Preface

The subject and the history. *Foliation theory* is about 60 years old. The notion of a foliation appeared in the 1940s in a series of papers of G. Reeb and Ch. Ehresmann, culminating in the book [40]. Since then, the subject has enjoyed a rapid development. Foliations relate with such topics as vector fields, integrable distributions, almost-product structures, submersions, fiber bundles, pseudogroups, Lie groups actions, and explicit constructions (Hopf and Reeb foliations).

Reeb also published a paper [41] on extrinsic geometry of foliation in which he proved that the integral of the mean curvature of the leaves of any codimension-one foliation on any closed Riemannian manifold equals zero. By *extrinsic geometry* we mean properties of foliations on Riemannian manifolds which can be expressed in terms of the second fundamental form of the leaves and its invariants (principal curvatures, scalar mean curvature, higher mean curvatures, and so on).

More precisely, if $\mathscr{F}$ is a smooth foliation of a Riemannian manifold (M,g) then the *second fundamental forms* B_L of all the leaves $\{L\}$ of $\mathscr{F}$ provide a vector-valued symmetric tensor B on M defined by:

$$B(X,Y) = (\nabla_X Y)^{\perp},$$

where ∇ is the Levi-Civita connection on (M,g), X and Y are tangent to $\mathscr{F}$, and $(\,\cdot\,)^{\perp}$ denotes the projection of the tangent bundle TM onto the orthogonal complement $T^{\perp}\mathscr{F}$ of the bundle $T\mathscr{F}$ consisting of all the vectors tangent to (the leaves of) $\mathscr{F}$. The tensor B can be extended to the whole tangent bundle of M by $B(N,\cdot) = 0$ whenever N is orthogonal to $\mathscr{F}$. If $\mathscr{F}$ is of codimension 1 and transversely oriented, B induces a symmetric scalar $(0,2)$-tensor field b (the *second fundamental form*) given by

$$b(X,Y) = g(B(X,Y),N)$$

for all X and Y. All the properties of $\mathscr{F}$ which can be expressed in terms of B (respectively, b) belong to extrinsic geometry. For example, a foliation $\mathscr{F}$ is called *totally geodesic* when $B \equiv 0$,

minimal when the mean curvature vector $H = \frac{1}{n}\,\mathrm{Tr}_g(B)$ of $\mathscr{F}$ vanishes,

umbilical when $B(X,Y) = H \cdot g(X,Y)$ for all $X,Y \in T\mathscr{F}$, and so on.

One of the principal problems of extrinsic geometry of foliations reads as follows: *Given a foliation $\mathscr{F}$ on a manifold M and an extrinsic geometric property (P), does there exist a Riemannian metric g on M such that $\mathscr{F}$ enjoys (P) with respect to g?*

Similarly, one may ask the following, analogous question:
Given a manifold M and an extrinsic geometric property (P), does there exist a foliation $\mathscr{F}$ and a Riemannian metric g on M such that $\mathscr{F}$ enjoys (P) with respect to g?

Such problems (first posed by H. Gluck for geodesic foliations) were studied already in the 1970s when Sullivan [50] provided a topological condition (called *topological tautness*) for a foliation, equivalent to *geometrical tautness*, that is existence of a Riemannian metric making all the leaves minimal. From classical theorem of Novikov [32] and results of Sullivan, it follows directly that the three-dimensional sphere S^3 admits no two-dimensional foliations which are minimal with respect to any Riemannian metric. For instance, there is no metric making a Reeb foliation $\mathscr{F}_R$ on a three-dimensional sphere minimal.

Umbilizable foliations on M^3 are transversely holomorphic, hence, see [11]: *If a closed orientable M^3 admits an umbilical foliation then it is diffeomorphic to the total space of a Seifert fibration (all one-dimensional leaves are closed) or of a torus bundle over the circle*. For example, since S^3 is the total space of a Seifert fibration, there exist metrics making a Reeb foliation $(S^3, \mathscr{F}_R)$ umbilical. Another example of this type may be found in a recent paper by Langevin and the second author [30]: closed Riemannian spaces of negative Ricci curvature admit no codimension-1 umbilical foliations.

In recent decades, several tools providing results of this sort have been developed. Among them, one may find Sullivan's [49] *foliated cycles* and several *Integral Formulae* ([3, 9, 45, 46, 54], etc.), the very first of which is G. Reeb's vanishing of the integral of the mean curvature mentioned earlier.

The authors also have been interested in extrinsic geometry of foliations for a long time (see, for example, [42–44, 54–58]) and this work is, in some sense, a continuation of this interest.

The contents. The book includes several topics in Extrinsic Geometry of Foliations. The first topic presented in the book (Chap. 1) is a series of new Integral Formulae, for a codimension-one foliation on a closed Riemannian manifold. The formulae depend on the Weingarten operator, the Riemannian curvature tensor (e.g., Jacobi operator), and their scalar invariants. Integral formulae begin with the classic formula by Reeb, for manifolds of constant curvature they reduce, to known formulae by Brito et al. [9], and Asimov [4]. Integral formulae can be useful for the following problems: prescribing higher mean curvatures (or other symmetric functions of principal curvatures) of foliations; minimizing volume and energy defined for vector or plane fields on manifolds; existence of foliations whose leaves enjoy a given geometric property such as being totally geodesic, umbilical, minimal, etc.

The central topic of the book is *Extrinsic Geometric Flow* (*EGF*, for short, see Chap. 3) on foliated manifolds $(M,\mathscr{F})$, $\operatorname{codim}\mathscr{F}=1$, which may provide more results on geometry of foliations. EGFs arise as solutions to the partial differential equation (PDE)

$$\partial_t g_t = h(b_t),$$

where (g_t), $t\in[0,T)$, are Riemannian metrics on M along the leaves and $h(b_t)$ the symmetric $(0,2)$-tensors along the leaves expressed in terms of the second fundamental form b_t of $\mathscr{F}$ on (M,g_t); $h(b_t)$ being identically zero in the direction orthogonal to $\mathscr{F}$. In particular, EGF – for suitable choice of the right-hand side in the EGF equation – may provide families (g_t) of Riemannian structures on a given foliated manifold $(M,\mathscr{F})$ converging as $t\to T$ to a metric g_T for which $\mathscr{F}$ satisfies a given geometric property (P), say, is umbilical, minimal, or just totally geodesic.

A *Geometric Flow* is an evolution of a given geometric structure under a differential equation associated to a functional on a manifold which has geometric interpretation, usually associated with some (either extrinsic or intrinsic) curvature. Geometric flows play an essential role in many fields of mathematics and physics. They all correspond to dynamical systems in the infinite dimensional space of all possible geometric structures (of given type) on a given manifold.

The strong interest of scientists in GF of various types is demonstrated by Annual International Workshops (*GF in Mathematics and Physics*, 2006 – 2011, BIRS Banff; *GF in finite or infinite dimension*, 2011, CIRM; *Geometric Evolution Equations*, 2011, University of Constance; *GF and Geometric Operators*, 2009, Centro De Giorgi, Pisa, and so on).

To some extent, the idea of EGF is analogous to that of the famous *Ricci flow*. In the Ricci flow equation, the configuration space is a single manifold and the Riemannian structures are deformed by quantities which belong to intrinsic geometry, in the case of EGFs, the configuration space is a foliated manifold while the Riemannian structures are deformed by invariants of extrinsic geometry. In both cases, the (EGF or Ricci flow) equation makes sense because both its sides are symmetric tensors of the same type. Notice that the study of the Ricci flow provided the proof of outstanding conjectures: Poincaré Conjecture and Thurston Geometrization Conjecture.

To apply EGF to various problems of extrinsic geometry, one needs *variational formulae* (see Chap. 2) which express variation of different quantities belonging to extrinsic geometry of a fixed foliation under variation of the Riemannian structure of the ambient manifold. Also, some special solutions (called *extrinsic geometric solitons* here, EGS, for short, see Sect. 3.8) of the EGF equation are of great interest because, in several cases, they provide Riemannian structures with very particular geometric properties of the leaves.

Throughout the book, (M^{n+1}, g_t) is a Riemannian manifold with a codimension one transversely oriented foliation $\mathscr{F}$, ∇^t the Levi-Civita connection of g_t,

$$2g_t(\nabla^t_X Y, Z) = X(g_t(Y,Z)) + Y(g_t(X,Z)) - Z(g_t(X,Y)) \\ + g_t([X,Y],Z) - g_t([X,Z],Y) - g_t([Y,Z],X)$$

for all the vector fields X, Y, Z on M, N the positively oriented unit normal to $\mathscr{F}$ with respect to any g_t, $A : X \in T\mathscr{F} \mapsto -\nabla^t_X N$ the Weingarten operator of the leaves, which we extend to a $(1,1)$-tensor field on TM by $A(N) = 0$.

Observe that the difference of two connections is always a tensor, hence $\Pi_t := \partial_t \nabla^t$ is a (1, 2)-tensor field on (M, g_t). Differentiating with respect to t the above classical formula yields the known formula, which allows us to express Π_t by:

$$2g_t(\Pi_t(X,Y),Z) = (\nabla^t_X S)(Y,Z) + (\nabla^t_Y S)(X,Z) - (\nabla^t_Z S)(X,Y),$$

where $S = \partial_t g_t$ is time-dependent symmetric (0,2)-tensor field and $X, Y, Z \in TM$.

The definition of the *$\mathscr{F}$-truncated (r,k)-tensor field* $\hat{S}$ (where $r = 0, 1$, and $\widehat{\ }$ denotes the $T\mathscr{F}$-component) will be helpful in Chaps. 2 and 3,

$$\hat{S}(X_1, \ldots, X_k) = S(\hat{X}_1, \ldots, \hat{X}_k) \qquad (X_i \in TM).$$

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Haifa – Łodz

Vladimir Rovenski
Paweł Walczak

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