Chapter 2
First Order Differential Equations

2.1 Problem A

The plan for this book is to apply the theory developed in Chapter 1 to a sequence of more or less increasingly complex differential equation problems. We begin with

$$\varepsilon y' + a(x,\varepsilon)y = b(x,\varepsilon), \tag{2.1}$$

where $a(x,\varepsilon), b(x,\varepsilon) \in C^\infty([0,1]\times[0,\varepsilon_o])$ for some $\varepsilon_o > 0$, and $a(x,\varepsilon) > 0$. We also assume $y(0,\varepsilon) = \alpha(\varepsilon) \in C^\infty([0,\varepsilon_o])$. The problem is to asymptotically approximate $y(x,\varepsilon)$ uniformly on $0 \le x \le 1$ as $\varepsilon \to 0^+$. If we put $z(x,\varepsilon) = y(x,\varepsilon) - \alpha(\varepsilon)$, then

$$z(x,\varepsilon) = \varepsilon^{-1}\int_0^x \hat{b}(x-t,\varepsilon)e^{-[k(x,\varepsilon)-k(x-t,\varepsilon)]/\varepsilon}\,dt, \tag{2.2}$$

where $\hat{b}(x,\varepsilon) = b(x,\varepsilon) - \alpha(\varepsilon)a(x,\varepsilon)$ and

$$k(x,\varepsilon) = \int_0^x a(t,\varepsilon)\,dt. \tag{2.3}$$

If we let $u(x,t,\varepsilon) = t^{-1}[k(x,\varepsilon) - k(x-t,\varepsilon)]$, then $u(x,t,\varepsilon) > 0$ on $[0,1]\times[0,x]\times[0,\varepsilon_o]$. In particular, $u(x,0,\varepsilon) = a(x,\varepsilon) > 0$. Thus we can rewrite (2.2) as

$$z(x,\varepsilon) = \varepsilon^{-1}\int_0^x f(x,t,t/\varepsilon,\varepsilon)\,dt, \tag{2.4}$$

where

$$f(x,t,T,\varepsilon) = \hat{b}(x-t,\varepsilon)e^{-Tu(x,t,\varepsilon)}, \tag{2.5}$$

and we can apply Corollary 2. In terms of $\phi(x,T,\varepsilon) = f(x,\varepsilon T,T,\varepsilon)$, it follows that

$$z(x,\varepsilon) = \sum_{n=0}^{N}\varepsilon^n\Phi_n(x,x/\varepsilon) + O(\varepsilon^N), \tag{2.6}$$

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where

$$\Phi_n(x, X) = \int_0^X \phi^{[0,0,n]}(x, T, 0)\, dT. \tag{2.7}$$

Furthermore,

$$\phi^{[0,0,n]}(x, T, 0) = p_n(x, T)e^{-a(x,0)T}, \tag{2.8}$$

where $p_n(x, T)$ is a polynomial in T, and therefore $\Phi_n(x, X) \in C^\infty([0, 1] \times [0, \infty])$. Hence, in accordance with Corollary 1, applied to the sum in (2.6), we see $y(x, \varepsilon) = z(x, \varepsilon) + \alpha(\varepsilon)$ has a uniformly valid asymptotic expansion of the form

$$y(x, \varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n [u_n(x) + v_n(x/\varepsilon)] + O(\varepsilon^N), \tag{2.9}$$

where $u_n(x) \in C^\infty([0, 1])$, $v_n(X) \in C^\infty([0, \infty])$, $v_n(\infty) = 0$, and these terms can be found by computing inner and outer expansions.

From (2.6), we have $O_1 z(x, \varepsilon) = \Phi_0(x, \infty)$ and since $p_0(x, T) = \hat{b}(x, 0)$, it follows that

$$O_1 z(x, \varepsilon) = \hat{b}(x, 0)/a(x, 0). \tag{2.10}$$

Similarly,

$$I_1 z(x, \varepsilon) = \Phi_0(0, x/\varepsilon) = [\hat{b}(0, 0)/a(0, 0)](1 - e^{-a(0,0)x/\varepsilon}) \tag{2.11}$$

and, from either (2.10) or (2.11),

$$O_1 I_1 z(x, \varepsilon) = I_1 O_1 z(x, \varepsilon) = \hat{b}(0, 0)/a(0, 0). \tag{2.12}$$

Also $C_1 y(x, \varepsilon) = C_1 z(x, \varepsilon) + \alpha(0)$. Hence, the first terms of (2.9) are

$$u_0(x) = b(x, 0)/a(x, 0), \qquad v_0(X) = [\alpha(0) - b(0, 0)/a(0, 0)]e^{-a(0,0)X}. \tag{2.13}$$

Of course, it is much easier to calculate the inner and outer expansions for (2.9) directly from the differential equation (2.1). For the N-term outer expansion,

$$O_N y(x, \varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n y_n(x), \tag{2.14}$$

(2.1) implies

$$y'_{n-1}(x) + \sum_{k=0}^{n} a_{n-k}(x) y_k(x) = b_n(x), \tag{2.15}$$

where $a_n(x) = a^{[0,n]}(x, 0)$, $b_n(x) = b^{[0,n]}(x, 0)$. Similarly, for the N-term inner expansion,

$$I_N y(x, \varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n Y_n(x/\varepsilon), \tag{2.16}$$

if we put $A(X,\varepsilon) = a(\varepsilon X, \varepsilon)$, $B(X,\varepsilon) = b(\varepsilon X, \varepsilon)$, in addition to $Y(X,\varepsilon) = y(\varepsilon X, \varepsilon)$, then (2.1) becomes

$$Y' + A(X,\varepsilon)Y = B(X,\varepsilon). \tag{2.17}$$

Therefore,

$$Y_n'(X) + \sum_{k=0}^{n} A_{n-k}(X)Y_k(X) = B_n(X), \tag{2.18}$$

where $A_n(X) = A^{[0,n]}(X,0)$, $B_n(X) = B^{[0,n]}(X,0)$. Also, $y(0,\varepsilon) = \alpha(\varepsilon)$ implies $Y_n(0) = \alpha^{[n]}(0)$. For the actual calculations we offer our first Maple program.

```
ProbA := proc (a, b, α, N)
ONy := sum(ε^n · y_n, n = 0..N − 1);
ONdy := sum(ε^n · dy_n, n = 0..N − 1);
ONeq := series(ε · ONdy + a · ONy − b, ε = 0, N);
for k from 0 to N − 1 do
temp := coeff(ONeq, ε, k); yk := solve(temp = 0, y_k);
dyk := diff(yk, x); ONeq := subs(y_k = yk, dy_k = dyk, ONeq);
ONy := subs(y_k = yk, ONy); print(u_k = yk);
end do;
A := subs(x = ε · X, a); B := subs(x = ε · X, b);
INy := sum(ε^n · Y_n, n = 0..N − 1);
INdy := sum(ε^n · dY_n, n = 0..N − 1);
INde := series(INdy + A ∗ INy − B, ε = 0, N);
Nα := series(α, ε = 0, N);
for k from 0 to N − 1 do
temp := coeff(INde, ε, k);
de := subs(Y_k = z(X), dY_k = diff(z(X), X), temp) = 0;
dsolve({de, z(0) = coeff(Nα, ε, k)});
Yk := rhs(%); dYk := diff(Yk, X);
INde := subs(Y_k = Yk, dY_k = dYk, INde); Y_k := Yk;
end do;
INONy := series(subs(x = ε · X, ONy), ε = 0, N);
for k from 0 to N − 1 do
vk := Y_k − coeff(INONy, ε, k);
print(v_k = simplify(expand(vk)));
end do;
end proc:
```

This program solves for the N-term outer expansion of $y(x,\varepsilon)$ using (2.15) and then computes the N-term inner expansion using (2.18). At the end, the program computes $I_N O_N y(x,\varepsilon)$. Normally there would be the matter of separating $I_N O_N y(x,\varepsilon)$ into two parts to form the functions $u_n(x)$ and $v_n(X)$.

In this problem, however, it is clear from (2.15) that $O_N y(x,\varepsilon)$ is continuous at $x=0$ and therefore $u_n(x)=y_n(x)$. This also means $v_n(x/\varepsilon)$ is the coefficient of ε^n in $[I_N - I_N O_N]y(x,\varepsilon)$. As an example,

$$a := 2+x+\varepsilon; \quad b := (2+x)^{-1}; \quad \alpha := \varepsilon; \quad ProbA(a,b,\alpha,2);$$

yields

$$u_0(x) = \frac{1}{(2+x)^2}, \qquad u_1(x) = \frac{-x}{(2+x)^4}, \tag{2.19}$$

and

$$v_0(X) = -\frac{1}{4}e^{-2X}, \qquad v_1(X) = \frac{1}{8}(8+2X+X^2)e^{-2X}. \tag{2.20}$$

2.2 Problem B

For our second differential equation we take

$$\varepsilon^2 y' + xa(x,\varepsilon)y = \varepsilon b(x,\varepsilon). \tag{2.21}$$

Again we assume $a(x,\varepsilon), b(x,\varepsilon) \in C^\infty([0,1]\times[0,\varepsilon_o])$ for some $\varepsilon_o > 0$. We also assume $a(x,\varepsilon) > 0$ on $[0,1]\times[0,\varepsilon_o]$, $y(0,\varepsilon) = \varepsilon\alpha(\varepsilon)$, where $\alpha(\varepsilon) \in C^\infty([0,\varepsilon_o])$, and, without loss of generality, we assume $a(0,0)=1$. The essential difference here from Problem A is the factor of x multiplying $a(x,\varepsilon)$.

In place of (2.3), we now have

$$k(x,\varepsilon) = \int_0^x ta(t,\varepsilon)\,dt \tag{2.22}$$

and therefore, as $t \to 0^+$,

$$k(x-t,0) = k(x,0) - txa(x,0) + \frac{1}{2}t^2[a(x,0) + xa^{[1,0]}(x,0)] + O(t^3). \tag{2.23}$$

If we put

$$u(x,t,\varepsilon) = t^{-2}[k(x,\varepsilon) - k(x-t,\varepsilon)] - (tx - t^2)a(x,0), \tag{2.24}$$

then $u(0,0,0) = 1/2$ and therefore $u(x,t,\varepsilon) > 0$ on $[0,x_o]\times[0,x]\times[0,\varepsilon_o]$ for some $x_o > 0$, and possibly a smaller $\varepsilon_o > 0$. Thus if we let $z(x,\varepsilon) = y(x,\varepsilon) - \varepsilon\alpha(\varepsilon)$ and $\hat{b}(x,\varepsilon) = b(x,\varepsilon) - x\alpha(\varepsilon)a(x,\varepsilon)$, then

$$z(x,\varepsilon) = \varepsilon^{-1} \int_0^x f(x,t,t/\varepsilon,\varepsilon) e^{-t(x-t)a(x,0)/\varepsilon^2}\, dt, \tag{2.25}$$

where,

$$f(x,t,T,\varepsilon) = \hat{b}(x-t,\varepsilon) e^{-T^2 u(x,t,\varepsilon)}, \tag{2.26}$$

and we have $f(x,t,T,\varepsilon) \in C^\infty([0,x_o] \times [0,x] \times [0,\infty] \times [0,\varepsilon_o])$. Also

$$0 < e^{-t(x-t)a(x,0)/\varepsilon^2} \le 1 \tag{2.27}$$

for all $(x,t,\varepsilon) \in [0,1] \times [0,x] \times (0,\varepsilon_o]$. Hence, applying Corollary 2 with $\phi(x,T,\varepsilon) = f(x,\varepsilon T,T,\varepsilon)$, from (2.25) we get

$$z(x,\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \Phi_n(x,x/\varepsilon) + O(\varepsilon^N) \tag{2.28}$$

uniformly as $\varepsilon \to 0^+$ for $0 \le x \le x_o$, where

$$\Phi_n(x,X) = \int_0^X \phi^{[0,0,n]}(x,T,0) e^{-T(X-T)a(x,0)}\, dT. \tag{2.29}$$

Also,

$$\phi^{[0,0,n]}(x,T,0) = p_n(x,T) e^{-T^2 u(x,0,0)}, \tag{2.30}$$

where $p_n(x,T)$ is a polynomial in T.

From Exercise 1.4, it is clear that

$$F(x,X) = \int_0^X T^m e^{-T^2 u(x,0,0)} e^{-T(X-T)a(x,0)}\, dT, \tag{2.31}$$

for any integer $m \ge 0$, is in $C^\infty([0,x_o] \times [0,\infty])$. Therefore $\Phi_n(x,X) \in C^\infty([0,x_o] \times [0,\infty])$ and thus from (2.28), by Corollary 1, we know $y(x,\varepsilon)$ has a uniformly valid expansion, at least for $0 \le x \le x_o$, of the form

$$y(x,\varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n [u_n(x) + v_n(x/\varepsilon)] + O(\varepsilon^N). \tag{2.32}$$

But also, from Problem A, we know that $y(x,\varepsilon) = O_N y(x,\varepsilon) + O(\varepsilon^N)$ uniformly as $\varepsilon \to 0^+$ for $x_o \le x \le 1$, since $xa(x,\varepsilon) > 0$ for $x_o \le x \le 1$. Furthermore, by Exercise 1.1, we know $[I_N - O_N I_N] y(x,\varepsilon) = O(\varepsilon^N)$ for $x_o \le x \le 1$. Hence, in conclusion, $y(x,\varepsilon)$ has a uniformly valid expansion of the form (2.32) as $\varepsilon \to 0^+$ on the full interval $0 \le x \le 1$, where $u_n(x) \in C^\infty([0,1])$, $v_n(X) \in C^\infty([0,\infty])$, $v_n(\infty) = 0$, and these functions can be determined by computing the corresponding outer and inner expansions for $y(x,\varepsilon)$ directly from the differential equation (2.21).

The inner expansion calculations for Problem B, if we just ask Maple to solve the associated sequence of differential equations, quickly gets bogged down with iterated error function integrals, so we have to help. It turns out that at each stage the equation to solve has the form

$$Y' + XY = \rho(X) + \sigma(X)P(X) + \tau(X)e^{-\frac{1}{2}X^2}, \tag{2.33}$$

where $\rho(X)$, $\sigma(X)$, $\tau(X)$ are polynomials and

$$P(X) = e^{-\frac{1}{2}X^2} \int_0^X e^{\frac{1}{2}T^2}\, dT. \tag{2.34}$$

The solution to this equation has the same form as its right hand side. That is,

$$Y(X) = p(X) + q(X)P(X) + r(X)e^{-\frac{1}{2}X^2}, \tag{2.35}$$

where $p(X)$, $q(X)$, $r(X)$ are polynomials. Indeed, if we substitute (2.35) into (2.33), we find

$$p' + q + Xp = \rho(X), \qquad q' = \sigma(X), \qquad r' = \tau(X). \tag{2.36}$$

Therefore

$$q(X) = q_0 + \int_0^X \sigma(T)\, dT, \qquad r(X) = r_0 + \int_0^X \tau(T)\, dT, \tag{2.37}$$

where $q_0 = q(0)$, $r_0 = r(0)$ have yet to be determined, and if d is the degree of $s(X) = \rho(X) - q(X)$, then, to satisfy (2.36a), the degree of $p(X)$ must be $d - 1$. Hence,

$$p(X) = \sum_{n=0}^{d-1} p_n X^n, \qquad s(X) = \sum_{n=0}^{d} s_n X^n \tag{2.38}$$

and (2.36a), together with $p_d = p_{d+1} = 0$, implies

$$p_{d-k-1} = s_{d-k} - (d-k+1)p_{d-k+1} \tag{2.39}$$

for $0 \le k \le d-1$. Also $p_1 = s_0$ so $q_0 = \rho(0) - p_1$ and finally, $Y(0) = p_0 + r_0$ determines r_0. This is all incorporated into our Maple program for this problem.

$ProbB :=$ **proc** (a, b, α, N)
$ONy := sum(\varepsilon^n \cdot y_n, n = 0..N-1);$
$ONdy := sum(\varepsilon^n \cdot dy_n, n = 0..N-1);$
$ONeq := series(\varepsilon^2 \cdot ONdy + x \cdot a \cdot ONy - \varepsilon \cdot b, \varepsilon = 0, N);$
for k **from** 0 **to** $N-1$ **do**
$temp := coeff(ONeq, \varepsilon, k);\ yk := solve(temp = 0, y_k);$
$dyk := diff(yk, x);\ ONeq := subs(y_k = yk, dy_k = dyk, ONeq);$

$ONy := subs(y_k = yk, ONy);\ y_k := yk;$
end do;
$A := subs(x = \varepsilon \cdot X, a);\ B := subs(x = \varepsilon \cdot X, b);$
$INy := sum(\varepsilon^n \cdot Y_n, n = 0..N-1);$
$N\varepsilon\alpha := series(\varepsilon \cdot \alpha, \varepsilon = 0, N);$
$rths := series(B \cdot I - X \cdot A \cdot INy + X \cdot INy, \varepsilon = 0, N+1);$
for k **from** 0 **to** $N-1$ **do**
$temp := coeff(rths, \varepsilon, k);$
$qcut := int(subs(X = T, coeff(temp, P)), T = 0..X);$
$s := coeff(temp, I) - qcut;$
if $s = 0$ **then** $d := 0$; **else** $d := degree(s)$; **end if**;
$p_d := 0;\ p_{d+1} := 0;$
for j **from** 0 **to** $d-1$ **do**
$p_{d-j-1} := coeff(s, X, d-j) - (d-j+1) \cdot p_{d-j+1};$
end do;
$q := qcut + subs(X = 0, coeff(temp, I)) - p_1;$
$r := coeff(N\varepsilon\alpha, \varepsilon, k) - p_0 + int(subs(X = T, coeff(temp, E)), T = 0..X);$
$p := sum(p_n \cdot X^n, n = 0..d-1);$
$Yk := p \cdot I + q \cdot P + r \cdot E;$
$rths := subs(Y_k = Yk, rths);\ Y_k := Yk;$
end do;
$series(subs(x = \varepsilon \cdot X, ONy), \varepsilon = 0, N);\ INONy := convert(\%, polynom);$
$vpart := sum(X^n \cdot coeff(INONy, X, n), n = 0..N);$
$upart := subs(X = \varepsilon^{-1} \cdot x, INONy - vpart);$
for k **from** 0 **to** $N-1$ **do**
$uk := y_k - coeff(upart, \varepsilon, k);$
$print(u_k = simplify(uk));\ u_k := uk;$
end do;
for k **from** 0 **to** $N-1$ **do**
$vk := Y_k - I \cdot coeff(vpart, \varepsilon, k);$
$print(v_k = simplify(coeff(vk, I) \cdot I + coeff(vk, P) \cdot P + coeff(vk, E) \cdot E);$
end do;
end proc:

In this program, the letters I, P, and E are used to denote 1, $P(X)$, and $\exp(-\frac{1}{2}X^2)$, respectively. Also, we have used the fact that $I_N O_N y(x,\varepsilon)$ is at most $O(X^{N-1})$ as $X \to \infty$ to split it into the two parts necessary to form $u_n(x)$ and $v_n(X)$ for $n = 0$ to $N-1$.

As an example, if we set

$$a(x,\varepsilon) = 1 + cx^2, \qquad b(x,\varepsilon) = 1, \qquad \alpha(\varepsilon) = 0, \tag{2.40}$$

then $ProbB(a, b, \alpha, 4)$ yields

$$y(x,\varepsilon) = P(x/\varepsilon) - \frac{\varepsilon c x}{1 + cx^2} + \varepsilon^2 v_2(x/\varepsilon) + \varepsilon^3 u_3(x) + O(\varepsilon^4) \tag{2.41}$$

with

$$v_2(X) = \frac{c}{4}[X + X^3 + (3 - X^4)P(X)], \qquad u_3(x) = -\frac{c^2x(3 + cx^2)}{(1 + cx^2)^3}. \quad (2.42)$$

As another, if

$$a(x, \varepsilon) = 1 + gx, \qquad b(x, \varepsilon) = h + kx, \qquad \alpha(\varepsilon) = 0, \qquad (2.43)$$

then

$$u_0(x) = 0, \qquad u_1(x) = \frac{k - gh}{1 + gx}, \qquad v_0(X) = hP(X), \qquad (2.44)$$

and

$$v_1(X) = \frac{1}{3}gh(1 + X^2) - \frac{1}{3}ghX^3P(X) + \frac{1}{3}(2gh - 3k)e^{-\frac{1}{2}X^2}. \qquad (2.45)$$

For

$$a(x, \varepsilon) = \cos(x) + x, \qquad b(x, \varepsilon) = \cos(x), \qquad \alpha(\varepsilon) = 1, \qquad (2.46)$$

a graph of

$$C_2y(x, \varepsilon) = P(x/\varepsilon) + \varepsilon[u_1(x) + v_1(x/\varepsilon)] \qquad (2.47)$$

when $\varepsilon = 0.2$ is shown in Figure 2.1, along with a portion of $O_2y(x, \varepsilon)$ and Maple's numerical solution of the differential equation. In this last example,

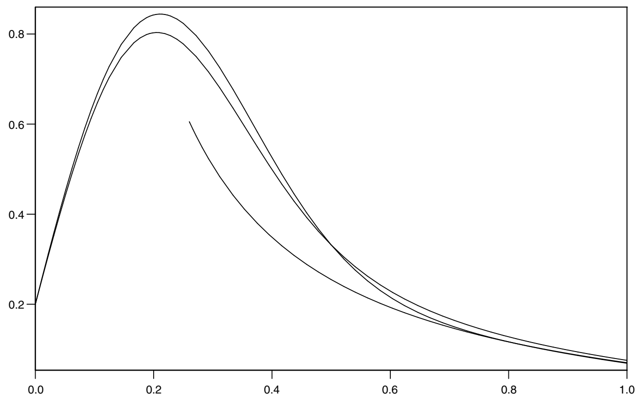

Fig. 2.1 Numerical solution of (2.21) and asymptotic approximations of the solution when $a(x, \varepsilon) = \cos(x) + x$, $b(x, \varepsilon) = \cos(x)$, $y(0, \varepsilon) = 1$ and $\varepsilon = 0.2$.

$$u_1(x) = \frac{-1}{\cos(x)+x}, \quad v_1(X) = \frac{1}{3}(1+X^2) - \frac{1}{3}X^3P(X) + \frac{5}{3}e^{-\frac{1}{2}X^2}. \quad (2.48)$$

2.3 Exercises

2.1. Let $y(x,\varepsilon)$ be the solution to

$$\varepsilon y' + x(1+x+\varepsilon)y = 2+x^2, \quad 1 \le x \le 2, \quad (2.49)$$

satisfying $y(1,\varepsilon) = 1+\varepsilon$. Use *ProbA* to show

$$y(x,\varepsilon) = \frac{2+x^2}{x(1+x)} + e^{-2(x-1)/\varepsilon} + \varepsilon[u_1(x) + v_1((x-1)/\varepsilon)] + O(\varepsilon^2) \quad (2.50)$$

for $1 \le x \le 2$, where

$$u_1(x) = \frac{-2-4x+3x^2+2x^3+x^4+x^5}{x^3(1+x)^3}, \quad (2.51)$$

$$v_1(X) = -\frac{1}{2}(1+2X+3X^2)e^{-2X}. \quad (2.52)$$

2.2. Note that $A_n(X)$ and $B_n(X)$ in (2.18) are polynomials and that therefore $Y_n(X) = p_n(X) + q_n(X)\exp[-a(0,0)X]$, where $p_n(X)$, $q_n(X)$ are polynomials. Therefore $v_n(X) = q_n(X)\exp[-a(0,0)X]$, since $v_n(\infty) = 0$. Furthermore,

$$v_n'(X) + \sum_{k=0}^{n} A_{n-k}(X)v_k(X) = 0, \quad (2.53)$$

and $v_n(0) = \alpha^{[n]}(0) - u_n(0)$. Write a new, shorter Maple program for computing the terms of (2.9).

2.3. Show that we could add ε^2 times $c(x,\varepsilon) \in C^\infty([0,1]\times[0,\varepsilon_o])$, to $xa(x,\varepsilon)$ in (2.21) without upsetting the basic analysis, and modify the program *ProbB* appropriately to include it.

2.4. Let $y(x,\varepsilon)$ be the solution to

$$\varepsilon^2 y' + x(1+x+\varepsilon)y = \varepsilon x(2+x^2), \quad 0 \le x \le 1, \quad (2.54)$$

satisfying $y(0,\varepsilon) = 1$. Use *ProbB* to show

$$y(x,\varepsilon) = \frac{2+x^2}{1+x} - e^{-\frac{1}{2}(x/\varepsilon)^2} + \varepsilon[u_1(x) + v_1(x/\varepsilon)] + O(\varepsilon^2), \quad (2.55)$$

where

$$u_1(x) = -\frac{2+x^2}{(1+x)^2}, \quad v_1(X) = 2P(X) + \frac{1}{6}(12+3X^2+2X^3)e^{-\frac{1}{2}X^2}. \quad (2.56)$$

2.5. Suppose $s(x) \in C^\infty([0,1])$, $s'(x) > 0$ and $g(x,\varepsilon) \in C^\infty([0,1] \times [0,\varepsilon_o])$ for some $\varepsilon_o > 0$. Let

$$F(x,\varepsilon) = \varepsilon^{-1} \int_0^x e^{-[s(x)-s(t)]/\varepsilon} g(x,t)\, dt, \tag{2.57}$$

$$G(x,\varepsilon) = \varepsilon^{-1} \int_x^1 e^{[s(x)-s(t)]/\varepsilon} g(x,\varepsilon)\, dt. \tag{2.58}$$

From the analysis at the beginning of Section 2.1, we know $F(x,\varepsilon)$ has a uniformly valid expansion of the form

$$F(x,\varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n [u_n(x) + v_n(x/\varepsilon)] + O(\varepsilon^N) \tag{2.59}$$

as $\varepsilon \to 0^+$, where $u_n(x) \in C^\infty([0,1])$, $v_n(X) \in C^\infty([0,\infty])$ and, as in Exercise 2.2, $v_n(X) = o(X^{-\infty})$ as $X \to \infty$. Show that $G(x,\varepsilon)$ has a uniformly valid expansion of the form

$$G(x,\varepsilon) = \sum_{n=0}^{N-1} \varepsilon^n [u_n(x) + w_n((1-x)/\varepsilon)] \tag{2.60}$$

as $\varepsilon \to 0^+$, where $u_n(x) \in C^\infty([0,1])$, $w_n(X) \in C^\infty([0,\infty])$ and $w_n(X) = o(X^{-\infty})$ as $X \to \infty$. In particular,

$$u_0(x) = g(x,0)/s'(x), \qquad w_0(X) = e^{-s'(1)X}. \tag{2.61}$$

2.6. Suppose $g(x,t)$, $h(x,t) \in C^\infty([0,1] \times [0,x])$, $h(x,t) > 0$ for $0 < t \le x \le 1$, $h(x,0) = 0$ for $0 \le x \le 1$, and $h^{[0,1]}(x,0) > 0$ for $0 < x \le 1$ but $h^{[0,1]}(0,0) = 0$. Let

$$F(x,\nu) = \int_0^x e^{-\nu h(x,t)} g(x,t)\, dt. \tag{2.62}$$

We know that for $x > 0$,

$$F(x,\nu) = \nu^{-1}[g(x,0)/h^{[0,1]}(x,0)] + O(\nu^{-1}). \tag{2.63}$$

For an expansion of $F(x,\nu)$ that is uniformly valid for $0 \le x \le 1$, note first that as $(x,t) \to (0,0)$,

$$h(x,t) = h_{11}xt + h_{02}t^2 + O((x^2+t^2)^{3/2}), \tag{2.64}$$

where we have begun using $h_{ij} = h^{[i,j]}(0,0)$, $g_{ij} = g^{[i,j]}(0,0)$, so it must be that $h_{11}x + h_{02}t \ge 0$ for $0 < t \le x \le 1$.

Assume $h_{11} > 0$ and $h_{02} > 0$. If we let

$$u(x,t) = t^{-2}[h(x,t) - th^{[0,1]}(x,0)], \qquad a(x) = x^{-1}h^{[0,1]}(x,0), \tag{2.65}$$

then $u(0,0) > 0$ and $a(0) > 0$, and hence there exists $x_o > 0$ such that $u(x,t) > 0$ on $[0,x_o] \times [0,x]$ and $a(x) > 0$ on $[0,x_o]$. Therefore

$$F(x,\nu) = \int_0^x f(x,t,t/\varepsilon)e^{-\nu xta(x)}\,dt, \tag{2.66}$$

where $\varepsilon = \nu^{-1/2}$,

$$f(x,t,T) = e^{-T^2u(x,t)}g(x,t) \tag{2.67}$$

and $f(x,t,T) \in C^\infty([0,x_o] \times [0,x] \times [0,\infty])$. In addition, $0 < e^{-\nu xta(x)} \le 1$ and thus, applying Corollary 2, with $\phi(x,T,\varepsilon) = f(x,\varepsilon T,T)$, we get

$$F(x,\nu) = \sum_{n=1}^{N-1} \varepsilon^n \Phi_n(x,x/\varepsilon) + O(\varepsilon^N) \tag{2.68}$$

uniformly as $\varepsilon \to 0^+$ for $0 \le x \le x_o$, where

$$\Phi_n(x,X) = \int_0^X \phi^{[0,0,n]}(x,T,0)e^{-XTa(x)}\,dT. \tag{2.69}$$

Show that $\Phi_n(x,X) \in C^\infty([0,x_o] \times [0,\infty])$, so we can apply Theorem 1 to each term of (2.68), and thereby determine

$$F(x,\nu) = \varepsilon v_1(x/\varepsilon) + \varepsilon^2[u_2(x) + v_2(x/\varepsilon)] + O(\varepsilon^3) \tag{2.70}$$

uniformly as $\varepsilon \to 0^+$ for $0 \le x \le x_o$, where

$$v_1(X) = g_{00}\int_0^X e^{-h_{02}T^2 - h_{11}XT}\,dT, \tag{2.71}$$

$$u_2(x) = \frac{g(x,0)}{h^{[0,1]}(x,0)} - \frac{g_{00}}{h_{11}x}, \tag{2.72}$$

and

$$v_2(X) = \int_0^X p_2(X,T)e^{-h_{02}T^2 - h_{11}XT}\,dT, \tag{2.73}$$

where

$$p_2(X,T) = g_{10} + (g_{01} - g_{00}h_{21}X^2)T - g_{00}h_{12}XT^2 - g_{00}h_{03}T^3. \qquad (2.74)$$

Note, furthermore, that for $x > 0$ the right side of (2.70) is asymptotically equal to $\varepsilon^2[g(x,0)/h^{[0,1]}(x,0)] + O(\varepsilon^3)$, the beginning of its outer expansion, and therefore, in view of (2.63), equation (2.70) actually holds uniformly for $0 \le x \le 1$.

The result (2.32) for the solution to Problem B is an example of an expansion for a case of $F(x,\nu)$ with $h_{11} > 0$ and $h_{02} < 0$.

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