Chapter 2
Multivariate Integral Inequalities Deriving from Sobolev Representations

Here we present very general multivariate tight integral inequalities of Chebyshev–Grüss, Ostrowski types and of comparison of integral means. These rely on the well-known Sobolev integral representation of a function. The inequalities engage ordinary and weak partial derivatives of the involved functions. We give also applications. On the way to prove the main results we obtain important estimates for the averaged Taylor polynomials and remainders of Sobolev integral representations. The exposed results are thoroughly discussed. This chapter relies on [4].

2.1 Introduction

This chapter is greatly motivated by the following theorems:

Theorem A (Chebychev, 1882, [7]). *Let $f, g : [a, b] \to R$ absolutely continuous functions. If $f', g' \in L_\infty([a, b])$, then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right)\right|$$

$$\leq \frac{1}{12}(b-a)^2 \left\|f'\right\|_\infty \left\|g'\right\|_\infty .$$

Theorem B (G. Grüss, 1935, [8]). *Let f, g integrable functions from $[a, b] \to R$, such that $m \leq f(x) \leq M$, $\rho \leq g(x) \leq \sigma$, for all $x \in [a, b]$, where m, M, ρ, $\sigma \in \mathbb{R}$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,g(x)\,\mathrm{d}x - \frac{1}{(b-a)^2}\left(\int_a^b f(x)\,\mathrm{d}x\right)\left(\int_a^b g(x)\,\mathrm{d}x\right)\right|$$

$$\leq \frac{1}{4}(M-m)(\sigma-\rho) .$$

In 1938, A. Ostrowski [9] proved.

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Theorem C. *Let $f : [a,b] \to R$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to R$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - f(x) \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \left\| f' \right\|_\infty ,$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

See also [1–3] for related works that inspired as well this chapter.

In this chapter using the Sobolev-type representation formulae, see Theorems 2.6, 2.8, 2.11 and 2.23, also Corollaries 2.12 and 2.13, we estimate first their remainders and then the involved averaged Taylor polynomials.

Based on these estimates we establish lots of very tight inequalities on $\mathbb{R}^n$, $n \in \mathbb{N}$, of Chebyshev–Grüss type, Ostrowski type and of Comparison of integral means with applications. The results involve ordinary and weak partial derivatives and they go to all possible directions using various norms. All of our machinery comes from the excellent monograph by V. Burenkov, [6].

2.2 Background

Here we follow [6].

For a measurable nonempty set $\Omega \subset \mathbb{R}^n$, $n \in \mathbb{N}$ we shall denote by $L_p^{\mathrm{loc}}(\Omega)$ $(1 \le p \le \infty)$ - the set of functions defined on Ω such that for each compact $K \subset \Omega$, $f \in L_p(K)$.

Definition 2.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, $\alpha \in \mathbb{Z}_+^n$, $\alpha \ne 0$ and $f, g \in L_1^{\mathrm{loc}}(\Omega)$. The function g is a weak derivative of the function f of order α on Ω (briefly $g = D_w^\alpha f$) if $\forall\, \varphi \in C_0^\infty(\Omega)$ (i.e., $\varphi \in C^\infty(\Omega)$ compactly supported in Ω)

$$\int_\Omega f D^\alpha \varphi \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g\varphi\, \mathrm{d}x. \tag{2.1}$$

Definition 2.2. $W_p^l(\Omega)$ $(l \in \mathbb{N}, 1 \le p \le \infty)$ – Sobolev space, which is the Banach space of functions $f \in L_p(\Omega)$ such that $\forall\, \alpha \in \mathbb{Z}_+^n$ where $|\alpha| \le l$ the weak derivatives $D_w^\alpha f$ exist on Ω and $D_w^\alpha f \in L_p(\Omega)$, with the norm

$$\|f\|_{W_p^l(\Omega)} = \sum_{|\alpha|\le l} \left\| D_w^\alpha f \right\|_{L_p(\Omega)} . \tag{2.2}$$

Definition 2.3. For $l \in \mathbb{N}$, we define the Sobolev-type local space $\left(W_1^l\right)^{(\mathrm{loc})}(\Omega) :=$ $\{f : \Omega \to \mathbb{R} : f \in L_{\mathrm{loc}}^1(\Omega)$ and all f-distributional partials of orders $\le l$ belong to $L_{\mathrm{loc}}^1(\Omega)\} = \{f \in L_1^{\mathrm{loc}}(\Omega)$: for each open set G compactly embedded into Ω, $f \in W_1^l(G)\}$.

Definition 2.4. A domain $\Omega \subset \mathbb{R}^n$ is called star-shaped with respect to the point $y \in \Omega$ if $\forall\ x \in \Omega$ the closed interval line segment $[x, y] \subset \Omega$. A domain $\Omega \subset \mathbb{R}^n$ is called star-shaped with respect to an open ball $B \subset \Omega$ if $\forall\ y \in B$ and $\forall\ x \in \Omega$ we have $[x, y] \subset \Omega$.

We call the set

$$V_x = V_{x,B} = \cup_{y \in B} (x, y) = \text{ convex hull of } \{x\} \cup B,$$

a conic body with vertex x constructed on the open ball B (if $x \in B$, then $V_x = B$; if $\overline{B} \subset \Omega$ and $x \in \overline{B}$, then $V_x = B$).

In fact V_x for otherwise is the region consisting of B and the part of the cone with vertex at x, tangent to the sphere of B, which lies between x and B.

Next comes the multidimensional Taylor's formula.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the point $x_0 \in \Omega$, $l \in \mathbb{N}$ and $f \in C^l(\Omega)$. Then $\forall\ x \in \Omega$*

$$f(x) = \sum_{|\alpha|<l} \frac{(D^\alpha f)(x_0)}{\alpha!}(x - x_0)^\alpha + l \sum_{|\alpha|=l} \times \frac{(x - x_0)^\alpha}{\alpha!} \int_0^1 (1-t)^{l-1} (D^\alpha f)(x_0 + t(x - x_0))\, dt \tag{2.3}$$

(here we mean $x_0 + t(x - x_0) = (x_{01} + t(x_1 - x_{01}), \ldots, x_{0n} + t(x_n - x_{0n}))$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$, $|\alpha| = \sum_{i=1}^n \alpha_i$, $\alpha! = \alpha_1! \ldots \alpha_n!$, $(x - x_0)^\alpha = (x_1 - x_{01})^{\alpha_1} \ldots (x_n - x_{0n})^{\alpha_n}$). Here $|\cdot|$ stands for the Euclidean norm: $|x| = \sqrt{\sum_{i=1}^n x_i^2}$, $x := (x_1, \ldots, x_n)$.

Next we mention the Sobolev representations.

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in L_1(\mathbb{R}^n)$, the support supp $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\, dx = 1$, $l \in \mathbb{N}$ and $f \in C^l(\Omega)$. Then for every $x \in \Omega$*

$$f(x) = \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_B (D^\alpha f)(y)(x - y)^\alpha \omega(y)\, dy + l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B (x - y)^\alpha \omega(y) \times \left(\int_0^1 (1-t)^{l-1} (D^\alpha f)(y + t(x - y))\, dt \right) dy. \tag{2.4}$$

Proof. We write (2.3) for $x, x_0 = y$, multiply it both sides by $\omega(y)$ and integrate on B with respect to y. ■

Call $\|D^\alpha f\|_{\infty,l,B}^{\max} := \max_{|\alpha|=l} \left\{ \|D^\alpha f\|_{\infty,B} \right\}$, where $\|\cdot\|_{\infty,B}$ is the supremum norm on B, $d :=$ diameter of B.

Proposition 2.7. *Same assumption as in Theorem 2.6, $x \in B$. Then*

$$R_1 := |\textit{Remainder (4)}| \leq \frac{(nd)^l \, \|\omega\|_{L_1(\mathbb{R}^n)} \, \|D^\alpha f\|_{\infty,l,B}^{\max}}{l!}. \tag{2.5}$$

Proof. We have that

$$\left| l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B (x-y)^\alpha \, \omega(y) \left(\int_0^1 (1-t)^{l-1} (D^\alpha f)(y + t(x-y)) \, dt \right) dy \right|$$

$$\leq l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B |(x-y)^\alpha| \cdot |\omega(y)| \left(\int_0^1 (1-t)^{l-1} |(D^\alpha f)(y+t(x-y))| \, dt \right) dy$$

$$\leq \|D^\alpha f\|_{\infty,l,B}^{\max} \left(\sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B |(x-y)^\alpha| \cdot |\omega(y)| \, dy \right)$$

$$\leq \|D^\alpha f\|_{\infty,l,B}^{\max} \cdot d^l \cdot \left(\sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B |\omega(y)| \, dy \right)$$

$$= \frac{\|D^\alpha f\|_{\infty,l,B}^{\max} \cdot d^l \cdot \|\omega\|_{L_1(B)}}{l!} \left(\sum_{|\alpha|=l} \frac{l!}{\alpha!} \right)$$

$$= \frac{\|D^\alpha f\|_{\infty,l,B}^{\max} \cdot (d \cdot n)^l \, \|\omega\|_{L_1(B)}}{l!}.$$

■

From [6], p. 104, we mention

Theorem 2.8. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$,*

$$\omega \in L_1(\mathbb{R}^n), \quad \textit{supp}\, \omega \subset \overline{B}, \quad \int_{\mathbb{R}^n} \omega(x) \, dx = 1, \tag{2.6}$$

$l \in \mathbb{N}$ and $f \in C^l(\Omega)$. Then for every $x \in \Omega$

$$f(x) = \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_B (D^\alpha f)(y) (x-y)^\alpha \, \omega(y) \, dy$$

$$+ \sum_{|\alpha|=l} \int_{V_x} \frac{(D^\alpha f)(y)}{|x-y|^{n-l}} w_\alpha(x,y) \, dy, \tag{2.7}$$

where for $x, y \in \mathbb{R}^n$, $x \neq y$,

$$w_\alpha(x, y) := \frac{|\alpha|}{\alpha!} \frac{(x-y)^\alpha}{|x-y|^{|\alpha|}} w(x, y), \tag{2.8}$$

and

$$w(x, y) := \int_{|x-y|}^{\infty} \omega\left(x + \rho \frac{y-x}{|y-x|}\right) \rho^{n-1} d\rho, \tag{2.9}$$

(for $x = y \in \Omega$ *we define* $w_\alpha(x, x) = w(x, x) = 0$*).*

Remark 2.9. By (2.4) and (2.7) we derive

$$l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B (x-y)^\alpha \omega(y) \left(\int_0^1 (1-t)^{l-1} (D^\alpha f)(y + t(x-y)) \, \mathrm{d}t \right) \mathrm{d}y$$

$$= \sum_{|\alpha|=l} \int_{V_x} \frac{(D^\alpha f)(y)}{|x-y|^{n-l}} w_\alpha(x, y) \, \mathrm{d}y. \tag{2.10}$$

By Proposition 2.7 we obtain

$$\left| \sum_{|\alpha|=l} \int_{V_x} \frac{(D^\alpha f)(y)}{|x-y|^{n-l}} w_\alpha(x, y) \, \mathrm{d}y \right| \leq \frac{(nd)^l \|\omega\|_{L_1(\mathbb{R}^n)} \|D^\alpha f\|_{\infty,l}^{\max}}{l!}. \tag{2.11}$$

Remark 2.10. Let D = diameter of Ω be finite, i.e., Ω is bounded. By [6] we have

$$\|w(x, y)\|_{C(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\omega\|_{L_\infty(\mathbb{R}^n)} D^{n-1} d, \tag{2.12}$$

and $\forall\ \alpha \in \mathbb{Z}_+^n$ satisfying $|\alpha| = l$

$$\|w_\alpha(x, y)\|_{C(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\omega\|_{L_\infty(\mathbb{R}^n)} n D^{n-1} d. \tag{2.13}$$

Notice $\|w(x, y)\|_{C(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\omega\|_{L_\infty(\mathbb{R}^n)} d^n$ and $\|w_\alpha(x, y)\|_{C(\mathbb{R}^n \times \mathbb{R}^n)} \leq \|\omega\|_{L_\infty(\mathbb{R}^n)} n d^n$, if $\Omega = B$. Hence, if ω is bounded, then for bounded Ω the functions w and w_α are bounded on $\mathbb{R}^n \times \mathbb{R}^n$. Also by [6], if Ω is unbounded, then w, w_α are bounded on $K \times \mathbb{R}^n$ for each compact K.

If $\omega \in C^\infty(\mathbb{R}^n)$, then $w(x, y)$, $w_\alpha(x, y)$ have continuous derivatives of all orders $\forall\ x, y \in \mathbb{R}^n : x \neq y$ and at the points (x, x), where $x \notin \overline{B}$ they are discontinuous, see [6].

Finally, we give the very general Sobolev representation, see [6].

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$,*

$$\omega \in L_\infty(\mathbb{R}^n), \quad supp\, \omega \subset \overline{B}, \quad \int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1, \tag{2.14}$$

$l \in \mathbb{N}$ and $f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. Then for almost every $x \in \Omega$

$$f(x) = \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_B \left(D_w^\alpha f\right)(y)(x-y)^\alpha \omega(y)\,\mathrm{d}y$$
$$+ \sum_{|\alpha|=l} \int_{V_x} \frac{\left(D_w^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y. \tag{2.15}$$

Corollary 2.12 ([6]). *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$,*

$$\omega \in C_0^\infty(\Omega), \quad supp\, \omega \subset \overline{B}, \quad \int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1. \tag{2.16}$$

Then $\forall\ f \in C^l(\Omega)$ for every $x \in \Omega$ and $\forall\ f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$ for almost every $x \in \Omega$

$$f(x) = \int_B \left(\sum_{|\alpha|<l} \frac{(-1)^{|\alpha|}}{\alpha!} D_y^\alpha \left[(x-y)^\alpha \omega(y)\right]\right) f(y)\,\mathrm{d}y$$
$$+ \sum_{|\alpha|=l} \int_{V_x} \frac{(D^\alpha f)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y \tag{2.17}$$

with $D_w^\alpha f$ replacing $D^\alpha f$ in the case of $f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$.

Next $\alpha \geq \beta$ means $\alpha_i \geq \beta_i$, $i = 1, ..., n$ and $\alpha - \beta \in \mathbb{Z}_+$.

Corollary 2.13 ([6]). *Under the assumptions of Corollary 2.12, let $\beta \in \mathbb{Z}_+^n$ and $0 < |\beta| < l$. Then $\forall\ f \in C^l(\Omega)$ for every $x \in \Omega$ and $\forall\ f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$ for almost every $x \in \Omega$*

$$\left(D^\beta f\right)(x) = \int_B \left(\sum_{|\alpha|<l-|\beta|} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha!} D_y^{\alpha+\beta} \left[(x-y)^\alpha \omega(y)\right]\right) f(y)\,\mathrm{d}y$$
$$+ \sum_{|\alpha|=l, \alpha\geq\beta} \int_{V_x} \frac{(D^\alpha f)(y)}{|x-y|^{n-l+|\beta|}} w_{\alpha-\beta}(x,y)\,\mathrm{d}y \tag{2.18}$$

with $D_w^\beta f$ replacing $D^\beta f$ and $D_w^\alpha f$ replacing $D^\alpha f$ if $f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$.

Remark 2.14. Again $d = diamB$, $D = diam\Omega$. We suppose $\|\omega\|_{L_\infty(\mathbb{R}^n)} < \infty$. Here $\overline{D}^\alpha f$ could mean either $D^\alpha f$ or $D_w^\alpha f$. Then

$$\begin{aligned}
R_2 &:= \left| \sum_{|\alpha|=l} \int_{V_x} \frac{\left(\overline{D}^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y \right| \\
&\leq \sum_{|\alpha|=l} \int_{V_x} |x-y|^{l-n} \left| \left(\overline{D}^\alpha f\right)(y) \right| |w_a(x,y)|\,\mathrm{d}y \\
&\leq \left(\sum_{|\alpha|=l} \int_{V_x} |x-y|^{l-n} \left| \left(\overline{D}^\alpha f\right)(y) \right| \mathrm{d}y \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}\, n d D^{n-1},
\end{aligned} \tag{2.19}$$

if $D < \infty$.

Next we assume $x \in B$, $l \geq n$, then we retake $\Omega = B$, i.e., $d = D$, etc., and thus

$$\begin{aligned}
R_2 &\leq \sum_{|\alpha|=l} \left(\int_B \left| \left(\overline{D}^\alpha f\right)(y) \right| \mathrm{d}y \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}\, n d^l \\
&= \left(\sum_{|\alpha|=l} \left\| \overline{D}^\alpha f \right\|_{L_1(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}\, n d^l .
\end{aligned}$$

That is, we proved for $l \geq n$, $x \in B$, that

$$\left| \sum_{|\alpha|=l} \int_B \frac{\left(\overline{D}^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y \right| \leq \left(\sum_{|\alpha|=l} \left\| \overline{D}^\alpha f \right\|_{L_1(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}\, n d^l . \tag{2.20}$$

Again we assume $x \in B$, $l \geq n$ and $\left\| \overline{D}^\alpha f \right\|_{L_\infty(B)} < \infty$ for all $\alpha : |\alpha| = l$ (which is true for $D^\alpha f$ always by $f \in C^l(\Omega)$). Then

$$R_2 \leq \left(\sum_{|\alpha|=l} \left\| \overline{D}^\alpha f \right\|_{L_\infty(B)} \right) Vol(B)\, \|\omega\|_{L_\infty(\mathbb{R}^n)}\, n d^l =: (*) .$$

We know that

$$\mathrm{Vol}(B) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} r^n = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \frac{d^n}{2^n},$$

where Γ is the gamma function.

Therefore

$$(*) = \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \frac{n\pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} d^{l+n}.$$

So we have proved if $l \geq n$, $x \in B$, and $\left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} < \infty$ for all $\alpha : |\alpha| = l$ that

$$\left| \sum_{|\alpha|=l} \int_B \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y \right| \leq \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \frac{n\pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} d^{l+n}. \tag{2.21}$$

We use Lemma (4.3.1), p. 100 of [5].
It follows

Lemma 2.15. *If $f \in L_p(\Omega^*)$, $1 < p < \infty$, Ω^* is a region of diameter $d_* > 0$, and $m > \frac{n}{p}$, then*

$$\int_\Omega |x-z|^{m-n} |f(z)|\,\mathrm{d}z \leq c_p d_*^{m-\frac{n}{p}} \|f\|_{L_p(\Omega^*)}, \tag{2.22}$$

$\forall\, x \in \Omega^*$, where c_p is a constant depending only on p.

We make

Remark 2.16 (continuing from Remark 2.14). We assume now that $\overline{D}^{\alpha} f \in L_p(B)$, $|\alpha| = l$, $1 < p < \infty$, $l > \frac{n}{p}$. Then by (2.22),

$$\int_B |x-y|^{l-n} \left| \left(\overline{D}^{\alpha} f\right)(y) \right| \mathrm{d}y \leq c_p d^{l-\frac{n}{p}} \left\| \overline{D}^{\alpha} f \right\|_{L_p(B)}, \quad x \in B. \tag{2.23}$$

Consequently, we derive

$$\left| \sum_{|\alpha|=l} \int_B \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y \right| \leq \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_p(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)} n c_p d^{l-\frac{n}{p}+n}, \quad x \in B. \tag{2.24}$$

Also we make

Remark 2.17. Again here $d = diamB$, assume $\omega \in C_0^{\infty}(\Omega)$, supp $\omega \subset \overline{B}$, i.e., $\|\omega\|_{\infty} < \infty$, $\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$. Here $\overline{D}^{\alpha} f$ is either $D^{\alpha} f$ or $D_w^{\alpha} f$. Then for $l \geq n + |\beta|$, $x \in B$, we obtain

$$\left| \sum_{|\alpha|=l, \alpha \geq \beta} \int_B \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l+|\beta|}} w_{\alpha-\beta}(x,y)\,\mathrm{d}y \right|$$

$$\leq \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_1(B)} \right) \|\omega\|_{\infty}\, n d^{l-|\beta|}. \tag{2.25}$$

Also for $l \geq n + |\beta|$, $\left\| \overline{D}^{\alpha} f \right\|_{L_{\infty}(B)} < \infty$, $|\alpha| = l$, $\alpha \geq \beta$, $x \in B$, we get

$$\left| \sum_{|\alpha|=l, \alpha \geq \beta} \int_B \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l+|\beta|}} w_{\alpha-\beta}(x,y)\,\mathrm{d}y \right|$$

$$\leq \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_{\infty}(B)} \right) \|\omega\|_{\infty} \frac{n\pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} d^{l-|\beta|+n}. \tag{2.26}$$

Next, suppose $\overline{D}^{\alpha} f \in L_p(B)$, all $\alpha : |\alpha| = l$, $\alpha \geq \beta$, $1 < p < \infty$, $l > \frac{n}{p} + |\beta|$, $x \in B$. Then

$$\left| \sum_{|\alpha|=l, \alpha \geq \beta} \int_B \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l+|\beta|}} w_{\alpha-\beta}(x,y)\,\mathrm{d}y \right|$$

$$\leq \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_p(B)} \right) \|\omega\|_{\infty}\, n c_p d^{l-|\beta|-\frac{n}{p}+n}. \tag{2.27}$$

We make

Remark 2.18. Here $\overline{D}^{\alpha} f$ denotes either $D^{\alpha} f$ or $D_w^{\alpha} f$, $d = diamB$. Suppose $\|\omega\|_{L_{\infty}(\mathbb{R}^n)} < \infty$. Denote by

$$Q^{l-1} f(x) := \sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \int_B \left(\overline{D}^{\alpha} f\right)(y)(x-y)^{\alpha} \omega(y)\,\mathrm{d}y, \quad \forall\, x \in \Omega, \tag{2.28}$$

the quasi-averaged Taylor polynomial. When $l = 1$, then $Q^0 f(x) := 0$.

In this chapter sums of the form $\sum_{1 \leq |\alpha| \leq 0} . = 0$.

Then for $x \in B$ we obtain

$$\begin{aligned}
\left|Q^{l-1} f(x)\right| &\leq \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{1}{\alpha!} \int_B \left|\left(\overline{D}^{\alpha} f\right)(y)\right| \cdot |(x-y)^{\alpha}| \, \mathrm{d}y\right)\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \\
&\leq \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{1}{\alpha!} \int_B \left|\left(\overline{D}^{\alpha} f\right)(y)\right| \cdot |x-y|^{|\alpha|} \, \mathrm{d}y\right)\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \\
&=: (**)\,. \qquad (2.29)
\end{aligned}$$

We notice that

$$\begin{aligned}
(**) &\leq \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left(\int_B \left|\left(\overline{D}^{\alpha} f\right)(y)\right| \mathrm{d}y\right)\right)\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \\
&= \left\{\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|\overline{D}^{\alpha} f\right\|_{L_1(B)}\right)\right\} \|\omega\|_{L_\infty(\mathbb{R}^n)}\,.
\end{aligned}$$

So we have proved for $x \in B$ that

$$\left|Q^{l-1} f(x)\right| \leq \left\{\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|\overline{D}^{\alpha} f\right\|_{L_1(B)}\right)\right\} \|\omega\|_{L_\infty(\mathbb{R}^n)}\,. \qquad (2.30)$$

Also, when $\left\|\overline{D}^{\alpha} f\right\|_{L_\infty(B)} < \infty$, for all $\alpha : 1 \leq |\alpha| \leq l-1$, we get

$$\begin{aligned}
&\left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left(\int_B \left|\left(\overline{D}^{\alpha} f\right)(y)\right| \mathrm{d}y\right)\right)\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \\
&\quad \leq \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|\overline{D}^{\alpha} f\right\|_{L_\infty(B)}\right)\right) Vol(B) \, \|\omega\|_{L_\infty(\mathbb{R}^n)} \\
&\quad = \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|\overline{D}^{\alpha} f\right\|_{L_\infty(B)}\right)\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}+1\right)} \frac{d^n}{2^n}.
\end{aligned}$$

So we proved, when $\left\|\overline{D}^{\alpha} f\right\|_{L_\infty(B)} < \infty$, for all $\alpha : 1 \leq |\alpha| \leq l-1$, $x \in B$, that

$$\left|Q^{l-1} f(x)\right| \leq \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{(n+|\alpha|)}}{\alpha!} \left\|\overline{D}^{\alpha} f\right\|_{L_\infty(B)}\right)\right) \frac{\|\omega\|_{L_\infty(\mathbb{R}^n)} \cdot \pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)}. \qquad (2.31)$$

We need

Lemma 2.19. *Let Ω^* be a region of $\mathbb{R}^n$ of finite diameter $d_* > 0$ and $f \in L_p(\Omega^*)$, $1 < p, q < \infty : \frac{1}{p} + \frac{1}{q} = 1$ and $m \in \mathbb{N}$, then*

$$\int_{\Omega^*} |x - z|^m |f(z)| \, \mathrm{d}z \le c_{q,m,n} d_*^{\left(m + \frac{n}{q}\right)} \|f\|_{L_p(\Omega^*)}, \quad \forall\, x \in \Omega^*. \tag{2.32}$$

Proof. We see that

$$\int_{\Omega^*} |x - z|^m |f(z)| \, \mathrm{d}z \le \left(\int_{\Omega^*} |x - z|^{mq} \, \mathrm{d}z \right)^{\frac{1}{q}} \|f\|_{L_p(\Omega^*)}$$

(using polar coordinates)

$$\le C_n \left(\int_0^{d_*} r^{mq+n-1} dr \right)^{\frac{1}{q}} \|f\|_{L_p(\Omega^*)}$$

$$= c_{q,m,n} d_*^{\frac{mq+n}{q}} \|f\|_{L_p(\Omega^*)} = c_{q,m,n} d_*^{\left(m + \frac{n}{q}\right)} \|f\|_{L_p(\Omega^*)}.$$

∎

Remark 2.20 (continuing from Remark 2.18). Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. Assume $\left(\overline{D}^\alpha f\right) \in L_p(B)$, for all $1 \le |\alpha| \le l - 1$, $x \in B$, then by Lemma 2.19 we obtain

$$\left| Q^{l-1} f(x) \right| \le \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{1}{\alpha!} \int_B \left| \left(\overline{D}^\alpha f\right)(y) \right| \cdot |x - y|^{|\alpha|} \, \mathrm{d}y \right) \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}$$

$$\le \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{1}{\alpha!} c_{q,|\alpha|,n} d^{\left(|\alpha| + \frac{n}{q}\right)} \left\| \overline{D}^\alpha f \right\|_{L_p(B)} \right) \right) \|\omega\|_{L_\infty(\mathbb{R}^n)}.$$

That is

$$\left| Q^{l-1} f(x) \right| \le c_{q,l,n} \|\omega\|_{L_\infty(\mathbb{R}^n)} \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{\left\| \overline{D}^\alpha f \right\|_{L_p(B)} \cdot d^{\left(|\alpha| + \frac{n}{q}\right)}}{\alpha!} \right) \right),$$

$$x \in B. \tag{2.33}$$

Remark 2.21. For $x \in B$, we consider here $\omega \in C_0^\infty(\Omega)$, $\operatorname{supp} \omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x) \, \mathrm{d}x = 1$; $f \in C^l(\Omega)$ or $f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. Here $\overline{D}^\alpha$ denotes any of D^α, D_w^α. We also consider

$$
\begin{aligned}
Q^{l-1} f(x) &= \int_B \left(\sum_{1 \le |\alpha| \le l-1} \frac{(-1)^{|\alpha|}}{\alpha!} D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right) f(y)\, \mathrm{d}y \\
&= \sum_{1 \le |\alpha| \le l-1} \frac{(-1)^{|\alpha|}}{\alpha!} \int_B D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] f(y)\, \mathrm{d}y. \qquad (2.34)
\end{aligned}
$$

Hence

$$
\left| Q^{l-1} f(x) \right| \le \sum_{1 \le |\alpha| \le l-1} \frac{1}{\alpha!} \int_B \left| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right| \cdot |f(y)|\, \mathrm{d}y. \qquad (2.35)
$$

We derive $\forall\, x \in B$,

$$
\left| Q^{l-1} f(x) \right| \le \begin{cases}
\left(\sum_{1 \le |\alpha| \le l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{\infty} \right) \|f\|_{L_1(B)}, \\
\left(\sum_{1 \le |\alpha| \le l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{L_1(B)} \right) \|f\|_{L_\infty(B)}, \\
\qquad \text{if } f \in L_\infty(B), \\
\text{when } p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1, \text{ we have} \\
\left(\sum_{1 \le |\alpha| \le l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{L_q(B)} \right) \|f\|_{L_p(B)}, \\
\qquad \text{if } f \in L_p(B).
\end{cases} \qquad (2.36)
$$

Let $\beta \in \mathbb{Z}_+^n$ and $0 < |\beta| < l$.

We consider here

$$
\begin{aligned}
Q_\beta^{l-1} f(x) &:= \int_B \left(\sum_{1 \le |\alpha| \le l-|\beta|-1} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha!} D_y^{\alpha+\beta} \left[(x-y)^{\alpha} \omega(y) \right] \right) f(y)\, \mathrm{d}y \\
&= \sum_{1 \le |\alpha| \le l-|\beta|-1} \frac{(-1)^{|\alpha|+|\beta|}}{\alpha!} \int_B D_y^{\alpha+\beta} \left[(x-y)^{\alpha} \omega(y) \right] \\
&\quad \times f(y)\, \mathrm{d}y. \qquad (2.37)
\end{aligned}
$$

When $l = |\beta| + 1$, then $Q_\beta^{l-1} f(x) := 0$. Hence

$$
\left| Q_\beta^{l-1} f(x) \right| \le \sum_{1 \le |\alpha| \le l-|\beta|-1} \frac{1}{\alpha!} \int_B \left| D_y^{\alpha+\beta} \left[(x-y)^{\alpha} \omega(y) \right] \right| \cdot |f(y)|\, \mathrm{d}y. \quad (2.38)
$$

We obtain $\forall\ x \in B$,

$$\left|Q_{\beta}^{l-1} f\,(x)\right| \leq \begin{cases} \left(\sum\limits_{1\leq|\alpha|\leq l-|\beta|-1} \frac{1}{\alpha!}\left\|D_{y}^{\alpha+\beta}\left[(x-y)^{\alpha}\,\omega\,(y)\right]\right\|_{\infty}\right)\|f\|_{L_1(B)}, \\ \left(\sum\limits_{1\leq|\alpha|\leq l-|\beta|-1} \frac{1}{\alpha!}\left\|D_{y}^{\alpha+\beta}\left[(x-y)^{\alpha}\,\omega\,(y)\right]\right\|_{L_1(B)}\right)\|f\|_{L_\infty(B)}, \\ \qquad \text{if } f \in L_{\infty}\,(B)\,, \\ \text{when } p, q > 1 : \frac{1}{p}+\frac{1}{q}=1, \text{ we have} \\ \left(\sum\limits_{1\leq|\alpha|\leq l-|\beta|-1} \frac{1}{\alpha!}\left\|D_{y}^{\alpha+\beta}\left[(x-y)^{\alpha}\,\omega\,(y)\right]\right\|_{L_q(B)}\right)\|f\|_{L_p(B)}, \\ \qquad \text{if } f \in L_{p}\,(B)\,. \end{cases} \tag{2.39}$$

The final remark follows.

Remark 2.22. Here $\overline{D}^{\alpha}$ denotes D^{α} or D_{w}^{α}, and $\overline{D}^{\beta}$ means D^{β} or D_{w}^{β}. We rewrite (2.4), (2.7), (2.15), and (2.17). For $x \in \Omega$ we get

$$f\,(x) = \int_{B} f\,(y)\,\omega\,(y)\,\mathrm{d}y + Q^{l-1} f\,(x) + R^{l} f\,(x)\,, \tag{2.40}$$

where

$$R^{l} f\,(x) := \sum_{|\alpha|=l} \int_{V_x} \frac{\overline{D}^{\alpha} f\,(y)}{|x-y|^{n-l}} w_{\alpha}\,(x,y)\,\mathrm{d}y \tag{2.41}$$

is equal to the remainders of ((2.4)and (2.44), respectively).

Also for $x \in \Omega$ we write (2.18) as follows:

$$\left(\overline{D}^{\beta} f\right)(x) = (-1)^{|\beta|} \int_{B} \left(D^{\beta}\omega\right)(y)\, f\,(y)\,\mathrm{d}y + Q_{\beta}^{l-1} f\,(x) + R_{\beta}^{l} f\,(x)\,, \tag{2.42}$$

where

$$R_{\beta}^{l} f\,(x) := \sum_{|\alpha|=l,\alpha\geq\beta} \int_{V_x} \frac{\left(\overline{D}^{\alpha} f\right)(y)}{|x-y|^{n-l+|\beta|}} w_{\alpha-\beta}\,(x,y)\,\mathrm{d}y. \tag{2.43}$$

Additionally we give

Theorem 2.23. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in L_\infty(\mathbb{R}^n)$, supp $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$, $l \in \mathbb{N}$ and $f \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. Then for almost every $x \in \Omega$*

$$\begin{aligned} f(x) = {} & \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_B \left(D_w^\alpha f\right)(y)(x-y)^\alpha \omega(y)\,\mathrm{d}y \\ & + l \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B (x-y)^\alpha \omega(y) \\ & \times \left(\int_0^1 (1-t)^{l-1} \left(D_w^\alpha f\right)(y + t(x-y))\,\mathrm{d}t \right) \mathrm{d}y. \end{aligned} \tag{2.44}$$

Proof. From the assumptions of the theorem, we get for almost every $x \in \Omega$ that

$$\begin{aligned} & f(x) - \sum_{|\alpha|<l} \frac{1}{\alpha!} \int_B \left(D_w^\alpha f\right)(y)(x-y)^\alpha \omega(y)\,\mathrm{d}y \\ & = \sum_{|\alpha|=l} \int_{V_x} \frac{\left(D_w^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y, \end{aligned}$$

implying that $\int_{V_x} \frac{\left(D_w^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y$ is finite for almost every $x \in \Omega$.

From [6], p. 105, (3.41) there, we know that $\forall\ x \in \mathbb{R}^n$

$$\sup p_y w_\alpha(x,y) = \sup p_y w(x,y) \subset \overline{K_x},$$

where K_x is the cone in $\mathbb{R}^n$ related to V_x, see again [6], pp. 93–100.

So acting similarly to [6], p. 107 and working backwards we derive

$$\sum_{|\alpha|=l} \int_{V_x} \frac{\left(D_w^\alpha f\right)(y)}{|x-y|^{n-l}} w_\alpha(x,y)\,\mathrm{d}y = l \sum_{|\alpha|=l} \frac{1}{\alpha!} J_\alpha,$$

where

$$J_\alpha = \int_{\mathbb{R}^n} \left(D_w^\alpha f\right)(z) \frac{(x-z)^\alpha}{|x-z|^n} \left(\int_{|x-z|}^\infty \omega\left(x + \rho \frac{z-x}{|z-x|}\right) \rho^{n-1} d\rho \right) \mathrm{d}z.$$

Replacing ρ by $\frac{|x-z|}{1-t}$, we obtain

$$J_\alpha = \int_{\mathbb{R}^n} \left(D_w^\alpha f\right)(z)(x-z)^\alpha \left(\int_0^1 \omega\left(\frac{z-tx}{1-t}\right) \frac{\mathrm{d}t}{(1-t)^{n+1}} \right) \mathrm{d}z.$$

Next, setting $z = y + t(x-y)$ and noticing $(x-z)^{\alpha} = (1-t)^{l}(x-y)^{\alpha}$ and $\mathrm{d}z = (1-t)^{n}\,\mathrm{d}y$ we find that

$$\begin{aligned}J_{\alpha} &= \int_0^1 (1-t)^{l-1}\left(\int_{\mathbb{R}^n} \left(D_w^{\alpha} f\right)(y+t(x-y))(x-y)^{\alpha}\,\omega(y)\,\mathrm{d}y\right)\mathrm{d}t\\ &= \int_{\mathbb{R}^n} (x-y)^{\alpha}\,\omega(y)\left(\int_0^1 (1-t)^{l-1}\left(D_w^{\alpha} f\right)(y+t(x-y))\,\mathrm{d}t\right)\mathrm{d}y\\ &= \int_B (x-y)^{\alpha}\,\omega(y)\left(\int_0^1 (1-t)^{l-1}\left(D_w^{\alpha} f\right)(y+t(x-y))\,\mathrm{d}t\right)\mathrm{d}y.\end{aligned}$$

We have proved that

$$\begin{aligned}&\sum_{|\alpha|=l}\int_{V_x}\frac{\left(D_w^{\alpha} f\right)(y)}{|x-y|^{n-l}}w_{\alpha}(x,y)\,\mathrm{d}y\\ &= l\sum_{|\alpha|=l}\frac{1}{\alpha!}\int_B (x-y)^{\alpha}\,\omega(y)\left(\int_0^1 (1-t)^{l-1}\left(D_w^{\alpha} f\right)(y+t(x-y))\,\mathrm{d}t\right)\mathrm{d}y,\end{aligned}$$

establishing the claim. ■

Proposition 2.24. *Same assumptions as in Theorem 2.23. Then for almost every $x \in B$ we get*

$$|\mathit{Remainder}\ (2.44)| \le l\,d^{l}\,\|\omega\|_{L_{\infty}(\mathbb{R}^n)}\sum_{|\alpha|=l}\frac{1}{\alpha!}\left\|\left(D_w^{\alpha} f\right)(y+t(x-y))\right\|_{L_1(B\times[0,1])}. \tag{2.45}$$

In (2.45) we assume for all $\alpha : |\alpha| = l$ that

$$\left\|\left(D_w^{\alpha} f\right)(y+t(x-y))\right\|_{L_1(B\times[0,1])} < \infty.$$

Proof. We have that

$$\begin{aligned}&\left|l\sum_{|\alpha|=l}\frac{1}{\alpha!}\int_B (x-y)^{\alpha}\,\omega(y)\left(\int_0^1 (1-t)^{l-1}\left(D_w^{\alpha} f\right)(y+t(x-y))\,\mathrm{d}t\right)\mathrm{d}y\right|\\ &\le l\sum_{|\alpha|=l}\frac{1}{\alpha!}\int_B \left|(x-y)^{\alpha}\right|\cdot|\omega(y)|\\ &\quad\times\left(\int_0^1 (1-t)^{l-1}\left|\left(D_w^{\alpha} f\right)(y+t(x-y))\right|\mathrm{d}t\right)\mathrm{d}y\end{aligned}$$

$$\leq l d^l \|\omega\|_{L_\infty(\mathbb{R}^n)} \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B \left(\int_0^1 (1-t)^{l-1} \left| \left(D_w^\alpha f \right) (y + t(x-y)) \right| \mathrm{d}t \right) \mathrm{d}y$$

$$\leq l d^l \|\omega\|_{L_\infty(\mathbb{R}^n)} \sum_{|\alpha|=l} \frac{1}{\alpha!} \int_B \left(\int_0^1 \left| \left(D_w^\alpha f \right) (y + t(x-y)) \right| \mathrm{d}t \right) \mathrm{d}y$$

$$= l d^l \|\omega\|_{L_\infty(\mathbb{R}^n)} \sum_{|\alpha|=l} \frac{1}{\alpha!} \left\| \left(D_w^\alpha f \right) (y + t(x-y)) \right\|_{L_1(B\times[0,1])},$$

proving the claim. ■

2.3 Main Results

On the way to prove the general Chebyshev–Grüss-type inequalities, we establish the general

Theorem 2.25. *For f, g under the assumptions of any of Theorems 2.6, 2.8, 2.11, 2.23 and Corollary 2.12 we obtain that*

$$\begin{aligned} \Delta(f,g) :=& \left| \int_B \omega(x) f(x) g(x) \mathrm{d}x - \left(\int_B \omega(x) f(x) \mathrm{d}x \right) \left(\int_B \omega(x) g(x) \mathrm{d}x \right) \right| \\ &\leq \frac{1}{2} \Bigg[\Bigg(\int_B |\omega(x)| |g(x)| \left| Q^{l-1} f(x) \right| \mathrm{d}x \\ &\qquad + \int_B |\omega(x)| |f(x)| \left| Q^{l-1} g(x) \right| \mathrm{d}x \Bigg) \\ &\qquad + \Bigg(\int_B |\omega(x)| |g(x)| \left| R^l f(x) \right| \mathrm{d}x \\ &\qquad + \int_B |\omega(x)| |f(x)| \left| R^l g(x) \right| \mathrm{d}x \Bigg) \Bigg]. \end{aligned} \tag{2.46}$$

Proof. For $x \in B$ we have

$$f(x) = \int_B f(y) \omega(y) \mathrm{d}y + Q^{l-1} f(x) + R^l f(x),$$

and

$$g(x) = \int_B g(y) \omega(y) \mathrm{d}y + Q^{l-1} g(x) + R^l g(x).$$

Hence

$$\omega(x) f(x) g(x) = \omega(x) g(x) \int_B f(y) \omega(y) \, \mathrm{d}y + \omega(x) g(x) Q^{l-1} f(x) + \omega(x) g(x) R^l f(x),$$

and

$$\omega(x) f(x) g(x) = \omega(x) f(x) \int_B g(y) \omega(y) \, \mathrm{d}y + \omega(x) f(x) Q^{l-1} g(x) + \omega(x) f(x) R^l g(x).$$

Therefore

$$\begin{aligned}\int_B \omega(x) f(x) g(x) \, \mathrm{d}x = &\left(\int_B \omega(x) g(x) \, \mathrm{d}x\right)\left(\int_B f(y) \omega(y) \, \mathrm{d}y\right) \\ &+ \int_B \omega(x) g(x) Q^{l-1} f(x) \, \mathrm{d}x \\ &+ \int_B \omega(x) g(x) R^l f(x) \, \mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}\int_B \omega(x) f(x) g(x) \, \mathrm{d}x = &\left(\int_B \omega(x) f(x) \, \mathrm{d}x\right)\left(\int_B g(y) \omega(y) \, \mathrm{d}y\right) \\ &+ \int_B \omega(x) f(x) Q^{l-1} g(x) \, \mathrm{d}x \\ &+ \int_B \omega(x) f(x) R^l g(x) \, \mathrm{d}x.\end{aligned}$$

Consequently, there hold

$$\begin{aligned}&\int_B \omega(x) f(x) g(x) \, \mathrm{d}x - \left(\int_B \omega(x) g(x) \, \mathrm{d}x\right)\left(\int_B f(x) \omega(x) \, \mathrm{d}x\right) \\ &= \int_B \omega(x) g(x) Q^{l-1} f(x) \, \mathrm{d}x + \int_B \omega(x) g(x) R^l f(x) \, \mathrm{d}x,\end{aligned}$$

and

$$\begin{aligned}&\int_B \omega(x) f(x) g(x) \, \mathrm{d}x - \left(\int_B \omega(x) f(x) \, \mathrm{d}x\right)\left(\int_B g(x) \omega(x) \, \mathrm{d}x\right) \\ &= \int_B \omega(x) f(x) Q^{l-1} g(x) \, \mathrm{d}x + \int_B \omega(x) f(x) R^l g(x) \, \mathrm{d}x.\end{aligned}$$

Adding the last two equalities we obtain

$$\int_B \omega(x) f(x) g(x) \, dx - \left(\int_B \omega(x) f(x) \, dx \right) \left(\int_B \omega(x) g(x) \, dx \right)$$
$$= \frac{1}{2} \left[\left(\int_B \omega(x) g(x) Q^{l-1} f(x) \, dx + \int_B \omega(x) f(x) Q^{l-1} g(x) \, dx \right) \right.$$
$$\left. + \left(\int_B \omega(x) g(x) R^l f(x) \, dx + \int_B \omega(x) f(x) R^l g(x) \, dx \right) \right],$$

hence proving the claim. ■

We give

Theorem 2.26. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in L_1(\mathbb{R}^n)$, supp $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x) \, dx = 1$, $l \in \mathbb{N}$ and $f, g \in C^l(\Omega)$. Then*

$$\left| \int_B \omega(x) f(x) g(x) \, dx - \left(\int_B \omega(x) f(x) \, dx \right) \left(\int_B \omega(x) g(x) \, dx \right) \right|$$
$$\leq \frac{\|\omega\|_{L_1(\mathbb{R}^n)}^2}{2} \left[\left[\|g\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^\alpha f\|_{\infty,B}}{\alpha!} \right) \right. \right.$$
$$\left. + \|f\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^\alpha g\|_{\infty,B}}{\alpha!} \right) \right]$$
$$\left. + \left[\frac{(nd)^l}{l!} \left(\|g\|_{\infty,B} \|D^\alpha f\|_{\infty,l,B}^{\max} + \|f\|_{\infty,B} \|D^\alpha g\|_{\infty,l,B}^{\max} \right) \right] \right], \tag{2.47}$$

where d is the diameter of B.

When $l = 1$ the sums in (2.47) collapse.

Proof. One also in general obtains ($x \in B$)

$$\left| Q^{l-1} f(x) \right| \leq \sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \int_B \left| \overline{D}^\alpha f(y) \right| \cdot |(x-y)^\alpha| \, |\omega(y)| \, dy$$
$$\leq \sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \left(\int_B |\omega(y)| \, dy \right) d^{|\alpha|} \left\| \overline{D}^\alpha f \right\|_{L_\infty(B)}$$
$$= \sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \|\omega\|_{L_1(B)} d^{|\alpha|} \left\| \overline{D}^\alpha f \right\|_{L_\infty(B)}.$$

So for $\left\| \overline{D}^{\alpha} f \right\|_{L_{\infty}(B)} < \infty$, for all $\alpha : 1 \leq |\alpha| \leq l-1$, we proved

$$\left| Q^{l-1} f(x) \right| \leq \|\omega\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \left\| \overline{D}^{\alpha} f \right\|_{L_{\infty}(B)}}{\alpha!} \right), \tag{2.48}$$

for $x \in B$.

By (2.46), Proposition 2.7 and (2.48) we obtain

$$\begin{aligned} \Delta(f,g) \leq \frac{1}{2} \Bigg[\|\omega\|^2_{L_1(B)} \Bigg[\|g\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^{\alpha} f\|_{\infty,B}}{\alpha!} \right) \\ + \|f\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^{\alpha} g\|_{\infty,B}}{\alpha!} \right) \Bigg] \\ + \|\omega\|_{L_1(B)} \Bigg[\|g\|_{\infty,B} \frac{(nd)^l \|\omega\|_{L_1(\mathbb{R}^n)} \|D^{\alpha} f\|^{\max}_{\infty,l,B}}{l!} \\ + \|f\|_{\infty,B} \frac{(nd)^l \|\omega\|_{L_1(\mathbb{R}^n)} \|D^{\alpha} g\|^{\max}_{\infty,l,B}}{l!} \Bigg] \Bigg] \\ = \frac{\|\omega\|^2_{L_1(\mathbb{R}^n)}}{2} \Bigg[\Bigg[\|g\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^{\alpha} f\|_{\infty,B}}{\alpha!} \right) \\ + \|f\|_{\infty,B} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \|D^{\alpha} g\|_{\infty,B}}{\alpha!} \right) \Bigg] \\ + \left[\frac{(nd)^l}{l!} \left(\|g\|_{\infty,B} \|D^{\alpha} f\|^{\max}_{\infty,l,B} + \|f\|_{\infty,B} \|D^{\alpha} g\|^{\max}_{\infty,l,B} \right) \right] \Bigg], \end{aligned}$$

proving the claim. ■

We present

Theorem 2.27. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in L_{\infty}(\mathbb{R}^n)$, supp $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\, dx = 1$, $l \in \mathbb{N}$ and $f, g \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. Suppose further that $l \geq n$. Then*

$$\left|\int_B \omega(x) f(x) g(x) \, dx - \left(\int_B \omega(x) f(x) \, dx\right)\left(\int_B \omega(x) g(x) \, dx\right)\right|$$
$$\leq \frac{\|\omega\|^2_{L_\infty(\mathbb{R}^n)}}{2} \left[\left\{\|g\|_{L_1(B)} \left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|D^\alpha_w f\right\|_{L_1(B)}\right)\right)\right.\right.$$
$$\left. + \|f\|_{L_1(B)} \left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|D^\alpha_w g\right\|_{L_1(B)}\right)\right)\right\}$$
$$+ (nd)^l \left[\|g\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\|D^\alpha_w f\right\|_{L_1(B)}\right)\right.$$
$$\left.\left. + \|f\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\|D^\alpha_w g\right\|_{L_1(B)}\right)\right]\right]. \quad (2.49)$$

Proof. Here we get by (2.43), (2.30), and (2.20) that

$$\Delta(f,g) \leq \frac{\|\omega\|^2_{L_\infty(\mathbb{R}^n)}}{2} \left[\left\{\|g\|_{L_1(B)} \left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|D^\alpha_w f\right\|_{L_1(B)}\right)\right)\right.\right.$$
$$\left. + \|f\|_{L_1(B)} \left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\|D^\alpha_w g\right\|_{L_1(B)}\right)\right)\right\}$$
$$+ \left\{\left[\|g\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\|D^\alpha_w f\right\|_{L_1(B)}\right)\right.\right.$$
$$\left.\left.\left. + \|f\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\|D^\alpha_w g\right\|_{L_1(B)}\right)\right] nd^l\right\}\right],$$

proving the claim. ∎

Based on (2.46), (2.31), and (2.21) we have

Theorem 2.28. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in L_\infty(\mathbb{R}^n)$, supp $\omega \subset \overline{B}$,*

$\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$, $l \in \mathbb{N}$ *and* $f, g \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. *Furthermore assume* $\left\| D_w^\alpha f \right\|_{L_\infty(B)}$, $\left\| D_w^\alpha g \right\|_{L_\infty(B)} < \infty$ *for all* $\alpha : 1 \le |\alpha| \le l$, $l \ge n$. *Then*

$$\left| \int_B \omega(x) f(x) g(x)\,\mathrm{d}x - \left(\int_B \omega(x) f(x)\,\mathrm{d}x\right)\left(\int_B \omega(x) g(x)\,\mathrm{d}x\right) \right|$$
$$\le \frac{\pi^{\frac{n}{2}} \|\omega\|_{L_\infty(\mathbb{R}^n)}^2}{2^{n+1}\Gamma\left(\frac{n}{2}+1\right)} \left\{ \left[\|g\|_{L_1(B)} \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{d^{n+|\alpha|}}{\alpha!} \left\| D_w^\alpha f \right\|_{L_\infty(B)} \right) \right) \right.\right.$$
$$\left. + \|f\|_{L_1(B)} \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{d^{n+|\alpha|}}{\alpha!} \left\| D_w^\alpha g \right\|_{L_\infty(B)} \right) \right) \right]$$
$$+ \left[\|g\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\| D_w^\alpha f \right\|_{L_\infty(B)} \right) \right.$$
$$\left.\left. + \|f\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\| D_w^\alpha g \right\|_{L_\infty(B)} \right) \right] n d^{l+n} \right\}. \tag{2.50}$$

Based on (2.46), (2.33), and (2.24) we get

Theorem 2.29. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain star-shaped with respect to the open ball* $B = B(x_0, r)$ *such that* $\overline{B} \subset \Omega$, $\omega \in L_\infty(\mathbb{R}^n)$, *supp* $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\,\mathrm{d}x = 1$, $l \in \mathbb{N}$ *and* $f, g \in \left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. *Furthermore suppose* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $l > \frac{n}{p}$, *for all* $\alpha : 1 \le |\alpha| \le l$, $D_w^\alpha f$, $D_w^\alpha g \in L_p(B)$. *Then*

$$\left| \int_B \omega(x) f(x) g(x)\,\mathrm{d}x - \left(\int_B \omega(x) f(x)\,\mathrm{d}x\right)\left(\int_B \omega(x) g(x)\,\mathrm{d}x\right) \right|$$
$$\le \frac{\|\omega\|_{L_\infty(\mathbb{R}^n)}^2}{2} \left\{ \left[\|g\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\| D_w^\alpha f \right\|_{L_p(B)} \right) \right.\right.$$
$$\left. + \|f\|_{L_1(B)} \left(\sum_{|\alpha|=l} \left\| D_w^\alpha g \right\|_{L_p(B)} \right) \right] n c_p d^{l-\frac{n}{p}+n}$$
$$+ c_{q,l,n} \left[\|g\|_{L_1(B)} \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{\left\| D_w^\alpha f \right\|_{L_p(B)} d^{|\alpha|+\frac{n}{q}}}{\alpha!} \right) \right) \right.$$
$$\left.\left. + \|f\|_{L_1(B)} \left(\sum_{1 \le |\alpha| \le l-1} \left(\frac{\left\| D_w^\alpha g \right\|_{L_p(B)} d^{|\alpha|+\frac{n}{q}}}{\alpha!} \right) \right) \right] \right\}. \tag{2.51}$$

Remark 2.30. When $f, g \in C^l(\Omega)$ the Theorems 2.27, 2.28, 2.29 are again valid. In this case we replace D_w^α by D_w in all inequalities (2.49), (2.50), and (2.51).

We give

Theorem 2.31. *Let $\Omega \subset \mathbb{R}^n$ be a domain star-shaped with respect to the open ball $B = B(x_0, r)$ such that $\overline{B} \subset \Omega$, $\omega \in C_0^\infty(\Omega)$, supp $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\, dx = 1$. Let f, g either in $C^l(\Omega)$ or in $\left(W_1^l\right)^{\mathrm{loc}}(\Omega)$. Here $\overline{D}^\alpha f$ denotes either $D^\alpha f$ or $D_w^\alpha f$ and $\Delta(f, g)$ is as in (2.46).*

We have the following cases:

(i) Here $l \geq n$. Then

$$\Delta(f, g) \leq \frac{\|\omega\|_\infty}{2}\left[\left\{2\|g\|_{L_1(B)}\|f\|_{L_1(B)} \times \left(\sum_{1\leq|\alpha|\leq l-1}\frac{1}{\alpha!}\left\|D_y^\alpha\left[(x-y)^\alpha w(y)\right]\right\|_{\infty, B^2}\right)\right\} + \|\omega\|_\infty n d^l \left[\|g\|_{L_1(B)}\left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha f\right\|_{L_1(B)}\right) + \|f\|_{L_1(B)}\left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha g\right\|_{L_1(B)}\right)\right]\right]. \tag{2.52}$$

(ii) Here $l \geq n$; $f, g \in L_\infty(B)$ and $\overline{D}^\alpha f$, $\overline{D}^\alpha g \in L_\infty(B)$ for all $\alpha : |\alpha| = l$. Then

$$\Delta(f, g) \leq \frac{\|\omega\|_\infty}{2}\left\{\left(\|f\|_{L_\infty(B)}\|g\|_{L_1(B)} + \|f\|_{L_1(B)}\|g\|_{L_\infty(B)}\right) \times \left(\sum_{1\leq|\alpha|\leq l-1}\frac{1}{\alpha!}\sup_{x\in B}\left\|D_y^\alpha\left[(x-y)^\alpha w(y)\right]\right\|_{L_1(B)}\right) + \frac{n d^{l+n}\pi^{\frac{n}{2}}\|\omega\|_\infty}{2^n\Gamma\left(\frac{n}{2}+1\right)}\left[\|g\|_{L_1(B)}\left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha f\right\|_{L_\infty(B)}\right) + \|f\|_{L_1(B)}\left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha g\right\|_{L_\infty(B)}\right)\right]\right\}. \tag{2.53}$$

(iii) *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $l > \frac{n}{p}$, *for all* $\alpha : |\alpha| = l$, $\overline{D}^{\alpha} f$, $\overline{D}^{\alpha} g$, $f, g \in L_p(B)$. *Then*

$$\Delta(f,g) \leq \frac{\|\omega\|_\infty}{2} \left[\left(\|g\|_{L_1(B)} \|f\|_{L_p(B)} + \|f\|_{L_1(B)} \|g\|_{L_p(B)} \right) \cdot \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \sup_{x \in B} \left\| D_y^\alpha \left[(x-y)^\alpha w(y) \right] \right\|_{L_q(B)} \right) + \|\omega\|_\infty c_{q,l,n} \left[\|g\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{\|\overline{D}^\alpha f\|_{L_p(B)}}{\alpha!} d^{|\alpha| + \frac{n}{q}} \right) \right) + \|f\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{\|\overline{D}^\alpha g\|_{L_p(B)}}{\alpha!} d^{|\alpha| + \frac{n}{q}} \right) \right) \right] \right]. \quad (2.54)$$

Proof. By use of (2.36) and Theorems 2.27–2.29. ■

We also present

Theorem 2.32. *Let* $\Omega \subset \mathbb{R}^n$ *be a domain star-shaped with respect to the open ball* $B = B(x_0, r)$ *such that* $\overline{B} \subset \Omega$, $\omega \in L_\infty(\mathbb{R}^n)$, *supp* $\omega \subset \overline{B}$, $\int_{\mathbb{R}^n} \omega(x)\,dx = 1$, $l \in \mathbb{N}$ *and* $f, g \in \left(W_1^l\right)^{\text{loc}}(\Omega)$. *Furthermore suppose for all* $\alpha : |\alpha| = l \geq n$ *that* $\left\| \left(D_w^\alpha f\right)(y + t(x-y)) \right\|_{L_1(B \times [0,1])}, \left\| \left(D_w^\alpha g\right)(y + t(x-y)) \right\|_{L_1(B \times [0,1])} < \infty$. *Then*

$$\Delta(f,g) \leq \frac{\|\omega\|^2_{L_\infty(\mathbb{R}^n)}}{2} \cdot \left[\left\{ \|g\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\| D_w^\alpha f \right\|_{L_1(B)} \right) \right) + \|f\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \left(\frac{d^{|\alpha|}}{\alpha!} \left\| D_w^\alpha g \right\|_{L_1(B)} \right) \right) \right\} + l d^l \left[\|g\|_{L_1(B)} \left(\sum_{|\alpha| = l} \frac{1}{\alpha!} \left\| \left(D_w^\alpha f\right)(y + t(x-y)) \right\|_{L_1(B \times [0,1])} \right) + \|f\|_{L_1(B)} \left(\sum_{|\alpha| = l} \frac{1}{\alpha!} \left\| \left(D_w^\alpha g\right)(y + t(x-y)) \right\|_{L_1(B \times [0,1])} \right) \right] \right]. \quad (2.55)$$

Proof. By Theorem 2.27 and Proposition 2.24. ■

Next, we give a series of Ostrowski-type inequalities.

Theorem 2.33. *Let all as in Theorem 2.6. Call again*

$$Q^{l-1}f(x) := \sum_{1\le|\alpha|\le l-1} \frac{1}{\alpha!}\int_B (D^\alpha f)(y)(x-y)^\alpha \omega(y)\,dy.$$

Then for every $x \in B$, we obtain

$$\left| f(x) - \int_B f(y)\,\omega(y)\,dy - Q^{l-1}f(x)\right| \le \frac{(nd)^l \|\omega\|_{L_1(\mathbb{R}^n)} \|D^\alpha f\|_{\infty,l,B}^{\max}}{l!} := A_1. \tag{2.56}$$

Also it holds, by additionally assuming $\|\omega\|_{L_\infty(\mathbb{R}^n)} < \infty$, that

$$\left| f(x) - \int_B f(y)\,\omega(y)\,dy\right| \le \left(\sum_{1\le|\alpha|\le l-1}\left(\frac{d^{n+|\alpha|}}{\alpha!}\|D^\alpha f\|_\infty\right)\right)\frac{\|\omega\|_{L_\infty(\mathbb{R}^n)}\pi^{\frac{n}{2}}}{2^n\Gamma\left(\frac{n}{2}+1\right)} + \frac{(nd)^l \|\omega\|_{L_1(\mathbb{R}^n)} \|D^\alpha f\|_{\infty,l,B}^{\max}}{l!} := B_1. \tag{2.57}$$

Proof. Use of Theorem 2.6, (2.40), (2.5), and (2.31). ■

We continue with

Theorem 2.34. *All as in Theorems 2.8 or 2.11. Assume $l \ge n$, $\|\omega\|_{L_\infty(\mathbb{R}^n)} < \infty$. Then for every $x \in B$ (almost every $x \in B$, respectively) we get*

$$E(f)(x) := \left| f(x) - \int_B f(y)\,\omega(y)\,dy - Q^{l-1}f(x)\right| \le \left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha f\right\|_{L_1(B)}\right)\|\omega\|_{L_\infty(\mathbb{R}^n)} nd^l =: A_2. \tag{2.58}$$

Also it holds

$$\Delta(f)(x) := \left| f(x) - \int_B f(y)\,\omega(y)\,dy\right| \le \left[\left\{\sum_{1\le|\alpha|\le l-1}\left(\frac{d^{|\alpha|}}{\alpha!}\left\|\overline{D}^\alpha f\right\|_{L_1(B)}\right)\right\} + \left(\sum_{|\alpha|=l}\left\|\overline{D}^\alpha f\right\|_{L_1(B)}\right) nd^l\right]\|\omega\|_{L_\infty(\mathbb{R}^n)} =: B_2. \tag{2.59}$$

Proof. Use of Theorems 2.8, 2.11; (2.40), (2.20), (2.28), and (2.30). ■

We give

Theorem 2.35. *All as in Theorems 2.8, 2.11. Suppose $l \geq n$, $\|\omega\|_{L_\infty(\mathbb{R}^n)} < \infty$, and $\left\|\overline{D}^\alpha f\right\|_{L_\infty(B)} < \infty$ for all $\alpha : |\alpha| = l$. Then for every $x \in B$ (almost every $x \in B$, respectively), it holds*

$$E(f)(x) \leq \left(\sum_{|\alpha|=l} \left\|\overline{D}^\alpha f\right\|_{L_\infty(B)}\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} \frac{n\pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} d^{l+n} =: A_3. \quad (2.60)$$

Additionally, assume that $\left\|\overline{D}^\alpha f\right\|_{L_\infty(B)} < \infty$ for all $\alpha : 1 \leq |\alpha| \leq l-1$. It holds

$$\begin{aligned} \Delta(f)(x) \leq & \left[\left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{d^{n+|\alpha|}}{\alpha!} \left\|\overline{D}^\alpha f\right\|_{L_\infty(B)}\right)\right)\right. \\ & \left. + \left(\sum_{|\alpha|=l} \left\|\overline{D}^\alpha f\right\|_{L_\infty(B)}\right) n d^{l+n}\right] \frac{\pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} \|\omega\|_{L_\infty(\mathbb{R}^n)} \\ =: & B_3. \end{aligned} \quad (2.61)$$

Proof. Use of Theorems 2.8, 2.11; (2.40), (2.21), and (2.31). ■

We present

Theorem 2.36. *All as in Theorems 2.8, 2.11. Assume $\|\omega\|_{L_\infty(\mathbb{R}^n)} < \infty$; $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $l > \frac{n}{p}$, $\overline{D}^\alpha f \in L_p(B)$ for $|\alpha| = l$. Then for every $x \in B$ (almost every $x \in B$, respectively), it holds*

$$E(f)(x) \leq \left(\sum_{|\alpha|=l} \left\|\overline{D}^\alpha f\right\|_{L_p(B)}\right) \|\omega\|_{L_\infty(\mathbb{R}^n)} n c_p d^{l-\frac{n}{p}+n} =: A_4. \quad (2.62)$$

Additionally, assume that $\overline{D}^\alpha f \in L_p(B)$, for $1 \leq |\alpha| \leq l-1$. Then

$$\begin{aligned} \Delta(f)(x) \leq & \left[c_{q,l,n} \left(\sum_{1\leq|\alpha|\leq l-1} \left(\frac{\left\|\overline{D}^\alpha f\right\|_{L_p(B)}}{\alpha!} d^{|\alpha|+\frac{n}{q}}\right)\right)\right. \\ & \left. + \left(\sum_{|\alpha|=l} \left\|\overline{D}^\alpha f\right\|_{L_p(B)}\right) n c_p d^{l-\frac{n}{p}+n}\right] \|\omega\|_{L_\infty(\mathbb{R}^n)} \\ =: & B_4. \end{aligned} \quad (2.63)$$

Proof. By Theorems 2.8, 2.11; (2.40), (2.24), and (2.33). ■

Proposition 2.37. *All as in Theorem 2.35. It holds (for every $x \in B$ and almost every $x \in B$, respectively)*

$$\Delta(f)(x) \leq \|\omega\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{d^{|\alpha|} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)}}{\alpha!} \right) + \frac{n d^{l+n} \pi^{\frac{n}{2}} \|\omega\|_{L_\infty(\mathbb{R}^n)}}{2^n \Gamma\left(\frac{n}{2}+1\right)} \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} \right) =: B_5. \tag{2.64}$$

Proof. By Theorem 2.35 and (2.48). ■

We also have

Theorem 2.38. *Here all as in Corollary 2.12. Assume $l \geq n$. Then for every $x \in B$ (almost every $x \in B$, respectively), it holds*

$$\Delta(f)(x) := \left| f(x) - \int_B f(y)\,\omega(y)\,dy \right| \leq \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{\infty} \right) \|f\|_{L_1(B)} + \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_1(B)} \right) \|\omega\|_{L_\infty(\mathbb{R}^n)} n d^l. \tag{2.65}$$

Proof. Based on Corollary 2.12, Theorem 2.34 and (2.36). ■

Theorem 2.39. *Here all as in Corollary 2.12. Suppose $l \geq n$ and $\left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} < \infty$ for all $\alpha : |\alpha| = l$; $f \in L_\infty(B)$. Then for every $x \in B$ (almost every $x \in B$, respectively), we find*

$$\Delta(f)(x) \leq \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{L_1(B)} \right) \|f\|_{L_\infty(B)} + \frac{n d^{l+n} \pi^{\frac{n}{2}} \|\omega\|_{L_\infty(\mathbb{R}^n)}}{2^n \Gamma\left(\frac{n}{2}+1\right)} \left(\sum_{|\alpha|=l} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} \right). \tag{2.66}$$

Proof. Based on Corollary 2.12, Theorem 2.35 and (2.36). ■

Theorem 2.40. *Here all as in Corollary 2.12. Assume* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $l > \frac{n}{p}$, $\overline{D}^{\alpha} f \in L_p(B)$ *for* $|\alpha| = l$ *and* $f \in L_p(B)$. *Then for every* $x \in B$ *(almost every* $x \in B$, *respectively), we derive*

$$\Delta(f)(x) \leq \left(\sum_{1 \leq |\alpha| \leq l-1} \frac{1}{\alpha!} \left\| D_y^{\alpha} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{L_q(B)} \right) \|f\|_{L_p(B)}$$
$$+ n c_p d^{l - \frac{n}{p} + n} \|\omega\|_{L_\infty(\mathbb{R}^n)} \left(\sum_{|\alpha| = l} \left\| \overline{D}^{\alpha} f \right\|_{L_p(B)} \right). \tag{2.67}$$

Proof. Based on Corollary 2.12, Theorem 2.36 and (2.36). ∎

We also give

Theorem 2.41. *Here all as in Corollary 2.13 with* $Q_{\beta}^{l-1} f(x)$ *as in (2.37). Let* $l \geq n + |\beta|$. *Then for every* $x \in B$ *(almost every* $x \in B$, *respectively) it holds*

$$E_{\beta}(f)(x) := \left| \left(\overline{D}^{\beta} f \right)(x) - (-1)^{|\beta|} \int_B \left(D^{\beta} \omega \right)(y) f(y) \, dy - Q_{\beta}^{l-1} f(x) \right|$$
$$\leq \left(\sum_{|\alpha| = l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_1(B)} \right) \|\omega\|_{\infty} n d^{l - |\beta|} =: A_5. \tag{2.68}$$

Also, we derive

$$\Delta_{\beta}(f)(x) := \left| \left(\overline{D}^{\beta} f \right)(x) - (-1)^{|\beta|} \int_B \left(D^{\beta} \omega \right)(y) f(y) \, dy \right|$$
$$\leq \left(\sum_{1 \leq |\alpha| \leq l - |\beta| - 1} \frac{1}{\alpha!} \left\| D_y^{\alpha + \beta} \left[(x-y)^{\alpha} \omega(y) \right] \right\|_{\infty} \right) \|f\|_{L_1(B)}$$
$$+ \left(\sum_{|\alpha| = l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_1(B)} \right) \|\omega\|_{\infty} n d^{l - |\beta|}. \tag{2.69}$$

Proof. By Corollary 2.13, (2.25), (2.39), and (2.42). ∎

We continue with

Theorem 2.42. *Here all as in Corollary 2.13. Assume* $l \geq n + |\beta|$; $\left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} < \infty$, *all* $\alpha : |\alpha| = l, \alpha \geq \beta$. *Then for every* $x \in B$ *(almost every* $x \in B$, *respectively) we find*

$$E_{\beta}(f)(x) \leq \left(\sum_{|\alpha| = l, \alpha \geq \beta} \left\| \overline{D}^{\alpha} f \right\|_{L_\infty(B)} \right) \|\omega\|_{\infty} \frac{n \pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2} + 1 \right)} d^{l - |\beta| + n} =: A_6. \tag{2.70}$$

Additionally, assume $f \in L_\infty(B)$. *It holds*

$$\Delta_\beta(f)(x) \leq \left(\sum_{1 \leq |\alpha| \leq l-|\beta|-1} \frac{1}{\alpha!} \left\| D_y^{\alpha+\beta} \left[(x-y)^\alpha \omega(y) \right] \right\|_{L_1(B)} \right) \|f\|_{L_\infty(B)}$$
$$+ \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^\alpha f \right\|_{L_\infty(B)} \right) \frac{\|\omega\|_\infty n \pi^{\frac{n}{2}}}{2^n \Gamma\left(\frac{n}{2}+1\right)} d^{l-|\beta|+n}. \tag{2.71}$$

Proof. By Corollary 2.13, (2.26), (2.39), and (2.42). ∎

We finish Ostrowski-type inequalities with

Theorem 2.43. *Here all as in Corollary 2.13. Assume* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\overline{D}^\alpha f \in L_p(B)$ *all* $\alpha : |\alpha| = l, \alpha \geq \beta, l > \frac{n}{p} + |\beta|$. *Then for every* $x \in B$ *(almost every* $x \in B$*, respectively), we derive*

$$E_\beta(f)(x) \leq \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^\alpha f \right\|_{L_p(B)} \right) \|\omega\|_\infty n c_p d^{l-|\beta|-\frac{n}{p}+n} =: A_7. \tag{2.72}$$

Additionally assume $f \in L_p(B)$. *It holds*

$$\Delta_\beta(f)(x) \leq \left(\sum_{1 \leq |\alpha| \leq l-|\beta|-1} \frac{1}{\alpha!} \left\| D_y^{\alpha+\beta} \left[(x-y)^\alpha \omega(y) \right] \right\|_{L_q(B)} \right) \|f\|_{L_p(B)}$$
$$+ \left(\sum_{|\alpha|=l, \alpha \geq \beta} \left\| \overline{D}^\alpha f \right\|_{L_p(B)} \right) \|\omega\|_\infty n c_p d^{l-|\beta|-\frac{n}{p}+n}. \tag{2.73}$$

Proof. By Corollary 2.13, (2.27), (2.39), and (2.42). ∎

We make

Remark 2.44. In preparation to present comparison of integral means inequalities we consider the open ball $B_1 = B_1(y_0, r_1) \subseteq B$. We consider also a weight function $\psi \geq 0$ which is Lebesgue integrable on $\mathbb{R}^n$ with $supp\ \psi \subset \overline{B_1} \subset \Omega$, and $\int_{\mathbb{R}^n} \psi(x)\, dx = 1$. Clearly here $\int_{B_1} \psi(x)\, dx = 1$. For example for x$\in B_1$, $\psi(x) := \frac{1}{Vol(B_1)}$, 0 elsewhere, etc.

We will apply the following principle.

In general a constraint of the form $|F(x) - G| \leq \varepsilon$, where F is a function and G, ε real numbers that all make sense, implies that $\left| \int_{\mathbb{R}^n} F(x)\psi(x)\, dx - G \right| \leq \varepsilon$.

Next we give a series of comparison of integral means inequalities based on Ostrowski-type inequalities presented in this chapter. We use Remark 2.44.

Theorem 2.45. *All as in Theorem 2.33. Then*

$$M(f) := \left| \int_{B_1} f(x)\,\psi(x)\,\mathrm{d}x - \int_B f(x)\,\omega(x)\,\mathrm{d}x \right.$$
$$\left. - \int_{B_1} Q^{l-1} f(x)\,\psi(x)\,\mathrm{d}x \right| \leq A_1, \tag{2.74}$$

and

$$m(f) := \left| \int_{B_1} f(x)\,\psi(x)\,\mathrm{d}x - \int_B f(x)\,\omega(x)\,\mathrm{d}x \right| \leq B_1. \tag{2.75}$$

Theorem 2.46. *All as in Theorem 2.34. Then*

$$M(f) \leq A_2, \tag{2.76}$$

and

$$m(f) \leq B_2. \tag{2.77}$$

Theorem 2.47. *All as in Theorem 2.35. Then*

$$M(f) \leq A_3, \tag{2.78}$$

and

$$m(f) \leq B_3. \tag{2.79}$$

Theorem 2.48. *All as in Theorem 2.36. Then*

$$M(f) \leq A_4, \tag{2.80}$$
$$m(f) \leq B_4. \tag{2.81}$$

Theorem 2.49. *All as in Proposition 2.37. Then*

$$m(f) \leq B_5. \tag{2.82}$$

Theorem 2.50. *All as in Theorem 2.41. Then*

$$M^{\beta}(f) := \left| \int_{B_1} \psi(x) \left(\overline{D}^{\beta} f \right)(x)\,\mathrm{d}x - (-1)^{|\beta|} \int_B \left(D^{\beta}\omega \right)(x)\, f(x)\,\mathrm{d}x \right.$$
$$\left. - \int_{B_1} \psi(x)\, Q_{\beta}^{l-1} f(x)\,\mathrm{d}x \right| \leq A_5. \tag{2.83}$$

Theorem 2.51. *All as in Theorem 2.42. Then*

$$M^{\beta}(f) \leq A_6. \tag{2.84}$$

We finish with

Theorem 2.52. *All as in Theorem 2.43. Then*

$$M^{\beta}(f) \leq A_7. \tag{2.85}$$

2.4 Applications

Example 2.53 (see also [5], p. 93). Let $B := \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$, and

$$\varphi(x) := \begin{cases} e^{-\left(1-\left(\frac{|x-x_0|}{\rho}\right)^2\right)^{-1}}, & \text{if } |x - x_0| < \rho, \\ 0, \quad \text{if } |x - x_0| \geq \rho. \end{cases} \tag{2.86}$$

Call $c := \int_{\mathbb{R}^n} \varphi(x)\,\mathrm{d}x > 0$, then $\Phi(x) := \frac{1}{c}\varphi(x) \in C_0^{\infty}(\mathbb{R}^n)$ with $\operatorname{supp}\Phi = \overline{B}$ and $\int_{\mathbb{R}^n} \Phi(x)\,\mathrm{d}x = 1$ and $\max|\Phi| \leq cons\tan t \cdot \rho^{-n}$.

We call Φ a cut-off function.

One for this chapter's results by choosing $\omega(x) = \Phi(x)$ or $\omega(x) = \frac{1}{Vol(B)}$, etc., can give lots of applications.

Here, selectively we give some special cases inequalities. We start with Chebyshev–Grüss-type inequalities.

Corollary 2.54 (to Theorem 2.26). *All assumptions as in Theorem 2.26. Case of $l = 1$. Then*

$$\left|\int_B \omega(x) f(x) g(x)\,\mathrm{d}x - \left(\int_B \omega(x) f(x)\,\mathrm{d}x\right)\left(\int_B \omega(x) g(x)\,\mathrm{d}x\right)\right|$$
$$\leq \frac{nd\,\|\omega\|^2_{L_1(\mathbb{R}^n)}}{2}\left[\|g\|_{\infty,B}\,\|D^{\alpha}f\|^{\max}_{\infty,1,B} + \|f\|_{\infty,B}\,\|D^{\alpha}g\|^{\max}_{\infty,1,B}\right]. \tag{2.87}$$

If $f = g$, then

$$\left|\int_B \omega(x) f^2(x)\,\mathrm{d}x - \left(\int_B \omega(x) f(x)\,\mathrm{d}x\right)^2\right|$$
$$\leq nd\,\|\omega\|^2_{L_1(\mathbb{R}^n)}\,\|f\|_{\infty,B}\,\|D^{\alpha}f\|^{\max}_{\infty,1,B}. \tag{2.88}$$

Corollary 2.55 (to Theorem 2.27). *All assumptions as in Theorem 2.27. Case of $f = g$, $l = n$ and $\omega(x) := \frac{1}{Vol(B)}$, for all $x \in \overline{B}$, $\omega(x) := 0$ on $\mathbb{R}^n - \overline{B}$. Then*

$$\left| \int_B f^2(x)\,dx - \frac{2^n \Gamma\left(\frac{n}{2}+1\right)}{d^n \pi^{\frac{n}{2}}} \left(\int_B f(x)\,dx\right)^2 \right|$$
$$\leq \frac{2^n \Gamma\left(\frac{n}{2}+1\right)}{\pi^{\frac{n}{2}}} \left[\left\{ \|f\|_{L_1(B)} \left(\sum_{1 \leq |\alpha| \leq n-1} \left(\frac{d^{|\alpha|-n}}{\alpha!} \left\| D_w^{\alpha} f \right\|_{L_1(B)} \right) \right) \right\} \right.$$
$$\left. + n \left(\|f\|_{L_1(B)} \left(\sum_{|\alpha|=n} \left\| D_w^{\alpha} f \right\|_{L_1(B)} \right) \right) \right]. \quad (2.89)$$

We continue the Ostrowski-type inequality.

Corollary 2.56 (to Theorem 2.33). *All as in Theorem 2.33. Case of $l = 1$. Then for every $x \in B$ it holds*

$$\left| f(x) - \int_B f(y)\,\omega(y)\,dy \right| \leq nd\, \|\omega\|_{L_1(\mathbb{R}^n)} \left\| D^{\alpha} f \right\|_{\infty,1,B}^{\max} := Z_1. \quad (2.90)$$

We finish chapter with a comparison of means inequality.

Corollary 2.57 (to Corollary 2.56). *All as in Corollary 2.56 and Remark 2.44. Then*

$$\left| \int_{B_1} f(x)\,\psi(x)\,dx - \int_B f(y)\,\omega(y)\,dy \right| \leq Z_1. \quad (2.91)$$

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