum\nolimits_{m=0}^{n-1} f_m(\overrightarrow{\tau})\,g(A^m e_a,e_c).$$

We neglect the third (and more) order terms with $\psi_{ac}=g_{ac,0}$. By Lemma 2.2 we have $(A^2)^j_i=\sum_r A^j_r A^r_i=\frac{1}{4\,g_{00}}\sum_{s_1,s_2,r}\psi_{rs_1}g^{s_1 j}\psi_{is_2}g^{s_2 r}$ and, in fact, $\tau_i=o([\psi_{ac}]^{i-1})$ for $i>0$. From above we obtain expansions for f_j $(j=0,1,2)$ as:

$$\begin{aligned}
f_0(\overrightarrow{\tau}) &= f_0(\tau_1,\tau_2,0,\dots,0)+o([\psi_{ac}]^2)=f_0(0)\\
&\quad -\frac{1}{2\sqrt{g_{00}}}\,f_{0,\tau_1}(0)\sum\nolimits_{i,j}g^{ij}\psi_{ij}+\frac{1}{8\,g_{00}}\,f_{0,\tau_1\tau_1}(0)\Big(\sum\nolimits_{i,j}g^{ij}\psi_{ij}\Big)^2\\
&\quad +\frac{1}{4\,g_{00}}\,f_{0,\tau_2}(0)\sum\nolimits_{i,r,s_1,s_2}\psi_{rs_1}\psi_{is_2}g^{s_1 i}g^{s_2 r}+o([\psi_{ac}]^2),\\
f_1(\overrightarrow{\tau}) &= f_1(0)-\frac{1}{2\sqrt{g_{00}}}\,f_{1,\tau_1}(0)\sum\nolimits_{i,j}g^{ij}\psi_{ij}+o([\psi_{ac}]),\\
f_2(\overrightarrow{\tau}) &= f_2(0)-\frac{1}{2\sqrt{g_{00}}}\,f_{2,\tau_1}(0)\sum\nolimits_{i,j}g^{ij}\psi_{ij}+o([\psi_{ac}]).
\end{aligned}$$

Substituting into the function $h(b)$ with b_m and τ's from Lemma 2.2,

$$h(b)_{ac}=\sum\nolimits_{m,s_i,r_j}\frac{(-1)^m f_m(\overrightarrow{\tau})}{(2\sqrt{g_{00}})^m}\,\psi_{ar_1}\psi_{s_1 r_2}\dots\psi_{s_{m-1}c}\,g^{s_1 r_1}\dots g^{s_{m-1}r_{m-1}},$$

we obtain the second-order approximation of $h(b)$, see (2.4),

$$\begin{aligned}
h(b)_{ac} &\approx f_0(0)\,g_{ac}-\frac{f_{0,\tau_1}(0)}{2\sqrt{g_{00}}}\Big(\sum\nolimits_{i,j}g^{ij}\psi_{ij}\Big)\,g_{ac}+f_1(0)\,\psi_{ac}\\
&\quad+\bigg(\frac{f_{0,\tau_2}(0)}{4\,g_{00}}\sum\nolimits_{i,r,s_1,s_2}g^{s_1 i}g^{s_2 r}\psi_{rs_1}\psi_{is_2}+\frac{f_{0,\tau_1\tau_1}(0)}{8\,g_{00}}\Big(\sum\nolimits_{i,j}g^{ij}\psi_{ij}\Big)^2\bigg)\,g_{ac}\\
&\quad-\frac{f_{1,\tau_1}(0)}{2\sqrt{g_{00}}}\Big(\sum\nolimits_{i,j}g^{ij}\psi_{ij}\Big)\,\psi_{ac}+\frac{f_2(0)}{4\,g_{00}}\sum\nolimits_{i,j}g^{ij}\,\psi_{ai}\psi_{cj}+o([\psi_{ac}]^2).
\end{aligned}$$

From above it follows the first-order approximation (2.3) of $h(b)$. □

Example 2.1. (a) If $h=f(\overrightarrow{\tau})\,\hat{b}_1$ and $f(0)\neq 0$ then (2.3) has a form:

$$h(b)_{ac}=f(0)\,\psi_{ac}+o([\psi_{ac}]).$$

(b) The extrinsic Ricci tensor (see (3.87), Sect. 3.9.1) is a second-degree polynomial

$$\mathrm{Ric}^{\mathrm{ex}}(g)_{ac} = -\frac{1}{4\,g_{00}}\sum\nolimits_{i,j} g^{ij}\,\psi_{ai}\psi_{cj} - \frac{1}{2\sqrt{g_{00}}}\Big(\sum\nolimits_{i,j} g^{ij}\psi_{ij}\Big)\,\psi_{ac}.$$

(c) The components of the Newton transformation $T_r(A)$ are given by $(T_r(A))^j_i = (r!)^{-1}\varepsilon^{j_1\ldots j_r j}_{i_1\ldots i_r i}A^{i_1}_{j_1}\ldots A^{i_r}_{j_r}$. Hence $T_r(b)$ is the rth degree polynomial

$$T_r(b)_{ac} = (r!)^{-1}\sum \varepsilon^{j_1\ldots j_r j}_{i_1\ldots i_r a}\left(\frac{-1}{2\sqrt{g_{00}}}\right)^s g^{i_1,m_1}\ldots g^{i_r,m_r}g_{jc}\,\psi_{m_1,j_1}\cdots\psi_{m_r,j_r}.$$

(d) For a $\mathscr{F}$-conformal metric $g_{ij} = \mathrm{e}^{u(x)}\,\delta_{ij}$ $(i,j>0)$, we approximate $h(b)$ as

$$h(b)_{ac} = f_0(0)\,\delta_{ac} + u_{,0}\left(f_1(0) - n\frac{f_{0,\tau_1}(0)}{2\sqrt{g_{00}}}\right)\delta_{ac} + o(u_{,0}). \tag{2.5}$$

If f_0, f_1, f_{0,τ_1} are zero at the origin 0, we use the second-order approximation

$$h(b)_{ac} = {u_{,0}}^2\left(\frac{f_2(0)}{4\,g_{00}} - n\frac{f_{1,\tau_1}(0)}{2\sqrt{g_{00}}}\right)\delta_{ac} + o({u_{,0}}^2). \tag{2.6}$$

Let $\hat{g}$ and $g^{\perp}$ be the components of g along the distributions $T\mathscr{F}$ and $T\mathscr{F}^{\perp}$, respectively. Because $T\mathscr{F}^{\perp}$ is one-dimensional, $g^{\perp}$ is determined by the value $g^{\perp}(N,N)=1$.

The next lemma is standard. For the convenience of the reader we give its proof.

Lemma 2.3. *Let $(M,\, g = \hat{g}\oplus g^{\perp})$ be a Riemannian manifold with a codimension-one foliation $\mathscr{F}$ and a unit normal N. Define a metric $\bar{g} = (\mathrm{e}^{2\varphi}\hat{g})\oplus g^{\perp}$, where $\varphi\in C^1(M)$. Then the second fundamental forms and the Weingarten operators of $\mathscr{F}$ with respect to $\bar{g}$ and g are related by*

$$\bar{b} = \mathrm{e}^{2\phi}(b - N(\varphi)\,\hat{g}), \qquad \bar{A} = A - N(\varphi)\,\widehat{\mathrm{id}}. \tag{2.7}$$

Proof. By the formula for Levi-Civita connection and $g(T\mathscr{F},N)=0$, we get

$$\begin{aligned} 2\mathrm{e}^{2\phi}g(\bar{\nabla}_X N,Y) = 2\,\bar{g}(\bar{\nabla}_X N,Y) &= X(\bar{g}(N,Y)) + N(\bar{g}(X,Y)) - Y(\bar{g}(N,X)) \\ &\quad + \bar{g}([X,N],Y) - \bar{g}([N,Y],X) + \bar{g}([Y,X],N) \\ &= 2\,\mathrm{e}^{2\phi}N(\phi)\,g(X,Y) + 2\,\mathrm{e}^{2\phi}g(\nabla_X N,Y) \end{aligned}$$

for any vector fields $X,Y\in T\mathscr{F}$. Hence

$$g(\bar{\nabla}_X N,Y) = N(\phi)\,g(X,Y) + g(\nabla_X N,Y).$$

From this and the definition $\bar{A}(X) = -\bar{\nabla}_X N$, it follows $(2.7)_2$.

Using the definition $\bar{b}(X,Y)=\bar{g}(\bar{\nabla}_X Y,N)$, we compute the tensor $\bar{b}$ on $T\mathscr{F}$

$$\begin{aligned}\bar{b}(X,Y) &= \bar{g}(\bar{\nabla}_X Y,N)=\mathrm{e}^{2\phi}g(A(X)-N(\phi)X,Y)\\ &=\mathrm{e}^{2\phi}b(X,Y)-\mathrm{e}^{2\phi}g(X,Y)N(\phi)=\mathrm{e}^{2\phi}\left(b(X,Y)-N(\phi)\,g(X,Y)\right).\end{aligned}$$

From this it follows $(2.7)_1$. □

Remark 2.1. If $N(\varphi)=0$ (e.g., φ is constant) then by Lemma 2.3 we have

(i) $\bar{b}=\mathrm{e}^{2\varphi}b$;
(ii) $\bar{A}=A$;
(iii) $\bar{\tau}_j=\tau_j$ (the power sums).

2.2.2 Foliations with a Time-Dependent Metric

Consider a foliation $(M,\mathscr{F})$ with a time-dependent metric g_t such that $S=\partial_t g$ is an $\mathscr{F}$-truncated tensor. We denote the bundle of $\mathscr{F}$-truncated (k,l)-tensors on M by $\widehat{\Lambda}_l^k(M)$. The *inner product of tensors* $F,G\in\widehat{\Lambda}_l^k(M)$, denoted by $\langle\cdot,\cdot\rangle$ on M, is given by the following sum:

$$\langle F,G\rangle=g^{a_1b_1}\ldots g^{a_lb_l}g_{c_1d_1}\ldots g_{c_kd_k}F^{c_1\ldots c_k}_{a_1\ldots a_l}G^{d_1\ldots d_k}_{b_1\ldots b_l}.$$

Recall that the *musical isomorphism* $\sharp:T^*M\to TM$ sends a covector $\omega=\omega_i\mathrm{d}x^i$ to $\omega^\sharp=\omega^i\partial_i=g^{ij}\omega_j\partial_i$, and $\flat:TM\to T^*M$ sends a vector $X=X^i\partial_i$ to $X^\flat=X_i\mathrm{d}x^i=g_{ij}X^j\mathrm{d}x^i$. In other words, $X^\flat=g(X,\cdot)$. We denote by $B^\sharp$ the $(1,1)$-tensor field on M which is g-dual to a symmetric $(0,2)$-tensor B,

$$B(X,Y)=g(B^\sharp(X),Y)\quad \text{for all vectors } X,Y.$$

For symmetric $(0,2)$-tensors B,S we have

$$\langle B,S\rangle=\mathrm{Tr}\,(B^\sharp S^\sharp)=\langle B^\sharp,S^\sharp\rangle. \tag{2.8}$$

Indeed, in a local coordinate basis (e_i) of TM we have

$$g_{ik}B^{ij}=B^j_k,\quad g^{kl}S_{lj}=S^k_j,\quad \langle B,S\rangle=B^{ij}S_{ij}.$$

From the above, in view of identity $g_{ik}g^{kj}=\delta^j_i$ (Kronecker's delta), (2.8) follows,

$$\mathrm{Tr}\,(B^\sharp S^\sharp)=B^j_kS^k_j=g_{ik}B^{ij}g^{kl}S_{lj}=B^{ij}S_{ij}.$$

For example,

$$\langle \hat{g}, \hat{g} \rangle = \mathrm{Tr}\,\widehat{\mathrm{id}} = n, \quad \langle \hat{g}, S^2 \rangle = \langle S, S \rangle,$$

where S^2 is the symmetric $(0,2)$-tensor dual to $(S^\sharp)^2$. If $S = s\,\hat{g}$ for some scalar function s then $S^\sharp = s\,\hat{\mathrm{id}}$ and has the trace $\mathrm{Tr}\,S^\sharp = ns$.

Lemma 2.4. *For a smooth family (B_t) of symmetric $(0,2)$-tensors on $(M, g_t, \mathscr{F})$ and $S = \partial_t g_t$, where $g_t \in \mathscr{M}(M, \mathscr{F}, N)$, we have*

$$(\partial_t B_t)^\sharp = \partial_t B_t^\sharp + S^\sharp \cdot B_t^\sharp, \qquad \partial_t (\mathrm{Tr}\,_{g_t} B_t) = \mathrm{Tr}\,_{g_t}(\partial_t B_t) - \langle B_t, S \rangle_{g_t}, \quad (2.9)$$

$$\partial_t \langle B_t, S \rangle_{g_t} = \langle \partial_t B_t, S \rangle_{g_t} - 2 \langle B_t, S^2 \rangle_{g_t} + \langle B_t, \partial_t S \rangle_{g_t}. \quad (2.10)$$

Proof. Because $\partial_t g_{ij} = S_{ij}$, we have

$$\partial_t g^{ij} = -S^{ij} := -S_{kl}\, g^{ik} g^{jl}.$$

To establish $(2.9)_1$, we write $(B^\sharp)^k_i = g^{kj} B_{ij}$ for any t and compute

$$\begin{aligned} (\partial_t B^\sharp)^k_i &= \partial_t (g^{kj} B_{ij}) = \partial_t g^{kj} B_{ij} + g^{kj} \partial_t B_{ij} = -S^{kj} B_{ij} + (\partial_t B)^{\sharp k}_{\ i} \\ &= -g^{ik} S^j_i\, g_{il} B^l_j + (\partial_t B)^{\sharp k}_{\ i} = -S^j_i\, B^k_j + (\partial_t B)^{\sharp k}_{\ i}. \end{aligned}$$

Notice that $(2.9)_2$ is a consequence of $(2.9)_1$ and the identity $\mathrm{Tr}\,(\partial_t B_t^\sharp) = \partial_t (\mathrm{Tr}\,B_t^\sharp)$. Then (2.10) directly follows from the calculation

$$\begin{aligned} \partial_t (B^{ij} S_{ij}) &= (\partial_t B)^{ij} S_{ij} + B^{ij} (\partial_t S)_{ij} = \partial_t (g^{ai} g^{bj} B_{ab}) S_{ij} + B^{ij} (\partial_t S)_{ij} \\ &= (\partial_t g)^{ai} g^{bj} B_{ab} S_{ij} + g^{ai} (\partial_t g)^{bj} B_{ab} S_{ij} + g^{ai} g^{bj} (\partial_t B)_{ab} S_{ij} + B^{ij} (\partial_t S)_{ij} \\ &= -B_{ab} (S^{ai} g^{bj} S_{ij} + g^{ai} S^{bj} S_{ij}) + (\partial_t B)^{ij} S_{ij} + B^{ij} (\partial_t S)_{ij} \\ &= -B_{ab} \left(g^{al} S^i_l S^b_i + g^{bl} S^j_l S^a_j \right) + (\partial_t B)^{ij} S_{ij} + B^{ij} (\partial_t S)_{ij} \\ &= -2 B_{ab} (S^2)^{ab} + (\partial_t B)^{ij} S_{ij} + B^{ij} (\partial_t S)_{ij}. \end{aligned}$$

□

Now let ∇^t denote the Levi-Civita connection on (M, g_t), where $t \in [0, \varepsilon)$. Observe that the difference of two connections is always a tensor, hence $\Pi_t := \partial_t \nabla^t$ is a $(1,2)$-tensor field on (M, g_t). Differentiating with respect to t, the classical formula for the Levi-Civita connection yields the known formula (see Preface)

$$2\, g_t(\Pi_t(X,Y), Z) = (\nabla^t_X S)(Y,Z) + (\nabla^t_Y S)(X,Z) - (\nabla^t_Z S)(X,Y), \quad (2.11)$$

where $X, Y, Z \in TM$. If the vector field $Y = Y(t)$ is time-dependent then

$$\partial_t \nabla^t_X Y = \Pi_t(X,Y) + \nabla_X (\partial_t Y). \quad (2.12)$$

Notice the symmetry of the tensor: $\Pi_t(X,Y) = \Pi_t(Y,X)$. If the tensor S is $\mathscr{F}$-truncated then by (2.11) we have $\Pi_t(N,\cdot) \perp N$.

Let E be the pull-back of the tangent bundle TM under the projection $M \times [0,\varepsilon) \to M$, i.e., the fiber of E over a point (x,t) is given by $E_{x,t} = T_xM$. There is a natural connection $\tilde{\nabla}$ on E, which extends the Levi-Civita connection ∇ on M. In order to define this connection, we need to specify the covariant t-derivative $\widetilde{\nabla}_{\partial_t}$. Given any section X of the vector bundle E, we define

$$\widetilde{\nabla}_{\partial_t} X = \partial_t X + \frac{1}{2} S^\sharp(X), \quad \text{in particular,} \quad \widetilde{\nabla}_{\partial_t} N = 0. \tag{2.13}$$

The connection $\tilde{\nabla}$ is compatible with the natural bundle metric on E, i.e.,

$$(\widetilde{\nabla}_{\partial_t} g)(X,Y) = 0, \qquad X,Y \in TM.$$

To show this, we calculate

$$(\widetilde{\nabla}_{\partial_t} g)(X,Y) = \partial_t(g(X,Y)) - g(\widetilde{\nabla}_{\partial_t} X,Y) - g(X,\widetilde{\nabla}_{\partial_t} Y) = (\partial_t g)(X,Y) - S(X,Y) = 0.$$

This connection is not symmetric: in general $\widetilde{\nabla}_{\partial_t}\partial_i \neq 0$, while $\widetilde{\nabla}_{\partial_i}\partial_t = 0$ always for $i > 0$. However, each submanifold $M \times \{t\}$ is totally geodesic, so computing derivatives of spatial tangent vector fields gives the same result as computing for sections of $T(M \times [0,\varepsilon))$. In particular, the corresponding Weingarten operators $\tilde{A}$ and A satisfy $\tilde{A} = A$. Clearly, the *torsion tensor*

$$\mathrm{Tor}(X,Y) := \widetilde{\nabla}_X Y - \widetilde{\nabla}_Y X - [X,Y]$$

vanishes if both arguments are spatial, so the only nonzero components of Tor are

$$\mathrm{Tor}(\partial_t,\partial_i) = \widetilde{\nabla}_{\partial_t}\partial_i - \widetilde{\nabla}_{\partial_i}\partial_t = \frac{1}{2} S^\sharp(\partial_i), \qquad i > 0.$$

2.2.3 A Differential Operator

To shorten later formulas, we introduce the differential operator

$$\mathscr{V}(F) := \tau_1 F - \nabla_N F,$$

where $F \in \widehat{\Lambda}^k_l(M)$ is either a smooth (k,l)-tensor or a function on M.

Lemma 2.5. *For a foliation $\mathscr{F}$ on a closed Riemannian manifold (M,g) we have*

$$\int_M \mathscr{V}(F)\,\mathrm{d\,vol} = 0 \quad \text{for any} \quad F \in C^1(M).$$

Proof. Using the equality $\operatorname{div} N = -\tau_1$, we have

$$\operatorname{div}(FN) = F \operatorname{div} N + N(F) = -\tau_1 F + N(F) = -\mathscr{V}(F).$$

By the Divergence Theorem, $\int_M \operatorname{div}(FN)\,\mathrm{d}\,\mathrm{vol} = 0$. From the above the required equality follows. □

The next lemmas (concerning the above operator $\mathscr{V}$) are also global and can be proved for arbitrary (k,l)-tensor fields. First, we show that the operator ∇_N is conjugate to $\mathscr{V}$. Notice that ∇_N commutes with traces of $\mathscr{F}$-truncated $(1,1)$-tensors:

$$\nabla_N(\operatorname{Tr} F) = \operatorname{Tr}(\nabla_N F) \quad \text{for any } (1,1)\text{-tensor field } F.$$

Lemma 2.6. *Let S, B be $\mathscr{F}$-truncated symmetric $(0,2)$-tensor fields on a closed M. Then*

$$\int_M \langle B, \nabla_N S\rangle\,\mathrm{d}\,\mathrm{vol} = \int_M \langle \mathscr{V}(B), S\rangle\,\mathrm{d}\,\mathrm{vol}. \tag{2.14}$$

In particular, for $F \in C^1(M)$ and $S = s\,\hat{g}$ we have

$$\int_M F\, N(s)\,\mathrm{d}\,\mathrm{vol} = \int_M s\,\mathscr{V}(F)\,\mathrm{d}\,\mathrm{vol}. \tag{2.15}$$

Proof. Notice that the $(1,1)$-tensor $\nabla_N S^\sharp$ is g-dual to $\nabla_N S$. To show (2.14), calculate with the help of (2.8) and Lemma 2.5,

$$\begin{aligned}\int_M \langle B, \nabla_N S\rangle\mathrm{d}\,\mathrm{vol} &= \int_M \operatorname{Tr}(B^\sharp\, \nabla_N S^\sharp)\,\mathrm{d}\,\mathrm{vol} = \int_M \Big(N(\operatorname{Tr}(B^\sharp S^\sharp)) - \operatorname{Tr}((\nabla_N B^\sharp) S^\sharp)\Big)\,\mathrm{d}\,\mathrm{vol}\\ &= \int_M \operatorname{Tr}(\tau_1 B^\sharp S^\sharp - (\nabla_N B^\sharp) S^\sharp)\,\mathrm{d}\,\mathrm{vol} = \int_M \langle \mathscr{V}(B), S\rangle\,\mathrm{d}\,\mathrm{vol}.\end{aligned}$$

Applying (2.14) to $S = s\,\hat{g}$, we obtain (2.15). □

Example 2.2. Let F and s be smooth functions on a closed M. One may use Lemma 2.6 to prove the following:

$$\int_M F\,s N(s)\,\mathrm{d}\,\mathrm{vol} = \frac{1}{2}\int_M \mathscr{V}(F) s^2\,\mathrm{d}\,\mathrm{vol}, \tag{2.16}$$

$$\int_M F\,s N(N(s))\,\mathrm{d}\,\mathrm{vol} = \int_M \left(\frac{1}{2}\mathscr{V}(\mathscr{V}(F))\,s^2 - F\,N(s)^2\right)\mathrm{d}\,\mathrm{vol}. \tag{2.17}$$

Indeed, for any $\mathscr{F}$-truncated symmetric $(0,2)$-tensor field S on M, we have

$$\int_M F\langle S, \nabla_N S\rangle\,\mathrm{d}\,\mathrm{vol} = \int_M \langle \mathscr{V}(FS), S\rangle\,\mathrm{d}\,\mathrm{vol} = \int_M \langle \mathscr{V}(F)S, S\rangle - F\langle S, \nabla_N S\rangle\,\mathrm{d}\,\mathrm{vol}.$$

From the above for $S = s\,\hat{g}$, one obtains (2.16). Using (2.15) with substitution $F \to sF$ and $s \to N(s)$, one has (2.17).

Given linear operator $\Phi : \widehat{\Lambda}^0_2(M) \to \widehat{\Lambda}^0_2(M)$, define

$$\mu(\Phi) := \inf_{S\in\widehat{\Lambda}^0_2(M)} \int_M \langle \Phi(\nabla_N S), \nabla_N S\rangle \,\mathrm{d\,vol} \Big/ \int_M \langle S, S\rangle \,\mathrm{d\,vol}.$$

Lemma 2.7. *Let a be supremum of the lengths of N-curves.*

(i) If $a = \infty$ and $\langle \Phi(\nabla_N S), \nabla_N S\rangle \ge 0$ for any $S \in \widehat{\Lambda}^0_2(M)$ then $\mu(\Phi) = 0$.
(ii) If $0 \le \langle \Phi(\nabla_N S), \nabla_N S\rangle \le b^2\langle S,S\rangle$ for any $S \in \widehat{\Lambda}^0_2(M)$ and some $b \ge 0$ then

$$\mu(\Phi) \in [0, \pi^2 b^2/a^2].$$

Proof. Consider the well-known constrained variation problem

$$J(s) = \int_0^a (s'(x))^2 \,\mathrm{d}x \to \min, \quad s(0) = s(a) = 0, \quad \int_0^a (s(x))^2 \,\mathrm{d}x = 1,$$

where s is smooth on $(0,a)$. We claim that the minimum of J is π^2/a^2. Indeed, the *Euler equation* for the functional $\tilde{J}(s) = \int_0^a \tilde{F}\,\mathrm{d}x$ with $\tilde{F} = (s')^2 + \mu(s^2 - 1)$ and $\mu \in \mathbb{R}$ is $s'' + \mu\, s = 0$. Using the boundary conditions at $x \in \{0,a\}$, we find

$$\mu = (\pi/a)^2, \qquad s = C\sin(\pi x/a).$$

From the constraint we calculate $C^2 = 2/a$. The solution (to the constrained variation problem) is

$$\bar{s} = \sqrt{2/a}\,\sin(\pi x/a), \qquad \min J = J(\bar{s}) = \pi^2/a^2.$$

Assuming $\bar{s} = 0$ on $\{x < 0\} \cup \{x > 1\}$, we build a piece-wise smooth function $\bar{s}$ on $\mathbb{R}$. Then, there exist functions $s_n \in C^1(\mathbb{R})$ vanishing outside of $(-\frac{1}{n}, a + \frac{1}{n})$ such that $s_n \to \bar{s}$ and $\int_{\mathbb{R}} (s'(x))^2 \,\mathrm{d}x \to \pi^2/a^2$ when $n \to \infty$, the claim is proved.

Let the lengths of N-curves be unbounded and $\langle \Phi(\nabla_N S),\ \nabla_N S\rangle \ge 0$. Then for any $a > 0$ there exist an N-curve $\gamma : [0,a] \to M$ of length a and a smooth function $s_1 \ge 0$ on γ with the properties

$$s_1(\gamma(0)) = s_1(\gamma(a)) = 0, \qquad \int_\gamma N(s_1)^2 < 2\pi^2/a^2.$$

There is a "thin" biregular foliated chart $U(q) \supset \gamma$ with $q = \gamma(0)$, $x = (x_0, \hat{x}) \in [0,1]^{n+1}$, and $\int_0^1 g_{00}(x_0, 0)\,\mathrm{d}x_0 = a$. Consider a smooth function $s = s_1(x_0)s_2(\hat{x})$ (with $0 \le s_2 \le 1$) supported in this chart. Define a tensor field $S = s\tilde{S}$ on M with an arbitrary $\mathscr{F}$-truncated symmetric $(0,2)$-tensor field $\tilde{S}$ satisfying $\nabla_N \tilde{S} = 0$ on γ, and let $F = \sup_M \langle \Phi(\tilde{S}), \tilde{S}\rangle$. As the volume form vol satisfies vol $< Q(a)\,\mathrm{d}x_0 \wedge \mathrm{d}\hat{x}$ along γ for some $Q(a) > 0$ (depending on γ), one may assume that vol $< 2\,Q(a)\,\mathrm{d}x_0 \wedge \mathrm{d}\hat{x}$ on $U(q) = I \times \hat{U}(q)$. Applying the above inequality and the *Fubini Theorem*, we have

$$\begin{aligned}
\int_{U(q)} \langle \Phi(\nabla_N S), \nabla_N S\rangle \,\mathrm{d}\,\mathrm{vol} &< F \int_{U(q)} N(s_1)^2 \mathrm{d}\,\mathrm{vol} \\
&< 2F \int_{U(q)} Q(a)\partial_0(s_1)^2/g_{00}(x)\,\mathrm{d}x_0 \wedge \mathrm{d}\hat{x} \\
&= 2F\,Q(a) \int_{\hat{U}(q)} \frac{1}{g_{00}(x)} \left(\int_I \partial_0(s_1)^2\,\mathrm{d}x_0\right) \mathrm{d}\hat{x} \\
&< 4\,Q(a)\left(\frac{\pi}{a}\right)^2 F \int_{\hat{U}(q)} \frac{1}{g_{00}(x)}\,\mathrm{d}\hat{x}.
\end{aligned}$$

One may take $\hat{U}(q)$ such that $4\pi^2 F\,Q(a) \int_{\hat{U}(q)} \frac{1}{g_{00}(x)}\,\mathrm{d}\hat{x} < 1$ for all x_0. Thus,

$$\int_{U(q)} \langle \Phi(\nabla_N S), \nabla_N S\rangle \,\mathrm{d}\,\mathrm{vol} < 1/a^2.$$

As $a > 0$ is arbitrary, it follows that $\mu(\Phi) = 0$, that completes the proof of (i).

Claim (ii) follows directly from the estimates above. □

Remark 2.2. Concerning Lemma 2.7(i), recall [20] that there exist compact manifolds (M^{n+1}, g), $n > 2$, foliated by closed curves whose lengths are unbounded.

Lemma 2.8. *Given linear operators* $\Phi_i : \widehat{\Lambda}^0_2(M) \to \widehat{\Lambda}^0_2(M)$ $(i = 1,2,3)$, *define*

$$J(S) := \int_M (\langle \Phi_1(S),\, S\rangle + \langle \Phi_2(\nabla_N S),\, \nabla_N S\rangle + \langle \Phi_3(S),\, \nabla_N S\rangle)\,\mathrm{d}\,\mathrm{vol}.$$

If $J \geq 0$ *for any symmetric tensor* $S \in \Lambda^0_2(M)$ *then* $\langle \Phi_2(\nabla_N S), \nabla_N S\rangle \geq 0$.

Moreover, if

$$\langle \Phi_2(\nabla_N S),\, \nabla_N S\rangle \geq 0, \qquad \langle (\Phi_1 + \mathscr{V} \circ \Phi_3)(S), S\rangle \geq -\mu(\Phi_2)\langle S, S\rangle, \tag{2.18}$$

for any symmetric tensor $S \in \widehat{\Lambda}^0_2(M)$ *then* $J \geq 0$.

Proof. By (2.14), we have

$$\int_M \langle \Phi_3(S), \nabla_N S\rangle \,\mathrm{d}\,\mathrm{vol} = \int_M \langle \mathscr{V}(\Phi_3(S)), S\rangle \,\mathrm{d}\,\mathrm{vol}.$$

Certainly, (2.18) with any symmetric tensor $S \in \widehat{\Lambda}^0_2(M)$ are suffices for $J \geq 0$.

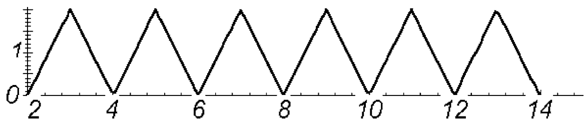

Fig. 2.1 A saw-shaped function s_1

In order to prove that $(2.18)_1$ is necessary, we shall use S with the support in a *biregular foliated chart* $U(q)$ adapted to $\mathscr{F}$ and N with coordinates $x=(x_0,\hat{x})\in[0,1]^{n+1}$, $\hat{x}=(x_1,\ldots x_n)$, see Sect. 2.2.1. Hence, $x_0=c=\text{const}$ on the leaves $x_i=c_i=\text{const}$ $(i>0)$ along N-curves, the coordinate vector fields $\partial_i=\partial_{x_i}$ $(i>0)$, are tangent to leaves and N is directed along $\partial_0=\partial_{x_0}$. In these coordinates, the metric g has the form:

$$g=g_{00}\,\mathrm{d}x_0^2+\sum\nolimits_{i,j>0}g_{ij}\,\mathrm{d}x_i\mathrm{d}x_j$$

with $g_{00}=1$ for $\hat{x}=0$, and $N=\beta\,\partial_0$ for $\beta=1/\sqrt{g_{00}}$ (because N is the unit normal to $\mathscr{F}$). One may assume that $\mathrm{vol}_g U(q)<1$. Take $s=s_1(x_0)s_2(\hat{x})$, where $0\le s_i\le 1$ and $\mathrm{supp}(s)\subset U(q)$. Notice that $N(s)=N(s_1)s_2$.

To prove $(2.18)_1$ assume the contrary: that $J\ge 0$ but $F_2:=\langle\Phi_2(\tilde{S}),\tilde{S}\rangle<0$ for some symmetric $(0,2)$-tensor $\tilde{S}$ at a point q. One may extend $\tilde{S}$ on a neighborhood $U(q)$ (of q) with the property $\nabla_N\tilde{S}=0$ at q and assume that $F_{2|U(q)}<-\delta$ for some $\delta>0$. Take a saw-shaped function $s_1=s_1(x_0)$ (Fig. 2.1) with a number of oscillations of slope ± 1, such that the values of s_1 belong to $[0,\varepsilon]$, where $\varepsilon>0$ can be chosen as small as necessary. Define a symmetric $(0,2)$-tensor field $S=s\tilde{S}$ on M.

Then $N(s_1)^2=|\partial_0 s_1|^2/g_{00}=1/g_{00}$ almost everywhere on $[0,1]$, and

$$\int_M\langle\Phi_2(\nabla_N S),\nabla_N S\rangle\mathrm{d}\,\mathrm{vol}=\int_M\frac{s_2^2}{g_{00}}\langle\Phi_2(\tilde{S}),\tilde{S}\rangle\mathrm{d}\,\mathrm{vol}<-\delta\int_M\frac{s_2^2}{g_{00}}\,\mathrm{d}\,\mathrm{vol}=-\delta\beta_1,$$

where $\beta_1=\int_M s_2^2/g_{00}\,\mathrm{d}\,\mathrm{vol}>0$. We also have

$$\int_M\big|\langle\Phi_1(S),S\rangle+\langle\Phi_3(S),\nabla_N S\rangle\big|\,\mathrm{d}\,\mathrm{vol}\le F_1\varepsilon^2+F_3\varepsilon,$$

where $F_1=\sup_{U(q)}|\langle\Phi_1(\tilde{S}),\tilde{S}\rangle|$, $F_3=\sup_{U(q)}|\langle\Phi_3(\tilde{S}),\tilde{S}\rangle|$. Hence,

$$J(S)<F_1\varepsilon^2+F_3\,\varepsilon-\beta_1\delta.$$

For $\varepsilon>0$ small enough we obtain $J(S)<0$, a contradiction. $\square$

Given the function F on M, define

$$\mu(F):=\inf_{s\in C^1(M)}\int_M F N(s)^2\,\mathrm{d}\,\mathrm{vol}\Big/\int_M s^2\,\mathrm{d}\,\mathrm{vol}\,.$$

Remark that (by Lemma 2.7) we have the following:

(a) If $a = \infty$ and $F \geq 0$, then $\mu(F) = 0$.
(b) If $0 \leq F \leq b^2$ for some $b \geq 0$, then $\mu(F) \in [0, \frac{\pi^2 b^2}{a^2}]$;

Here a is supremum of the lengths of N-curves.

From Lemma 2.8 we obtain the following

Corollary 2.1. *Let F_i $(i = 1,2)$ be continuous functions on M and*

$$J(s) := \int_M (F_1 s^2 + F_2 N(s)^2)\,\mathrm{d}\,\mathrm{vol}.$$

If $J(s) \geq 0$ for any $s \in C^1(M)$ then $F_2 \geq 0$.

Moreover, if $F_2 \geq 0$ and $F_1 \geq -\mu(F_2)$ then $J \geq 0$.

2.3 Variational Formulae for Codimension-One Foliations

2.3.1 Variations of Extrinsic Geometric Quantities

In order to calculate variations of the functional I_f with respect to metrics $g_t \in \mathscr{M}$, we find the variational formula for A, and apply it to the Newton transformations $T_i(A)$ and to symmetric functions τ_j, σ_j of A. For short, we shall omit the index t for the time-dependent tensors $S, A, \hat{b}_j$ and functions τ_i, σ_i.

Let $\hat{b}$ be the extension of b to the $\mathscr{F}$-truncated symmetric $(0,2)$-tensor field on M. Notice that $\hat{b}(N,\cdot) = 0$ and

$$\hat{b}(X,Y) = g(A(X),Y).$$

In other words, $\hat{b}(N,\cdot) = 0$ and $\hat{b}$ is dual to the extended Weingarten operator A. Denote by $\hat{b}_j$ the symmetric $(0,2)$-tensor fields on M dual to powers A^j of extended Weingarten operator,

$$\hat{b}_0(X,Y) = \hat{g}(X,Y), \quad \hat{b}_j(X,Y) = \hat{g}(A^j(X),Y) \quad (j > 0, \quad X,Y \in TM).$$

Lemma 2.9. *Let $g_t \in \mathscr{M}$ be a family of $\mathscr{F}$-truncated metrics and $S = \partial_t g_t$. Then the Weingarten operator A of $\mathscr{F}$ and the symmetric functions τ_i and σ_i of A evolve by*

$$\partial_t A = \frac{1}{2}\left([A, S^\sharp] - \nabla^t_N S^\sharp\right), \tag{2.19}$$

$$\partial_t \tau_i = -\frac{i}{2}\,\mathrm{Tr}\,(A^{i-1}\nabla^t_N S^\sharp), \quad \partial_t \sigma_i = -\frac{1}{2}\,\mathrm{Tr}\,(T_{i-1}(A)\nabla^t_N S^\sharp), \quad i > 0. \tag{2.20}$$

For $S = s\,\hat{g}$ $(s \in C^1(M))$ *we get*

$$\partial_t A = -\frac{1}{2} N(s)\,\hat{\rm id},$$
$$\partial_t \tau_i = -\frac{i}{2}\,\tau_{i-1} N(s), \quad \partial_t \sigma_i = -\frac{1}{2}(n-i+1)\,\sigma_{i-1} N(s), \quad i > 0.$$

Proof. Using (2.11) and $S(\cdot, N) = 0$ for $\mathscr{F}$-truncated tensors, we obtain

$$\begin{aligned}
\partial_t b(X,Y) &= \partial_t g_t(\nabla^t_X Y, N) = (\partial_t g_t)(\nabla^t_X Y, N) + g_t(\partial_t \nabla^t_X Y, N)\\
&= S(\nabla^t_X Y, N) + (1/2)\big((\nabla^t_X S)(Y,N) + (\nabla^t_Y S)(X,N) - (\nabla^t_N S)(X,Y)\big)\\
&= (1/2)\big(S(AX,Y) + S(AY,X) - (\nabla^t_N S)(X,Y)\big)
\end{aligned}$$

for all $X, Y \in T\mathscr{F}$.

Because $S(AX,Y) = g_t(S^\sharp AX, Y)$ and $(\nabla^t_N S)(X,Y) = g_t((\nabla^t_N S^\sharp)X, Y)$, we have

$$\begin{aligned}
g_t((\partial_t A)X, Y) &= g_t(\partial_t(AX), Y) = \partial_t b(X,Y) - S(AX,Y)\\
&= \frac{1}{2}[g_t(S^\sharp AY, X) - g_t(S^\sharp AX, Y) - (\nabla^t_N S)(X,Y)]\\
&= \frac{1}{2}[g_t([A, S^\sharp]X, Y) - g_t((\nabla^t_N S^\sharp)X, Y)].
\end{aligned}$$

Formula (2.19) follows from the above and the freedom of choice of $X, Y \in T\mathscr{F}$.

Multiplying (2.19) from the left by A^{i-1}, we get

$$2A^{i-1}\partial_t A = A^{i-1}[A, S^\sharp] - A^{i-1}\nabla^t_N S^\sharp, \quad i > 0.$$

Notice that ${\rm Tr}\,(A^{i-1}\cdot[A, S^\sharp]) = 0$, see Remark 2.3. Then, using the identity

$$i\,{\rm Tr}\,(A^{i-1}\partial_t A) = {\rm Tr}\,(\partial_t A^i) = \partial_t \tau_i,$$

see (1.11) of the following Remark 1.2, we deduce $(2.20)_1$.

Substituting $\partial_t A$ from (2.19) into the formula (1.12) of Remark 1.2, we obtain

$$\partial_t \sigma_i = \frac{1}{2}\,{\rm Tr}\,\Big(T_{i-1}(A)([A, S^\sharp] - \nabla^t_N S^\sharp)\Big) = -\frac{1}{2}\,{\rm Tr}\,(T_{i-1}(A)\nabla^t_N S^\sharp),$$

that proves $(2.20)_2$. For $S = s\,\hat{g}$, we have, respectively, $\nabla^t_N S^\sharp = N(s)\,\widehat{\rm id}$, and

$${\rm Tr}\,(A^{i-1}\nabla^t_N S^\sharp) = \tau_{i-1}N(s), \qquad {\rm Tr}\,(T_{i-1}(A)\nabla^t_N S^\sharp) = (n-i+1)\,\sigma_{i-1}N(s).$$

From the above, the case $S = s\,\hat{g}$ of lemma follows. $\square$

Remark 2.3. For any n-by-n matrices A,B, and C such that $AB=BA$ one has

$$\mathrm{Tr}\,(A\cdot[B,C]) = \mathrm{Tr}\,(ABC) - \mathrm{Tr}\,((AC)B) = \mathrm{Tr}\,(BAC) - \mathrm{Tr}\,(B(AC)) = 0.$$

Example 2.3. (a) We shall find the evolution of tensors A^i and $\nabla^t_N A^i$ and their dual. Using (2.19) and the definition $S=\partial_t g_t$, we generalize (2.19) and find for $i>0$

$$2\,\partial_t A^i = \sum\nolimits_{j=0}^{i-1} A^j([A,S^\sharp]-\nabla^t_N S^\sharp)A^{i-1-j} = [A^i,S^\sharp] - \sum\nolimits_{j=0}^{i-1} A^j(\nabla^t_N S^\sharp)A^{i-1-j}.$$

As $A^i=(\hat b_i)^\sharp$, from the above and (2.9) we obtain the evolution of $\hat b_i$ $(i>0)$,

$$(\partial_t \hat b_i)^\sharp = \partial_t A^i + S^\sharp A^i = \frac12\left(S^\sharp A^i + A^i S^\sharp - \sum\nolimits_{j=0}^{i-1} A^j(\nabla^t_N S^\sharp)A^{i-1-j}\right).$$

Observe that tracing $\partial_t A^i$ we get $(2.20)_1$.

From (2.19), using (2.11), (2.12), and the following calculations:

$$\begin{aligned}
2\,g_t(\partial_t(\nabla^t_N A^i)X,Y) &= 2\,g_t(\partial_t(\nabla^t_N(A^iX) - A^i\nabla^t_N X),\,Y)\\
&= 2\,g_t(\partial_t\nabla^t_N(A^iX) - (\partial_t A^i)\nabla^t_N X - A^i\partial_t(\nabla^t_N X),\,Y)\\
&= 2\,g_t(\nabla^t_N((\partial_t A^i)X) - (\partial_t A^i)\nabla^t_N X, Y) + (\nabla^t_N S)(A^iX,Y)\\
&\quad +(\nabla^t_{A^iX}S)(N,Y) - (\nabla^t_Y S)(N,A^iX) - (\nabla^t_N S)(X,A^iY)\\
&\quad -(\nabla^t_X S)(N,A^iY) + (\nabla^t_{A^iY}S)(N,X)\\
&= 2\,g_t((\nabla^t_N(\partial_t A^i))X,Y) - g_t([\,[A^i,S^\sharp],A]X,Y)\\
&\quad +g_t([(\nabla^t_N S^\sharp),A^i]X,Y),
\end{aligned}$$

we obtain

$$2\,\partial_t(\nabla^t_N A^i) = 2\,\nabla^t_N(\partial_t A^i) - [\,[A^i,S^\sharp],A] + [\nabla^t_N S^\sharp,\,A^i].$$

From the above and the formula for $\partial_t A^i$ we find the evolution of $\nabla^t_N A^i$ $(i>0)$:

$$\partial_t(\nabla^t_N A^i) = \frac12\left([\nabla^t_N A^i,\,S^\sharp] - [\,[A^i,S^\sharp],\,A] - \nabla^t_N\sum\nolimits_{j=0}^{i-1} A^j(\nabla^t_N S^\sharp)A^{i-1-j}\right).$$

Notice that

$$\partial_t(\nabla^t_N \hat b_i)(X,Y) = \partial_t(g_t(\nabla^t_N A^i)X,Y) = S((\nabla^t_N A^i)X,Y) + g_t(\partial_t(\nabla^t_N A^i)X,Y).$$

Putting these facts together yield the evolution equation for $\nabla^t_N\,\hat b_i$ $(i>0)$:

$$(\partial_t\nabla^t_N \hat b_i)^\sharp = \frac12\left(S^\sharp\nabla^t_N A^i + (\nabla^t_N A^i)S^\sharp - [\,[A^i,S^\sharp],A] - \nabla^t_N\sum\nolimits_{j=0}^{i-1} A^j(\nabla^t_N S^\sharp)A^{i-1-j}\right).$$

(b) Next, we shall find the evolution of tensors $T_i(A)$ and $\nabla^t_N T_i(A)$ and their duals.
By Lemma 2.9 and the method of Example 2.3(a) we find the evolution of $T_i(A)$,

$$2\,\partial_t T_i(A) = [T_i(A), S^\sharp] - \sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}\sum\nolimits_{p=0}^{j-1}A^p(\nabla^t_N S^\sharp)A^{j-1-p}$$
$$-\sum\nolimits_{j=0}^{i-1}(-1)^j\,\mathrm{Tr}\,(T_{i-j-1}(A)\nabla^t_N S^\sharp)A^j.$$

For $S = s\,\hat{g}$ $(s\in C^1(M))$ and $i>0$ we certainly have

$$\partial_t A^i = -\frac{i}{2}A^{i-1}N(s), \qquad \partial_t T_i(A) = \frac{1}{2}N(s)(n-i)\sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}A^{j-1}.$$

Notice that $\det A\neq 0$ provides

$$\sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}A^{j-1} = (T_i(A)-\sigma_i\,\widehat{\mathrm{id}})A^{-1}.$$

Similar to the result for $\nabla^t_N A^i$ in (a), we find the evolution of $\nabla^t_N T_i(A)$,

$$\begin{aligned}\partial_t(\nabla^t_N T_i(A)) &= \nabla^t_N(\partial_t T_i(A)) + (1/2)\left([\nabla^t_N S^\sharp,\, T_i(A)] - [[T_i(A),S^\sharp],A]\right)\\
&= -\frac{1}{2}\nabla^t_N\Big(\sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}\sum\nolimits_{p=0}^{j-1}A^p(\nabla^t_N S^\sharp)A^{j-1-p}\\
&\qquad +\sum\nolimits_{j=0}^{i-1}(-1)^j\,\mathrm{Tr}\,(T_{i-j-1}(A)\nabla^t_N S^\sharp)A^j\Big)\\
&\quad +\frac{1}{2}\left([\nabla^t_N T_i(A),\, S^\sharp] - [[T_i(A),S^\sharp],A]\right).\end{aligned}$$

As $T_i(A) = (T_i(b))^\sharp$, $\nabla^t_N T_i(A) = (\nabla^t_N T_i(b))^\sharp$, from the above and (2.9) we find the evolution of $T_i(b)$ and $\nabla^t_N T_i(b)$ for $i>0$,

$$\begin{aligned}(\partial_t T_i(b))^\sharp &= \partial_t T_i(A) + S^\sharp T_i(A) = \frac{1}{2}\Big(S^\sharp T_i(A) + T_i(A)S^\sharp\\
&\quad -\sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}\sum\nolimits_{p=0}^{j-1}A^p(\nabla^t_N S^\sharp)A^{j-1-p}\\
&\quad -\sum\nolimits_{j=0}^{i-1}(-1)^j\,\mathrm{Tr}\,(T_{i-j-1}(A)\nabla^t_N S^\sharp)A^j\Big),\\
(\partial_t\nabla^t_N T_i(b))^\sharp &= \partial_t\nabla^t_N T_i(A) + S^\sharp\nabla^t_N T_i(A)\\
&= \frac{1}{2}\Big(S^\sharp\nabla^t_N T_i(A) + (\nabla^t_N T_i(A))S^\sharp - [[T_i(A),\, S^\sharp],\, A]\\
&\quad -\nabla^t_N\Big(\sum\nolimits_{j=0}^{i-1}(-1)^j\,\mathrm{Tr}\,(T_{i-j-1}(A)\nabla^t_N S^\sharp)A^j\\
&\qquad +\sum\nolimits_{j=1}^{i}(-1)^j\sigma_{i-j}\sum\nolimits_{p=0}^{j-1}A^p(\nabla^t_N S^\sharp)A^{j-1-p}\Big)\Big).\end{aligned}$$

Notice that using the connection (2.13), we also have

$$(\widetilde{\nabla}_{\partial_t} b)(X,Y) = \partial_t b(X,Y) - b(\widetilde{\nabla}_{\partial_t} X,Y) - b(X,\widetilde{\nabla}_{\partial_t} Y) = -\frac{1}{2}(\nabla^t_N S)(X,Y).$$

Proposition 2.2. *Let $g_t \in \mathscr{M}$ be a family of $\mathscr{F}$-truncated metrics and $S = \partial_t g_t$. Then*

$$\begin{aligned}
\widetilde{\nabla}_{\partial_t} \hat{b}_i &= -\frac{1}{2} \sum\nolimits_{j=0}^{i-1} \Big(A^j (\nabla^t_N S^\sharp) A^{i-1-j}\Big)^\flat, \\
\widetilde{\nabla}_{\partial_t} (\nabla^t_N \hat{b}_i) &= -\frac{1}{2}\Big([[A^i, S^\sharp], A] + \nabla^t_N \sum\nolimits_{j=0}^{i-1} A^j (\nabla^t_N S^\sharp) A^{i-1-j}\Big)^\flat, \\
\widetilde{\nabla}_{\partial_t} T_i(b) &= -\frac{1}{2}\Big(\sum\nolimits_{j=1}^{i} (-1)^j \sigma_{i-j} \sum\nolimits_{p=0}^{j-1} A^p (\nabla^t_N S^\sharp) A^{j-1-p} \\
&\qquad + \sum\nolimits_{j=0}^{i-1} (-1)^j \operatorname{Tr}(T_{i-j-1}(A) \nabla^t_N S^\sharp) A^j\Big)^\flat, \\
\widetilde{\nabla}_{\partial_t} (\nabla^t_N T_i(b)) &= -\frac{1}{2}\Big([[T_i(A), S^\sharp], A] + \nabla^t_N \Big(\sum\nolimits_{j=0}^{i-1} (-1)^j \operatorname{Tr}\Big(T_{i-j-1}(A) \nabla^t_N S^\sharp\Big) A^j \\
&\qquad + \sum\nolimits_{j=1}^{i} (-1)^j \sigma_{i-j} \sum\nolimits_{p=0}^{j-1} A^p (\nabla^t_N S^\sharp) A^{j-1-p}\Big)\Big)^\flat.
\end{aligned}$$

Proof. Using Example 2.3(a), (2.13), and equalities

$$\begin{aligned}
\widetilde{\nabla}_{\partial_t} \hat{b}_i(X,Y) &= \partial_t \hat{b}_i(X,Y) - \hat{b}_i(\widetilde{\nabla}_{\partial_t} X,Y) - \hat{b}_i(X,\widetilde{\nabla}_{\partial_t} Y), \\
\widetilde{\nabla}_{\partial_t} (\nabla^t_N \hat{b}_i)(X,Y) &= \partial_t((\nabla^t_N \hat{b}_i)(X,Y)) - (\nabla^t_N \hat{b}_i)(\widetilde{\nabla}_{\partial_t} X,Y) - (\nabla^t_N \hat{b}_i)(X,\widetilde{\nabla}_{\partial_t} Y),
\end{aligned}$$

we find $\widetilde{\nabla}_{\partial_t} \hat{b}_i$ and $\widetilde{\nabla}_{\partial_t} (\nabla^t_N \hat{b}_i)$. Similarly, from Example 2.3(b) and the definition (2.13), the formulae for $\widetilde{\nabla}_{\partial_t} T_i(b)$ and $\widetilde{\nabla}_{\partial_t} (\nabla^t_N T_i(b))$ follow. □

2.3.2 Variations of General Functionals

Here we develop *variational formulae* for the functional $I_f(g)$ of (2.1), restricted to metrics in $\mathscr{M}$ and $\mathscr{M}_1$, respectively. Let

$$\pi : \mathscr{M} \to \mathscr{M}_1, \qquad \pi(g) = \bar{g} = \Big(\mathrm{vol}(M,g)^{-2/n}\, \hat{g}\Big) \oplus g^\perp$$

be the $\mathscr{F}$-conformal projection. Metrics $\bar{g}_t = (\phi_t \hat{g}_t) \oplus g_t^\perp$ with dilating factors $\phi_t = \mathrm{vol}(M,g_t)^{-2/n}$, belong to $\mathscr{M}_1$, i.e., $\int_M \mathrm{d}\,\overline{\mathrm{vol}}_t = 1$.

For a family of metrics $g_t \in \mathscr{M}$, we denote $g = g_0$, $S = \partial_t g_t$, and $\dot S = \partial_t S$. Recall [53] that the volume form of g_t evolves as:

$$\partial_t(\mathrm{vol}_t) = \frac{1}{2}(\mathrm{Tr}\, S^\sharp)\,\mathrm{vol}_t\,. \tag{2.21}$$

We certainly have $\mathrm{Tr}\, S^\sharp = \mathrm{Tr}_{\,g_t} S = \langle \hat g_t, S\rangle$. For conformal metrics $g_t = \mathrm{e}^{\varphi_t} g$, where (φ_t) is a smooth family of continuous (smooth whenever needed) functions on M, we obtain a conformal tensor $S_t = s_t\,\hat g_t$, where $s_t = \partial_t\varphi_t$.

Remark 2.4. (a) Let S and $\dot S$ be $\mathscr{F}$-truncated symmetric $(0,2)$-tensor fields. If

$$g_t = g + tS + \frac{1}{2}t^2 \dot S$$

is a “quadratic in t” variation of the metric $g = \hat g \oplus g^\perp \in \mathscr{M}$ then the metrics

$$\bar g_t = \pi(g_t) = \left(\phi_t\left(\hat g + tS + \frac{1}{2}t^2\dot S\right)\right) \oplus g^\perp$$

belong to $\mathscr{M}_1$. For a conformal tensor $S = s\,\hat g$ ($s : M \to \mathbb{R}$), the above metrics are

$$g_t = g + \left(ts + \frac{1}{2}t^2\dot s\right)\hat g, \qquad \bar g_t = \pi(g_t) = \left(\phi_t\left(1 + ts + \frac{1}{2}t^2\dot s\right)\hat g\right) \oplus g^\perp.$$

One may use the above approximations for finding the 1st and second variations of our functionals at $t = 0$ with respect to general families g_t and $\bar g_t$, respectively.

(b) Let $\tilde f : M \to (0,\infty)$ be a smooth function constant on the leaves of $\mathscr{F}$, and

$$\tilde g(X,Y) = \tilde f^2 g(X,Y), \quad X,Y \in T\mathscr{F}.$$

If at least one of vectors X,Y is perpendicular to $\mathscr{F}$, we set $\tilde g(X,Y) = g(X,Y)$. A foliated Riemannian manifold $(M,\mathscr{F},\tilde g)$ is called a *warped foliation* (with a *warping function* $\tilde f$), see [59]. They were studied from the point of view of the Gromov–Hausdorff convergence. The warped foliation generalizes the Bergers modification of a metric of S^3 along the fibers of the Hopf fibration (called *Berger spheres*). For warped foliations $g_t = ((1+ct)\hat g) \oplus g^\perp$ ($c \in \mathbb{R}$), A and τ_j do not depend on t, and we have

$$\mathrm{vol}_t = (1+ct)^{n/2}\,\mathrm{vol}, \quad \phi = (1+tc)^{-1}, \quad \bar g_t = g.$$

Hence, see (2.1),

$$I_f(g_t) = (1+ct)^{\frac{n}{2}} I_f(g), \quad I_f'(g) = \frac{n}{2} I_f(g).$$

We conclude that *if g is a critical metric for I_f with respect to $\mathscr{F}$-conformal variations g_t then $I_f(g)=0$.*

In Theorem 2.1 and its corollaries below, we shall find the variations of functionals I_f with respect to metrics $\bar g_t\in\mathscr{M}_1$ and $g_t\in\mathscr{M}$. In the g_t-case, the variations/gradients are given by the same formulae as in the first one but with *underlined terms* deleted. This can be explained as follows: under the π_*-projection (from $T\mathscr{M}$ onto $T\mathscr{M}_1$), the gradient of the functional contains additional (underlined) component. So,

$$\nabla I_f(g)=\frac12 f\,\hat g-\mathscr{V}(B_f)\quad(\text{in}\quad T\mathscr{M}),$$

while its projection onto $T\mathscr{M}_1$, see (2.22), is

$$\bar\nabla I_f(g)=\nabla I_f(g)-\frac12 I_f(g)\,\hat g=\frac12(f\underline{-I_f(g)})\,\hat g-\mathscr{V}(B_f).$$

The scalar product in $T\mathscr{M}$ is given by $\langle\langle\dot g_1,\dot g_2\rangle\rangle=\int_M\langle\dot g_1,\dot g_2\rangle\,\mathrm{d}\,\mathrm{vol}$. By Lemma 2.5, $\int_M\operatorname{Tr}\mathscr{V}(B^\sharp_f)\,\mathrm{d}\,\mathrm{vol}=0$, hence $\langle\langle\bar\nabla I_f(g),\hat g\rangle\rangle=0$.

Theorem 2.1. *The gradient of the functional $I_f:\mathscr{M}\to\mathbb{R}$, see (2.1), and its projection via $\pi_*:T\mathscr{M}\to T\mathscr{M}_1$ are given by:*

$$\nabla I_f(g)=\frac12(f\underline{-I_f(g)})\,\hat g-\mathscr{V}(B_f),\tag{2.22}$$

where $B_f=\sum_{i=1}^n\frac{i}{2}f_{,\tau_i}\hat b_{i-1}$. The $\mathscr{F}\mathscr{M}_1$- and $\mathscr{F}\mathscr{M}$- components of the gradients are

$$\nabla^{\mathscr{F}}I_f(g)=\left(\frac12\,(f\underline{-I_f(g)})-\frac1n\,\mathscr{V}(\operatorname{Tr}B^\sharp_f)\right)\hat g,\tag{2.23}$$

where $\operatorname{Tr}B^\sharp_f=\sum_{i=1}^n\frac{i}{2}f_{,\tau_i}\tau_{i-1}$.

The second variation of $I_f(\bar g_t)$ (when $S=\partial_t g_t$) at a critical metric $g=\bar g_0$ and its restriction to the $\mathscr{F}$-conformal variations (i.e., $S=s\,\hat g,\ s:M\to\mathbb{R}$) are given by:

$$\begin{aligned}I''_f(\bar g_t)_{|t=0}&=\int_M(\langle\Phi_1(S),S\rangle+\langle\Phi_2(\nabla_N S),\nabla_N S\rangle+\langle\Phi_3(S),\nabla_N S\rangle)\,\mathrm{d}\,\mathrm{vol},\\ I''_f(\bar g_t)_{|t=0}&=\int_M\Phi_f\,N(s)^2\mathrm{d}\,\mathrm{vol},\end{aligned}\tag{2.24}$$

respectively, where

$$\Phi_f=\frac14\sum\nolimits_{i=2}^n i(i-1)\,\tau_{i-2}f_{,\tau_i}+\frac14\sum\nolimits_{i,j=1}^n ij\,\tau_{i-1}\tau_{j-1}f_{,\tau_i\tau_j},$$

and the linear operators $\Phi_i : \widehat{\Lambda}^0_2(M) \to \widehat{\Lambda}^0_2(M)$ $(i=1,2,3)$ *are defined by:*

$$\Phi_1(S^\sharp) = \frac{1}{4}(f \underline{- I_f(g)})(\operatorname{Tr} S^\sharp)\,\widehat{\operatorname{id}} - B^\sharp_f[S^\sharp,\, A], \qquad \Phi_3(S^\sharp) = -(\operatorname{Tr} S^\sharp) B^\sharp_f,$$
$$\Phi_2(S^\sharp) = \sum\nolimits_{i,j=1}^n \frac{ij}{4} f_{,\tau_i\tau_j} \operatorname{Tr}(A^{i-1}S^\sharp)A^{j-1} + \sum\nolimits_{i=2}^n \frac{i}{4} f_{,\tau_i} \sum\nolimits_{j=0}^{i-2} A^j S^\sharp A^{i-2-j}.$$

Proof. As the metrics $\bar g_t$ and g_t are $\mathscr{F}$-conformal with constant scale ϕ_t, by Lemma 2.3 we have $\tau_j(\bar g_t) = \tau_j(g_t)$ and $\overline{\mathrm{vol}}_t = \phi_t^{n/2}\,\mathrm{vol}_t$. Differentiating the last equality and using (2.21), we obtain

$$\partial_t\, \overline{\mathrm{vol}}_t = (\phi^{\frac{n}{2}})'\,\mathrm{vol}_t + \phi^{\frac{n}{2}}\partial_t\,\mathrm{vol}_t = \frac{1}{2}\left(\operatorname{Tr} S^\sharp - \int_M (\operatorname{Tr} S^\sharp)\,\mathrm{d}\,\overline{\mathrm{vol}}_t\right)\overline{\mathrm{vol}}_t.$$

Here, we used the fact that $\phi_0 = 1$ and

$$\phi' = -\frac{2}{n}\,\mathrm{vol}^{-\frac{2}{n}-1}\int_M \partial_t(\mathrm{d}\,\mathrm{vol}_t) = -\frac{1}{n}\,\phi^{\frac{n}{2}+1}\int_M (\operatorname{Tr} S^\sharp)\,\mathrm{d}\,\mathrm{vol}_t = -\frac{\phi}{n}\int_M (\operatorname{Tr} S^\sharp)\,\mathrm{d}\,\overline{\mathrm{vol}}_t.$$

Differentiating the functional $I_f(\bar g_t)$, we obtain

$$\begin{aligned} I'_f(\bar g_t) &= \int_M \left(\partial_t f + \frac{1}{2} f\left(\operatorname{Tr} S^\sharp - \int_M (\operatorname{Tr} S^\sharp)\,\mathrm{d}\,\overline{\mathrm{vol}}_t\right)\right)\mathrm{d}\,\overline{\mathrm{vol}}_t \\ &= \int_M \left(\partial_t f + \frac{1}{2}(f \underline{- I_f(\bar g_t)})\operatorname{Tr} S^\sharp\right)\mathrm{d}\,\overline{\mathrm{vol}}_t. \end{aligned} \tag{2.25}$$

Now, we simplify (2.25): for $f = f(\overrightarrow{\tau})$, by Lemma 2.9, we have

$$\partial_t f = \sum\nolimits_{i=1}^n f_{,\tau_i}\partial_t \tau_i = -\operatorname{Tr}\left(\nabla^t_N S^\sharp \sum\nolimits_{i=1}^n \frac{i}{2} f_{,\tau_i} A^{i-1}\right) = -\operatorname{Tr}\left(B^\sharp_f \nabla^t_N S^\sharp\right), \tag{2.26}$$

where $B^\sharp_f = \sum_{i=1}^n \frac{i}{2} f_{,\tau_i} A^{i-1}$ is dual to B_f. From (2.26), by Lemma 2.6, we have

$$\int_M (\partial_t f)\mathrm{d}\,\overline{\mathrm{vol}}_t = \int_M [\operatorname{Tr}((\nabla^t_N B^\sharp_f)S^\sharp) - N(\operatorname{Tr}(B^\sharp_f S^\sharp))]\mathrm{d}\,\overline{\mathrm{vol}}_t = \int_M \langle -\mathscr{V}(B_f), S\rangle_t \mathrm{d}\,\overline{\mathrm{vol}}_t.$$

Thus, (2.25) yields (2.22):

$$I'_f(\bar g_t) = \int_M \left\langle \frac{1}{2}(f \underline{- I_f(\bar g_t)})\,\hat g_t - \mathscr{V}(B_f),\, S\right\rangle_t \mathrm{d}\,\overline{\mathrm{vol}}_t. \tag{2.27}$$

For a $(0,2)$-tensor $\tilde B_t := \frac{1}{2}(f \underline{- I_f(\bar g_t)})\,\hat g_t - \mathscr{V}(B_f)$ in (2.27), by Lemma 2.4 we have

$$\begin{aligned} \partial_t \langle \tilde B_t, S\rangle_{g_t} &= \langle \partial_t \tilde B_t, S\rangle_{g_t} - 2\langle \tilde B_t, S^2\rangle_{g_t} + \langle \tilde B_t, \partial_t S\rangle_{g_t}, \\ \partial_t \tilde B_t &= \frac{1}{2}(\partial_t f)\,\hat g_t - \frac{1}{2} I'_f(\bar g_t)\,\hat g_t + \frac{1}{2}(f \underline{- I_f(\bar g_t)})S - \partial_t \mathscr{V}(B_f). \end{aligned}$$

Differentiating (2.27) at a critical metric $\bar g_0$ and using $\tilde B_0 = 0$, we get

$$I''_f(\bar g_t)_{\,|t=0} = \int_M \Big\langle \frac12\,(\partial_t f)\,\hat g + \frac12 (f \underline{- I_f(g)})S - \partial_t \mathscr{V}(B_f),\ S\Big\rangle\,\mathrm{d}\,\mathrm{vol}. \tag{2.28}$$

One may compute $I''_f(\bar g_t)_{\,|t=0}$ explicitly using Lemma 2.9 and (2.26). We have

$$\begin{aligned}
\partial_t \mathscr{V}(B_f) &= (\partial_t \tau_1) B_f + \tau_1 \partial_t B_f - \partial_t \nabla^t_N B_f, \qquad \text{where for } t=0\\
(\partial_t B_f)^\sharp &= \textstyle\sum_i \frac{i}{2}\left(f_{,\tau_i}\partial_t \hat b_{i-1} + (\partial_t f_{,\tau_i})\hat b_{i-1}\right)^\sharp = \frac12 (S^\sharp B^\sharp_f + B^\sharp_f S^\sharp)\\
&\quad - \textstyle\sum_i \frac{i}{2}\left(\frac12 f_{,\tau_i}\sum_{j=0}^{i-2} A^j (\nabla_N S^\sharp) A^{i-2-j}\right.\\
&\qquad\qquad \left. + \textstyle\sum_{j=1}^n \frac{j}{2} f_{,\tau_i\tau_j}\,\mathrm{Tr}\left(A^{j-1}(\nabla_N S^\sharp)\right) A^{i-1}\right),\\
(\partial_t(\nabla_N B_f))^\sharp &= \textstyle\sum_i \frac{i}{2}\left(N(f_{,\tau_i})\partial_t \hat b_{i-1} + f_{,\tau_i}\partial_t \nabla_N \hat b_{i-1} + \nabla_N((\partial_t f_{,\tau_i})\hat b_{i-1})\right)^\sharp\\
&= \frac12\left(S^\sharp \nabla_N B^\sharp_f + (\nabla_N B^\sharp_f) S^\sharp\right)\\
&\quad - \textstyle\sum_i \frac{i}{2}\left(\frac12 \nabla_N\left(f_{,\tau_i}\sum_{j=0}^{i-2} A^j(\nabla_N S^\sharp)A^{i-2-j}\right)\right.\\
&\quad \left. + \nabla_N\left(A^{i-1}\textstyle\sum_j \frac{j}{2} f_{,\tau_i\tau_j}\,\mathrm{Tr}\,(A^{j-1}\nabla_N S^\sharp)\right)\right) - (1/2)[[B^\sharp_f, S^\sharp], A].
\end{aligned}$$

Here we used the t-derivatives of $\hat b_{i-1}$ and $\nabla^t_N \hat b_{i-1}$ of Example 2.3 and the equality

$$\partial_t(f_{,\tau_i}) = -\textstyle\sum_{j=1}^n \frac{j}{2} f_{,\tau_i\tau_j}\,\mathrm{Tr}\,(A^{j-1}\nabla^t_N S^\sharp).$$

Substituting the above into (2.28), and using the equalities

$$\begin{aligned}
\mathrm{Tr}\,([\,[B^\sharp_f, S^\sharp], A]S^\sharp) &= 2\,\mathrm{Tr}\,(B^\sharp_f [S^\sharp, A] S^\sharp) \qquad \text{for commuting } B^\sharp_f \text{ and } A,\\
\mathscr{V}(B^\sharp_f) &= \frac12 (f \underline{- I_f(g)})\,\widehat{\mathrm{id}} \qquad \text{at a critical metric},\\
\int_M \mathrm{Tr}\,(\nabla_N S^\sharp)\,\mathrm{Tr}\,(B^\sharp_f S^\sharp)\mathrm{d}\,\mathrm{vol} &= \int_M \langle \mathscr{V}(\mathrm{Tr}\,(B^\sharp_f S^\sharp))\,\hat g,\, S\rangle\,\mathrm{d}\,\mathrm{vol}\\
&= \int_M \mathscr{V}(\mathrm{Tr}\,(B^\sharp_f S^\sharp))\,\mathrm{Tr}\,S^\sharp\,\mathrm{d}\,\mathrm{vol}\\
&= \int_M \left(\mathrm{Tr}\,(\mathscr{V}(B^\sharp_f)S^\sharp)\,\mathrm{Tr}\,S^\sharp - \mathrm{Tr}\,(B^\sharp_f \nabla_N S^\sharp)\,\mathrm{Tr}\,S^\sharp\right)\mathrm{d}\,\mathrm{vol},
\end{aligned}$$

(the last one is based on Lemma 2.6) we obtain

$$
\begin{aligned}
I''_f(\bar g_t)_{|t=0} = \int_M \Big\{ & \frac14 (f - \underline{I_f(g)})(\operatorname{Tr} S^\sharp)^2 - \operatorname{Tr}(B^\sharp_f[S^\sharp, A]S^\sharp) - \operatorname{Tr}(B^\sharp_f \nabla_N S^\sharp)\operatorname{Tr} S^\sharp \\
& + \tau_1 \sum_i \frac{i}{2}\Big(\frac12 f_{,\tau_i} \sum_{j\le i-2} \operatorname{Tr}(A^j(\nabla_N S^\sharp)A^{i-2-j}S^\sharp) + \operatorname{Tr}(A^{i-1}S^\sharp) \\
& \sum_j \frac{j}{2} f_{,\tau_i\tau_j} \operatorname{Tr}(A^{j-1}\nabla_N S^\sharp)\Big) \\
& - \sum\nolimits_i \frac{i}{2}\Big(\frac12 \operatorname{Tr}\Big(S^\sharp \nabla_N \Big(f_{,\tau_i} \sum\nolimits_{j=0}^{i-2} A^j(\nabla_N S^\sharp)A^{i-2-j}\Big)\Big) \\
& + \operatorname{Tr}\Big(S^\sharp \nabla^t_N \Big(A^{i-1}\sum\nolimits_j \frac{j}{2} f_{,\tau_i\tau_j} \operatorname{Tr}(A^{j-1}\nabla_N S^\sharp)\Big)\Big)\Big)\Big\} \, d\,\mathrm{vol}.
\end{aligned}
$$

Simplifying terms with $f_{,\tau_i\tau_j}$ and sums "$\sum_{j=0}^{i-2}$" (by Lemma 2.5), we finally obtain

$$
\begin{aligned}
I''_f(\bar g_t)_{|}t = 0 = \int_M \Big\{ & \frac14 (f - \underline{I_f(g)})(\operatorname{Tr} S^\sharp)^2 - \operatorname{Tr}(B^\sharp_f[S^\sharp, A]S^\sharp) - \operatorname{Tr}(B^\sharp_f \nabla_N S^\sharp)\operatorname{Tr} S^\sharp \\
& + \sum\nolimits_{i=2}^n \sum\nolimits_{j=0}^{i-2} \frac{i}{4} f_{,\tau_i} \operatorname{Tr}(A^j(\nabla_N S^\sharp)A^{i-2-j}\nabla_N S^\sharp) \\
& + \sum\nolimits_{i,j=1}^n \frac{ij}{4} f_{,\tau_i\tau_j} \operatorname{Tr}(A^{i-1}\nabla_N S^\sharp)\operatorname{Tr}(A^{j-1}\nabla_N S^\sharp)\Big\} \, d\,\mathrm{vol}. \qquad (2.29)
\end{aligned}
$$

By (2.29), the integrand of $I''_f(\bar g_t)_{|t=0}$ has the form $(2.24)_1$.

Let $S = s\,\hat g$. Although the result follows from the above (the RHS of the formula for Φ_2 reads as $\Phi_f\, N(s)^2$ and $\int_M \langle(\Phi_1 + \mathscr{V}\circ\Phi_3)(S), S\rangle \, d\,\mathrm{vol} = 0$), we shall prove it independently. In this case, $\operatorname{Tr} B^\sharp_f = \sum_{i=1}^n \frac{i}{2} f_{,\tau_i}\tau_{i-1}$, and (2.27) reads:

$$
\begin{aligned}
I'_f(\bar g_t) &= \int_M s\,\Big(\frac{n}{2}(f - \underline{I_f(\bar g_t)}) - \mathscr{V}(\operatorname{Tr} B^\sharp_f)\Big)\, d\,\overline{\mathrm{vol}}_t \\
&= \int_M \Big\langle \Big(\frac12 (f - \underline{I_f(\bar g_t)}) - \frac1n \mathscr{V}(\operatorname{Tr} B^\sharp_f)\Big)\,\hat g,\ s\,\hat g\Big\rangle \, d\,\overline{\mathrm{vol}}_t. \qquad (2.30)
\end{aligned}
$$

From (2.30) with $t = 0$ (or, from (2.22)), (2.23) follows. Differentiating (2.30) at a critical metric $g = \bar g_0$, or applying $S = s\,\hat g$ to (2.28), we find

$$
I''_f(\bar g_t)_{|t=0} = \int_M \Big(s\frac{n}{2}\partial_t f + s^2\frac{n}{2}(f - \underline{I_f(g)}) - s\operatorname{Tr}_g(\partial_t \mathscr{V}(B_f))\Big)\, d\,\mathrm{vol}.
$$

From the above formula and (2.23) at a critical metric, we have

$$\int_M \Big(s\,\frac{n}{2}\,\partial_t f + s^2\frac{n}{2}(f - \underline{I_f(\bar g_t)})\Big)\,\mathrm{d\,vol} = \int_M \Big(\mathscr{V}(\mathrm{Tr}\,B_f^\sharp)\,s^2 - \frac{n}{2}\,(\mathrm{Tr}\,B_f^\sharp) s N(s)\Big)\,\mathrm{d\,vol}$$
$$= \Big(1-\frac{n}{4}\Big)\int_M \mathscr{V}(\mathrm{Tr}\,B_f^\sharp)\,s^2\,\mathrm{d\,vol}\,.$$

By Lemma 2.9, we have

$$\partial_t(\mathrm{Tr}\,B_f^\sharp)_{|t=0} = \sum\nolimits_i \frac{i}{2}\,((\partial_t f_{,\tau_i})\tau_{i-1} + f_{,\tau_i}\partial_t\tau_{i-1}) = -\Phi_f\,N(s).$$

Using this, $(2.9)_2$ and the identity

$$\partial_t\mathscr{V}(\phi_t) = (\partial_t\tau_1)\,\phi_t + \mathscr{V}(\partial_t\phi_t), \quad \forall\,\phi_t\in C^1(M),$$

we calculate

$$\begin{aligned}
\int_M s\,\mathrm{Tr}_{\,g}(\partial_t\mathscr{V}(B_f))\,\mathrm{d\,vol} &= \int_M s\,\big(\partial_t(\mathrm{Tr}_{\,g_t}\mathscr{V}(B_f))_{|t=0} + s\,\mathrm{Tr}_{\,g}\mathscr{V}(B_f)\big)\,\mathrm{d\,vol}\\
&= \int_M \Big(\mathscr{V}(\mathrm{Tr}\,B_f^\sharp)\,s^2 + s\,\partial_t(\mathscr{V}(\mathrm{Tr}\,B_f^\sharp))_{|t=0}\Big)\,\mathrm{d\,vol}\\
&= \int_M \Big(\mathscr{V}(\mathrm{Tr}\,B_f^\sharp)\,s^2 + s\,\Big(\mathscr{V}(-\Phi_f N(s))\\
&\qquad\qquad + (\partial_t\tau_1)\,\mathrm{Tr}\,B_f^\sharp\Big)\Big)\,\mathrm{d\,vol}\\
&= \int_M \Big(\mathscr{V}(\mathrm{Tr}\,B_f^\sharp)\,s^2 - \mathscr{V}(\Phi_f)\,sN(s) + \Phi_f\,sN(N(s))\\
&\qquad\qquad - \frac{n}{2}(\mathrm{Tr}\,B_f^\sharp)\,sN(s)\Big)\,\mathrm{d\,vol}\\
&= \int_M \Big(\Big(1-\frac{n}{4}\Big)\,\mathscr{V}(\mathrm{Tr}\,B_f^\sharp)s^2 - \Phi_f\,N(s)^2\Big)\,\mathrm{d\,vol}\,.
\end{aligned}$$

The formula $(2.24)_2$ follows from the above. □

Example 2.4 (Totally geodesic foliations). Let $\mathscr{F}$ be a totally geodesic foliation on (M,g) of unit volume. Then

$$A=0, \quad \overrightarrow{\tau}=0, \quad B_f = \frac{1}{2}f_{,\tau_1}(0)\hat g, \quad \mathscr{V}(B_f)=0, \quad I_f(g)=f(0).$$

By (2.22) of Theorem 2.1, g is a critical metric for the functional I_f with respect to variations $\bar g_t\in\mathscr{M}_1$. We also have $\Phi_f = \frac{n}{2}f_{,\tau_2}(0) + \frac{n^2}{4}f_{,\tau_1\tau_1}(0)$ and

$$\Phi_1(S^\sharp)=0, \qquad \Phi_3(S^\sharp) = -\frac{1}{2}f_{,\tau_1}(0)(\mathrm{Tr}\,S^\sharp)\,\widehat{\mathrm{id}},$$
$$\Phi_2(S^\sharp) = \frac{1}{4}f_{,\tau_1\tau_1}(0)(\mathrm{Tr}\,S^\sharp)\,\widehat{\mathrm{id}} + \frac{1}{2}f_{,\tau_2}(0)S^\sharp.$$

Using Lemmas 2.5 and 2.6, we calculate

$$I''_f(\bar{g}_t)_{|t=0} = \int_M \left(\frac{1}{4} f_{,\tau_1\tau_1}(0)(N(\operatorname{Tr} S^\sharp))^2 + \frac{1}{2} f_{,\tau_2}(0) \operatorname{Tr}((\nabla_N S^\sharp)^2)\right) \mathrm{d}\,\mathrm{vol}.$$

We conclude that $I''_f \ge 0$, when $f_{,\tau_1\tau_1}(0) \ge 0$ and $f_{,\tau_2}(0) \ge 0$.

Question. Under what conditions on a smooth function $f(\tau_1, \dots, \tau_n)$ is the form

$$\langle \Phi_2(S), S\rangle = \sum_{i,j=1}^{n} \frac{ij}{4} f_{,\tau_i\tau_j} \operatorname{Tr}(A^{i-1}S^\sharp)\operatorname{Tr}(A^{j-1}S^\sharp) + \sum_{i=2}^{n} \frac{i}{4} f_{,\tau_i} \sum_{j=0}^{i-2} \operatorname{Tr}(A^j S^\sharp A^{i-2-j} S^\sharp)$$

positive definite for all $\mathscr{F}$-truncated symmetric $(0,2)$-tensors S?

Example. For $f = \tau_2$, we have $\langle \Phi_2(S), S\rangle = \frac{1}{2}\langle S, S\rangle$. Notice that for $S = s\,\widehat{\mathrm{id}}$ the condition reads:

$$\sum\nolimits_{i,j=1}^{n} ij\, f_{,\tau_i\tau_j} \tau_{i-1}\tau_{j-1} + \sum\nolimits_{i=2}^{n} i(i-2) f_{,\tau_i} \tau_{i-2} > 0.$$

Remark 2.5. Consider the function

$$F = \sum\nolimits_{i,j=1}^{n-1} \tilde{H}_{ij}\tau_i\tau_j + 2\sum\nolimits_{i=1}^{n-1} b_i\tau_i + c, \text{ where } c = n^2 f_{,\tau_1\tau_1} + 2n f_{,\tau_2},$$

while the $(n-1)\times(n-1)$ matrix $\tilde{H}$ and $(n-1)$-vector b are given by:

$$\begin{aligned}
\tilde{H}_{ij} &= (i+1)(j+1) f_{,\tau_{i+1}\tau_{j+1}} \quad (1 \le i, j < n),\\
b_i &= n(i+1) f_{,\tau_1\tau_{i+1}} - n^2 f_{,\tau_1\tau_n}\delta_{i,n-1} + \frac{1}{2}(i+2)(i+1) f_{,\tau_{i+2}} \quad (1 \le i < n).
\end{aligned}$$

Critical points of F are solutions $\tilde{\tau} = (\tau_1, \dots, \tau_{n-1})$ to the system $\tilde{H}\tilde{\tau} = -b$. If $\tilde{H}$ is positive definite and $b^T(\tilde{H}^{-1})^T(\tilde{H} - 2\,\mathrm{id})\,b > -c$ for all $\tilde{\tau}$, then $F > 0$. Indeed, under above conditions, Φ_f of Theorem 2.1 is strictly positive and for any $g \in \mathscr{M}_1$ the functional I_f, when restricted on $\mathscr{F}$-conformal metrics of unit volume, has at most one critical point.

Proposition 2.3. *Let a metric g on a closed foliated manifold $(M, \mathscr{F})$ be a stable local maximum on the space $\mathscr{M}_1$ for the functional I_f (with a fixed $f \in C^2(\mathbb{R}^n)$). If $\int_M \langle \Phi_2(\nabla_N S), \nabla_N S\rangle \mathrm{d}\,\mathrm{vol} \ge 0$ for any $\mathscr{F}$-truncated symmetric $(0,2)$-tensor S, and*

$$\sum\nolimits_{m=1}^{n} m f_{,\tau_m}(k_i^{m-1} - k_j^{m-1})(k_i - k_j) \ge 0 \tag{2.31}$$

for any principal curvatures $k_i \ne k_j$, then $\mathscr{F}$ is umbilical.

Proof. One may take $S^\sharp$ with the property $\operatorname{Tr} S^\sharp = 0$. Then, by Theorem 2.1,

$$I''_f(g) = \int_M \left(-\operatorname{Tr}(B_f^\sharp[S^\sharp, A]S^\sharp) + \langle \Phi_2(\nabla_N S), \nabla_N S\rangle\right) \mathrm{d}\,\mathrm{vol}.$$

Let $S^\sharp = (s_{ij})$ in the frame of principal directions (for A). One may show that

$$-\operatorname{Tr}(A^m [S^\sharp,\, A]\, S^\sharp) = \sum\nolimits_{i<j} s_{ij}^2 (k_i^{m-1} - k_j^{m-1})(k_i - k_j)$$

for all $m > 0$. Hence, by the condition (2.31), for $B_f^\sharp = \sum_{m=1}^n f_{,\tau_m} A^{m-1}$ we have

$$-\operatorname{Tr}(B_f^\sharp\, [S^\sharp,\, A]\, S^\sharp) \geq 0.$$

and the equality holds only when $k_i = k_j$ for all $i \neq j$. As the second variation $I''_f(g) \leq 0$, from the above we conclude that $\mathscr{F}$ is umbilical. $\square$

Notice that the condition $f_{,\tau_m} \begin{cases} \geq 0, & \text{for } m \text{ even} \\ = 0, & \text{for } m \text{ odd} \end{cases}$ yields the inequality (2.31).

2.3.3 Variations of Particular Functionals

The following functionals on $\mathscr{M}$ for particular cases of f were introduced in (1.1):

$$I_{\tau,k}(g) = \int_M \tau_k \,\mathrm{d}\,\mathrm{vol}_g, \quad I_{\sigma,k}(g) = \int_M \sigma_k \,\mathrm{d}\,\mathrm{vol}_g, \quad k = 1, 2, \ldots.$$

From [41], see (1.2), it is known that $I_{\tau,1} = I_{\sigma,1} = 0$ for any $\mathscr{F}$ and g on a closed M.

From Theorem 2.1 with $f = \tau_k$ it follows

Corollary 2.2. *The gradient of the functional $I_{\tau,k} : \mathscr{M} \to \mathbb{R}$ for $k > 1$ and its projection via $\pi_* : T\mathscr{M} \to T\mathscr{M}_1$ are given by:*

$$\nabla I_{\tau,k}(g) = \frac{1}{2}\,(\tau_k \underline{- I_{\tau,k}(g)})\,\hat{g} - \frac{k}{2}\,\mathscr{V}(\hat{b}_{k-1}).$$

The $\mathscr{F}\mathscr{M}$- and $\mathscr{F}\mathscr{M}_1$- components of above gradient are given, respectively, by

$$\nabla^{\mathscr{F}} I_{\tau,k}(g) = \frac{1}{2}\left(\tau_k \underline{- I_{\tau,k}(g)} - \frac{k}{n}\,\mathscr{V}(\tau_{k-1})\right)\hat{g}.$$

The second variation of $I_{\tau,k}$ at a critical metric $g = \bar{g}_0$, and its restriction to the $\mathscr{F}$-conformal variations $S = s\,\hat{g}$ $(s : M \to \mathbb{R})$ are given by (2.24), where $S = \partial_t \bar{g}_t$, $f = \tau_k$, $\Phi_f = \frac{1}{4}k(k-1)\tau_{k-2}$, and

$$\Phi_1(S^\sharp) = \frac{1}{4}\,(\tau_k \underline{- I_{\tau,k}(g)})(\operatorname{Tr} S^\sharp)\,\widehat{\mathrm{id}} - \frac{k}{2} A^{k-1} [S^\sharp,\, A],$$

$$\Phi_2(S^\sharp) = \frac{k}{4} \sum\nolimits_{j=0}^{k-2} A^j\, S^\sharp A^{k-2-j}, \qquad \Phi_3(S^\sharp) = -\frac{k}{2} (\operatorname{Tr} S^\sharp) A^{k-1}.$$

Proof. As in the proof of Theorem 2.1 (see (2.25) with $f=\tau_k$), we obtain

$$I'_{\tau,k}(\bar g_t)=\int_M\Big(\partial_t\,\tau_k+\frac12\,(\tau_k-I_{\tau,k}(\bar g_t))\,\mathrm{Tr}\,S^\sharp\Big)\,\mathrm d\,\overline{\mathrm{vol}}_t\,.$$

By $(2.20)_1$ and Lemma 2.6, we have at $t=0$

$$\int_M(\partial_t\,\tau_k)\,\mathrm d\,\overline{\mathrm{vol}}=-\frac k2\int_M\langle A^{k-1},\,\nabla_N S^\sharp\rangle\mathrm d\,\overline{\mathrm{vol}}=-\frac k2\int_M\big\langle\,\mathscr V(A^{k-1}),\,S^\sharp\big\rangle\mathrm d\,\overline{\mathrm{vol}}.$$

The above (or Theorem 2.1 with $B_f=\frac k2\hat b_{k-1}$ and $\mathrm{Tr}\,B^\sharp_f=\frac k2\tau_{k-1}$) yield the formula for the gradient $\nabla I_{\tau,k}(g)$.

We shall comment about $I''_{\tau,k}(\bar g_t)_{\,|t=0}$. By (2.28), we obtain

$$I''_{\tau,k}(\bar g_t)_{\,|t=0}=\frac12\int_M\Big\langle(\partial_t\,\tau_k)\,\hat g+(\tau_k\underline{-I_{\tau,k}(g)})\,S-k\,\partial_t\mathscr V(\hat b_{k-1}),\;S\Big\rangle\,\mathrm d\,\mathrm{vol}\,. \qquad (2.32)$$

Observe that by $(2.20)_1$ the first term in (2.32) yields:

$$\frac12\int_M\langle(\partial_t\,\tau_k)\,\hat g,\;S\rangle\,\mathrm d\,\mathrm{vol}=-\frac k4\int_M(\mathrm{Tr}\,S^\sharp)\,\mathrm{Tr}\,(A^{k-1}\nabla_N S^\sharp)\,\mathrm d\,\mathrm{vol}\,.$$

We have $\partial_t\mathscr V(\hat b_{k-1})=(\partial_t\,\tau_1)\hat b_{k-1}+\tau_1\partial_t\hat b_{k-1}-\partial_t\nabla^t_N\hat b_{k-1}$, where by Example 2.3(a),

$$\begin{aligned}\int_M\langle\partial_t\hat b_{k-1},S\rangle\,\mathrm d\,\mathrm{vol}&=\int_M\Big(\mathrm{Tr}\,(S^\sharp A^{k-1}S^\sharp)\\&\qquad-\frac12\,\mathrm{Tr}\,\Big(S^\sharp\textstyle\sum_{j=0}^{k-2}A^j(\nabla_N S^\sharp)A^{k-2-j}\Big)\Big)\,\mathrm d\,\mathrm{vol},\\ \int_M\langle\partial_t(\nabla_N\hat b_{k-1}),\;S\rangle\,\mathrm d\,\mathrm{vol}&=\int_M\Big(\mathrm{Tr}\,(S^\sharp(\nabla^t_N A^{k-1})S^\sharp)-\mathrm{Tr}\,(A^{k-1}[S^\sharp,A]S^\sharp)\\&\qquad-\frac12\,\mathrm{Tr}\,\Big(S^\sharp\nabla^t_N\textstyle\sum_{j=0}^{k-2}A^j(\nabla^t_N S^\sharp)A^{k-2-j}\Big)\Big)\,\mathrm d\,\mathrm{vol}\,.\end{aligned}$$

Using the above, and the identities

$$\int_M\mathrm{Tr}\,(\nabla_N S^\sharp)\,\mathrm{Tr}\,(S^\sharp A^{k-1})\,\mathrm d\,\mathrm{vol}=\int_M\Big(\mathrm{Tr}\,\Big(S^\sharp\,\mathscr V(A^{k-1})-A^{k-1}\nabla_N S^\sharp\Big)\,\mathrm{Tr}\,S^\sharp\Big)\,\mathrm d\,\mathrm{vol},$$

$$k\mathscr V(A^{k-1})=(\tau_k\underline{-I_{\tau,k}(g)})\,\widehat{\mathrm{id}}\quad\text{(at a critical metric)}$$

(or directly from (2.29) with $f=\tau_k$) we obtain

$$I''_f(\bar g_t)_{|t=0}=\int_M\Big\{\frac14\,(\tau_k\underline{-I_{\tau,k}(g)})(\operatorname{Tr}S^\sharp)^2-\frac{k}{2}\operatorname{Tr}(A^{k-1}[S^\sharp,\,A]S^\sharp)$$
$$-\frac{k}{2}\operatorname{Tr}(A^{k-1}\nabla_N S^\sharp)\operatorname{Tr}S^\sharp+\frac{k}{4}\operatorname{Tr}\Big((\nabla_N S^\sharp)\sum\nolimits_{j=0}^{k-2}A^j(\nabla_N S^\sharp)A^{k-2-j}\Big)\Big\}\mathrm{d}\,\mathrm{vol}.$$

From the above the required formulae for Φ_i $(i=1,2,3)$ follow. □

Notice that for $f=\tau_2$ the form $\langle\Phi_2(\nabla_N S^\sharp),\,\nabla_N S^\sharp\rangle$ is non-negative definite. By Lemma 2.7(i), if the lengths of N-curves are unbounded then $\mu(\Phi_2)=0$.

In the next consequence of Theorem 2.1 (for $f=\sigma_k$) we represent the operators Φ_k explicitly using Newtonian transformations.

Corollary 2.3. *The gradient of the functional $I_{\sigma,k}:\mathscr{M}\to\mathbb{R}$ for $k>1$ and its projection via $\pi_*:T\mathscr{M}\to T\mathscr{M}_1$ are given by:*

$$\nabla I_{\sigma,k}(g)=\frac12(\sigma_k\underline{-I_{\sigma,k}(g)})\,\hat g-\frac12\,\mathscr{V}(T_{k-1}(b)).$$

The $\mathscr{F}\mathscr{M}$- and $\mathscr{F}\mathscr{M}_1$- components of above gradients are given, respectively, by:

$$\nabla^{\mathscr{F}}I_{\sigma,k}(g)=\frac12\left(\sigma_k\underline{-I_{\sigma,k}(g)}-\frac{n-k+1}{n}\,\mathscr{V}(\sigma_{k-1})\right)\hat g.$$

The second variation of $I_{\sigma,k}$ $(k>1)$ at a critical metric $g=\bar g_0$, and its restriction to the $\mathscr{F}$-conformal variations $S=s\,\hat g$ $(s:M\to\mathbb{R})$ are given by (2.24), where $S=\partial_t\bar g_t$, $f=\sigma_k$, $\Phi_f=\frac14\,(n-k+1)(n-k+2)\,\sigma_{k-2}$, and

$$\Phi_1(S^\sharp)=\frac14(\sigma_k\underline{-I_{\sigma,k}(g)})(\operatorname{Tr}S^\sharp)\,\widehat{\mathrm{id}}-\frac12\,T_{k-1}(A)[S^\sharp,\,A],$$
$$\Phi_2(S^\sharp)=\frac14\sum\nolimits_{j=0}^{k-2}(-1)^j\left(\operatorname{Tr}(T_{k-j-2}(A)S^\sharp)A^j-\sigma_{k-j-2}\sum\nolimits_{p=0}^{j}A^pS^\sharp A^{j-p}\right),$$
$$\Phi_3(S^\sharp)=-\frac12(\operatorname{Tr}S^\sharp)\,T_{k-1}(A).$$

Proof. Using only Proposition 2.2 and the identity $\operatorname{Tr}T_k(A)=(n-k)\,\sigma_k$ we obtain $B_f=\frac12 T_{k-1}(b)$. As in the proof of Theorem 2.1 (see (2.25) with $f=\sigma_k$), we get

$$I'_{\sigma,k}(\bar g_t)=\int_M\left(\partial_t\sigma_k+\frac12\,(\sigma_k-I_{\sigma,k}(\bar g_t))\operatorname{Tr}S^\sharp\right)\mathrm{d}\,\overline{\mathrm{vol}}_t.$$

By $(2.20)_2$ and Lemma 2.6, we have at $t=0$

$$\int_M(\partial_t\sigma_k)\,\mathrm{d}\,\overline{\mathrm{vol}}=-\frac12\int_M\langle T_{k-1}(A),\,\nabla_N S^\sharp\rangle\mathrm{d}\,\overline{\mathrm{vol}}=-\frac12\int_M\big\langle\mathscr{V}(T_{k-1}(A)),\,S^\sharp\big\rangle\mathrm{d}\,\overline{\mathrm{vol}}.$$

The above (or Theorem 2.1 with $B_f = \frac{1}{2}T_{k-1}(b)$ and $\operatorname{Tr} B_f^\sharp = \frac{1}{2}(n-k+1)\sigma_{k-1}$) yield the formula for the gradient $\nabla I_{\sigma,k}(g)$. Concerning the second variation of $I_{\sigma,k}$, as in the proof of Theorem 2.1 (with $f = \sigma_k$) or by (2.28), one has

$$I''_{\sigma,k}(\bar g_t)_{\,|t=0} = \frac{1}{2}\int_M \Big\langle (\partial_t \sigma_k)\,\hat g + (\sigma_k \underline{- I_{\sigma,k}(g)})S - \partial_t \mathscr{V}(T_{k-1}(b)),\ S\Big\rangle\,\mathrm{d}\,\mathrm{vol}. \quad (2.33)$$

We have

$$\partial_t \mathscr{V}(T_{k-1}(b)) = (\partial_t \tau_1)T_{k-1}(b) + \tau_1 \partial_t T_{k-1}(b) - \partial_t \nabla^t_N T_{k-1}(b),$$

where $\partial_t T_{k-1}(b)$ and $\partial_t \nabla^t_N T_{k-1}(b)$ are given in Example 2.3(b). Using the identities

$$\mathscr{V}(T_{k-1}(A)) = (\sigma_k \underline{- I_{\sigma,k}(g)})\widehat{\mathrm{id}} \quad \text{(at a critical metric)},$$

$$\frac{1}{2}\operatorname{Tr}([\,[T_{k-1}(A), S^\sharp], A]\,S^\sharp) = \operatorname{Tr}(T_{k-1}(A)[S^\sharp, A]\,S^\sharp),$$

from (2.33), as in the proof of Theorem 2.1, we obtain

$$\begin{aligned} I''_{\sigma,k}(\bar g_t)_{\,|t=0} = \frac{1}{2}\int_M \Big(&\frac{1}{2}(\sigma_k \underline{- I_{\sigma,k}(g)})(\operatorname{Tr} S^\sharp)^2 - \operatorname{Tr}(T_{k-1}(A)[S^\sharp, A]S^\sharp) \\ &- \operatorname{Tr}(T_{k-1}(A)\nabla_N S^\sharp)\operatorname{Tr} S^\sharp + \frac{1}{2}\sum\nolimits_{j=0}^{k-2}(-1)^j \operatorname{Tr}(T_{k-j-2}(A)\nabla_N S^\sharp)\operatorname{Tr}(A^j \nabla_N S^\sharp) \\ &+ \frac{1}{2}\sum\nolimits_{j=1}^{k-1}(-1)^j \sigma_{k-j-1}\sum\nolimits_{p=0}^{j-1}\operatorname{Tr}(A^p(\nabla^t_N S^\sharp)A^{j-p-1}(\nabla_N S^\sharp))\Big)\,\mathrm{d}\,\mathrm{vol}. \end{aligned}$$

The formulae for Φ_i $(i = 1,2,3)$ follow from the above. □

Although the Ricci tensor is the notion of intrinsic geometry, the functional

$$E_N(g) = \int_M \operatorname{Ric}(N,N)\,\mathrm{d}\,\mathrm{vol}_g, \qquad g \in \mathscr{M}$$

(the total *normal Ricci curvature*) belongs to extrinsic geometry of a foliation $\mathscr{F}$ on (M,g): by known integral formula (1.3) we have $E_N = I_{\sigma,2}$.

Example 2.5. For the function $f = \sigma_2$, we have the following particular case of Corollary 2.3. The gradient of the functional $E_N : \mathscr{M} \to \mathbb{R}$ and its projection via $\pi_* : T\mathscr{M} \to T\mathscr{M}_1$ are given by

$$\bar\nabla E_N(g) = \frac{1}{2}(\sigma_2 \underline{- E_N(g)})\,\hat g - \frac{1}{2}\mathscr{V}(T_1(b)).$$

Recall that $T_1(b) = \sigma_1\,\hat g - \hat b_1$. The $\mathscr{F}\mathscr{M}_1$- and $\mathscr{F}\mathscr{M}$- components of the above gradient are

$$\nabla^{\mathscr{F}} E_N(g) = \frac{1}{2}\left(\sigma_2 \underline{- E_N(g)} - \frac{n-1}{n}\mathscr{V}(\sigma_1)\right)\hat{g}.$$

The second variation of E_N at a critical metric $g = \bar{g}_0$, and its restriction to the $\mathscr{F}$-conformal variations are given by (2.24), where $S = \partial_t \bar{g}_t$, $f = \sigma_2$, and

$$\Phi_1(S^\sharp) = \frac{1}{4}(\sigma_2 \underline{- E_N(g)})(\operatorname{Tr} S^\sharp)\,\widehat{\mathrm{id}} - \frac{1}{2}\,T_1(A)[S^\sharp,\,A],$$

$$\Phi_2(S^\sharp) = \frac{1}{4}(\operatorname{Tr} S^\sharp)\,\widehat{\mathrm{id}} - \frac{1}{4}S^\sharp, \qquad \Phi_3(S^\sharp) = -\frac{1}{2}(\operatorname{Tr} S^\sharp)\,T_1(A).$$

Notice that the form $\langle \Phi_2(\nabla_N S^\sharp), \nabla_N S^\sharp\rangle$ is not definite. By Lemmas 2.7 and 2.8, if the lengths of N-curves are unbounded, there are no stable critical metrics for $E_N(g)$.

Example 2.6. The property "being umbilical" (or being close, in some sense, to such), relates to the measure of "nonumbilicity" for foliations, see [30]. The measure of "nonumbilicity" for foliations is expressed by the functional (2.1) with

$$f = \textstyle\sum_{i<j}(k_i - k_j)^2 = n\,\tau_2 - \tau_1^2.$$

Metrics with minimal total "nonumbilicity" (if they exist) are critical for the functional $U_{\mathscr{F}}(g) = \int_M (n\,\tau_2 - \tau_1^2)\,\mathrm{d\,vol}$. Using notations of Theorem 2.1, one has $B_f = n\,\hat{b}_1 - \tau_1 \hat{g}$. (Notice that $\operatorname{Tr} B_f = 0$). In this case,

$$\nabla U_{\mathscr{F}} = \left(\mathscr{V}(\tau_1) + \frac{1}{2}(n\tau_2 - \tau_1^2 \underline{- U_{\mathscr{F}}(g)})\right)\hat{g} - n\mathscr{V}(\hat{b}_1).$$

We also have

$$\Phi_1(S^\sharp) = \frac{1}{4}(\operatorname{Tr} S^\sharp)\left(n\tau_2 - \tau_1^2 \underline{- U_{\mathscr{F}}(g)}\right)\widehat{\mathrm{id}} - nA\,[S^\sharp, A],$$

$$\Phi_2(S^\sharp) = \frac{n}{2}\,S^\sharp, \qquad \Phi_3(S^\sharp) = -(\operatorname{Tr} S^\sharp)(nA - \tau_1\,\widehat{\mathrm{id}}).$$

Hence, $\langle \Phi_2(\nabla_N S), \nabla_N S\rangle \geq 0$. By Lemma 2.7(i), if the lengths of N-curves on M are unbounded then $\mu(\Phi_2)$.

2.4 Applications and Examples

2.4.1 Variational Formulae for Umbilical Foliations

Let $\mathscr{F}$ be an *umbilical* foliation on (M, g) with the normal curvature $\lambda : M \to \mathbb{R}$. One may show that $\mathscr{F}$-conformal variations $g_t \in \mathscr{M}$ preserve this property (i.e., $\lambda = H = \frac{1}{n}\tau_1$).

Proposition 2.4. *Let $\mathscr{F}$ be an umbilical foliation on (M, g_0). If $g_t \in \mathscr{M}$ $(0 \le t < \varepsilon)$ is an $\mathscr{F}$-conformal variation of g_0 then $\mathscr{F}$ is umbilical for any g_t.*

Proof. The claim follows from Lemma 2.3 (see also Lemma 2.9 for $S = s\,\hat{g}$). □

Given function $\psi \in C(\mathbb{R})$, consider the functional $I_\psi : \mathscr{M}_{|\mathscr{U}} \to \mathbb{R}$ on the space $\mathscr{U}$ of all Riemannian metrics with respect to which $\mathscr{F}$ is umbilical,

$$I_\psi(g) = \int_M \psi(\lambda)\,\mathrm{d}\,\mathrm{vol}_g\,. \tag{2.34}$$

Corollary 2.4. *Let $\mathscr{F}$ be an umbilical foliation on (M, g) with the normal curvature λ, and $\psi \in C^2(\mathbb{R})$. The $\mathscr{F}\mathscr{M}_1$- and $\mathscr{F}\mathscr{M}$- components of the gradient of the functional I_ψ are given by:*

$$\nabla^{\mathscr{F}} I_\psi(g) = \frac{1}{2}\left(\psi(\lambda)\,\underline{-I_\psi(g)} - \frac{1}{n}\mathscr{V}(\psi'(\lambda))\right)\hat{g}. \tag{2.35}$$

The second variation of I_ψ at a critical metric $g = \bar{g}_0 \in \mathscr{M}_1$, with respect to $\mathscr{F}$-conformal (i.e., of $T\mathscr{F}\mathscr{M}$ or $T\mathscr{F}\mathscr{M}_1$) variations $\bar{g}_t \in \mathscr{M}_1$ with $S = s\,\hat{g}$ $(s \in C^1(M))$ is

$$I''_\psi(\bar{g}_t)_{\,|t=0} = \frac{1}{4}\int_M \psi''(\lambda) N(s)^2\,\mathrm{d}\,\mathrm{vol}\,. \tag{2.36}$$

Proof. Because $\tau_i = n\lambda^i$, we set $f = \psi(\tau_1/n)$ and apply Theorem 2.1. In this case, $B_f = \frac{1}{2n}\,\psi'\,\hat{g}$, and, see (2.35),

$$I'_\psi(\bar{g}_t) = \frac{1}{2}\int_M s(\psi\,\underline{-I_\psi(g)} - \frac{1}{n}\mathscr{V}(\psi'))\,\mathrm{d}\,\overline{\mathrm{vol}}.$$

We prove (2.36) directly. To find I''_ψ we differentiate the above and get

$$I''_\psi(\bar{g}_t) = \frac{1}{2}\int_M s\left(n\,\partial_t(\psi') - \partial_t(\mathscr{V}(\psi'))\right)\,\mathrm{d}\,\overline{\mathrm{vol}}. \tag{2.37}$$

Using $\partial_t\lambda = -\frac{1}{2}N(s),\ \partial_t(\psi) = -\frac{1}{2}\psi' N(s)$, and so on, we compute

$$\partial_t(\mathscr{V}(\psi')) = \frac{1}{2}\left((-n\psi' - n\lambda\,\psi'' + N(\psi''))\,sN(s) + \psi'' sN(N(s))\right).$$

Hence, the components of the integral (2.37) are

$$\int_M s\,\partial_t(\mathscr{V}(\psi'))\,\mathrm{d}\,\overline{\mathrm{vol}} = \int_M \left(\frac{1}{2}\,\psi'' N(s)^2 - \frac{n}{4}\mathscr{V}(\psi')s^2\right)\mathrm{d}\,\overline{\mathrm{vol}},$$

$$\int_M s\,n\,\partial_t(\psi')\,\mathrm{d}\,\overline{\mathrm{vol}} = -\frac{n}{4}\int_M \mathscr{V}(\psi')\,s^2\,\mathrm{d}\,\overline{\mathrm{vol}}.$$

This yields the formula for I''_ψ. □

Remark 2.6. If $\psi'' > 0$ then the functional $I_\psi : \mathscr{M}_1 \to \mathbb{R}$ restricted to umbilical metrics has at most one critical point. See also Remark 2.5.

2.4.2 The Energy and Bending of the Unit Normal Vector Field

The *energy of a unit vector field* N on (M^{n+1}, g) can be expressed by the formula

$$\mathscr{E}_N(g) = \frac{1}{2}(n+1)\,\mathrm{vol}(M,g) + \int_M \|\nabla N\|^2 \,\mathrm{d}\,\mathrm{vol}_g,$$

see, for example, [10]. The last integral,

$$\mathscr{B}_N(g) = \int_M \|\nabla N\|^2 \,\mathrm{d}\,\mathrm{vol}_g,$$

up to the constant $c_{n+1} = \frac{1}{n}\mathrm{vol}(S^{n+1}(1))$, is called the *total bending* of N. The problems of minimizing $\mathscr{E}_N(g)$ and $\mathscr{B}_N(g)$ with respect to variations $\bar g_t \in \mathscr{M}_1$ are equivalent.

Let $e_0 = N$ and $e_1, \ldots, e_n$ be a local orthonormal basis of $(M, \mathscr{F}, g)$. We calculate

$$\|\nabla N\|^2 = \textstyle\sum_{i=1}^n g(\nabla_{e_i} N, \nabla_{e_i} N) + g(Z,Z) = \tau_2 + \|Z\|^2,$$

where $Z = \nabla_N N$ for short. Thus, we decompose the bending into two parts,

$$\mathscr{B}_N = I_{\tau,2} + \mathscr{B}_N^\perp, \quad \text{where} \quad \mathscr{B}_N^\perp(g) = \int_M \|Z\|^2 \,\mathrm{d}\,\mathrm{vol}.$$

Notice that $\mathscr{B}_N^\perp = 0$ for Riemannian foliations, i.e., $Z = 0$.

Lemma 2.10. *The vector field $Z = \nabla^t_N N$ is evolved by $g_t \in \mathscr{M}$ with $S = \partial_t g_t$ as*

$$(i)\ \partial_t Z = -S^\sharp(Z), \qquad (ii)\ \partial_t Z = -sZ \quad \text{for} \quad S = s\,\hat g. \tag{2.38}$$

In particular, all variations $g_t \in \mathscr{M}$ preserve Riemannian foliations.

Proof. We use (2.11) to compute for any $X \in T\mathscr{F}$

$$g_t(\partial_t Z, X) = \frac{1}{2}\big(2(\nabla^t_N S)(X,N) - (\nabla^t_X S)(N,N)\big) = -S(\nabla^t_N N, X) = -g_t(S^\sharp(Z), X).$$

From this, all of (2.38) follow. If $Z = 0$ at $t = 0$ then by uniqueness of a solution to the linear ODE (2.38)(i) along N-curves, we have $Z = 0$ for all t. □

By Lemma 2.10, we have $\partial_t(Z^\flat) = 0$. Indeed, we calculate (for any vector X)

$$\partial_t(Z^\flat)(X) = \partial_t(g(Z,X)) = S(Z,X) + g(\partial_t Z, X) = 0.$$

Notice that $\langle Z^\flat \odot Z^\flat, S\rangle = S(Z,Z)$, in particular, $\langle Z^\flat \odot Z^\flat, \hat g\rangle = \|Z\|^2$.

As the components of 1-form $Z^\flat = g(Z,\cdot)$ are $(Z^\flat)_i = Z^a g_{ia}$, by definition of the tensor product, we have

$$(Z^\flat \odot Z^\flat)_{ij} = Z^a g_{ia} Z^b g_{jb}.$$

From this we obtain

$$\langle Z^\flat \odot Z^\flat, S\rangle = Z^a g_{ia} Z^b g_{jb} S^{ij} = Z^a Z^b S_{ab} = S(Z,Z).$$

By Lemma 2.10, we have $\partial_t (Z^\flat \odot Z^\flat) = 0$. Indeed, we find (for any vectors X,Y)

$$\begin{aligned}\partial_t (Z^\flat \odot Z^\flat)(X,Y) &= \partial_t (g(Z,X) g(Z,Y))\\ &= S(Z,X) g(Z,Y) + g(Z,X) S(Z,Y) + g(\partial_t Z,X) g(Z,Y)\\ &\quad + g(Z,X) g(\partial_t Z,Y) = 0.\end{aligned}$$

Theorem 2.2. *The gradient of the bending functional $\mathscr{B}_N : \mathscr{M} \to \mathbb{R}$ (and its projection via $\pi_* : T\mathscr{M} \to T\mathscr{M}_1$) is given by*

$$\nabla \mathscr{B}_N(g) = \frac{1}{2}\left(\|Z\|_g^2 + \tau_2 \underline{- \mathscr{B}_N(g)}\right)\hat{g} - Z^\flat \odot Z^\flat - \mathscr{V}(\hat{b}_1),$$

where $Z = \nabla_N^t N$. The $\mathscr{F}\mathscr{M}_1$- (and $\mathscr{F}\mathscr{M}$-) component of the gradient is

$$\nabla^{\mathscr{F}} \mathscr{B}_N(g) = \left(\frac{1}{2}\left(\|Z\|_g^2 + \tau_2 \underline{- \mathscr{B}_N(g)}\right) - \frac{1}{n}\|Z\|_g^2 - \frac{1}{n}\mathscr{V}(\tau_1)\right)\hat{g}.$$

The second variation of $\mathscr{B}_N$ at a critical metric $g = \bar{g}_0$, where $S = \partial_t \bar{g}_t$, and its restriction to the $\mathscr{F}$-conformal variations (i.e., $S = s\,\hat{g}$, $s : M \to \mathbb{R}$) are, respectively,

$$\begin{aligned}\mathscr{B}_N''(\bar{g}_t)_{|t=0} &= \int_M (\langle \Phi_1(S), S\rangle + \langle \Phi_2(\nabla_N S), \nabla_N S\rangle + \langle \Phi_3(S), \nabla_N S\rangle)\,\mathrm{d\,vol},\\ \bar{\mathscr{B}}_N''(\bar{g}_t)_{|t=0} &= \frac{n}{2}\int_M \left(\left(\frac{n}{2} - 1\right)\|Z\|_g^2 s^2 + N(s)^2\right)\mathrm{d\,vol},\end{aligned}$$

where

$$\begin{aligned}\Phi_1(S^\sharp) &= -\frac{1}{4}\left(\|Z\|_g^2 + \tau_2 \underline{- \mathscr{B}_N(g)}\right)(\mathrm{Tr}\, S^\sharp)\,\widehat{\mathrm{id}} - A[S^\sharp, A] - S^2(Z,Z)\,\widehat{\mathrm{id}},\\ \Phi_2(S^\sharp) &= \frac{1}{2} S^\sharp, \qquad \Phi_3(S^\sharp) = \mathrm{Tr}\,(A\,S^\sharp)\,\widehat{\mathrm{id}}.\end{aligned}$$

Proof. First, using (2.11) and case (i) in Lemma 2.10, we compute

$$\partial_t \|Z\|_{g_t}^2 = (\partial_t g_t)(Z,Z) + 2\,g_t(\partial_t Z, Z) = S(Z,Z) - 2\,S(Z,Z) = -S(Z,Z).$$

For $f=\|Z\|^2_{g_t}+\tau_2$ we have

$$\partial_t f=-S(Z,Z)-\operatorname{Tr}(A\nabla^t_N S^\sharp).$$

Hence

$$\int_M(\partial_t f)\,\mathrm{d}\,\mathrm{vol}=-\int_M\big(S(Z,Z)+\langle\mathscr{V}(\hat b_1),\,S\rangle\big)\,\mathrm{d}\,\mathrm{vol}\,.$$

Then, similarly to (2.25) and (2.27), we obtain

$$\mathscr{B}'_N(\bar g_t)=\int_M\Big\langle\frac12\,\big(\|Z\|^2_{g_t}+\tau_2\underline{-\mathscr{B}_N(\bar g_t)}\big)\,\hat g-Z^\flat\odot Z^\flat-\mathscr{V}(\hat b_1),\,S\Big\rangle\,\mathrm{d}\,\mathrm{vol}\,. \quad (2.39)$$

In order to find the second variations, using Lemma 2.4, as for (2.28) we compute

$$\begin{aligned}\mathscr{B}''_N(\bar g_t)_{|t=0}=\int_M&\Big(\frac12\,(\|Z\|^2_g+\tau_2-\mathscr{B}_N(g))\langle S,S\rangle+\operatorname{Tr}(A\nabla^t_N S^\sharp)\Big)\,(\operatorname{Tr}S^\sharp)\\&-\frac12\,\big(S(Z,Z)-\langle\partial_t\mathscr{V}(\hat b_1),\,S\rangle\big)\,\mathrm{d}\,\mathrm{vol}\,.\end{aligned}$$

We have $\partial_t\mathscr{V}(\hat b_1)=(\partial_t\tau_1)\hat b_1+\tau_1\partial_t\hat b_1-\partial_t\nabla^t_N\hat b_1$, where by Example 2.3(a),

$$\int_M\langle\tau_1\partial_t\hat b_1,S\rangle\,\mathrm{d}\,\mathrm{vol}=\int_M\tau_1\,\Big(\operatorname{Tr}(S^\sharp A S^\sharp)-\frac14 N\,\Big(\operatorname{Tr}({S^\sharp}^2)\Big)\Big)\,\mathrm{d}\,\mathrm{vol},$$

$$\begin{aligned}\int_M\langle\partial_t(\nabla_N\hat b_1),\,S\rangle\,\mathrm{d}\,\mathrm{vol}=\int_M\Big(&\operatorname{Tr}\Big(S^\sharp(\nabla^t_N A)S^\sharp\Big)-\operatorname{Tr}(A[S^\sharp,\,A]S^\sharp)\\&-\frac14\tau_1 N(\operatorname{Tr}({S^\sharp}^2))+\frac12\operatorname{Tr}((\nabla^t_N S^\sharp)^2)\Big)\,\mathrm{d}\,\mathrm{vol}\,.\end{aligned}$$

Hence

$$\begin{aligned}\int_M\langle\partial_t\mathscr{V}(\hat b_1),S\rangle\mathrm{d}\,\mathrm{vol}=\int_M\Big(&-\frac12 N(\operatorname{Tr}S^\sharp)\operatorname{Tr}(AS^\sharp)+\operatorname{Tr}(S^\sharp\mathscr{V}(A)S^\sharp)\\&+\operatorname{Tr}(A[S^\sharp,A]S^\sharp)-\frac12\operatorname{Tr}((\nabla^t_N S^\sharp)^2)\Big)\,\mathrm{d}\,\mathrm{vol}\,.\end{aligned}$$

Finally, we obtain (see also Corollary 2.2 for $k=2$)

$$\begin{aligned}\mathscr{B}''_N(\bar g_t)_{|t=0}=\int_M\Big(&-\frac14\,\big(\|Z\|^2_g+\tau_2-\mathscr{B}_N(g)\big)\,(\operatorname{Tr}S^\sharp)^2-\operatorname{Tr}(A[S^\sharp,A]S^\sharp)\\&-S^2(Z,Z)+\operatorname{Tr}(A\,S^\sharp)N(\operatorname{Tr}S^\sharp)+\frac12\operatorname{Tr}((\nabla_N S^\sharp)^2)\Big)\,\mathrm{d}\,\mathrm{vol}\,.\end{aligned}$$

Formulae for $\mathscr{F}$-conformal case follow directly from above. □

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