

Chapter 2

Hypercyclic and chaotic operators

In this chapter we begin our investigation of linear dynamical systems, that is, dynamical systems that are defined by linear maps. As a simple example one may think of the differentiation operator

$$D : f \rightarrow f'.$$

In the language of dynamical systems, the exponential function, for example, is a fixed point of D , while the sine function is periodic with period 4. We will see in this chapter that D is in fact a chaotic operator.

The linearity of a map only makes sense if the underlying space carries, besides its topological structure, a linear structure also. Familiar examples of such spaces are Hilbert spaces and Banach spaces. But some of our main examples demand that we go beyond Banach spaces and allow so-called Fréchet spaces. These spaces will be introduced in the first section of this chapter.

In the subsequent sections we revisit the various topics discussed in Chapter 1 under the influence of linearity, and we contrast linear with nonlinear dynamics.

2.1 Linear dynamical systems

The dynamical systems studied in Chapter 1 were defined by continuous maps on metric spaces. For linear dynamical systems, the underlying space must in addition have a linear structure, as is the case for Hilbert spaces and Banach spaces. In this book we will assume a certain familiarity with such spaces; some of their basic properties are collected in Appendix A.

However, some interesting examples of linear dynamical systems are defined on spaces of a more general type, the so-called Fréchet spaces. In this section we introduce these spaces and describe their operators. The main purpose will be to familiarize the reader with this new concept. Some more

advanced results that are only used occasionally in this book will be covered in Appendix A.

The following two examples will motivate the concept of a Fréchet space.

Example 2.1. The process of taking derivatives, that is, the operator $D : f \rightarrow f'$, provides an interesting linear dynamical system. In order to have the powerful tools of complex analysis at our disposal we regard D as acting on the space of entire functions,

$$H(\mathbb{C}) = \{f : \mathbb{C} \rightarrow \mathbb{C} ; f \text{ holomorphic}\}.$$

The natural concept of convergence for entire functions is that of local uniform convergence, that is, the uniform convergence on all compact sets. In contrast to Banach spaces, convergence is described here by a countably infinite collection of conditions. More precisely, we have that $f_k \rightarrow f$ in $H(\mathbb{C})$ if and only if, for all $n \in \mathbb{N}$, $p_n(f_k - f) \rightarrow 0$ as $k \rightarrow \infty$, where

$$p_n(f) := \sup_{|z| \leq n} |f(z)|.$$

Here, $(p_n)_n$ is an increasing sequence of norms.

Example 2.2. Many natural spaces of sequences are Banach spaces. But the space of *all* (real or complex) sequences,

$$\omega := \mathbb{K}^{\mathbb{N}} = \{(x_n)_n ; x_n \in \mathbb{K}, n \in \mathbb{N}\},$$

lies outside this framework, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . The natural concept of convergence is that of coordinatewise convergence; that is, we have that $x^{(\nu)} \rightarrow x$ in ω if and only if, for all $n \in \mathbb{N}$, $p_n(x^{(\nu)} - x) \rightarrow 0$ as $\nu \rightarrow \infty$, where

$$p_n(x) := \sup_{1 \leq k \leq n} |x_k|, \quad x = (x_k)_k.$$

Here, $(p_n)_n$ is an increasing sequence of seminorms.

We recall the notion of a seminorm.

Definition 2.3. A functional $p : X \rightarrow \mathbb{R}_+$ on a vector space X over $\mathbb{K} = \mathbb{R}$ or \mathbb{C} is called a *seminorm* if it satisfies, for all $x, y \in X$ and $\lambda \in \mathbb{K}$,

- (i) $p(x + y) \leq p(x) + p(y)$,
- (ii) $p(\lambda x) = |\lambda|p(x)$.

A *norm* is a seminorm p for which $p(x) = 0$ implies that $x = 0$. A *Banach space* is a vector space X endowed with a norm, usually denoted by $\|\cdot\|$, whose topology is defined via the metric

$$d(x, y) := \|x - y\|, \quad x, y \in X,$$

and which is complete in that metric. If, moreover, the norm derives from an inner product $\langle \cdot, \cdot \rangle$ via

$$\|x\| := \sqrt{\langle x, x \rangle}, \quad x \in X,$$

then X is called a *Hilbert space*.

We recall here some classical Banach and Hilbert spaces. Further spaces will be introduced as the need arises.

Example 2.4. (a) Let $1 \leq p < \infty$. Then the space

$$\ell^p := \left\{ x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} ; \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}$$

of p -summable sequences, endowed with the norm $\|x\| := (\sum_{n=1}^{\infty} |x_n|^p)^{1/p}$, is a Banach space. In particular, ℓ^2 is a Hilbert space with inner product defined by $\langle x, y \rangle = \sum_{n=1}^{\infty} x_n \overline{y_n}$. Occasionally we let the index start with 0. The *finite sequences*, that is, sequences of the form $(x_1, \dots, x_n, 0, 0, \dots)$, $n \geq 1$, constitute a dense subset. Considering only the finite sequences with entries from \mathbb{Q} or $\mathbb{Q} + i\mathbb{Q}$ we see that any ℓ^p , $1 \leq p < \infty$, is separable. The space $\ell^p(\mathbb{Z})$ of p -summable sequences, indexed over \mathbb{Z} , is defined analogously.

(b) The space $\ell^\infty := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} ; \sup_{n \in \mathbb{N}} |x_n| < \infty\}$ of bounded sequences, endowed with the sup-norm $\|x\| := \sup_{n \in \mathbb{N}} |x_n|$, is a Banach space. Since it is not separable it will be of less interest to us. Instead, its closed subspace

$$c_0 := \{x = (x_n)_n \in \mathbb{K}^{\mathbb{N}} ; \lim_{n \rightarrow \infty} x_n = 0\}$$

of null sequences is a separable Banach space under the induced norm.

(c) Let $a < b$ and $1 \leq p < \infty$. Then the space

$$L^p[a, b] := \left\{ f : [a, b] \rightarrow \mathbb{K} ; f \text{ measurable and } \int_a^b |f(t)|^p dt < \infty \right\}$$

of p -integrable functions, endowed with the norm $\|f\| := (\int_a^b |f(t)|^p dt)^{1/p}$, is a Banach space; as usual, we identify functions that are equal almost everywhere. In particular, $L^2[a, b]$ is a Hilbert space with inner product defined by $\langle f, g \rangle = \int_a^b f(t) \overline{g(t)} dt$. We will occasionally need the fact that the functions $t \rightarrow \frac{1}{\sqrt{2\pi}} e^{int}$, $n \in \mathbb{Z}$, form an orthonormal basis in $L^2[0, 2\pi]$.

(d) Let $a < b$. Then the space

$$C[a, b] := \left\{ f : [a, b] \rightarrow \mathbb{K} ; f \text{ continuous} \right\}$$

of continuous functions, endowed with the sup-norm $\|f\| := \sup_{t \in [a, b]} |f(t)|$, is a Banach space.

The concept of a Fréchet space generalizes that of a Banach space by defining the topology via a sequence $(p_n)_n$ of seminorms, which we can always assume to be increasing (by considering $\max_{k \leq n} p_k$, if necessary). Moreover, the sequence is supposed to be *separating*, that is, $p_n(x) = 0$ for all $n \geq 1$ implies that $x = 0$. Then it is easy to see that

$$d(x, y) := \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x - y)), \quad x, y \in X \quad (2.1)$$

defines a metric on X ; see Exercise 2.1.1. An important feature of this metric is that it is *translation-invariant*, that is,

$$d(x, y) = d(x + z, y + z) \quad \text{for all } x, y, z \in X.$$

Definition 2.5. A *Fréchet space* is a vector space X , endowed with a separating increasing sequence $(p_n)_n$ of seminorms, which is complete in the metric given by (2.1).

The following result will be of constant use. We leave its proof as a useful exercise to the reader; see Exercise 2.1.2.

Lemma 2.6. Let X be a Fréchet space with a defining increasing sequence $(p_n)_n$ of seminorms. Let $x_k, x \in X$, $k \geq 1$, and $U \subset X$. Then:

- (i) $x_k \rightarrow x$ if and only if $p_n(x_k - x) \rightarrow 0$ as $k \rightarrow \infty$, for all $n \geq 1$;
- (ii) $(x_k)_k$ is a Cauchy sequence if and only if $p_n(x_k - x_l) \rightarrow 0$ as $k, l \rightarrow \infty$, for all $n \geq 1$;
- (iii) U is a neighbourhood of x if and only if there are $n \geq 1$ and $\varepsilon > 0$ such that $\{y \in X ; p_n(y - x) < \varepsilon\} \subset U$.

Example 2.7. (a) Let X be a Banach space with norm $\|\cdot\|$. Setting $p_n = \|\cdot\|$, $n \geq 1$, it follows from (i) and (ii) that X is also a Fréchet space according to Definition 2.5.

(b) With the seminorms defined in Example 2.1, the space $H(\mathbb{C})$ of entire functions is a Fréchet space; in view of Lemma 2.6(ii), completeness follows in the usual way. Since the Taylor series expansion of an entire function converges on every compact set, the polynomials form a dense subset of $H(\mathbb{C})$. Considering polynomials with coefficients from $\mathbb{Q} + i\mathbb{Q}$ we see that $H(\mathbb{C})$ is separable.

(c) The space $\omega = \mathbb{K}^{\mathbb{N}}$ of all real or complex sequences, endowed with the seminorms given in Example 2.2, is a Fréchet space. Since the finite sequences with entries from \mathbb{Q} or $\mathbb{Q} + i\mathbb{Q}$ form a dense subset, ω is separable. More generally, if X is a separable Fréchet space then, in a canonical way, also $X^{\mathbb{N}}$ is a separable Fréchet space; see Exercise 2.1.3.

Further Fréchet spaces will be introduced in the course of the book.

Looking at the way the metric is defined in a Banach space it is tempting to introduce, also in a Fréchet space X , a norm-like functional by setting

$$\|x\| := \sum_{n=1}^{\infty} \frac{1}{2^n} \min(1, p_n(x)), \quad x \in X, \quad (2.2)$$

so that $d(x, y) = \|x - y\|$. We summarize its characteristic properties; see Exercise 2.1.4.

Proposition 2.8. *The functional $\|\cdot\| : X \rightarrow \mathbb{R}_+$ given by (2.2) satisfies, for all $x, y \in X$ and $\lambda \in \mathbb{K}$,*

- (i) $\|x + y\| \leq \|x\| + \|y\|$;
- (ii) $\|\lambda x\| \leq \|x\|$ if $|\lambda| \leq 1$;
- (iii) $\lim_{\lambda \rightarrow 0} \|\lambda x\| = 0$;
- (iv) $\|x\| = 0$ implies that $x = 0$.

Definition 2.9. A functional $\|\cdot\| : X \rightarrow \mathbb{R}_+$ on a vector space X that satisfies conditions (i)–(iv) of Proposition 2.8 is called an *F-norm*.

The notion of an F-norm has the advantage that one can largely argue as if one was working in a Banach space. One need only be aware of the fact that the positive homogeneity of a norm is no longer available. In fact, in many cases, this property is not needed at all or it can be replaced by the following weaker property that follows directly from conditions (i) and (ii): for all $x \in X$ and $\lambda \in \mathbb{K}$,

$$\|\lambda x\| \leq (|\lambda| + 1) \|x\|. \quad (2.3)$$

Having discussed Fréchet spaces and their topology we now turn to the concept of operators on them.

Definition 2.10. Let X and Y be Fréchet spaces. Then a continuous linear map $T : X \rightarrow Y$ is called an *operator*. The space of all such operators is denoted by $L(X, Y)$. If $Y = X$ we say that T is an *operator on X* , with $L(X) = L(X, X)$.

The following extends a familiar result from Banach spaces to Fréchet spaces; see also Exercise 2.1.7.

Proposition 2.11. *Let X and Y be Fréchet spaces with defining increasing sequences of seminorms $(p_n)_n$ and $(q_n)_n$, respectively. Then a linear map $T : X \rightarrow Y$ is an operator if and only if, for any $m \geq 1$, there are $n \geq 1$ and $M > 0$ such that*

$$q_m(Tx) \leq Mp_n(x), \quad x \in X.$$

Proof. The condition is obviously sufficient because, by Lemma 2.6, it implies that when $x_k \rightarrow x$ in X then $Tx_k \rightarrow Tx$ in Y .

Conversely, let $m \geq 1$. By Lemma 2.6, the set $W := \{y \in Y ; q_m(y) < 1\}$ is a 0-neighbourhood in Y . By continuity there is a 0-neighbourhood W' in X such that $T(W') \subset W$. Hence there are $n \geq 1$ and $\varepsilon > 0$ such that $p_n(x) < \varepsilon$

implies that $x \in W'$, and therefore $q_m(Tx) < 1$. Now let $x \in X$. Then, for any $\delta > 0$, we have that

$$p_n\left(\frac{\varepsilon}{p_n(x) + \delta}x\right) < \varepsilon$$

and hence

$$q_m(Tx) < \frac{p_n(x) + \delta}{\varepsilon}.$$

Since $\delta > 0$ is arbitrary we obtain the result with $M = 1/\varepsilon$. \square

In contrast to Banach space operators one cannot associate a norm with a Fréchet space operator.

Example 2.12. (a) The map

$$D : f \rightarrow f'$$

is an operator on $H(\mathbb{C})$. This follows from the Cauchy estimates by which, for any $n \geq 1$, $\sup_{|z| \leq n} |f'(z)| \leq \sup_{|z| \leq n+1} |f(z)|$.

(b) Let $X = H(\mathbb{C})$. Then the translation map T_a is defined by

$$T_a f(z) = f(z + a), \quad a \in \mathbb{C}.$$

This is clearly an operator on X .

(c) Let $X = \ell^p$, $1 \leq p < \infty$, or c_0 . Then the *backward shift* $B : X \rightarrow X$, defined by

$$B(x_1, x_2, \dots) = (x_2, x_3, \dots)$$

is an operator on X , of norm $\|B\| = 1$, and it is also an operator on ω .

In Chapter 1 we associated to any two dynamical systems $S : X \rightarrow X$ and $T : Y \rightarrow Y$ a new dynamical system $S \times T$ on $X \times Y$. In a linear setting one usually employs a different, additive notation. More specifically, let X and Y be Fréchet spaces with defining increasing sequences of seminorms $(p_n)_n$ and $(q_n)_n$, respectively. Then the space

$$X \oplus Y := \{(x, y) ; x \in X, y \in Y\}$$

will be endowed with the seminorms $(x, y) \rightarrow p_n(x) + q_n(y)$, $n \geq 1$, which induce the product topology on $X \oplus Y$. This space then becomes a Fréchet space, which is separable if X and Y are.

Definition 2.13. Let $S : X \rightarrow X$ and $T : Y \rightarrow Y$ be operators on Fréchet spaces X and Y . Then the operator $S \oplus T$ is defined by

$$S \oplus T : X \oplus Y \rightarrow X \oplus Y, \quad (S \oplus T)(x, y) = (Sx, Ty).$$

With this we end our introduction to Fréchet spaces and their operators. We will show in Chapter 12 that several important results in linear dynamics

hold true in the wider context of operators on so-called topological vector spaces. However, other results require the existence of a complete metric, and most operators in linear dynamics are naturally defined on Fréchet spaces. In addition, since we are primarily interested in operators with a dense orbit, we will assume that the space is separable. Thus, with Chapter 12 the only exception, we adopt the following point of view throughout this book.

Definition 2.14. A *linear dynamical system* is a pair (X, T) consisting of a separable Fréchet space X and an operator $T : X \rightarrow X$.

Usually we simply call T or $T : X \rightarrow X$ a linear dynamical system. *From now on, all operators will be defined on separable Fréchet spaces, if nothing else is said.*

2.2 Hypercyclic operators

We begin our study of the dynamics of linear operators by considering dynamical systems with a dense orbit. In the presence of linearity, such systems are given their own name.

Definition 2.15. An operator $T : X \rightarrow X$ is called *hypercyclic* if there is some $x \in X$ whose orbit under T is dense in X . In such a case, x is called a *hypercyclic vector* for T . The set of hypercyclic vectors for T is denoted by $HC(T)$.

The origin of this terminology is easily explained. For a long time, operator theorists have been studying so-called cyclic vectors in connection with the invariant subspace problem. Vectors with a more restrictive property were then called supercyclic.

Definition 2.16. Let $T : X \rightarrow X$ be an operator. A vector $x \in X$ is called *cyclic* for T if the linear span of its orbit,

$$\text{span} \{T^n x ; n \geq 0\}$$

is dense in X . A vector $x \in X$ is called *supercyclic* for T if its projective orbit,

$$\{\lambda T^n x ; n \geq 0, \lambda \in \mathbb{K}\}$$

is dense in X .

Operators that possess a cyclic (or supercyclic) vector are called *cyclic* (or *supercyclic*, respectively).

This suggested the name of hypercyclicity for the case when the orbit itself is dense. Cyclic and supercyclic vectors will not be studied in detail in this book.

The *invariant subspace problem*, which is open to this day, asks whether every Hilbert space operator possesses an invariant closed subspace other than the trivial ones given by $\{0\}$ and the whole space. Counterexamples do exist for operators on non-reflexive spaces like ℓ^1 .

Obviously, the smallest closed T -invariant subspace of X that contains a given point x coincides with the closure of the span of its orbit. Therefore, an operator has no nontrivial invariant closed subspace precisely if every nonzero vector is cyclic. By the same token we have a link between hypercyclicity and the *invariant subset problem*: does every Hilbert space operator possess an invariant closed subset other than the trivial ones given by $\{0\}$ and the whole space?

Observation 2.17. *An operator has no nontrivial invariant closed subsets if and only if every nonzero vector is hypercyclic.*

Having explained the historical interest in hypercyclicity, our first question has to be if hypercyclic operators exist. That is, does the additional requirement of linearity still allow us to find maps with dense orbits? Indeed, a very simple operator on the Hilbert space ℓ^2 turns out to be hypercyclic.

Example 2.18. Let $T : \ell^2 \rightarrow \ell^2$ be twice the backward shift, that is,

$$T = 2B : (x_1, x_2, x_3, \dots) \rightarrow 2(x_2, x_3, x_4, \dots).$$

The space ℓ^2 has a countable dense set $\{y^{(k)} ; k \geq 1\}$ consisting of finite sequences; for each $k \geq 1$, let m_k be the greatest index with $y_{m_k}^{(k)} \neq 0$. By S we denote half the forward shift operator,

$$S = \frac{1}{2}F : (x_1, x_2, x_3, \dots) \rightarrow \frac{1}{2}(0, x_1, x_2, \dots).$$

Then, by induction, we can find a sequence $(n_k)_k$ of positive integers such that, for any $k > j \geq 1$,

$$n_k \geq m_j + n_j \quad \text{and} \quad 2^{n_k} \geq 2^{n_j+k} \|y^{(k)}\|.$$

We claim that the vector

$$x := \sum_{k=1}^{\infty} S^{n_k} y^{(k)}$$

is hypercyclic for T . First, since $\|S^{n_k} y^{(k)}\| = 2^{-n_k} \|y^{(k)}\| \leq 2^{-k}$ for $k \geq 2$, the series converges and $x \in \ell^2$. Now, let $k \geq 1$. Then

$$\begin{aligned} T^{n_k} x &= \sum_{j=1}^{k-1} 2^{n_k-n_j} B^{n_k-n_j} y^{(j)} + y^{(k)} + \sum_{j=k+1}^{\infty} 2^{n_k-n_j} F^{n_j-n_k} y^{(j)} \\ &= y^{(k)} + \sum_{j=k+1}^{\infty} 2^{n_k-n_j} F^{n_j-n_k} y^{(j)}, \end{aligned}$$

where we have used that $n_k - n_j \geq m_j$ for $j < k$. From

$$\sum_{j=k+1}^{\infty} 2^{n_k - n_j} \|F^{n_j - n_k} y^{(j)}\| = \sum_{j=k+1}^{\infty} 2^{n_k - n_j} \|y^{(j)}\| \leq \sum_{j=k+1}^{\infty} 2^{-j} = 2^{-k}$$

we deduce that $\|T^{n_k} x - y^{(k)}\| \leq 2^{-k}$. Since the $y^{(k)}$ form a dense set, x has a dense orbit under T .

Instead of an explicit construction of a hypercyclic vector one can also apply the Birkhoff transitivity theorem to show that an operator is hypercyclic. This leads to more transparent proofs, in particular when the operator is complicated. For ease of reference we restate Birkhoff's theorem in our new setting; note that Fréchet spaces clearly have no isolated points.

Theorem 2.19 (Birkhoff transitivity theorem). *An operator T is hypercyclic if and only if it is topologically transitive. In that case, the set $HC(T)$ of hypercyclic vectors is a dense G_δ -set.*

Directly or indirectly, the transitivity theorem will be our main tool for proving the hypercyclicity of an operator.

The first examples of hypercyclic operators were found by G.D. Birkhoff in 1929, G.R. MacLane in 1952 and S. Rolewicz in 1969. These operators will accompany us throughout the book as they will serve as a testing ground for any new concept in linear dynamics; indeed, Example 2.18 was already a special Rolewicz operator.

Example 2.20. (Birkhoff's operators) On the space $H(\mathbb{C})$ of entire functions we consider the translation operators given by

$$T_a f(z) = f(z + a), \quad a \neq 0.$$

Let $U, V \subset H(\mathbb{C})$ be arbitrary nonempty open sets, and fix $f \in U$, $g \in V$. By the definition of the topology on $H(\mathbb{C})$ there is a closed disk K centred at 0 and an $\varepsilon > 0$ such that an entire function h belongs to U (or to V) whenever $\sup_{z \in K} |f(z) - h(z)| < \varepsilon$ (or $\sup_{z \in K} |g(z) - h(z)| < \varepsilon$, respectively). Let $n \in \mathbb{N}$ be any integer such that K and $K + na$ are disjoint disks. Considering the function that is defined as f on a neighbourhood of K and by $z \rightarrow g(z - na)$ on a neighbourhood of $K + na$, Runge's theorem (see Appendix A), tells us that there exists a polynomial p such that

$$\sup_{z \in K} |f(z) - p(z)| < \varepsilon \quad \text{and} \quad \sup_{z \in K + na} |g(z - na) - p(z)| < \varepsilon,$$

and hence also

$$\sup_{z \in K} |g(z) - (T_a^n p)(z)| = \sup_{z \in K} |g(z) - p(z + na)| < \varepsilon.$$

This shows that $p \in U$ and $T_a^n p \in V$, so that T_a is topologically transitive. Since $H(\mathbb{C})$ is a separable Fréchet space, T_a is hypercyclic.

Example 2.21. (MacLane's operator) We next consider the differentiation operator

$$D : f \rightarrow f'$$

on $H(\mathbb{C})$. Since the polynomials are dense in $H(\mathbb{C})$, given arbitrary nonempty open sets $U, V \subset H(\mathbb{C})$, there are polynomials $p \in U$ and $q \in V$, $p(z) = \sum_{k=0}^N a_k z^k$ and $q(z) = \sum_{k=0}^N b_k z^k$. Let $n \geq N + 1$ be arbitrary. Then the polynomial

$$r(z) = p(z) + \sum_{k=0}^N \frac{k! b_k}{(k+n)!} z^{k+n}$$

has the property that $D^n r = q$. Moreover, for any $R > 0$ we have that

$$\sup_{|z| \leq R} |r(z) - p(z)| \leq \sum_{k=0}^N \frac{k! |b_k|}{(k+n)!} R^{k+n} \rightarrow 0$$

as $n \rightarrow \infty$. Thus, if n is sufficiently large, then $r \in U$ and $D^n r \in V$. This implies that D is hypercyclic.

Example 2.22. (Rolewicz's operators) On the spaces $X := \ell^p$, $1 \leq p < \infty$, or $X := c_0$ we consider the multiple

$$T = \lambda B : X \rightarrow X, \quad (x_1, x_2, x_3, \dots) \rightarrow \lambda(x_2, x_3, x_4, \dots)$$

of the backward shift, where $\lambda \in \mathbb{K}$. First, if $|\lambda| \leq 1$ then $\|T^n x\| = |\lambda|^n \|B^n x\| \leq \|x\|$ for all $x \in X$ and $n \geq 0$. Thus T cannot be hypercyclic in this case.

On the other hand, T is hypercyclic whenever $|\lambda| > 1$. Indeed, if $U, V \subset X$ are nonempty open sets, we can find $x \in U$ and $y \in V$ of the form

$$x = (x_1, x_2, \dots, x_N, 0, 0, \dots), \quad y = (y_1, y_2, \dots, y_N, 0, 0, \dots),$$

for some $N \in \mathbb{N}$. Let $n \geq N$ be arbitrary. Defining $z \in X$ by $z_k = x_k$ if $1 \leq k \leq N$, $z_k = \lambda^{-n} y_{k-n}$ if $n+1 \leq k \leq n+N$, and $z_k = 0$ otherwise, we obtain a sequence with $T^n z = y$. Moreover, $\|x - z\| = |\lambda|^{-n} \|y\| \rightarrow 0$ as $n \rightarrow \infty$. Thus, if n is sufficiently large, then $z \in U$ and $T^n z \in V$. This shows that T is topologically transitive; since the underlying spaces are separable Banach spaces, T is hypercyclic.

There are various ways to derive the hypercyclicity of one operator from that of another. The first result of this type follows from Proposition 1.14 by the Birkhoff transitivity theorem. Recall that by the inverse mapping theorem (see Appendix A), any bijective operator has a continuous inverse and is therefore.

Proposition 2.23. *Let T be an invertible operator. Then T is hypercyclic if and only if T^{-1} is.*

Birkhoff's operators provide examples of invertible hypercyclic operators. Next, in our present context, Proposition 1.19 reads as follows.

Proposition 2.24. *Hypercyclicity is preserved under quasiconjugacy.*

We emphasize that the map ϕ defining the quasiconjugacy need not be linear; it may, for instance, be defined between a complex space and a real space; see Exercise 2.2.5.

We also formulate part of Proposition 1.42 in the linear setting.

Proposition 2.25. *Let $S : X \rightarrow X$ and $T : Y \rightarrow Y$ be operators. If $S \oplus T$ is hypercyclic then so are S and T .*

We will see in Remark 4.17 that the converse fails in general. As an application of Proposition 2.25 we obtain an interesting transference principle that is specific to the linear setting. Let X be a real separable Fréchet space. Then the *complexification* \tilde{X} of X is defined formally as

$$\tilde{X} = \{x + iy \ ; \ x, y \in X\},$$

which will be identified with $X \oplus X$. If multiplication by complex scalars is defined by $(a + ib)(x + iy) = (ax - by) + i(ay + bx)$, then \tilde{X} becomes a complex separable Fréchet space.

Moreover, let $T : X \rightarrow X$ be a (real-linear) operator on X . Then its *complexification* $\tilde{T} : \tilde{X} \rightarrow \tilde{X}$ is defined by

$$\tilde{T}(x + iy) = Tx + iTy.$$

An easy computation shows that \tilde{T} is a (complex-linear) operator on \tilde{X} . For all of these statements see Exercise 2.2.7. Since \tilde{T} is nothing but the operator $T \oplus T$ on $X \oplus X$, Proposition 2.25 implies the following.

Proposition 2.26. *Let T be an operator on a real separable Fréchet space. If its complexification \tilde{T} is hypercyclic then so is T .*

In fact, since \tilde{T} coincides with $T \oplus T$, hypercyclicity of \tilde{T} is equivalent to weak mixing of T ; see also Section 2.5.

Example 2.27. The complexification of the real Rolewicz operator $T = \lambda B$, $\lambda \in \mathbb{R}$, $|\lambda| > 1$, on the spaces $X := \ell^p$, $1 \leq p < \infty$, or $X := c_0$, of real sequences can be identified with the same operator as understood on the corresponding spaces of complex sequences. Therefore, hypercyclicity for the complex Rolewicz operators implies the same for the real operators.

As a second application of Proposition 2.25 we consider restrictions of hypercyclic operators to invariant subspaces. Let M_1 and M_2 be closed subspaces of a (real or complex) Fréchet space X such that, algebraically, $X = M_1 \oplus M_2$, that is, $X = M_1 + M_2$ and $M_1 \cap M_2 = \{0\}$.

Note that, at the end of Section 2.1, we had defined $M_1 \oplus M_2$ as the topological product of the two spaces. But since the map $\phi : (x_1, x_2) \rightarrow x_1 + x_2$ defines an algebraic isomorphism between $M_1 \times M_2$ and $M_1 + M_2$, it is also a topological isomorphism by the inverse mapping theorem (see Appendix A), so that the two forms of $M_1 \oplus M_2$ can be identified.

Now suppose that T is an operator on X that leaves M_1 and M_2 invariant. Then we have that

$$Tx = Tx_1 + Tx_2 \text{ if } x = x_1 + x_2, x_1 \in M_1, x_2 \in M_2.$$

In the sense of the isomorphism ϕ we therefore have that

$$T = T|_{M_1} \oplus T|_{M_2};$$

hence Proposition 2.25 implies the following.

Proposition 2.28. *Let $T : X \rightarrow X$ be a hypercyclic operator, and let M_1 and M_2 be T -invariant closed subspaces of X such that $X = M_1 \oplus M_2$. Then the restrictions $T|_{M_1}$ and $T|_{M_2}$ are hypercyclic.*

2.3 Linear chaos

As defined in Chapter 1, chaos in the sense of Devaney consists in demanding topological transitivity and the density of the set of periodic points. In view of the Birkhoff transitivity theorem we can rephrase this definition in our present setting.

Definition 2.29 (Linear chaos). An operator T is said to be *chaotic* if it satisfies the following conditions:

- (i) T is hypercyclic;
- (ii) T has a dense set of periodic points.

We recall that sensitive dependence on initial conditions was a consequence of chaos for metric spaces without isolated points; see Theorem 1.29. For operators, hypercyclicity in itself already implies sensitive dependence.

Proposition 2.30. *Let T be a hypercyclic operator. Then T has sensitive dependence on initial conditions (with respect to any translation-invariant metric defining the topology of X).*

Proof. Let d be any translation-invariant metric on X that induces its topology. Let $\delta, \varepsilon > 0$ and $x \in X$ be arbitrary. We then consider the nonempty open sets

$$U = \{z \in X ; d(0, z) < \varepsilon\}, \quad V = \{z \in X ; d(0, z) > \delta\}.$$

By the topological transitivity of T , there are $n \in \mathbb{N}_0$ and $z \in U$ such that $T^n z \in V$. For the point $y := x + z$ we then obtain that $d(x, y) = d(0, z) < \varepsilon$ and $d(T^n x, T^n y) = d(0, T^n z) > \delta$, which implies the result. \square

Remark 2.31. Devaney's notion of chaos has been generally accepted in linear dynamics. There are, however, also other definitions of chaos. We mention here that a continuous map $T : X \rightarrow X$ on a metric space (X, d) is called *chaotic in the sense of Auslander and Yorke* if it is topologically transitive and it has sensitive dependence on initial conditions. By the previous proposition, every hypercyclic operator is Auslander–Yorke chaotic.

In some cases, the periodic points of an operator are easily determined. As a first example we consider the multiples of backward shifts.

Example 2.32. (Rolewicz's operators) Let $T = \lambda B$, $|\lambda| > 1$, be Rolewicz's operator on $X = \ell^p$, $1 \leq p < \infty$, or $X = c_0$. One easily verifies that $x \in X$ is periodic if and only if there are $N \in \mathbb{N}$ and $x_k \in \mathbb{K}$, $k = 1, \dots, N$, such that

$$x = (x_1, \dots, x_N, \lambda^{-N}x_1, \dots, \lambda^{-N}x_N, \lambda^{-2N}x_1, \dots, \lambda^{-2N}x_N, \dots).$$

In order to see that the set of periodic points is dense in X it suffices to approximate any finite sequence $y = (y_1, \dots, y_n, 0, \dots)$. By choosing a periodic point whose $N \geq n$ first coordinates coincide with those of y we see that $\|x - y\| \leq \sum_{j=1}^{\infty} |\lambda|^{-jN} \|y\| \rightarrow 0$ as $N \rightarrow \infty$. Therefore Rolewicz's operators are chaotic.

Let us observe that, for linear maps T on arbitrary vector spaces X , the set of periodic points of T is a subspace of X . Indeed, let $x, y \in X$ be periodic points for T . Then we have that $T^n x = x$ and $T^m y = y$ for certain $n, m \in \mathbb{N}$. Thus $T^{nm}(ax + by) = a(T^n)^m x + b(T^m)^n y = ax + by$, for any $a, b \in \mathbb{K}$, so that also $ax + by$ is periodic.

There is, in fact, a nice and very useful description of the space of periodic points in terms of eigenvectors to unimodular eigenvalues, that is, eigenvalues of absolute value 1, provided that we have a complex space. The corresponding result is of a purely algebraic nature.

Proposition 2.33. *Let T be a linear map on a complex vector space X . Then the set of periodic points of T is given by*

$$\text{Per}(T) = \text{span}\{x \in X ; Tx = e^{\alpha\pi i}x \text{ for some } \alpha \in \mathbb{Q}\}.$$

Proof. If $Tx = e^{\alpha\pi i}x$ with $\alpha = \frac{k}{n}$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$, then $T^{2n}x = x$, so that x is periodic. This yields one inclusion.

For the other one, suppose that $T^n x = x$, $n \in \mathbb{N}$. We then decompose the polynomial $z^n - 1$ into a product of monomials,

$$z^n - 1 = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n).$$

Since all the roots λ_k , $k = 1, \dots, n$, are different, the system $\{p_1, \dots, p_n\}$ of polynomials with $p_k(z) := \prod_{j \neq k} (z - \lambda_j)$, $1 \leq k \leq n$, is a basis of the space of polynomials of degree strictly less than n . In particular, there are $\alpha_k \in \mathbb{C}$, $k = 1, \dots, n$, such that

$$1 = \sum_{k=1}^n \alpha_k p_k(z), \quad z \in \mathbb{C}.$$

This means that, when we substitute z by T , then

$$I = \sum_{k=1}^n \alpha_k p_k(T).$$

We therefore have that $x = \sum_{k=1}^n \alpha_k y_k$ with $y_k := p_k(T)x$, $k = 1, \dots, n$. Since $(T - \lambda_k)y_k = (T^n - I)x = 0$ with $\lambda_k^n = 1$, we see that x belongs to the desired span. \square

In many concrete situations, this proposition leads to a simple verification that a given operator has a dense set of periodic points. As an example we consider Birkhoff's and MacLane's operators. We first need the following result, where e_λ denotes the exponential function

$$e_\lambda(z) = e^{\lambda z}, \quad z \in \mathbb{C}.$$

Lemma 2.34. *Let $\Lambda \subset \mathbb{C}$ be a set with an accumulation point. Then the set*

$$\text{span}\{e_\lambda ; \lambda \in \Lambda\}$$

is dense in $H(\mathbb{C})$.

Proof. By assumption there are $\lambda \in \mathbb{C}$ and $\lambda_n \in \Lambda$ with $\lambda_n \rightarrow \lambda$ and $\lambda_n \neq \lambda$ for all $n \geq 1$. Writing

$$e^{\lambda_n z} = e^{\lambda z} e^{(\lambda_n - \lambda)z} = e^{\lambda z} + e^{\lambda z}(\lambda_n - \lambda)z + e^{\lambda z} \frac{(\lambda_n - \lambda)^2 z^2}{2!} + \dots \quad (2.4)$$

we see that

$$e^{\lambda_n z} \rightarrow e^{\lambda z} \quad \text{uniformly on compact sets,}$$

which, incidentally, also follows directly. Therefore, $e_\lambda \in \overline{\text{span}}\{e_{\lambda_n} ; n \geq 1\}$.

But now (2.4) also shows that

$$\frac{e^{\lambda_n z} - e^{\lambda z}}{\lambda_n - \lambda} = e^{\lambda z} z + e^{\lambda z} \frac{(\lambda_n - \lambda)z^2}{2!} + \dots,$$

and hence

$$\frac{e^{\lambda_n z} - e^{\lambda z}}{\lambda_n - \lambda} \rightarrow ze^{\lambda z} \quad \text{uniformly on compact sets,}$$

so that also the function $z \rightarrow ze^{\lambda z}$ belongs to $\overline{\text{span}}\{e_{\lambda_n} ; n \geq 1\}$.

Continuing in this way we find that all functions $z \rightarrow z^k e^{\lambda z}$, $k \geq 0$, belong to $\overline{\text{span}}\{e_{\lambda_n} ; n \geq 1\}$.

Now let $f \in H(\mathbb{C})$. Then we have that

$$f(z) = e^{\lambda z} (e^{-\lambda z} f(z)) = e^{\lambda z} \left(\sum_{k=0}^{\infty} a_k z^k \right) = \sum_{k=0}^{\infty} a_k z^k e^{\lambda z}$$

with suitable coefficients $a_k \in \mathbb{C}$, $k \geq 0$, where convergence takes place in $H(\mathbb{C})$. Thus we also have that $f \in \overline{\text{span}}\{e_{\lambda_n} ; n \geq 1\}$, which had to be shown.

□

The lemma allows us to show that Birkhoff's and MacLane's operators are chaotic on $H(\mathbb{C})$.

Example 2.35. (Birkhoff's and MacLane's operators) For the differentiation operator D , any function e_λ is an eigenvector of D to the eigenvalue λ . Thus, since the subspace

$$\text{span}\{e_\lambda ; \lambda = e^{\alpha\pi i} \text{ for some } \alpha \in \mathbb{Q}\}$$

is dense in $H(\mathbb{C})$ by Lemma 2.34, Proposition 2.33 tells us that $\text{Per}(T)$ is dense. Since we already know that D is hypercyclic, it is also chaotic.

For the translation operators T_a , $a \in \mathbb{C} \setminus \{0\}$, any function e_λ is an eigenvector of T_a to the eigenvalue $e^{a\lambda}$. Thus, since also the subspace

$$\text{span}\{e_\lambda ; e^{a\lambda} = e^{\alpha\pi i} \text{ for some } \alpha \in \mathbb{Q}\} = \text{span}\{e_\lambda ; \lambda = \frac{\alpha}{a}\pi i, \alpha \in \mathbb{Q}\}$$

is dense in $H(\mathbb{C})$, we conclude as before that each T_a is chaotic.

The restriction that Proposition 2.33 only holds for complex spaces can sometimes be overcome by using complexifications: an operator on a real space is chaotic if and only if its complexification is, as we will see in Corollary 2.51 below.

2.4 Mixing operators

In this and the following section we study the mixing and weak mixing properties in the light of linearity.

Since we are in the setting of Fréchet spaces, the proofs could be formulated in terms of their seminorms or their metric. However, the arguments become particularly transparent when we use the topological language of open sets and 0-neighbourhoods. For this we only need the following simple result.

As usual, we set $A + B = \{a + b ; a \in A, b \in B\}$ for subsets A, B of a vector space.

Lemma 2.36. *Let X be a Fréchet space. If $U \subset X$ is a nonempty open set then there is a nonempty open subset $U_1 \subset U$ and a 0-neighbourhood W such that $U_1 + W \subset U$. If W is a 0-neighbourhood then there is a 0-neighbourhood W_1 such that $W_1 + W_1 \subset W$.*

Proof. Let $\|\cdot\|$ be an F-norm defining the topology of X . Then there is some $x_0 \in U$ and some $\varepsilon > 0$ such that $U_\varepsilon(x_0) = \{x \in X ; \|x - x_0\| < \varepsilon\}$ is contained in U . One may then take $U_1 = U_{\varepsilon/2}(x_0)$ and $W = U_{\varepsilon/2}(0)$. The second claim follows similarly upon taking $x_0 = 0$. \square

We recall that the mixing property consists in demanding the cofiniteness of the return sets $N(U, V)$ for each pair U, V of nonempty open subsets of X . For operators this requirement can be weakened.

Proposition 2.37. *An operator T is mixing if and only if, for any nonempty open set $U \subset X$ and any 0-neighbourhood W , the return sets*

$$N(U, W) \text{ and } N(W, U)$$

are cofinite.

Proof. It suffices to show sufficiency of the condition. Let $U, V \subset X$ be nonempty open sets. By Lemma 2.36 there are nonempty open sets U_1, V_1 and a 0-neighbourhood W such that $U_1 + W \subset U$ and $V_1 + W \subset V$. By hypothesis, there exists some $N \in \mathbb{N}$ such that, for any $n \geq N$, there are $u \in U_1$ and $w \in W$ so that $T^n u \in W$ and $T^n w \in V_1$. But then $u + w \in U$ and $T^n(u + w) = T^n u + T^n w \in V$, which implies that $N(U, V)$ is cofinite. \square

By following the proofs of the hypercyclicity of Rolewicz's, Birkhoff's and MacLane's operators, one immediately obtains that they are even mixing.

Example 2.38. (Birkhoff's, MacLane's and Rolewicz's operators) The three classical hypercyclic operators are mixing.

We have also seen that these mixing operators are even chaotic. This is not always the case, as the following example shows.

Example 2.39. We consider the weighted shift $T : \ell^1 \rightarrow \ell^1$ given by

$$T(x_1, x_2, \dots) = (2x_2, \frac{3}{2}x_3, \frac{4}{3}x_4, \dots).$$

Let U be a nonempty open subset of ℓ^1 and W a 0-neighbourhood. By density of the finite sequences in ℓ^1 there is a sequence of the form $u = (u_1, \dots, u_N, 0, 0, \dots)$ in U .

Since $T^n u = 0$ whenever $n \geq N$, the set $N(U, W)$ is cofinite. On the other hand, we have that

$$T^n x = ((n+1)x_{n+1}, (\frac{n+2}{2})x_{n+2}, (\frac{n+3}{3})x_{n+3}, \dots), \quad n \geq 1.$$

Thus, if we define $w \in \ell^1$ by $w_k = \frac{k-n}{k}u_{k-n}$ for $k = n+1, \dots, n+N$, and $w_k = 0$ otherwise, then $T^n w = u$ and

$$\|w\| \leq \frac{N}{n+1} \|u\|,$$

so that also $N(W, U)$ is cofinite. Hence T is a mixing operator.

We now show that T has no nontrivial periodic points and therefore cannot be chaotic. Indeed, let us suppose that $x \neq 0$ is periodic for T , that is, there is some $n \in \mathbb{N}$ with $T^n x = x$, hence also $T^{jn} x = x$ for all $j \in \mathbb{N}$. Using the above formula for $T^n x$ we obtain that $\frac{jn+k}{k}x_{jn+k} = x_k$, for all $k, j \in \mathbb{N}$. Now, since $x \neq 0$ there is some $k \in \mathbb{N}$ with $x_k \neq 0$. Hence

$$\|x\| \geq \sum_{j=1}^{\infty} |x_{jn+k}| = |x_k| \sum_{j=1}^{\infty} \frac{k}{jn+k} = \infty,$$

which is a contradiction.

We reformulate part of a previous result, Proposition 1.42, for operators.

Proposition 2.40. *Let $S : X \rightarrow X$ and $T : Y \rightarrow Y$ be hypercyclic operators. If at least one of them is mixing then $S \oplus T$ is hypercyclic. Moreover, $S \oplus T$ is mixing if and only if both S and T are.*

For a later application (see Proposition 8.5), we also need to consider direct sums of countably many operators on Banach spaces. Thus, let T_n be operators on separable Banach spaces X_n , $n \geq 1$. For $1 \leq p < \infty$, we define the *direct ℓ^p -sum* of these spaces as

$$\left(\bigoplus_{n=1}^{\infty} X_n \right)_{\ell^p} = \left\{ (x_n)_{n \geq 1} ; \quad x_n \in X_n, n \geq 1, \text{ and } \sum_{n=1}^{\infty} \|x_n\|^p < \infty \right\};$$

endowed with the norm $\|(x_n)_n\| = (\sum_{n=1}^{\infty} \|x_n\|^p)^{1/p}$ this space turns into a separable Banach space. The *direct c_0 -sum* $(\bigoplus_{n=1}^{\infty} X_n)_{c_0}$ is defined similarly.

Now suppose that $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$. Then the direct sum of the operators T_n , defined by

$$\left(\bigoplus_{n=1}^{\infty} T_n\right)(x_n)_n = (T_n x_n)_n,$$

is an operator on $(\bigoplus_{n=1}^{\infty} X_n)_{\ell^p}$ and on $(\bigoplus_{n=1}^{\infty} X_n)_{c_0}$.

Proposition 2.41. *Let T_n be operators on separable Banach spaces X_n , $n \geq 1$, with $\sup_{n \in \mathbb{N}} \|T_n\| < \infty$. Let $1 \leq p < \infty$. Then $\bigoplus_{n=1}^{\infty} T_n$ is mixing on $(\bigoplus_{n=1}^{\infty} X_n)_{\ell^p}$ if and only if each operator T_n , $n \geq 1$, is mixing.*

The same result holds for the direct c_0 -sum.

Proof. For the necessity part one need only note that each T_n , $n \geq 1$, is quasiconjugate to $\bigoplus_{k=1}^{\infty} T_k$ via the map $\phi : (x_k)_k \rightarrow x_n$.

Now suppose that each T_n , $n \geq 1$, is mixing. Let $U, V \subset (\bigoplus_{n=1}^{\infty} X_n)_{\ell^p}$ be nonempty open sets. It follows from the definition of the norm on this space that there are $\varepsilon > 0$, $m \geq 1$, and points $x := (x_1, \dots, x_m, 0, 0, \dots) \in U$ and $y := (y_1, \dots, y_m, 0, 0, \dots) \in V$ such that the open balls of radius ε around these points belong to U and V , respectively. Since each T_k is mixing there is some $N \geq 1$ such that, for each $1 \leq k \leq m$ and $n \geq N$, there are $x_k^{(n)} \in X_k$ such that $\|x_k^{(n)} - x_k\| < \varepsilon/m^{1/p}$ and $\|T_k^n x_k^{(n)} - y_k\| < \varepsilon/m^{1/p}$. Then, for all $n \geq N$, $x^{(n)} := (x_1^{(n)}, \dots, x_m^{(n)}, 0, 0, \dots) \in U$ and $(\bigoplus_{k=1}^{\infty} T_k)^n x^{(n)} \in V$, which implies that $\bigoplus_{k=1}^{\infty} T_k$ is mixing. The proof for direct c_0 -sums is similar. \square

2.5 Weakly mixing operators

In our present context, an operator $T : X \rightarrow X$ is weakly mixing if and only if $T \oplus T$ is hypercyclic, and if and only if, for any nonempty open subsets U_1, U_2, V_1 and V_2 of X , $N(U_1, V_1) \cap N(U_2, V_2) \neq \emptyset$.

Observation 2.42. *For any linear dynamical system,*

$$\text{mixing} \implies \text{weak mixing} \implies \text{hypercyclicity}.$$

The study of specific hypercyclic operators in Chapter 4 will lead to many simple examples of weakly mixing, non-mixing operators; see Remark 4.10.

In contrast, the strictness of the second implication turned out to be much more delicate and was posed as an open problem by D. Herrero in 1992. The problem has only recently been solved.

Theorem 2.43 (De la Rosa–Read). *There are hypercyclic operators on Banach spaces that are not weakly mixing.*

Refining the techniques of De la Rosa and Read, Bayart and Matheron have shown that such operators even exist on any of the spaces ℓ^p , $1 \leq p < \infty$, and c_0 , in particular on Hilbert spaces. The proof is, however, beyond the scope of this book.

The answer to Herrero's question raises the problem of finding (weak) conditions on a hypercyclic operator to be weakly mixing.

To this end we first derive a useful property of hypercyclic operators involving open sets and 0-neighbourhoods.

Lemma 2.44. *Let T be a hypercyclic operator. Then, for any nonempty open sets U and V in X and any 0-neighbourhood W , there is a nonempty open set $U_1 \subset U$ and a 0-neighbourhood $W_1 \subset W$ such that*

$$N(U_1, W_1) \subset N(V, W) \quad \text{and} \quad N(W_1, U_1) \subset N(W, V).$$

Proof. Using topological transitivity and continuity of T one finds $m \in \mathbb{N}_0$, a nonempty open set $U_1 \subset U$ and a 0-neighbourhood $W_1 \subset W$ such that $T^m(U_1) \subset V$ and $T^m(W_1) \subset W$. Now, if $n \in N(U_1, W_1)$, then there exists some $x \in U_1$ with $T^n x \in W_1$. It follows that $T^n T^m x = T^m T^n x \in W$, so that $n \in N(V, W)$. In the same way we also obtain that $N(W_1, U_1) \subset N(W, V)$. \square

This proof copies our proof of the 4-set trick (see Lemma 1.50); a direct application of that trick would not have given us that W_1 contains 0.

Theorem 2.45. *Let T be a hypercyclic operator. If, for any nonempty open set $U \subset X$ and any 0-neighbourhood W , there is a continuous map $S : X \rightarrow X$ commuting with T such that*

$$S(U) \cap W \neq \emptyset \quad \text{and} \quad S(W) \cap U \neq \emptyset, \quad (2.5)$$

then T is weakly mixing.

Proof. First, the 4-set trick and topological transitivity of T yield that, for any nonempty open set $U \subset X$ and for any 0-neighbourhood W ,

$$N(U, W) \cap N(W, U) \neq \emptyset.$$

By Proposition 1.53 it suffices to show that, given any pair U, V of nonempty open subsets of X , there is $n \in N(U, U) \cap N(U, V)$. To do this, using Lemma 2.36, we fix nonempty open sets $U_1 \subset U$, $V_1 \subset V$ and a 0-neighbourhood W_1 such that $U_1 + W_1 \subset U$ and $V_1 + W_1 \subset V$. Lemma 2.44 implies the existence of a 0-neighbourhood $W_2 \subset W_1$ and a nonempty open set $U_2 \subset U_1$ such that $N(W_2, U_2) \subset N(W_1, V_1)$. We fix $n \in N(U_2, W_2) \cap N(W_2, U_2)$; then there are $u_2 \in U_2$ with $T^n u_2 \in W_2$, $w_1 \in W_1$ with $T^n w_1 \in V_1$, and $w_2 \in W_2$ with $T^n w_2 \in U_2$. If we set $u_3 = u_2 + w_2 \in U$ and $u_4 = u_2 + w_1 \in U$, then we obtain that $T^n u_3 \in W_2 + U_2 \subset U$ and $T^n u_4 \in W_2 + V_1 \subset V$. That is, $n \in N(U, U) \cap N(U, V)$. \square

An operator $T : X \rightarrow X$ is called *flip transitive* if, for any pair U, V of nonempty open subsets of X ,

$$N(U, V) \cap N(V, U) \neq \emptyset;$$

see also Exercises 1.5.4 and 1.5.5. By the Birkhoff transitivity theorem, such an operator is hypercyclic. It follows from the previous result that even more is true.

Corollary 2.46. *Every flip transitive operator is weakly mixing.*

As another consequence we obtain a useful characterization of the weak mixing property; this result should also be compared with Proposition 2.37.

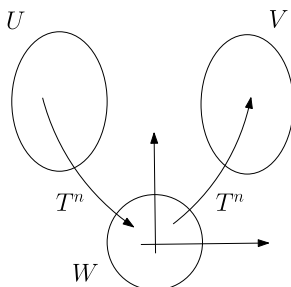


Fig. 2.1 Weak mixing (Theorem 2.47)

Theorem 2.47. *An operator T is weakly mixing if and only if, for any nonempty open sets $U, V \subset X$ and any 0-neighbourhood W ,*

$$N(U, W) \cap N(W, V) \neq \emptyset.$$

Proof. It suffices to show sufficiency. As in the proof of Proposition 2.37 the condition implies that T is topologically transitive, hence hypercyclic. Then an application of Theorem 2.45 yields the result. \square

Theorem 2.45 provides us with a rather weak condition, in terms of open sets and 0-neighbourhoods, for making a hypercyclic operator weakly mixing. As an application we can show that weak mixing is already implied by the existence of a dense set of points with tame orbits. For the precise formulation we need the notion of bounded sets in Fréchet spaces; see Appendix A.

Theorem 2.48. *Let T be a hypercyclic operator. If there exists a dense subset X_0 of X such that the orbit of each $x \in X_0$ is bounded, then T is weakly mixing.*

Proof. Let U be a nonempty open subset of X and W a 0-neighbourhood. If $(p_n)_n$ is an increasing sequence of seminorms defining the topology of X then there is some $k \in \mathbb{N}$ and some $\varepsilon > 0$ such that $p_k(x) < \varepsilon$ implies that

$x \in W$. Now, by assumption, we can find a point $x \in X_0 \cap U$; it follows that $M := \sup_{n \in \mathbb{N}_0} p_k(T^n x) < \infty$. Hence $\frac{\varepsilon}{2M} T^n x \in W$, for all $n \in \mathbb{N}_0$. On the other hand, by topological transitivity of T , there is some $n \in \mathbb{N}_0$ with $(\frac{\varepsilon}{2M} T^n(W)) \cap U = T^n(\frac{\varepsilon}{2M} W) \cap U \neq \emptyset$. Thus, condition (2.5) is satisfied for $S = \frac{\varepsilon}{2M} T^n$. \square

Of course, every periodic point has bounded orbit, as does every point whose orbit converges. The latter holds, a fortiori, for all points from the *generalized kernel*

$$\bigcup_{n=0}^{\infty} \ker T^n$$

of T . Thus we have the following.

Corollary 2.49. *Any of the following operators are weakly mixing:*

- (i) *chaotic operators;*
- (ii) *hypercyclic operators that have a dense set of points for which the orbits converge;*
- (iii) *hypercyclic operators with dense generalized kernel.*

A typical class of operators with dense generalized kernel is the class of unilateral weighted shifts that will be studied in detail in Section 4.1.

For an additional characterization of weakly mixing operators in terms of multiples of the iterates of T we refer to Theorem 12.29.

We end this section with an application to complexifications.

Proposition 2.50. *Let $T : X \rightarrow X$ be an operator. Then:*

- (i) *$T \oplus T$ is weakly mixing if and only if T is;*
- (ii) *$T \oplus T$ is chaotic if and only if T is.*

Proof. Suppose that T is chaotic. Then, by Corollary 2.49, $T \oplus T$ is hypercyclic. Moreover, the set of all points $(x, y) \in X \oplus X$ with periodic points x and y for T provides a dense set of periodic points for $T \oplus T$. Thus $T \oplus T$ is chaotic. The remaining implications are special cases of Propositions 1.42 and 1.48 and Theorem 1.51. \square

More generally, an arbitrary direct sum $S \oplus T$ is chaotic if and only if both S and T are; see Exercise 2.5.7. In contrast, assertion (i) cannot be generalized in the same way; see Remark 4.17.

The discussion before Proposition 2.26, together with Proposition 2.40, yields the following.

Corollary 2.51. *An operator T on a real separable Fréchet space is mixing, weakly mixing or chaotic, respectively, if and only if its complexification \tilde{T} is.*

This result can be applied, for instance, to Rolewicz's operators, just as in Example 2.27; see also Example 3.2.

For extensions of the results of the last two sections to sequences of operators we refer to Section 3.4.

2.6 The set of hypercyclic vectors

A natural question that arises in hypercyclicity is this: which kind of structures can we find in the set of hypercyclic vectors? By the Birkhoff transitivity theorem we already know that the set $HC(T)$ of hypercyclic vectors of a hypercyclic operator T is always a dense G_δ -set. Almost trivially this leads to a somewhat surprising representation result.

Proposition 2.52. *Let T be a hypercyclic operator on X . Then*

$$X = HC(T) + HC(T),$$

that is, every vector $x \in X$ can be written as the sum of two hypercyclic vectors.

Proof. Let $x \in X$. Since both $HC(T)$ and $x - HC(T)$ are dense G_δ -sets, their intersection must be nonempty by the Baire category theorem, which implies that $x \in HC(T) + HC(T)$. \square

As a consequence, the set $HC(T)$ of hypercyclic vectors can only then be a linear subspace, except for the zero vector, if any nonzero vector is hypercyclic, in which case the operator has no nontrivial invariant closed subset; see Observation 2.17. Such an operator exists, for example, on ℓ^1 , but the construction is highly nontrivial.

Weakening the requirement, it is natural to ask if, for a general hypercyclic operator T , $HC(T)$ contains a large linear subspace, except for 0. In this section we will interpret largeness as being dense. A different sense of largeness will be studied in Chapter 10.

We first need some auxiliary results that are also important in their own right. For the definition of the adjoint of an operator and the notation $\langle x, x^* \rangle$ we refer to Appendix A.

Lemma 2.53. (a) *Let T be a hypercyclic operator. Then its adjoint T^* has no eigenvalues. Equivalently, every operator $T - \lambda I$, $\lambda \in \mathbb{K}$, has dense range.*

(b) *Let T be a hypercyclic operator on a real separable Fréchet space. Then the adjoint \tilde{T}^* of its complexification \tilde{T} has no eigenvalues. Equivalently, every operator $\tilde{T} - \lambda I$, $\lambda \in \mathbb{C}$, has dense range.*

Proof. (a) Let $x \in X$ be a hypercyclic vector for T . Suppose, by way of contradiction, that T^* has an eigenvalue λ , that is,

$$T^* x^* = \lambda x^*$$

for some $x^* \in X^*$, $x^* \neq 0$. Then we have that, for any $n \geq 0$,

$$\langle T^n x, x^* \rangle = \langle x, (T^*)^n x^* \rangle = \lambda^n \langle x, x^* \rangle.$$

Since $x^* \neq 0$, the hypercyclicity of x implies that the left-hand side is dense in \mathbb{K} , while the right-hand side clearly is not, which is the desired contradiction.

Moreover, by the Hahn–Banach theorem (see Appendix A), $T - \lambda I$ has dense range precisely when

$$\langle x, T^*x^* - \lambda x^* \rangle = \langle (T - \lambda I)x, x^* \rangle = 0 \quad \text{for all } x \in X$$

entails that $x^* = 0$, which is equivalent to λ not being an eigenvalue of T^* .

(b) Now let X be a space over the real scalar field, and let \tilde{T} be the complexification of T . Let $x \in X$ be hypercyclic for T , and suppose that \tilde{T}^* has an eigenvector $\tilde{x}^* \in \tilde{X}^*$, $\tilde{x}^* \neq 0$, to an eigenvalue λ . Then we have that

$$|\langle T^n x, \tilde{x}^* \rangle| = |\langle \tilde{T}^n x, \tilde{x}^* \rangle| = |\langle x, (\tilde{T}^*)^n x^* \rangle| = |\lambda|^n |\langle x, \tilde{x}^* \rangle|, \quad (2.6)$$

for $n \geq 0$. Since $\langle x_1 + ix_2, \tilde{x}^* \rangle = \langle x_1, \tilde{x}^* \rangle + i\langle x_2, \tilde{x}^* \rangle$ for all $x_1, x_2 \in X$, and since $\tilde{x}^* \neq 0$, there is some $y \in X$ such that $|\langle y, \tilde{x}^* \rangle| > 0$. Hence $|\tilde{x}^*|$ can take every positive value on X . By the hypercyclicity of x , the left-hand side of (2.6) is dense in \mathbb{R}_+ , while the right-hand side clearly is not, which is a contradiction. The remainder of the proof can be given as in (a). \square

As a consequence we obtain one of the cornerstones of the theory of linear dynamical systems. We recall that for any polynomial $p(z) = \sum_{n=0}^N a_n z^n$ the operator $p(T)$ is defined as $p(T) = \sum_{n=0}^N a_n T^n$.

Theorem 2.54 (Bourdon). *If T is a hypercyclic operator and p is a nonzero polynomial, then the operator $p(T)$ has dense range.*

Proof (complex case). We can assume that $p(z) = \sum_{n=0}^N a_n z^n$ with $a_N \neq 0$, $N \geq 1$. For spaces X over the complex field the result follows immediately from Lemma 2.53(a) and the fact that p can be written as a product of linear factors, so that

$$p(T) = a_N (T - \lambda_1 I) \cdots (T - \lambda_N I)$$

with certain $\lambda_k \in \mathbb{C}$, $k = 1, \dots, N$.

(Real case). If X is a real space we consider the complexification \tilde{T} of T . With Lemma 2.53(b), it follows as in the complex case that, for any complex polynomial p , $p(\tilde{T})$ has dense range on \tilde{X} . Now if p has real coefficients, then

$$p(\tilde{T})(x + iy) = p(T)x + ip(T)y, \quad x, y \in X,$$

which implies that also $p(T) : X \rightarrow X$ has dense range. \square

We are now ready to deduce an important result on the algebraic structure of the set of hypercyclic vectors.

Theorem 2.55 (Herrero–Bourdon). *If x is a hypercyclic vector for T , then*

$$\{p(T)x ; p \text{ is a polynomial}\} \setminus \{0\}$$

is a dense set of hypercyclic vectors.

In particular, any hypercyclic operator admits a dense invariant subspace consisting, except for zero, of hypercyclic vectors.

Proof. Let $x \in X$ be a hypercyclic vector for T . Then

$$M = \{p(T)x ; p \text{ is a polynomial}\} = \text{span orb}(x, T)$$

is a dense T -invariant subspace of X . Moreover, if $y = p(T)x \in M \setminus \{0\}$ then $p \neq 0$ and

$$T^n y = p(T)(T^n x), \quad n \in \mathbb{N}_0.$$

Since x is hypercyclic and, by Theorem 2.54, $p(T)$ has dense range, also y has dense orbit under T . \square

The Herrero–Bourdon theorem allows us to deduce an additional topological structure of the set of hypercyclic vectors: it is always a connected set. This observation comes from the fact that, if $A \subset B \subset \overline{A} \subset X$ and A is connected, then also B is connected. We apply this to $A = M \setminus \{0\}$, and $B = HC(T)$, where M is the dense subspace of the Herrero–Bourdon theorem. Note that M is of dimension greater than 1 because, otherwise, x would be an eigenvector, which is not hypercyclic; hence $A = M \setminus \{0\}$ is connected.

Corollary 2.56. *The set $HC(T)$ of hypercyclic vectors for a hypercyclic operator T is a connected subset of X .*

2.7 Linear vs nonlinear maps, and finite vs infinite dimension

We have seen that chaotic linear operators exist. This is contrary to the common belief that (deterministic) chaos is necessarily connected to the nonlinearity of a system. In this section we want to explore the connection between linear and nonlinear chaos.

The dynamics of linear operators on a finite-dimensional space $X = \mathbb{K}^N$ are easy to describe, thanks to the Jordan decomposition theorem. We assume that \mathbb{K}^N is endowed with the Euclidean norm.

Proposition 2.57. *Let T be a linear operator on \mathbb{K}^N , $N \geq 1$. Then, for any $x \in \mathbb{K}^N$, either $T^n x \rightarrow 0$ or $\|T^n x\| \rightarrow \infty$ or there are $m, M > 0$ such that $m \leq \|T^n x\| \leq M$ for all $n \geq 0$.*

Proof. Since every operator $T : \mathbb{R}^N \rightarrow \mathbb{R}^N$ can be regarded as an operator on \mathbb{C}^N it suffices to consider the complex case.

By the Jordan decomposition theorem, \mathbb{C}^N has a basis with respect to which the matrix of T is in Jordan block form. Since all norms on \mathbb{C}^N are

equivalent we can assume that this basis is the canonical basis of \mathbb{C}^N , and it suffices to show the result for each operator given by a Jordan block

$$T = \begin{pmatrix} \lambda & 1 & 0 & \dots & \dots \\ 0 & \lambda & 1 & 0 & \dots \\ & \ddots & \ddots & \ddots & \vdots \\ & & 0 & \lambda & 1 \\ & & & 0 & \lambda \end{pmatrix} : \mathbb{C}^N \rightarrow \mathbb{C}^N,$$

with $N \geq 1$. For $n \geq N - 1$ we have that

$$T^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \dots & \dots & \binom{n}{N-1}\lambda^{n-N+1} \\ 0 & \lambda^n & n\lambda^{n-1} & \dots & \dots & \binom{n}{N-2}\lambda^{n-N+2} \\ & \ddots & \ddots & \ddots & \dots & \vdots \\ & & 0 & \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} \\ & & & 0 & \lambda^n & n\lambda^{n-1} \\ & & & & 0 & \lambda^n \end{pmatrix}.$$

We apply T^n to the vector $x = (x_1, \dots, x_N) \in \mathbb{C}^N$, $x \neq 0$.

Case 1: $|\lambda| > 1$. Let x_k be the last nonzero entry. Then the k th entry of $T^n x$ is $\lambda^n x_k$, hence $\|T^n x\| \rightarrow \infty$.

Case 2: $|\lambda| < 1$. Then all entries of $T^n x$ tend to zero, hence $T^n x \rightarrow 0$.

Case 3: $|\lambda| = 1$. If $x = (x_1, 0, \dots, 0)$ then $\|T^n x\| = \|\lambda^n x\| = \|x\|$ for $n \geq 0$. Otherwise, the first entry of $T^n x$ is

$$\lambda^n x_1 + n\lambda^{n-1} x_2 + \dots + \binom{n}{N-1} \lambda^{n-N+1} x_N,$$

which tends to infinity in absolute value; hence, again, $\|T^n x\| \rightarrow \infty$. \square

As an immediate consequence we obtain the following.

Theorem 2.58. *There are no hypercyclic operators on \mathbb{K}^N , $N \geq 1$.*

Of course, this also follows directly from Lemma 2.53 since every operator on \mathbb{C}^N has an eigenvalue. Further proofs of this result are suggested in Exercises 2.7.1 and 2.7.2.

Since every finite-dimensional Fréchet space is isomorphic to some \mathbb{K}^N , $N \geq 1$ (see Appendix A), the theorem extends to such spaces.

Corollary 2.59. *There are no hypercyclic operators on a finite-dimensional Fréchet space.*

The result also implies an interesting property of the orbit of a hypercyclic vector.

Proposition 2.60. *The orbit of any hypercyclic vector forms a linearly independent set.*

Proof. Let x be a hypercyclic vector for T and suppose that there are scalars $\alpha_k \in \mathbb{K}$, $k = 0, \dots, N$, such that

$$T^{N+1}x = \sum_{k=0}^N \alpha_k T^k x.$$

Then $F := \text{span}\{T^k x ; k = 0, \dots, N\}$ is a finite-dimensional T -invariant subspace of X . Since x is hypercyclic for T , it is also hypercyclic for $T|_F : F \rightarrow F$, which contradicts Corollary 2.59. \square

Alternatively, the proof can be based on Bourdon's theorem; see Exercise 2.7.3.

Proposition 2.57 tells us that the dynamics of linear maps on finite-dimensional spaces are quite restrictive. On the other hand, in an infinite-dimensional setting, linear dynamics can be arbitrarily complicated. Indeed, as we now show, every continuous map on a compact metric space is conjugate to the restriction of a linear operator on some invariant set. Even more strikingly, the same operator can be taken for all nonlinear systems, and the operator is even chaotic. In other words: the dynamics of any (compact) nonlinear dynamical system can be described by the dynamics of a single chaotic operator.

Theorem 2.61. *There exists a chaotic operator T on a separable Hilbert space H with the following property.*

For any continuous map f on any compact metric space K there exists a T -invariant subset L of H such that f is conjugate to the restriction $T|_L$ of T to L .

In other words, there is a homeomorphism $\phi : K \rightarrow L$ so that the diagram

$$\begin{array}{ccc} K & \xrightarrow{f} & K \\ \phi \downarrow & & \downarrow \phi \\ L & \xrightarrow{T|_L} & L \end{array}$$

commutes.

Proof. Let $H = (\bigoplus_{n=0}^{\infty} \ell^2)_{\ell^2}$ be the space of all sequences $x = (x_n)_{n \geq 0}$ of elements $x_n = (x_{n,k})_{k \geq 0}$ in ℓ^2 such that

$$\|x\| := \left(\sum_{n=0}^{\infty} \|x_n\|^2 \right)^{1/2} < \infty;$$

see the discussion before Proposition 2.41. Then H is a separable Hilbert space when endowed with the canonical inner product. On H we consider the multiple $T = 2B$ of the backward shift B given by

$$B(x_0, x_1, x_2, \dots) = (x_1, x_2, x_3, \dots).$$

The proof that T has the desired properties will be split into three steps.

Step 1. We define a suitable embedding $\phi : K \rightarrow H$.

First, we can assume that the metric d on K is bounded by 1, since otherwise we can replace it by the equivalent metric $d'(x, y) = \min(1, d(x, y))$. We then fix a dense sequence $(y_k)_{k \geq 0}$ in K ; this is possible because K is a compact metric space. Based on this sequence we define, for any $x \in K$,

$$\phi(x) = \left(\left(\frac{1}{2^{k+n}} d(y_k, f^n(x)) \right)_k \right)_n \in H.$$

Then we have for any $x, y \in K$ and $N \geq 0$ that

$$\begin{aligned} \|\phi(x) - \phi(y)\|^2 &= \sum_{k,n=0}^{\infty} \frac{1}{2^{2(k+n)}} |d(y_k, f^n(x)) - d(y_k, f^n(y))|^2 \\ &\leq \sum_{k,n \leq N} \frac{1}{2^{2(k+n)}} |d(y_k, f^n(x)) - d(y_k, f^n(y))|^2 + \sum_{\substack{k > N \text{ or} \\ n > N}} \frac{4}{2^{2(k+n)}}. \end{aligned}$$

This can be made arbitrarily small by first choosing N sufficiently large and then y sufficiently close to x . Thus $\phi : K \rightarrow H$ is continuous.

Moreover, ϕ is injective; indeed, $\phi(x) = \phi(y)$ implies that

$$\frac{1}{2^k} d(y_k, x) = \frac{1}{2^k} d(y_k, y) \quad \text{for all } k \geq 0,$$

hence $x = y$ by density of the y_k .

As a continuous injection on a compact space, ϕ is a homeomorphism onto a (compact) subset, L say, of H .

Step 2. We show that $\phi \circ f = T \circ \phi$ and that L is invariant under T .

In fact, for any $x \in K$ we have that

$$\begin{aligned} \phi(f(x)) &= \left(\left(\frac{1}{2^{k+n}} d(y_k, f^{n+1}(x)) \right)_k \right)_n \\ &= 2 \left(\left(\frac{1}{2^{k+n+1}} d(y_k, f^{n+1}(x)) \right)_k \right)_n = T(\phi(x)), \end{aligned}$$

which also implies that

$$T(L) = T(\phi(K)) = \phi(f(K)) \subset \phi(K) = L.$$

Step 3. The operator T is chaotic on H .

The proof is the same as the proof that Rolewicz's operators are chaotic; see Examples 2.22 and 2.32. \square

In summary we have found that

- *linear chaos exists;*
- *linear chaos is an infinite-dimensional phenomenon;*
- *linear dynamics can be as complicated as nonlinear dynamics.*

2.8 Hypercyclicity and complex dynamics

In this section we present a connection between the hypercyclicity of weighted backward shifts and the Julia sets of polynomials in one complex variable. In some sense we continue the theme of the previous section by comparing infinite-dimensional dynamics with nonlinear but finite-dimensional dynamics.

Let us first give a brief introduction to complex dynamics. Given a polynomial p in one complex variable, of degree $m \geq 2$, a periodic point z of p is called *repelling* if $|p'(z)| > 1$. The *Julia set* of p can be defined as

$$\mathcal{J}(p) = \overline{\{z \in \mathbb{C} ; z \text{ is a repelling periodic point of } p\}}.$$

While this is not the common definition of the Julia set, which is more complicated and involves the behaviour of the iterates of p near $\mathcal{J}(p)$, a result by Fatou and Julia tells us that the two definitions are equivalent.

For instance, for the doubling map $p(z) = z^2$ on \mathbb{C} (see Example 1.37), the periodic points are given by $e^{2\pi i\alpha}$ with $\alpha = \frac{k}{2^n-1}$, $n, k \in \mathbb{N}$, and all of them are repelling, so that $\mathcal{J}(p) = \mathbb{T}$. We have also seen that $p|_{\mathbb{T}}$ is chaotic.

This feature is shared by any complex polynomial of degree $m \geq 2$. More precisely one always has that $\mathcal{J}(p)$ is a compact p -invariant set such that $p|_{\mathcal{J}(p)}$ is chaotic.

The dynamical behaviour of p near the Julia set consists in spreading points. Indeed, the following property, which can be regarded as a multi-point approximation by the iterates of p on points near $\mathcal{J}(p)$, characterizes the Julia set: a point $z \in \mathbb{C}$ belongs to $\mathcal{J}(p)$ if and only if

$$\begin{aligned} \forall \varepsilon > 0, \forall z_1, \dots, z_k \in \mathbb{C}, \exists z'_1, \dots, z'_k \in \mathbb{C}, \exists n \in \mathbb{N} \text{ such that} \\ |z'_j - z| < \varepsilon \text{ and } |p^n(z'_j) - z_j| < \varepsilon, \quad j = 1, \dots, k. \end{aligned} \quad (2.7)$$

Now, one can deduce this property, for certain polynomials and at the point 0, from the hypercyclic behaviour of Rolewicz's operators.

Example 2.62. Let $p(z) = (z + 1)^m - 1$, where $m \geq 2$. We consider the following commutative diagram,

$$\begin{array}{ccc}
c_0 & \xrightarrow{mB} & c_0 \\
\phi \downarrow & & \downarrow \phi \\
c_0 & \xrightarrow{P} & c_0,
\end{array}$$

where mB is the Rolewicz operator with $\lambda = m$ on the complex space c_0 , $P(x_1, x_2, \dots) = (p(x_2), p(x_3), \dots)$, and $\phi(x_1, x_2, \dots) = (e^{x_1} - 1, e^{x_2} - 1, \dots)$.

It is easy to see that P and ϕ are continuous maps and that ϕ has dense range. By Proposition 1.19, it follows from the hypercyclicity of Rolewicz's operators that P has a dense orbit.

In order to verify condition (2.7) at $z = 0$, we fix $\varepsilon > 0$ and arbitrary $z_j \in \mathbb{C}$, $j = 1, \dots, k$. Since P has a dense orbit we can find $w \in c_0$ and $n \geq 0$ such that

$$\|w\| < \varepsilon \quad \text{and} \quad \|P^n w - (z_1, \dots, z_k, 0, 0, \dots)\| < \varepsilon.$$

By considering the $(j+n)$ th coordinates w_{j+n} of w , we deduce that

$$|w_{j+n}| < \varepsilon \quad \text{and} \quad |p^n(w_{j+n}) - z_j| < \varepsilon, \quad j = 1, \dots, k.$$

This shows that condition (2.7) holds for $z = 0$.

Let us mention that the map P in this example is a polynomial on c_0 ; see Exercise 2.8.1.

More generally, we have the following connection between infinite-dimensional dynamics and the dynamics of arbitrary complex polynomials; its proof is left to the reader: see Exercise 2.8.2.

Proposition 2.63. *Let p be a complex polynomial of degree $m \geq 2$ such that $p(0) = 0$. Let $P : c_0 \rightarrow c_0$ be the continuous map given by $P(x_1, x_2, \dots) := (p(x_2), p(x_3), \dots)$. Then P is topologically transitive if and only if 0 belongs to the Julia set $\mathcal{J}(p)$ of p .*

Exercises

Exercise 2.1.1. Let $(p_n)_n$ be a separating increasing sequence of seminorms on a vector space X . Show that the map d defined in (2.1) is a metric on X . Moreover, show that for any $x, y \in X$ and $n \in \mathbb{N}$:

- (i) if $p_n(x - y) < \frac{1}{2^n}$ then $d(x, y) < \frac{2}{2^n}$;
- (ii) if $d(x, y) < \frac{1}{2^n}$ then $p_k(x - y) < \frac{1}{2^n}$ for all $k \leq n$.

Exercise 2.1.2. Prove Lemma 2.6. (*Hint:* Use the previous exercise.)

Exercise 2.1.3. Show that the spaces $H(\mathbb{C})$ and ω are Fréchet spaces. Show that for every separable Fréchet space X , $X^{\mathbb{N}}$ is also a separable Fréchet space.

Exercise 2.1.4. Prove Proposition 2.8.

Exercise 2.1.5. Let $X = C^\infty(\mathbb{R})$ be the space of infinitely differentiable (real or complex) functions $f : \mathbb{R} \rightarrow \mathbb{K}$, endowed with the seminorms

$$p_n(f) = \max_{0 \leq k \leq n} \sup_{|x| \leq n} |f^{(k)}(x)|.$$

Show that X is a separable Fréchet space. (*Hint:* Use the Weierstrass approximation theorem to approximate $f^{(n)}$ on $[-n, n]$ by a polynomial.)

Exercise 2.1.6. Let $1 \leq p < \infty$. Let $v : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a strictly positive continuous function. We define the space of weighted p -integrable functions by

$$X = L_v^p(\mathbb{R}_+) = \{f : \mathbb{R}_+ \rightarrow \mathbb{K} ; f \text{ is measurable and } \|f\| < \infty\},$$

where $\|f\| := (\int_0^\infty |f(x)|^p v(x) dx)^{1/p}$. Then X is a separable Banach space. Show that the translation operator $T : X \rightarrow X$, $(Tf)(x) = f(x+1)$, is a well-defined operator if and only if

$$\sup_{x \in \mathbb{R}_+} \frac{v(x)}{v(x+1)} < \infty.$$

Exercise 2.1.7. Let X be a Fréchet space with defining increasing sequence of seminorms $(p_n)_n$. Then a seminorm $p : X \rightarrow \mathbb{R}$ is continuous if and only if there are $n \geq 1$ and $M > 0$ such that

$$p(x) \leq M p_n(x), \quad x \in X.$$

Show that this immediately implies the nontrivial part of Proposition 2.11.

Exercise 2.2.1. Let $X = C_0(\mathbb{R}_+) = \{f : \mathbb{R}_+ \rightarrow \mathbb{R} ; f \text{ is continuous and } \lim_{x \rightarrow \infty} f(x) = 0\}$, endowed with the sup-norm $\|f\| = \sup_{x \in \mathbb{R}_+} |f(x)|$. Given $a > 0$ and $\lambda > 1$, consider the operator $T : X \rightarrow X$, $(Tf)(x) = \lambda f(x+a)$, $x \in \mathbb{R}_+$. Show that T is hypercyclic. (*Hint:* Use the fact that the continuous functions with compact support form a dense subset of X .)

Exercise 2.2.2. Let X be a Banach space. Show that there are no operators T on X for which λT is hypercyclic for all $\lambda \neq 0$. (*Hint:* Consider $|\lambda| \leq 1/\|T\|$.)

Exercise 2.2.3. Given any sequence of nonzero scalars $(w_n)_n$, we define the operator $B_w : \omega \rightarrow \omega$, $(x_1, x_2, x_3, \dots) \rightarrow (w_2 x_2, w_3 x_3, w_4 x_4, \dots)$. Prove that B_w is hypercyclic.

Exercise 2.2.4. Let T be the translation operator on the space $L_v^p(\mathbb{R}_+)$ defined in Exercise 2.1.6. We assume that there are constants $M \geq 1$ and $w \in \mathbb{R}$ such that

$$v(x) \leq M e^{w(y-x)} v(y) \quad \text{whenever } y \geq x \geq 0;$$

in this case, v is called an *admissible weight function*. Show that T is hypercyclic if and only if $\liminf_{x \rightarrow \infty} v(x) = 0$. (*Hint:* If the condition does not hold then define $g(x) = v(x)^{-1/p}$ on $[0, 1]$, and 0 otherwise, and show that, if $\|f\|$ is small enough, then $\|T^n f - g\| \geq \frac{1}{2}$ for $n \geq 1$. For the sufficiency, use the density of the continuous functions of compact support.)

Exercise 2.2.5. Let $X = C^\infty(\mathbb{R})$ be the space of infinitely differentiable real functions $f : \mathbb{R} \rightarrow \mathbb{R}$; see Exercise 2.1.5. Show that the (real) differentiation operator $D : X \rightarrow X$, $f \rightarrow f'$ is hypercyclic by defining a suitable quasiconjugacy $\phi : H(\mathbb{C}) \rightarrow C^\infty(\mathbb{R})$.

Exercise 2.2.6. Let $T : X \rightarrow X$ be an operator. Suppose that $Y \subset X$ is a T -invariant dense subspace of X . Furthermore, suppose that Y carries a Fréchet space topology such

that the embedding $Y \rightarrow X$ is continuous and such that $T|_Y : Y \rightarrow Y$ is continuous. Show that T is quasiconjugate to $T|_Y$. In particular, if $T|_Y$ is hypercyclic, then so is T , and T has a hypercyclic vector belonging to Y . (*Remark:* this result is known as the *hypercyclic comparison principle*; it shows the interest of hypercyclicity on small spaces.)

Exercise 2.2.7. Show that the complexification \tilde{X} of a real separable Fréchet space X is a complex separable Fréchet space and that the complexification of a (real-linear) operator on X is a (complex-linear) operator on \tilde{X} .

Exercise 2.2.8. Let $T : X \rightarrow X$ be an operator, and let M_1 and M_2 be T -invariant closed subspaces of X such that $X = M_1 \oplus M_2$; see Proposition 2.28. Let P_{M_1} be the projection $X \rightarrow M_1$, $x = x_1 + x_2 \rightarrow x_1$, where $x_1 \in M_1$, $x_2 \in M_2$, and similarly for M_2 . Show that, for $j = 1, 2$, $T|_{M_j}$ is quasiconjugate to T via P_{M_j} .

Exercise 2.2.9. Let T be an operator on a Banach space X and $x \in X$. Given $d > 0$, let us call the orbit of x under T *d-dense* if for each $y \in X$ we can find $n \in \mathbb{N}_0$ such that $\|T^n x - y\| < d$. Show that if T admits a d -dense orbit then it is hypercyclic. (*Hint:* First observe that X is separable. Then, given a vector x whose orbit is d -dense, prove that, for each $\varepsilon > 0$, the vector $\frac{\varepsilon}{d}x$ has an ε -dense orbit and conclude the result by Exercise 1.2.5 and the Birkhoff transitivity theorem).

Exercise 2.2.10. Let $\varepsilon > 0$. An operator T on a Banach space X is called *ε -hypercyclic* if it admits a vector $x \in X$ such that, for any nonzero vector $y \in X$, we can find $n \in \mathbb{N}_0$ satisfying $\|T^n x - y\| \leq \varepsilon\|y\|$; the vector x is then also called *ε -hypercyclic*. Show that if an operator is ε -hypercyclic for all $\varepsilon > 0$ then it is hypercyclic. (*Hint:* First observe that X is separable. Then conclude the result by Exercise 1.2.5 and the Birkhoff transitivity theorem).

Exercise 2.2.11. An operator T on a Fréchet space X is called a *J-class operator* if there is a vector $x \neq 0$ in X with $J(x) = X$, where $J(x)$ is the J -set of x . By Exercise 1.2.8, every hypercyclic operator is J -class.

(a) Let B be the backward shift on ℓ^2 . Show that the operator $T = 2I \oplus 2B$ on $\mathbb{K} \oplus \ell^2$ is a J -class operator that is not hypercyclic. Deduce also that being J -class is not preserved under quasiconjugacy.

(b) Let T be a hypercyclic operator on X . Show that, for $\lambda \in \mathbb{K}$, the operator $\lambda I \oplus T$ is J -class on $\mathbb{K} \oplus X$ if and only if $|\lambda| > 1$. Deduce that there is an invertible J -class operator T whose inverse T^{-1} is not J -class.

(c) Show that the multiple $T = 2B$ of the backward shift is J -class on ℓ^∞ . Therefore there exist J -class operators on non-separable Banach spaces. (*Hint:* Consider $(1, 0, 0, \dots)$.)

Exercise 2.3.1. Let T be an operator on a Banach space X . Show that the following assertions are equivalent:

- (i) T has sensitive dependence on initial conditions with respect to the usual metric;
- (ii) $\sup_{n \geq 0} \|T^n\| = \infty$;
- (iii) T has an unbounded orbit.

(*Hint:* Use the Banach–Steinhaus theorem; see Appendix A.)

Exercise 2.3.2. Let T be the translation operator on $L_v^p(\mathbb{R}_+)$ with an admissible weight function v ; see Exercise 2.2.4. Show that T is chaotic if and only if $\int_0^\infty v(x) dx < \infty$.

Exercise 2.3.3. Let $T : \mathbb{K}^N \rightarrow \mathbb{K}^N$, $N \geq 1$, be an operator.

(a) Show that T has a dense set of periodic points if and only if $T^n = I$ for some $n \geq 1$. (*Hint:* Show that \mathbb{K}^N has a basis consisting of periodic points.)

(b) Deduce from (a) that no operator on \mathbb{K}^N can be chaotic.

Exercise 2.3.4. Let $H(\Omega)$ be the Fréchet space of all holomorphic functions on a domain Ω in \mathbb{C} ; see Section 4.3. Then $D : f \rightarrow f'$ is an operator on $H(\Omega)$. Show that the following assertions are equivalent:

- (i) D is chaotic on $H(\Omega)$;
- (ii) D is hypercyclic on $H(\Omega)$;
- (iii) Ω is simply connected.

(Hint: By Runge's theorem, the polynomials are dense in $H(\Omega)$ if Ω is simply connected. On the other hand, if Ω is not simply connected, approximate a suitable function $\frac{1}{z-a}$ by derivatives, and integrate both over closed curves in Ω .)

Exercise 2.4.1. Modify Example 2.39 to obtain a non-chaotic mixing operator on any space ℓ^p , $1 < p < \infty$.

Exercise 2.4.2. Let T be the translation operator on $L_v^p(\mathbb{R}_+)$ with an admissible weight function v ; see Exercise 2.2.4. Show that T is mixing if and only if $\lim_{x \rightarrow \infty} v(x) = 0$.

Exercise 2.4.3. Using Proposition 2.37, show that λD is mixing on $H(\mathbb{C})$ for any $\lambda \neq 0$. (This should be contrasted with Exercise 2.2.2.)

Exercise 2.4.4. Let T be an operator on a separable Banach space. It can be shown that $T \oplus T^*$ is never hypercyclic on $X \oplus X^*$; see Remark 4.17. Deduce that if T is mixing then T^* cannot be hypercyclic. (See Exercise 5.1.1 for a better result.)

Exercise 2.5.1. Let T be a weakly mixing (or mixing) operator and $\lambda \in \mathbb{K}$, $|\lambda| = 1$. Show that λT is also weakly mixing (or mixing, respectively). (Hint: Use Proposition 2.37 and Theorem 2.47.)

Exercise 2.5.2. Establish Theorem 2.45 without the use of Proposition 1.53, that is, show the weak mixing property directly. (Hint: See Figure 2.2.)

Exercise 2.5.3. Let T be an operator such that, for any nonempty open sets $U, V \subset X$ and any 0-neighbourhood W , $N(U, W)$ is nonempty and $N(W, V)$ is syndetic. Then prove that T is weakly mixing. Do likewise if the sets $N(U, W)$ are syndetic and the sets $N(W, V)$ are nonempty. (Hint: Apply Theorem 2.47.)

Exercise 2.5.4. A subset A of \mathbb{N}_0 is called *thickly syndetic* if, for any $n \in \mathbb{N}$, there is a syndetic sequence $(n_k)_k$ of positive integers such that $\{n_k + j ; k \in \mathbb{N}, j = 0, \dots, n\} \subset A$. Show the following:

- (i) the intersection of two thickly syndetic sets is thickly syndetic;
- (ii) let T be an operator and $U \subset X$ a nonempty open set; if, for any 0-neighbourhood W , $N(U, W)$ is syndetic then these sets are all thickly syndetic; and similarly for $N(W, U)$.

Deduce that, for any topologically ergodic operator T , $N(U, V)$ is thickly syndetic for any nonempty open sets $U, V \subset X$; see Exercise 1.5.6 for the definition of topological ergodicity.

Exercise 2.5.5. If $S : X \rightarrow X$ and $T : Y \rightarrow Y$ are topologically ergodic operators, show that $S \oplus T$ is topologically ergodic on $X \oplus Y$. In particular, topologically ergodic operators are weakly mixing. (Hint: Use the previous exercise.)

Exercise 2.5.6. An operator T is called *hereditarily ergodic* if, for any pair U, V of nonempty open subsets of X and any syndetic sequence $(n_k)_k$, there exists a subsequence $(n_{k_j})_j$ of $(n_k)_k$ that is also syndetic such that $T^{n_{k_j}}(U) \cap V \neq \emptyset$ for all $j \geq 1$. Show that an operator is hereditarily ergodic if and only if it is topologically ergodic. Deduce that every power T^p , $p \geq 1$, of a topologically ergodic operator is topologically ergodic. (Hint: Use Exercise 2.5.4.)

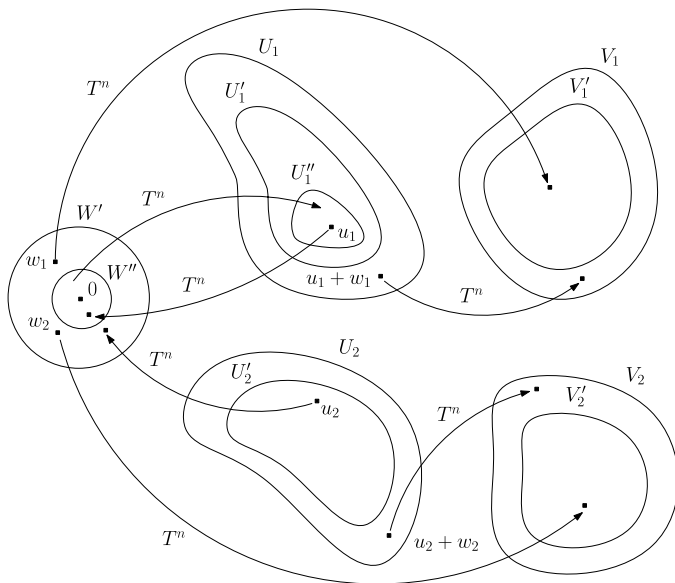


Fig. 2.2 Exercise 2.5.2

Exercise 2.5.7. Prove that the direct sum $S \oplus T$ of two chaotic operators is chaotic. (*Hint:* Use Exercises 1.5.6 and 2.5.5.)

Exercise 2.5.8. Show the following variant of Theorem 2.45: let T be an operator such that, for any nonempty open sets $U, V \subset X$ and any 0-neighbourhood W , there is a continuous map $S : X \rightarrow X$ commuting with T such that $S(U) \cap V \neq \emptyset$, $S(W) \cap W \neq \emptyset$ and

$$N(U, W) \cap N(W, U) \neq \emptyset.$$

Then T is weakly mixing. (*Hint:* Use the 4-set trick.)

Exercise 2.5.9. Let T be an operator such that $\lambda_1 T \oplus \lambda_2 T$ is hypercyclic (or mixing), where $|\lambda_1| \leq |\lambda_2|$. Show that then λT is weakly mixing (or mixing, respectively) whenever $|\lambda_1| \leq |\lambda| \leq |\lambda_2|$. (*Hint:* Use Proposition 2.37 and Theorem 2.47.)

Exercise 2.5.10. Let \mathcal{P} denote the set of nonzero polynomials. If T is a hypercyclic operator such that

$$\bigcup_{p \in \mathcal{P}} \ker p(T)$$

is dense in X , then prove that T is weakly mixing. (*Hint:* Given U and W , pick $u \in U$ and $p \in \mathcal{P}$ such that $p(T)u = 0$. Since $p(T)$ has a dense range by Theorem 2.54, there is some $r > 0$ with $p(T)(rW) \cap U \neq \emptyset$. Then define $S = rp(T)$ and apply Theorem 2.45.)

Exercise 2.5.11. An operator T on X is called *upper triangular* if it admits an increasing sequence $(E_n)_n$ of invariant subspaces such that $\dim E_n = n$ for every $n \geq 1$ and X is the closed linear span of the finite-dimensional spaces E_n , $n \geq 1$. If X is a Hilbert space, T is upper triangular if and only if T has an upper triangular matrix with respect to some orthonormal basis of X . Show that every hypercyclic upper triangular operator

is weakly mixing. (*Hint*: Use the previous exercise and a well-known result from linear algebra.)

Exercise 2.6.1. Show constructively, as in Example 2.18, that every vector $x \in \ell^2$ is the sum of two vectors that are hypercyclic for $T = 2B$.

Exercise 2.6.2. Let T be a hypercyclic operator on a complex space. Show that its adjoint T^* has no finite-dimensional invariant subspace.

Exercise 2.6.3. Let S and T be commuting operators on X such that T has dense range. Show that $HC(S)$ is T -invariant.

Exercise 2.6.4. (a) Let us call an operator T (on a real or complex space) *2-hypercyclic* if there are vectors $x, y \in X$ such that

$$\{T^n x + T^m y ; n, m \geq 0\}$$

is dense in X . Show that the adjoint T^* of a 2-hypercyclic operator has no eigenvalues. (*Hint*: Proceed as in the proof of Lemma 2.53(a).)

(b) Now let T be a hypercyclic operator on a real space. Show that its complexification \tilde{T} is 2-hypercyclic, and deduce Lemma 2.53(b).

(c) Show that there are 2-hypercyclic operators that are not hypercyclic. (*Hint*: Consider $T = T_1 \oplus T_2$, where T_1 and T_2 are hypercyclic but T is not; see Remark 4.17.)

Exercise 2.6.5. Here is an alternative proof of Bourdon's theorem in the real case. Supply the details.

(a) By the Jordan decomposition theorem, any operator on \mathbb{R}^2 has a matrix representation in one of the forms $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ or $c \begin{pmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{pmatrix}$ with $a, b, c, \varphi \in \mathbb{R}$. Therefore no operator on \mathbb{R}^2 can be hypercyclic.

(b) Let T be hypercyclic on X . In view of Lemma 2.53, it remains to prove that every operator $p(T) = T^2 + bT + cI$ has dense range. If this is not the case there is a nonzero $x^* \in X^*$ with $((T^*)^2 + bT^* + cI)x^* = 0$. By Lemma 2.53, x^* and T^*x^* are linearly independent. Then $\phi : X \rightarrow \mathbb{R}^2$, $\phi(x) = (\langle x, x^* \rangle, \langle x, T^*x^* \rangle)$ defines a quasiconjugacy from T to an operator on \mathbb{R}^2 , which is impossible.

Exercise 2.6.6. Let T be a hypercyclic operator on X . Show that $T \oplus I$ is supercyclic on $X \oplus \mathbb{K}$. Show that there is no subspace of $X \oplus \mathbb{K}$ of dimension 2 in which every nonzero vector is supercyclic; in particular, the Herrero–Bourdon theorem fails for supercyclicity.

Exercise 2.7.1. Let T be an operator on \mathbb{K}^N , $N \geq 1$, and let $y \in \mathbb{K}^N$ be a hypercyclic vector for T .

(a) Show that the vectors $y, Ty, T^2y, \dots, T^{N-1}y$ form a basis for \mathbb{K}^N .

(b) Choose $(n_k)_k$ and $(m_k)_k$ such that $T^{n_k}y \rightarrow 0$ and $T^{m_k}y \rightarrow y$. Show that $T^{n_k}x \rightarrow 0$ and $T^{m_k}x \rightarrow x$ for all $x \in \mathbb{K}^N$. Deduce that $(\det T)^{n_k} = \det(T^{n_k}) \rightarrow 0$ and $(\det T)^{m_k} \rightarrow 1$.

(c) Based on (b), give a new proof of Theorem 2.58.

Exercise 2.7.2. Use the Cayley–Hamilton theorem to give a new proof of Theorem 2.58.

Exercise 2.7.3. Use Bourdon's theorem to give a new proof of Proposition 2.60. (*Hint*: In the notation of the stated proof, consider $T^{N+1} - \sum_{k=0}^N \alpha_k T^k$.)

Exercise 2.7.4. Let T be a hypercyclic operator. Show that its adjoint T^* has no finite-dimensional invariant subspace. (This complements Exercise 2.6.2.) (*Hint*: Let $M = \text{span}\{x_1^*, \dots, x_N^*\}$ be T^* -invariant; construct $K = \text{span}\{x_1, \dots, x_N\}$ such that $x_k^*(x_j) = \delta_{j,k}$; show that $\phi(x) = \sum_{k=1}^N x_k^*(x)x_k$ provides a quasiconjugacy between T and $\phi \circ T|_K$ on K .)

Exercise 2.7.5. Show that there are supercyclic operators on \mathbb{R} and on \mathbb{R}^2 . (*Hint:* For \mathbb{R}^2 , use a rotation.)

Exercise 2.8.1. Let X be a Fréchet space. A map $Q : X \rightarrow X$ is called an *m-homogeneous polynomial*, $m \geq 0$, if there is a continuous multilinear map $A : X^m \rightarrow X$ such that $Q(x) = A(x, \dots, x)$, $x \in X$ (where, for $m = 0$, Q is understood to be a constant map). A map $P : X \rightarrow X$ is called a *polynomial* if it can be written as $P = \sum_{m=0}^N Q_m$ with m -homogeneous polynomials Q_m . Show that the map P in Example 2.62 is a polynomial on c_0 .

Exercise 2.8.2. Prove Proposition 2.63.

Sources and comments

Section 2.1. For introductory texts on functional analysis that also cover Fréchet spaces we refer to Rudin [271] and Meise and Vogt [237]. The notion of an F-norm can be found in Kalton, Peck and Roberts [212].

Section 2.2. The term “hypercyclic” for vectors with a dense orbit was apparently first used around 1986 (Beauzamy [46], [47], [48]) and then extended around 1988 to operators with a dense orbit (Bourdon, Godefroy, Shapiro [94], [165]). Supercyclic vectors were introduced by Hilden and Wallen [202] in 1973.

Beauzamy’s work was motivated by the invariant subspace problem. The negative solution for non-Hilbert spaces is due to Enflo [142] and Read [265]. Subsequently, Read [266] even constructed a counterexample to the invariant subset problem.

Theorem 2.64 (Read). *There exists an operator on ℓ^1 all of whose nonzero vectors are hypercyclic.*

Both problems remain open for Hilbert spaces.

The three classical hypercyclic operators were found by Birkhoff [75] in 1929, MacLane [225] in 1952 and Rolewicz [268] in 1969. Example 2.18 reproduces Rolewicz’s original proof; Birkhoff and MacLane used very similar constructions.

The first systematic studies of hypercyclicity are due to Kitai [215] in 1982 and Godefroy and Shapiro [165] in 1991. Though never published, Kitai’s thesis was widely circulated. Between them, Kitai, Godefroy and Shapiro laid the foundations for what was to become the theory of linear dynamical systems: they clarified the basic concepts, provided a wealth of examples and introduced important techniques like criteria for hypercyclicity that will be discussed in the next chapter. It is difficult to overemphasize the importance of their work for the further development of linear dynamics.

Quasiconjugacies were introduced in hypercyclicity by Herrero [195], Martínez and Peris [229]; the hypercyclic comparison principle (see Exercise 2.2.6), was formulated by Shapiro [279]. Complexifications were first studied in hypercyclicity by Bès and Peris [71].

Section 2.3. Godefroy and Shapiro [165] suggested acceptance of Devaney’s definition of chaos for linear operators, and they showed that the three classical operators are chaotic. Chaos in the sense of Auslander and Yorke was introduced for continuous maps on metric spaces in [18]. Proposition 2.30 is due to Godefroy and Shapiro [165]. Proposition 2.33 seems to be folklore, but see Herrero [195] and Bonet, Martínez and Peris [83].

We have taken the proof of Lemma 2.34 from Aron and Markose [15].

Sections 2.4. Proposition 2.37 appears in Grosse-Erdmann and Peris [187].

It does not seem to be easy to find a chaotic operator that is not mixing. The first such operator was constructed by Badea and Grivaux [19].

Sections 2.5. In 1992, Herrero [195] posed the problem of whether every hypercyclic operator (on a Hilbert space) is weakly mixing. De la Rosa and Read [126] constructed a counterexample in a suitable Banach space, while Bayart and Matheron [43] showed that counterexamples exist on many classical Banach spaces like ℓ^p , $1 \leq p < \infty$, c_0 , $C[0, 1]$ and $L^1[0, 1]$, as well as on the Fréchet space $H(\mathbb{C})$.

The results of the section appear in Grosse-Erdmann and Peris [187]; see also Bayart and Matheron [44], [45] and Moothathu [245]. Theorem 2.47 is due to Bernal and Grosse-Erdmann [62] and León [219]; the characterizing condition first appeared in Godefroy and Shapiro [165]. Unlike the proofs in [62] and [219], our argument avoids the Baire category theorem, as does another proof by Yousefi and Rezaei [303].

Theorem 2.48 is due to Grivaux [172]. The fact that every chaotic operator is weakly mixing, Corollary 2.49, is due Bès and Peris [71]; it also follows directly from a result that is due to Bauer and Sigmund [32] and to Stacey (see [28]) using Ansari's theorem (see Section 6.1).

Section 2.6. Proposition 2.52 was observed, for example, by Grosse-Erdmann [177], Godefroy (see [94]) and Kahane (see [249]). Lemma 2.53(a) is due to Kitai [215], while part (b) can be found in Bonet and Peris [85]. Herzog and Lemmert [199] have characterized the hypercyclic operators on $\omega = \mathbb{C}^{\mathbb{N}}$; Conejero [108, p. 123] noted that their condition can be rephrased as $\sigma_p(T^*) = \emptyset$.

Theorem 2.65 (Herzog–Lemmert). *An operator T on $\omega = \mathbb{C}^{\mathbb{N}}$ is hypercyclic if and only if T^* has no eigenvalues.*

In the complex case, Theorem 2.54 is due to Bourdon [90]. The real case was added by Bès [66]; our proof is due to Martínez [227]. Theorem 2.55 is due to Herrero [194] and Bourdon [90].

A considerable improvement of Corollary 2.56 is due to Fathi [146] and Godefroy (see [44, p. 16]). They showed that for any hypercyclic operator T on a Fréchet space X , the set $HC(T)$ of hypercyclic vectors for T is homeomorphic to X .

Section 2.7. Kitai [215] and Rolewicz [268] observed that there are no hypercyclic operators on finite-dimensional spaces. This prompted Rolewicz to pose the problem of whether every infinite-dimensional separable Banach space admits a hypercyclic operator, to which we will turn in Chapter 8. Theorem 2.61 is due to Feldman [147].

The title of this section was inspired by Protopopescu [262], where one finds discussions on the relationship between linear and nonlinear chaos; see also Protopopescu and Azmy [263].

Section 2.8. In this section we follow Peris [255]. For an introduction to the dynamics of complex polynomials we refer to Devaney [132] and Carleson and Gamelin [98].

Exercises. Exercise 2.2.1 can be found in Aron, Seoane and Weber [16]. The result of Exercise 2.2.4 is essentially due to Desch, Schappacher and Webb [131]. Exercise 2.2.9 is taken from Feldman [148], Exercise 2.2.10 from Badea, Grivaux and Müller [21], and Exercise 2.2.11 from Costakis and Manoussos [119]. Exercise 2.3.1 can be found in Feldman [147], Exercise 2.3.2 in deLaubenfels and Emamirad [128], Exercise 2.3.4 in Shapiro [280], and Exercise 2.4.2 in Bermúdez, Bonilla, Conejero, and Peris [49]. Exercises 2.5.1, 2.5.3 and 2.5.8 are extracted from Grosse-Erdmann and Peris [187]. The fact that topologically ergodic operators are weakly mixing (part of Exercise 2.5.5) is implicit in Grosse-Erdmann and Peris [185]. The main part of Exercise 2.5.5 is due to

Desch and Schappacher [129], who introduce the concept of operators satisfying the Recurrent Hypercyclicity Criterion, which is equivalent to topological ergodicity. Hereditarily ergodic operators (see Exercise 2.5.6), were introduced as hereditarily syndetic operators by Badea and Grivaux [19]. Exercise 2.5.9 is taken from Badea, Grivaux and Müller [20], Exercises 2.5.10 and 2.5.11 from Grivaux [172], Exercises 2.6.5 and 2.7.1 from Bès [66], and Exercise 2.6.6 from Bourdon [90].

Extensions. Let us add again some remarks on the setting chosen for this chapter (and for most of the book). Typically, the basic results in linear dynamics either use a Baire category argument, in which case they are often valid in all F-spaces (see below), or they hold in all topological vector spaces (see Chapter 12). Only in more specialized results do structural properties such as being locally convex, being normable or having an inner product play a role. Since our aim has not been to always provide the best possible result but to offer a widely accessible introduction to the main ideas of linear dynamics we have chosen to restrict ourselves to the setting of Fréchet spaces.

The larger class of F-spaces consists of all vector spaces that are endowed with an F-norm and that are complete under the induced metric. For example, the spaces ℓ^p with $0 < p < 1$ are F-spaces. One can show that a vector space is an F-space if and only if it carries a complete translation-invariant metric; see [212].

We finish with a citation that is representative of the common, and as we now know erroneous, belief that chaos and nonlinearity go hand in hand.

Chaotic systems not only exhibit sensitive dependence, but two other properties as well: they are *deterministic*, and they are *nonlinear*.

(L.A. Smith, Chaos: A very short introduction, [295, p. 1]; emphasis in the original.)

Linear Chaos

Grosse-Erdmann, K.; Peris Manguillot, A.

2011, XII, 388 p. 28 illus., Softcover

ISBN: 978-1-4471-2169-5