Chapter 2
Elementary Facts about Baire and Baire-Type Spaces

Abstract This chapter contains classical results about Baire-type conditions (Baire-like, b-Baire-like, CS-barrelled, s-barrelled) on tvs. We include applications to closed graph theorems and $C(X)$ spaces. We also provide the first proof in book form of a remarkable result of Saxon (extending earlier results of Arias de Reyna and Valdivia), that states that, under Martin's axiom, every lcs containing a dense hyperplane contains a dense non-Baire hyperplane. This part also contains analytic characterizations of certain completely regular Hausdorff spaces X. For example, we show that X is pseudocompact, is Warner bounded, or $C_c(X)$ is a (df)-space if and only if for each sequence $(\mu_n)_n$ in the dual $C_c(X)'$ there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is weakly bounded, strongly bounded, or equicontinuous, respectively. These characterizations help us produce a (df)-space $C_c(X)$ that is not a (DF)-space, solving a basic and long-standing open question.

2.1 Baire spaces and Polish spaces

Let A be a subset of a nonvoid Hausdorff topological space X. We shall say that A is *nowhere dense* (or rare) if its closure $\overline{A}$ has a void interior. Clearly, every subset of a nowhere dense set is nowhere dense.

A is called of *first category* if it is a countable union of nowhere dense subsets of X. Clearly, every subset of a first category set is again of first category. A is said to be of *second category* in X if it is not of first category. If A is of second category and $A \subset B$, then B is of second category.

The classical Baire category theorem states the following.

Theorem 2.1 *If E is either a complete metric or a locally compact Hausdorff space, then the intersection of countably many dense, open subsets of X is dense in E.*

Proof We show only that the intersection of countably many dense open sets in every metric complete space (E, d) is nonvoid. If this were false, then we would have $E = \bigcup_n E_n$, where each E_n is a closed subset with empty interior. Hence there exists x_1 and $0 < \varepsilon_1 < 1$ such that $B(x_1, \varepsilon_1) \subset E \setminus E_1$, where $B(x_1, \varepsilon_1)$ is the open ball at x_1 with radius ε_1. Next there exists $x_2 \in B(x_1, 2^{-1}\varepsilon_1)$ and $0 < \varepsilon_2 < 2^{-1}\varepsilon_1$

J. Kąkol et al., *Descriptive Topology in Selected Topics of Functional Analysis*, Developments in Mathematics 24,
DOI 10.1007/978-1-4614-0529-0_2,

such that $B(x_2, \varepsilon_2) \subset E \setminus E_2$. Continuing this way, one obtains a shrinking sequence of open balls $B(x_n, \varepsilon_n)$ with radius less than 2^{-n} disjoint with E_n. Clearly, $(x_n)_n$ is a Cauchy sequence in (E, d), so it converges to $x \in E \setminus E_n$, $n \in \mathbb{N}$, and we have reached a contradiction. Similarly, one gets that each open subset of (E, d) is of second category. □

This deep theorem is a principal one in analysis and topology, providing many applications for the closed graph theorems and the uniform boundedness theorem.

A topological space X is called a *Baire space* if every nonvoid open subset of X is of second category (equivalently, if the conclusion of the Baire theorem holds). Clearly, every Baire space is of second category. Although there exist topological spaces of second category that are not Baire spaces, we note that all tvs's considered in the sequel are assumed to be real or complex if not specified otherwise.

Proposition 2.1 *If a tvs E is of second category, E is a Baire space.*

Proof Let A be a nonvoid open subset of E. If $x \in A$, then there exists a balanced neighborhood of zero U in E such that $x + U \subset A$. Since $E = \bigcup_n nU$ and E is of second category, there exists $m \in \mathbb{N}$ such that mU is of second category. Then U is of second category, too. This implies that $x + U$ is of second category and A, containing $x + U$, is also of second category. □

We shall also need the following classical fact; see [328, 10.1.26]. For a completely regular Hausdorff space X, by $C_c(X)$ and $C_p(X)$ we denote the space of real-valued continuous functions on X endowed with the compact-open and pointwise topologies, respectively.

Proposition 2.2 *Let X be a paracompact and locally compact topological space. Then $C_c(X)$ is a Baire space.*

Proof Since X is a paracompact locally compact space, X can be represented as the topological direct sum of a disjoint family $\{X_t : t \in T\}$ of locally compact σ-compact spaces X_t, and we have a topological isomorphism of $C_c(X)$ and the product $\prod_{t \in T} C_c(X_t)$. It is known that each space $C_c(X_t)$ is a Fréchet space (i.e., a metrizable and complete lcs) and since products of Fréchet spaces are Baire spaces (see Theorem 14.2 below for an alternative proof), we conclude that $\prod_{t \in T} C_c(X_t)$ is a Baire space. □

The following general fact is a simple consequence of definitions above.

Proposition 2.3 *If E is a tvs and F is a vector subspace of E, then F is either dense or nowhere dense in E. If F is dense in E and F is Baire, then E is Baire.*

Proof Assume that F is not dense in E. Let G be its closure in E, a proper closed subspace of E. If G is not nowhere dense in E, then there exists a balanced neighborhood of zero U in E and a point $x \in G$ such that $x + U \subset G$. Then $E = \bigcup_n nU \subset G$, providing a contradiction. The other part is clear. □

Recall that every *Čech-complete* space E (i.e., E can be represented as a countable intersection of open subsets of a compact space) is a Baire space. Arkhangel'skii proved [31] that if E is a topological group and F is a Čech-complete subspace of E, then either F is nowhere dense in E or E is Čech-complete as well. This, combined with Proposition 2.3, shows that if a tvs E contains a dense Čech-complete vector subspace, then E is Čech-complete.

A subset A of a topological space X is said to have the *Baire property* in X if there exists an open subset U of X such that $U \setminus A$ and $A \setminus U$ are of first category.

Let $D(A)$ be the set of all $x \in X$ such that each neighborhood $U(x)$ of x intersects A in a set of second category. Set $O(A) := \operatorname{int} D(A)$.

Proposition 2.4 *A subset A of a topological space X has the Baire property if and only if $O(A) \setminus A$ is of first category.*

Proof Assume A has the Baire property, and let U be an open set in X such that $U \setminus A$ and $A \setminus U$ are sets of first category. Note that $D(A) \subset \overline{U}$. Indeed, if $x \in D(A) \setminus \overline{U}$, then (by definition) the set

$$(X \setminus \overline{U}) \cap A \ (= A \setminus \overline{U})$$

is of second category. By the assumption, $A \setminus \overline{U} \subset A \setminus U$ is of first category, a contradiction. Since $\overline{U} \setminus U$ is nowhere dense, one concludes that $\overline{U} \setminus A$ is of first category. Finally, since

$$O(A) \setminus A \subset D(A) \setminus A \subset \overline{U} \setminus A,$$

then $O(A) \setminus A$ is of first category as claimed.

Now assume that $O(A) \setminus A$ is a set of first category. It is enough to prove that $A \setminus O(A)$ is of first category. Let $C(A)$ be the union of the family $\mathfrak{L} := \{A_i : i \in I\}$ of all the open subsets of X that intersect A in a set of first category. Note that $O(A) = X \setminus \overline{C(A)}$. We show that

$$A \cap \overline{C(A)} \ (= A \setminus O(A))$$

is of first category. Let $\{A_i : i \in J\}$ be a maximal pairwise disjoint subfamily of $\mathfrak{L}$. Then (as is easily seen) $A \cap (\bigcup_{i \in J} A_i)$ is of first category. Then the set $A \cap \overline{\bigcup_{i \in J} A_i}$ is also of first category. By the maximality condition, we deduce that $\bigcup_{i \in I} A_i \subset \overline{\bigcup_{i \in J} A_i}$, which completes the proof. □

Since $A \cap \overline{C(A)}$ is of the first category, we note the following simple fact; see, for example, [421, p. 4].

Proposition 2.5 *Let E be a topological space, and let B be a subset of E that is the union of a sequence $(U_n)_n$ of subsets of E. Then $D(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is nowhere dense. Therefore $O(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is also nowhere dense.*

Proof Assume that the interior A of the closed set $D(B) \setminus \bigcup\{O(U_n) : n \in \mathbb{N}\}$ is non-void. Then $A \cap B$ is of second category. Hence there exists $m \in \mathbb{N}$ such that $A \cap U_m$ is of second category. Therefore, as $U_m \cap \overline{C(U_m)}$ is of first category, we have $A \not\subset \overline{C(U_m)}$, and hence $A \cap O(U_m)$ is nonvoid, a contradiction. □

Every Borel set in a topological space E has the Baire property. This easily follows from the following well-known fact; see [371].

Proposition 2.6 *Let E be a topological space. The family of all subsets of E with the Baire property forms a σ-algebra.*

Now we are ready to formulate the following useful fact.

Proposition 2.7 *Let U be a subset of a topological vector space E. If U is of second category and has the Baire property, $U - U$ is a neighborhood of zero.*

Proof Since U is of second category, we have $O(U) \neq \emptyset$. If $x \notin U - U$, then clearly $(x + U) \cap U = \emptyset$. The Baire property of U implies that $(x + O(U)) \cap O(U)$ is a set of first category. On the other hand, since nonvoid open subsets of $O(U)$ are of second category, it follows that $(x + O(U)) \cap O(U) = \emptyset$. Then $x \notin O(U) - O(U)$. This proves that $U - U$ contains the neighborhood of zero $O(U) - O(U)$. □

It is clear that the fact above has a corresponding variant for topological groups (called the Philips lemma); see, for example, [346].

In general, even the self-product $X \times X$ of a Baire space X need not be Baire; see [324], [106], [164]. Nevertheless, the product $\prod_{i \in I} X_i$ of metric complete spaces is Baire; see [328]. Also, the product $\prod_{i \in I} X_i$ of any family $\{X_i : i \in I\}$ of separable Baire spaces is a Baire space; see [421].

Arias de Reina [16] proved the following remarkable theorem.

Theorem 2.2 (Arias de Reina) *The Hilbert space $\ell^2(\omega_1)$ contains a family $\{X_t : t < \omega_1\}$ of different Baire subspaces such that for all $t, u < \omega_1$, $t \neq u$, the product $X_t \times X_u$ is not Baire.*

Valdivia [424] generalized this result by proving the same conclusion in each space $c_0(I)$ and $\ell^p(I)$, for uncountable set I, and $0 < p < \infty$. Lemma 1 in [16] has been improved by Drewnowski [127].

Every Polish space is a Baire space. A topological space E is called a *Polish space* if E is separable and if there exists a metric d on E generating the same topology such that (E, d) is complete.

Proposition 2.8 (*i*) *The intersection of any countable family of Polish subspaces of a topological space E is a Polish space.*

(*ii*) *Every open* (*closed*) *subspace V of a Polish space E is a Polish space. Hence a subspace of a Polish space that is a G_δ-set is a Polish space.*

Proof (i) Let $(E_n)_n$ be a sequence of Polish subspaces of E, and let $G := \bigcap_n E_n$. Then the product $\prod_n E_n$ (endowed with the product topology) and the diagonal $\Delta \subset \prod_n E_n$ (as a closed subset) are Polish spaces. Since Δ is homeomorphic to the intersection G, the conclusion follows.

(ii) Let d be a complete metric on E. Let V be open and $V^c := E \setminus V$. Define the function $d(x, V^c)$ by $d(x, V^c) := \inf\{d(x, y) : y \in V^c\}$. Set $\xi(x) := d(x, V^c)^{-1}$ and

$$D(x, y) := |\xi(x) - \xi(y)| + d(x, y)$$

for all $x, y \in V$. It is easy to see that $D(x, y)$ defines a complete metric on V giving the original topology of E restricted to V. Hence V is a Polish space. □

The following characterizes Polish subspaces of a Polish space; see [371].

Proposition 2.9 *A subspace F of a Polish space E is Polish if and only if F is a G_δ-set in E.*

Proof If F is a G_δ-set in E, then F is a Polish space by the previous proposition. To prove the converse, let d (resp. d_1) be a compatible (resp. complete) metric on E (resp. F). For each $x \in F$ and each $n \in \mathbb{N}$, there exists $0 < t_n(x) < n^{-1}$ such that $z \in F$ and $d(x, z) < t_n(x)$ imply $d_1(x, z) < n^{-1}$. Define $U_n(x) := \{z \in E : d(x, z) < t_n(x)\}$, $U_n := \bigcup_{x \in F} U_n(x)$ and $W := \bigcap_n U_n$. Then $W \subset F$. Indeed, if $y \in W$, then there exists a sequence $(x_n)_n$ in F with $d(x_n, y) < t_n(x_n) < n^{-1}$. Therefore $x_n \to y$. Fix $m \in \mathbb{N}$. Since $d(x_m, y) < t_m(x_m)$, there exists $k_m \in \mathbb{N}$ such that $k_m^{-1} + d(x_m, y) < t_m(x_m)$. Then, if $n > k_m$, one gets

$$d(x_n, x_m) \leq d(x_n, y) + d(y, x_m) < n^{-1} + d(y, x_m) < k_m^{-1} + d(x_m, y) < t_m(x_m).$$

By construction, the inequality $d(x_n, x_m) < t_m(x_m)$ yields $d_1(x_n, x_m) < m^{-1}$. Clearly, $(x_n)_n$ is Cauchy in (F, d_1), and from the completeness it follows that $y \in F$. □

Corollary 2.1 *A topological space E is a Polish space if and only if it is homeomorphic to a G_δ-set contained in the compact space* $[0, 1]^{\mathbb{N}}$.

Proof Since the space $[0, 1]^{\mathbb{N}}$ is a metric compact space, it is a Polish space and Proposition 2.8 applies. To get the converse, assume that E is a Polish space. Hence it is separable and metrizable, and consequently E is homeomorphic to a subspace of $[0, 1]^{\mathbb{N}}$. Proposition 2.9 applies to complete the proof. □

2.2 A characterization of Baire topological vector spaces

The following characterization of a Baire tvs is due to Saxon [353]; see also [328, Theorem 1.2.2].

Theorem 2.3 (Saxon) *The following are equivalent for a tvs E.*

(i) *E is a Baire space.*
(ii) *Every absorbing balanced and closed subset of E is a neighborhood of some point.*

The scalar field $\mathbb{K}$ is either the reals or the complexes. After a little preparation motivated by annular regions in the complex plane, we present a single proof that simultaneously solves both the real and complex cases.

Claim 2.1 Let U be a nonempty set in a tvs E. If $U - U \neq E$, then $\bigcap_{n=1}^{\infty} n^{-1}U$ has an empty interior.

Proof Let $x \in E \setminus (U - U)$. If V is a nonempty open set contained in $\bigcap_n n^{-1}U$, then the open neighborhood of zero $V - V$ contains $m^{-1}x$ for some sufficiently large $m \in \mathbb{N}$ and is contained in $n^{-1}(U - U)$ for every $n \in \mathbb{N}$. In particular, $m^{-1}x \in m^{-1}(U - U)$, which implies $x \in U - U$, a contradiction. □

For $0 < r_1 < r_2$, we define the annulus Λ_{r_1,r_2} in $\mathbb{K}$ by writing

$$\Lambda_{r_1,r_2} = \{t \in \mathbb{K} : r_1 \leq |t| \leq r_2\}.$$

For each $\varepsilon > 0$, let β_ε denote the open ball $\{t \in \mathbb{K} : |t| < \varepsilon\}$. For $0 < \varepsilon < 1$, compactness provides a finite subset Γ_ε of $\Lambda_{\varepsilon,1}$ such that

$$\Gamma_\varepsilon + \beta_{\varepsilon^2} \supset \Lambda_{\varepsilon,1}.$$

Claim 2.2 If $0 < \delta < \varepsilon < 1$ and $z + \beta_\varepsilon \subset \Lambda_{\varepsilon,1}$, then $\Gamma_\delta \cdot (z + \beta_\varepsilon) \supset \Lambda_{\delta,\varepsilon}$.

Proof

$$\begin{aligned}\Gamma_\delta \cdot (z + \beta_\varepsilon) &= \bigcup_{t\in\Gamma_\delta} t \cdot (z + \beta_\varepsilon) = \bigcup_{t\in\Gamma_\delta} \left(t \cdot z + \beta_{\varepsilon|t|}\right) \supset \bigcup_{t\in\Gamma_\delta} (t \cdot z + \beta_{\varepsilon\delta}) \\ &= \left(\Gamma_\delta + \beta_{\delta\varepsilon/|z|}\right) \cdot z \supset \left(\Gamma_\delta + \beta_{\delta^2}\right) \cdot z \supset \Lambda_{\delta,1} \cdot z = \Lambda_{\delta|z|,|z|} \supset \Lambda_{\delta,\varepsilon}. \quad □\end{aligned}$$

Claim 2.3 Let $(B_n)_n$ be a sequence of subsets of a tvs E. Fix $0 < r_1 < r_2$ and $y \in E$. If $\Lambda_{r_1,r_2} \cdot y \subset \bigcap_n B_n$, then there exists $\delta > 0$ such that $\{ty : 0 < |t| \leq \delta\} \subset \bigcup_n n^{-1}B_n$.

Proof For each n, we have $\Lambda_{r_1/n,r_2/n} \cdot y \subset n^{-1}B_n$. There is a natural number m such that $r_1/n \leq r_2/(n+1)$ for every $n \geq m$. The claim follows for $\delta = r_2 m^{-1}$. □

Now we are ready to prove Theorem 2.3.

Proof Only the implication (ii) $\Rightarrow$ (i) needs a proof. Indeed, if there exists an absorbing, balanced and nowhere dense set B, then $E = \bigcup_n nB$ is of first category and not Baire. We assume that E is not Baire and construct such a set B.

Since E is of first category, its topology is nontrivial and it contains a closed balanced neighborhood U of zero with $U - U \neq E$. Since U is also of first category in E, there exists a sequence $(A_n)_n$ of closed nowhere dense sets in E whose union is U. With the notation of Claim 2.2, we observe that each set

$$B_n := \bigcup_{j,k \leq n} \Gamma_{1/k} \cdot A_j$$

is closed and nowhere dense, being a finite union of such sets. Furthermore, each B_n is contained in the balanced set U.

We wish to see that $A := \bigcup_n n^{-1} B_n$ is nowhere dense. Suppose some nonempty open set W is contained in $\overline{A}$. Because each finite union $\bigcup_{n<k} n^{-1} B_n$ is nowhere dense, the closure of each $\bigcup_{n \geq k} n^{-1} B_n$ must contain W. But the closure of the kth union is clearly contained in $k^{-1} U$ for each $k \in \mathbb{N}$, so that $W \subset \bigcap_k k^{-1} U$, which contradicts Claim 2.1.

Next, we prove that A absorbs any given point y in E. Since U absorbs y, there exists $r > 0$ such that the set $\beta_r \cdot y$ is contained in U and is thus covered by $(A_n)_n$. We may harmlessly assume that $r \leq 1$. The set $\beta_r \cdot y$ either has the trivial topology or is a topological copy of the subset β_r of $\mathbb{K}$. Either way, $\beta_r \cdot y$ is of second category in itself, and there exist $p \in \mathbb{N}$, $\varepsilon > 0$ and $z \in \beta_r$ such that

$$z + \beta_\varepsilon \subset \beta_r \quad \text{and} \quad (z + \beta_\varepsilon) \cdot y \subset A_p.$$

In fact, rechoosing z and ε if needed, we may additionally insist that $z \neq 0$, and then we may yet again refine the choice of ε in the interval $(0, 1)$ so that

$$z + \beta_\varepsilon \subset \Lambda_{\varepsilon,1} \quad \text{and} \quad (z + \beta_\varepsilon) \cdot y \subset A_p.$$

Let q be a natural number larger than both p and ε^{-1}. For $n \geq q$, we apply Claim 2.2 to obtain

$$B_n \supset \Gamma_{1/q} \cdot A_p \supset \Gamma_{1/q} \cdot (z + \beta_\varepsilon) \cdot y \supset \Lambda_{1/q,\varepsilon} \cdot y.$$

Now Claim 2.3 shows that, for some $\delta > 0$,

$$\{ty : 0 < |t| \leq \delta\} \cdot y \subset \bigcup_{n \geq q} n^{-1} B_n \subset A.$$

Therefore A absorbs y, given that zero is (obviously) in A. The balanced core of A (i.e., the largest balanced set A_0 contained in A) is absorbing and nowhere dense because A has these properties. Therefore $B := \overline{A_0}$ is a closed, balanced, absorbing nowhere dense set in E, as promised. $\square$

Theorem 2.3 provides the following corollary.

Corollary 2.2 *Every Hausdorff quotient of a Baire tvs is a Baire space.*

We also have the following useful corollary.

Corollary 2.3 *If a Baire tvs E is covered by a sequence $(E_n)_n$ of vector subspaces of E, then E_m is dense and Baire for some $m \in \mathbb{N}$.*

Proof By hypothesis there exists $m \in \mathbb{N}$ such that E_m is of second category in E; therefore it cannot be nowhere dense in E. By Proposition 2.3, E_m is dense in E, thus of second category in itself and thus Baire by Proposition 2.1. □

When $\dim(E)$ is infinite, E_m may satisfy $\dim(E/E_m) = \dim(E)$, an extreme.

Proposition 2.10 *Every infinite-dimensional Baire tvs E contains a dense Baire subspace F whose dimension equals the codimension in E.*

Proof By $(x_t)_{t\in T}$ denote a Hamel basis of E. Fix a partition $(T_n)_n$ of T such that $\operatorname{card} T = \operatorname{card} T_n$ for all $n \in \mathbb{N}$. Set $E_n := \operatorname{span}\{x_t : t \in \bigcup_{i=1}^{n} T_i\}$. Then $(E_n)_n$ covers E and $\dim E = \dim E_n = \dim(E/E_n)$ for $n \in \mathbb{N}$. By Corollary 2.3, there exists a dense Baire subspace $F := E_m$ of E, as desired. □

At the other algebraic extreme, hyperplanes of E are also Baire when closed, and those that contain F are dense and Baire.

2.3 Arias de Reyna–Valdivia–Saxon theorem

For many years, the following question remained: *When does an infinite-dimensional Baire tvs E admit a non-Baire (necessarily dense) hyperplane?* In 1966, Wilansky and Klee conjectured: *Never, for E a Banach space*. This conjecture was denied in 1980 by Arias de Reyna [328, Theorem 1.2.12], who proved, under Martin's axiom, the answer: *Always, when E is a separable Banach space*. In 1983, Valdivia [427] proved, under Martin's axiom, the more general answer: *Always, when E is a separable tvs.* In 1987, Pérez Carreras and Bonet [328, Question 13.1.1] repeated the question for E a (not necessarily separable) Banach space.

Finally, in 1991, Saxon [355] provided a complete answer in the general locally convex setting. He proved the following theorem.

Theorem 2.4 (Arias de Reyna–Valdivia–Saxon) *Assume $\mathfrak{c}$-A. Every tvs E with an infinite-dimensional dual contains a non-Baire hyperplane. Consequently,* (1) *every infinite-dimensional lcs admits a non-Baire hyperplane and* (2) *an lcs E admits a dense non-Baire hyperplane iff $E' \neq E^*$.*

For a tvs E, by the dual of E we mean its topological dual E', a linear subspace of its algebraic dual E^*. A subset A of a tvs E is called *bornivorous* if A absorbs every bounded set in E. Recall that an lcs E is *barrelled* (*quasibarrelled*) if every closed absolutely convex and absorbing (and bornivorous) subset of E is a neighborhood of zero of E or, equivalently, if every bounded set in the weak dual $(E', \sigma(E', E))$ (strong dual $(E', \beta(E', E))$) is equicontinuous.

The axiom $\mathfrak{c}$-A ($\mathfrak{c}$-additivity, where $\mathfrak{c} := 2^{\aleph_0}$) proclaims: *The union of less than* $\mathfrak{c}$ *subsets of* $\mathbb{R}$*, each of measure zero, itself has measure zero.*

Note that CH $\Rightarrow$ Martin's axiom $\Rightarrow$ $\mathfrak{c}$-A, and the converse implications fail in general; see [172, Corollary 32(G)(c)].

To prove Theorem 2.4, we will need the following technical fact from [328, Theorem 1.2.11].

Lemma 2.1 *Let* e_1 *and* e_2 *be the canonical unit vectors in the real Euclidean space* $\mathbb{R}^2$ *endowed with its usual inner product* $(.,.)$ *and the corresponding norm* $\|.\|$*. By* $m(x) := \arccos\left((x, e_1)\|x\|^{-1}\right)$*, for* $x \neq 0$*, we denote the angle between* x *and* e_1*. If* $0 < b < 1$*,* q *is a positive integer,* $\|u\|, \|v\| > q^{-1}$ *and* $\|u - v\| < bq^{-1}$*, then* $|m(u) - m(v)| < 2b$*.*

Now we prove Theorem 2.4.

Proof It is enough to prove the initial claim. Clearly, (1) then follows and (2) as well. Indeed, if E is an lcs with $E' \neq E^*$, then E contains a non-Baire hyperplane H by (1). If (a) E is Baire, then all its closed hyperplanes are Baire, and H must be dense. If (b) E is non-Baire, then so are all its hyperplanes, including dense hyperplanes, which exist by the hypothesis $E' \neq E^*$. Conversely, no dense hyperplanes exist if $E' = E^*$.

We prove the real scalar case only. One may then easily dispatch the complex case by a standard procedure. We also assume, without loss of generality, that E is Baire.

The hypothesis implies a biorthogonal sequence $(x_n, h_n)_n \subset E \times E'$, so the map $T : E \longrightarrow \mathbb{R}^{\mathbb{N}}$ defined by $T(x) := (h_n(x))_n$ is continuous and linear. Therefore E admits a quotient $F := E/Q$ of dimension at most $\mathfrak{c}$ isomorphic to a subspace of $\mathbb{R}^{\mathbb{N}}$ containing the canonical unit vectors and endowed with a vector topology finer than the one inherited from the usual product topology. Hence each unit vector e_n and each coordinate functional f_n belong to F and F', respectively. For each $n \in \mathbb{N}$, set $M_n := \operatorname{span}\{e_i : i \leq n\}$. If $(w_{n,k})_k$ is an enumeration of the countable set $\{\sum_{i=1}^n a_i e_i : a_i \in \mathbb{Q},\ a_n \neq 0\}$, then $(w_{n,k})_k$ is dense in M_n for all $n \in \mathbb{N}$. Set

$$U_i := \bigcap_{n=1}^{i} \{x \in F : |f_n(x)| < 2^{-i}\}$$

for each $i \in \mathbb{N}$. Then

$$\bigcap_i U_i = \{0\},\ \ U_{i+1} + U_{i+1} \subset U_i,\ \ i \in \mathbb{N}. \tag{2.1}$$

For each fixed $n \in \mathbb{N}$, we choose a sequence $(\varepsilon_{n,k})_k$ of numbers such that

$$0 < 2\varepsilon_{n,k+1} \leq \varepsilon_{n,k} < 2^{-n} \wedge |f_n(w_{n,k})|.$$

Note that

$$w_{i,k} \notin V_{i,k} := \bigcap_{n=1}^{i} \{x \in F : |f_n(x)| < \varepsilon_{i,k}\} \subset U_i$$

for all $i, k \in \mathbb{N}$.

Moreover,

$$V_{n,k+1} + V_{n,k+1} \subset V_{n,k} \subset U_n, \ n, k \in \mathbb{N} \tag{2.2}$$

and

$$L_n := \bigcup_k (w_{n,k} + V_{n,k}) \subset F \setminus \{0\}, \ n \in \mathbb{N}. \tag{2.3}$$

Each L_n is dense and open in F $(= M_n \oplus (\bigcap_{i \leq n} f_i^{\perp}))$. To complete the proof, we need only find a non-Baire hyperplane G in F; indeed, the hyperplane H in E satisfying $H \supset Q$ and $G = H/Q$ would also be non-Baire by Corollary 2.2. Note that $\aleph_0 \leq \dim(F) =: \alpha \leq \mathfrak{c}$. From the previous section, we know that the Baire space F contains a dense Baire hyperplane P. Let

$$B := \left\{ y_\beta : \beta \text{ is an ordinal} < \alpha \right\}$$

be a Hamel basis for P. If A is the absolutely convex envelope of B, then $\overline{A} \cap P$ is a barrel in the dense barrelled subspace P and, consequently, $\overline{A}$ is a neighborhood of zero in F. Hence we can find a point $z \in F \setminus P$ such that $2z \in \overline{A}$. We observe from (2.3) that the set $F \setminus L$ is of first category in F, where $L := \bigcap_n L_n$. The proof will be complete if we find a dense hyperplane G contained in $F \setminus L$. The space G will take the form

$$G := \operatorname{span}\{y_\gamma + a_\gamma z : \gamma < \alpha\}$$

for scalars a_γ suitably chosen with each $|a_\gamma| \leq 1$.

Formula (2.3) ensures that L misses

$$\{0\} = \operatorname{span} \emptyset = \operatorname{span}\{y_\gamma + 1 \cdot z : \gamma \text{ is an ordinal } < 0\}.$$

Zorn's lemma provides a maximal subspace M of F of the form

$$M := \operatorname{span}\{y_\gamma + a_\gamma z : \gamma < \beta\}$$

subject to the conditions that L misses M, the ordinal β does not exceed α and each $|a_\gamma| \leq 1$.

Claim 2.4 We have $\beta = \alpha$.

Indeed, assume that $\alpha \neq \beta$. The set $\{\gamma : \gamma < \beta\}$ and the family $\mathfrak{F}$ of all its finite subsets have less than $\mathfrak{c}$ elements. Now we will apply Lemma 2.1 to our situation. Identify y_β and z with the unit vectors e_1 and e_2, respectively, and $\mathbb{R}^2$ with the linear span X of y_β and z.

Let $h : M + X \to X$ be the projection onto X along M. Let $\mathfrak{D}$ be the family of all subspaces of the form $X + \operatorname{span}\{y_\gamma + a_\gamma z : \gamma \in J\}$ for $J \in \mathfrak{F}$. Note that $\mathfrak{D}$ covers $M + X$ and $|\mathfrak{D}| < \mathfrak{c}$. Next observe that

$$L \cap (M + X) = \bigcup_{n \in \mathbb{N}, D \in \mathfrak{D}} L_{D,n},$$

where $L_{D,n} := L \cap D \cap \{x : \|h(x)\| > n^{-1}\}$. This follows from the fact that $h(x) \neq 0$ if $x \in L \cap (M + X)$, because then $x \notin M$ (since M misses L).

Fix arbitrary $n \in \mathbb{N}$ and $D \in \mathfrak{D}$ as well as $0 < t < 1$. Since D is finite-dimensional, we deduce from (2.1) that $(D \cap U_i)_i$ is a base of neighborhoods of zero in D. Since the restriction map $h|D$ is continuous, there exists $r > 1$ such that $\|h(x)\| < tn^{-1}$ for $x \in D \cap U_{r-1}$. Note that

$$L_{D,n} \subset L_r \cap D \cap \{x : \|h(x)\| > n^{-1}\} =$$

$$\bigcup_k D \cap (w_{r,k} + V_{r,k}) \cap \{x : \|h(x)\| > n^{-1}\}.$$

Assume for the moment that x and y are in the kth set. Then $\|h(x)\| > n^{-1}$ and $\|h(y)\| > n^{-1}$. Moreover, $2^{k-1}(x - y) \in D$. Since $x - y \in V_{r,k} + V_{r,k}$, we use (2.2) and (2.3) and $k - 1$ more steps to obtain

$$2^{k-1}(x - y) \in V_{r,1} + V_{r,1} \subset U_r + U_r \subset U_{r-1}.$$

Hence, by continuity of $h|D$, we have

$$\|h(x - y)\| < 2^{1-k} t n^{-1}.$$

Now we can apply Lemma 2.1 to obtain that

$$|m(h(x)) - m(h(y))| < 2^{2-k} t.$$

This proves that the set $m(h(L_{D,n}))$ is covered by a sequence of intervals whose union has measure less than $4t = \sum_k 2^{2-k} t$. Since the scalar t was arbitrary, we conclude that $m(h(L_{D,n}))$ has measure zero. Now axiom $\mathfrak{c}$-A declares that

$$C := m(h(L \cap (M + X)))$$

has measure zero. Therefore we can find an angle θ such that $0 < \theta < \pi/4$ and $\theta \notin C$ and $\pi - \theta \notin C$. Now set $a_\beta := \tan\theta$ and $v = y_\beta + a_\beta z$. Hence $0 < a_\beta < 1$. Note that

$$m(h(x + bv)) = m(bv) = m(v) = \theta, \;\; m(h(x - bv)) = m(-v) = \pi - \theta$$

if $x \in M$ and $b > 0$. This implies that $x \pm bv \notin L$. We know already that M misses L. Now $M + \operatorname{span}\{v\}$ also misses L, which contradicts maximality of M, proving the claim.

Finally, we prove that $G := M$ is dense in F. Since G is a hyperplane, it is enough to show that G is not closed. Now $tz \notin M$ for each $t \neq 0$. Since $2z \in \overline{A}$, there exists a net $(z_u)_u = (\sum_{\gamma \in J_u} b_{u,\gamma} y_\gamma)_u$ in A that converges to $2z$, where J_u is finite and $\sum_{\gamma \in J_u} |b_{u,\gamma}| \leq 1$. The corresponding net $(\sum_{\gamma \in J_u} b_{u,\gamma} a_\gamma z)_u$ has an adherent point pz in the compact interval $\{bz : |b| \leq 1\}$. Let U and V be neighborhoods of $2z$ and pz, respectively. V contains a cofinal subnet of the second net, and U contains points of the corresponding cofinal subnet of the first net. Hence $U + V$ contains points of a subnet of $(\sum_{\gamma \in J_u} b_{u,\gamma}(y_\gamma + a_\gamma z))_u \subset M$. It follows that $2z + pz \in \overline{M} \setminus M$, and $G = M$ is not closed. The proof is complete. □

Remark 2.1 If E is a tvs with a separable quotient F such that $F' \neq F^*$, one can find points $w_{n,k}$ in F and balanced open neighborhoods U_i and $V_{i,k}$ of zero satisfying (2.1) and (2.2) such that each L_n defined as in (2.3) is dense (and open) in F. It then follows from the proof of Theorem 2.4 that, under the assumption of $\mathfrak{c}$-A, there exists in E a dense non-Baire hyperplane.

2.4 Locally convex spaces with some Baire-type conditions

A new line of research concerning Baire-type conditions started with the Amemiya–Kōmura theorem (see [328, Theorem 8.2.12] and [419]), stating that if $(A_n)_n$ is an increasing sequence of absolutely convex closed subsets covering a metrizable and barrelled lcs E, then there exists $m \in \mathbb{N}$ such that A_m is a neighborhood of zero in E.

Saxon [354], motivated by the Amemiya–Kōmura result, defined an lcs E to be *Baire-like* if for any increasing sequence $(A_n)_n$ of closed absolutely convex subsets of E covering E there is an integer $n \in \mathbb{N}$ such that A_n is a neighborhood of zero. If the sequence $(A_n)_n$ is required to be *bornivorous* (i.e., for every bounded set B in E there exists A_m that absorbs B), then Ruess defines E to be *b-Baire-like*. Clearly, for an lcs, Baire implies Baire-like, Baire-like implies b-Baire-like and barrelled, and (b)-Baire-like implies (quasi)barrelled.

The main purpose of the research started by Saxon was to study stable lcs properties inherited by products and small-codimensional subspaces of Baire spaces. The Baire-like property is such an example. Although products [424] and countable-codimensional subspaces (even hyperplanes) [16] of Baire locally convex spaces need not be fully Baire, they are always Baire-like since countable-codimensional subspaces of Baire-like spaces are Baire-like and topological products of Baire-like spaces are Baire-like; see [354].

These weak Baireness/strong barrelledness properties have well occupied other authors and books; see [421], [328], [260], [162], [356], [362], [361]. Notwithstanding, our selection and treatment is unique.

Metrizable barrelled spaces are Baire-like (Amemiya–Kōmura) and include all metrizable (LF)-spaces [328, Proposition 4.2.6]. Conversely, as we shall soon see, Baire-like (LF)-spaces must be metrizable.

(i) Let E be a vector space, and let $(E_n, \tau_n)_n$ be an increasing sequence of vector subspaces of E covering E, each E_n endowed with a locally convex topology τ_n, such that $\tau_{n+1}|E_n \le \tau_n$ for each $n \in \mathbb{N}$. Then on E there exists the finest locally convex topology τ such that $\tau|E_n \le \tau_n$ for each $n \in \mathbb{N}$. If τ is Hausdorff, we say that (E, τ) is the *inductive limit space* of the sequence $(E_n, \tau_n)_n$, and the latter is a *defining sequence* for (E, τ).

(ii) If each (E_n, τ_n) is metrizable, then (E, τ) is an (LM)*-space*.

(iii) If each (E_n, τ_n) is a Fréchet space (a Banach space), then (E, τ) is an (LF)-*space* (an (LB)*-space*).

(iv) If $\tau_{n+1}|E_n = \tau_n$ for each $n \in \mathbb{N}$, then $\tau|E_n = \tau_n$ for each $n \in \mathbb{N}$, (E, τ) is the *strict inductive limit* of $(E_n, \tau_n)_n$ and (E, τ) is complete if each (E_n, τ_n) is complete; see [328, Proposition 8.4.16].

(v) An (LF)-space E is called *proper* if it has a defining sequence of proper subspaces of E.

The Bairelikeness of some concrete normed vector-valued function spaces was studied by several specialists; see [129], [130] for details. We note only that the normed spaces of Pettis or Bochner integrable functions are not Baire spaces but Baire-like; see also [420] for more examples of normed Baire-like spaces that are not Baire. We also have the following simple proposition.

Proposition 2.11 *Every metrizable lcs E is b-Baire-like.*

Proof Let $(U_n)_n$ be a decreasing base of absolutely convex neigborhoods of zero in E. Assume that E is not b-Baire-like. Then there exists a bornivorous sequence $(A_n)_n$ of absolutely convex closed sets such that $U_n \not\subset nA_n$ for each $n \in \mathbb{N}$. Choose

$$x_n \in U_n \setminus nA_n.$$

Since the null sequence $(x_n)_n$ is bounded, it is contained in mA_k for some $k, m \in \mathbb{N}$. Hence, for all $n \ge \max\{m, k\}$, we have

$$\{x_n : n \in \mathbb{N}\} \subset mA_k \subset nA_n,$$

a contradiction. □

Proposition 2.12 *Every barrelled b-Baire-like space E is Baire-like.*

Proof Let $(A_n)_n$ be an increasing sequence of absolutely convex closed subsets of E covering E. The proof will be finished if we show that $(A_n)_n$ is bornivorous. Indeed, then we apply that E is b-Baire-like, and some A_m will be a neighborhood of zero.

Assume, by way of contradiction, that there exists a bounded set $B \subset E$ such that $B \not\subset nA_n$ for each $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, select

$$x_n \in n^{-1}B \setminus A_n.$$

Since each set A_n is closed, for each $n \in \mathbb{N}$ there exists a closed and absolutely convex neighborhood of zero U_n such that

$$U_{n+1} + U_{n+1} \subset U_n, \;\; x_n \notin \overline{A_n + U_n}.$$

Set

$$U := \bigcap_n \overline{A_n + U_n}.$$

Then U is a barrel in E and is a neighborhood of zero to which almost all elements of the null sequence $(x_n)_n$ must belong, a contradiction. □

Proposition 2.12 shows that every metrizable barrelled space is Baire-like, the Amemiya–Kōmura result. A more general fact, due to Saxon, is known [354, Theorem 2.1]: A barrelled lcs that does not contain a (isomorphic) copy of φ (i.e., an $\aleph_0$-dimensional vector space with the finest locally convex topology) is Baire-like. Saxon actually proved the following [354, Corollary 2.2], which is even a bit more general; see also [222].

Theorem 2.5 *Let E be ℓ^∞-barrelled (i.e., every $\sigma(E', E)$-bounded sequence in E' is equicontinuous). Assume that E is covered by an increasing sequence $(A_n)_n$ of absolutely convex closed subsets of E such that no set A_n is absorbing in E. Then E contains a copy of φ.*

Proof By considering subsequences, we may assume that the span of each A_n is a proper subspace of the span of A_{n+1}. Arbitrarily select $(x_n)_n$ in E such that $x_n \in A_{n+1} \setminus \operatorname{span}(A_n)$. We show that the span $\mathbb{S}$ of the necessarily linearly independent sequence $(x_n)_n$ is a copy of φ. It suffices to show that if p is an arbitrary seminorm on $\mathbb{S}$, there exists a continuous seminorm q on E such that $p \le q|\mathbb{S}$. Without loss of generality, we may assume that

$$p\left(\sum_{j=1}^n a_j x_j\right) = \sum_{j=1}^n |a_j| p(x_j), \;\; p(x_n) \ge 1, \;\; n \in \mathbb{N}.$$

We proceed inductively to find a sequence $(f_n)_n$ in E' such that

$$f_n \in A_n^\circ, \tag{2.4}$$

and if $n \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n$ are scalars, then

$$\max_{1 \le r \le n} \left| f_r\left(\sum_{j=1}^n a_j x_j\right)\right| \ge (1 + 2^{-n}) p\left(\sum_{j=1}^n a_j x_j\right). \tag{2.5}$$

The Hahn–Banach separation theorem provides $f_1 \in A_1^\circ$ such that

$$|f_1(x_1)| \ge (1 + 2^{-1}) p(x_1).$$

Let $k \in \mathbb{N}$, and assume there exist $f_1, f_2, \ldots, f_k$ in E' such that the conditions above are satisfied for $n \le k$. Define

$$D := \left\{ x = \sum_{j=1}^{k} a_j x_j : \max_{1 \le r \le k} |f_r(x + x_{k+1})| < (1 + 2^{-k-1}) p(x + x_{k+1}) \right\}.$$

If D is empty, we complete the induction step by letting $f_{k+1} = 0$. Assume that D is nonempty. For $x \in D$, we have

$$(1 + 2^{-k}) p(x) - \max_{1 \le r \le k} |f_r(x_{k+1})| \le \max_{1 \le r \le k} |f_r(x + x_{k+1})|$$
$$< (1 + 2^{-k-1})[p(x) + p(x_{k+1})].$$

This yields

$$2^{-k-1} p(x) \le \max_{1 \le r \le k} |f_r(x_{k+1})| + (1 + 2^{-k-1}) p(x_{k+1}).$$

Hence $\gamma := \sup_{x \in D} \; p(x) < \infty$. For $A := A_{k+1} + \operatorname{span}\{x_1, \ldots, x_k\}$, we have

$$A_{k+1} \subset A, \;\; x_{k+1} \notin \operatorname{span} A = \operatorname{span} A_{k+1}.$$

Moreover, A is absolutely convex and closed. The Hahn–Banach theorem provides $f_{k+1} \in A^\circ \subset A^\circ_{k+1}$ such that

$$f_{k+1}(x_{k+1}) = (1 + 2^{-k-1})(\gamma + p(x_{k+1})).$$

Thus $x \in D$ implies that

$$f_{k+1}(x + x_{k+1}) = (1 + 2^{-k-1})(\gamma + p(x_{k+1})) \ge (1 + 2^{-k-1}) p(x + x_{k+1}).$$

To prove (2.5) for $n = k + 1$, consider an arbitrary element $z = y + a x_{k+1} \in \mathbb{S}$ with $y = \sum_{j=1}^{k} a_j x_j$ and

$$|f_r(z)| < (1 + 2^{-k-1}) p(z)$$

for $1 \le r \le k$. By the induction assumption, $a \ne 0$ and $a^{-1} y \in D$. Therefore, by the above, we have

$$|f_{k+1}(a^{-1} z)| \ge (1 + 2^{-k-1}) p(a^{-1} z).$$

Thus (2.4) and (2.5) hold for $n \le k + 1$; the induction is complete. Now fix $x \in E$. Since $f_n \in A_n^\circ$, the fact that $x \in A_n$ for almost all $n \in \mathbb{N}$ means that $|f_n(x)| \le 1$ for almost all $n \in \mathbb{N}$. Thus $(f_k)_k$ is $\sigma(E', E)$-bounded, and equicontinuous by hypothesis on E. The formula

$$x \mapsto q(x) := \sup_k |f_k(x)|$$

defines a continuous seminorm on E. By (2.5) we have $p \le q|\mathbb{S}$. This shows that p is continuous on $\mathbb{S}$. We proved that each seminorm on $\mathbb{S}$ is continuous, which shows that $\mathbb{S}$ is as desired. □

This yields the following corollary.

Corollary 2.4 (Saxon) *Every barrelled lcs that does not contain a copy of φ is a Baire-like space.*

Hence, if a barrelled lcs E admits a finer metrizable locally convex topology, then E is Baire-like. The converse implication fails. Indeed, let $E = (E, \vartheta)$ be an uncountable product of a metrizable and complete lcs. Clearly E is a nonmetrizable Baire lcs for which every finer locally convex topology ξ is nonmetrizable, for if we assume that ξ has a countable base $(U_n)_n$ of absolutely convex neighborhoods of zero, then, because ϑ is barrelled, $(\overline{U_n}^{\vartheta})_n$ is a countable base of neighborhoods of zero for ϑ, an impossibility.

Corollary 2.5 *Let E be an lcs such that each $\sigma(E', E)$-bounded sequence in E' is equicontinuous and E is covered by a strictly increasing sequence of closed subspaces. Then E contains a copy of φ.*

In fact, an ℓ^{∞}-barrelled space E contains a complemented copy of φ if and only if E is covered by a strictly increasing sequence of closed subspaces (see [356, Theorem 1]).

Corollary 2.6 *Every proper (LB)-space E contains a copy of φ.*

Proof We may assume that the unit ball U_n in the nth defining Banach space (E_n, τ_n) is contained in U_{n+1} for each $n \in \mathbb{N}$. No $\overline{U_n}$ is absorbing in the barrelled space E. Otherwise, E_n would be a dense barrelled subspace of E, consequently complete by the open mapping theorem, so that $E_n = E$, a contradiction. The theorem applies with $A_n = \overline{U_n}$. □

Hence no proper (LB)-space is metrizable. Nevertheless, metrizable and even normable proper (LF)-spaces do exist; see [328] for details and references.

Corollary 2.7 *For an (LF)-space E, the following conditions are equivalent*:

(i) *E is Baire-like.*
(ii) *E is metrizable.*
(iii) *E does not contain φ.*

Proof (i) $\Rightarrow$ (ii): Let $(E_n)_n$ be a defining sequence of Fréchet spaces for E, and for each $n \in \mathbb{N}$ let $\mathscr{F}_n$ be a countable base of neighborhoods of zero in the Fréchet space E_n. The set $\mathscr{F}$ of all unions of finite subsets of $\bigcup_n \mathscr{F}_n$ is countable. Let $\mathscr{M}$ be the set of all absolutely convex closed neighborhoods of zero in E. Given any $A \subset E$, let $\overline{\text{ac}}A$ denote the absolutely convex closed envelope of A in E. The set

$$\mathscr{N} := \mathscr{M} \cap \{\overline{\text{ac}}A : A \in \mathscr{F}\}$$

is countable. Given an arbitrary U in $\mathscr{M}$, for each $n \in \mathbb{N}$ there exists $U_n \in \mathscr{F}_n$ with $U_n \subset U$, and we set

$$A_n := \bigcup_{j=1}^{n} U_j \in \mathscr{F}.$$

Clearly, the increasing sequence $(n \cdot \overline{\text{ac}}A_n)_n$ covers E, and each $\overline{\text{ac}}A_n \subset U$. If E is Baire-like, some $\overline{\text{ac}}A_n$ is in $\mathscr{M}$ and hence in $\mathscr{N}$, which proves that the countable $\mathscr{N}$ is a base of neighborhoods of zero in E (i.e., E is metrizable).

(ii) $\Rightarrow$ (iii) is clear since φ is nonmetrizable.

(iii) $\Rightarrow$ (i) follows from Corollary 2.4. □

We provide later a general result that in particular shows also that nonmetrizable (LF)-spaces are not Baire-like.

We shall need also in the sequel the following result due to Valdivia [419]; see also [328, Proposition 8.2.27].

Proposition 2.13 (Valdivia) *Let F be a dense barrelled (quasibarrelled) subspace of an lcs E, and assume that $(B_n)_n$ is an increasing sequence of absolutely convex sets such that each point (each bounded set) of F is absorbed by some B_n. Then $\bigcap_{\varepsilon>0} \bigcup_n (1+\varepsilon)\overline{B_n} = \overline{\bigcup_n B_n}$, where the closure is in E. In particular, $E = \bigcup_n n\overline{B_n}$, and if $(B_n)_n$ is a covering of F, then $E = \bigcup_n \overline{B_n}$.*

Proof We prove only the barrelled case. It is enough to show that

$$\overline{\bigcup_n B_n} \subset \bigcup_n (1+\varepsilon)\overline{B_n}$$

for each $\varepsilon > 0$. If there exists $\varepsilon > 0$ and $x \notin \bigcup_n (1+\varepsilon)\overline{B_n}$, we may choose for each n an absolutely convex neighborhood of zero U_n in E such that

$$x \notin (1+\varepsilon)\, B_n + 2U_n.$$

Set

$$U := \bigcap_n \overline{\varepsilon B_n + U_n},$$

and observe that $U \cap F$ is a barrel in the barrelled space F. Thus $U \cap F$ is a neighborhood of zero in F, and U is a neighborhood of zero in E by density of F. But, for each n, we have

$$x \notin (1+\varepsilon)\, B_n + 2U_n \supset \overline{B_n + \varepsilon B_n + U_n} \supset B_n + \overline{\varepsilon B_n + U_n} \supset B_n + U,$$

so that $x \notin \bigcup_n B_n + U$. Hence $x \notin \overline{\bigcup_n B_n}$. □

We note the following corollary.

Corollary 2.8 *For any completely regular Hausdorff space X, the space $C_p(X)$ is b-Baire-like.*

Proof Let $(A_n)_n$ be a bornivorous increasing sequence of absolutely convex closed subsets of $C_p(X)$ covering $C_p(X)$. For each $n \in \mathbb{N}$, let B_n be the closure of A_n in the space $\mathbb{R}^X$. Since the space $C_p(X)$ is dense in $\mathbb{R}^X$ and $C_p(X)$ is quasibarrelled by [213, Theorem 2], we apply Proposition 2.13 to get $\mathbb{R}^X = \overline{C_p(X)} = \bigcup_n B_n$. Since $\mathbb{R}^X$ is a Baire space, and the sets B_n are closed and absolutely convex, there exists $m \in \mathbb{N}$ such that B_m is a neighborhood of zero in $\mathbb{R}^X$. Consequently, A_m is a neighborhood of zero in $C_p(X)$. □

Corollary 2.9 *If E is a barrelled space covered by an increasing sequence of absolutely convex complete subsets, then E is complete.*

An lcs E is called *locally complete* if every bounded closed absolutely convex set B is a Banach disc (i.e., the linear span of B endowed with the Minkowski functional norm $\|x\|_B := \inf\{\varepsilon > 0 : \varepsilon^{-1}x \in B\}$ is a Banach space). An lcs E is *docile* if every infinite-dimensional subspace of E contains an infinite-dimensional bounded set; see [229].

It is known by the Saxon–Levin–Valdivia theorem that every countable-codimensional subspace of a barrelled space is barrelled; see [328, Theorem 4.3.6]. Theorem 2.6 is due to Saxon; see [352]. In fact, Saxon [358] proved that even F is totally barrelled. An extension of this result to a barrelled lcs E such that $(E', \mu(E', E))$ is complete was obtained by Valdivia; see [328, Proposition 4.3.11].

Theorem 2.6 (Saxon) *If F is a subspace of an (LF)-space E such that* $\dim(E/F) < 2^{\aleph_0}$, *then F is barrelled.*

For the proof, we need the following lemma.

Lemma 2.2 (Sierpiński) *Every denumerable set $\mathbb{S}$ admits $2^{\aleph_0}$ denumerable subsets $\mathbb{S}_\iota$ ($\iota \in I$ and $|I| = 2^{\aleph_0}$) that are almost disjoint (i.e., if ι and τ are distinct members of the indexing set I, then $\mathbb{S}_\iota \cap \mathbb{S}_\tau$ is finite).*

Proof Take $\mathbb{S}$ to be all rational numbers, let I be the irrationals, and for each $\iota \in I$ choose $\mathbb{S}_\iota$ to be a sequence of rationals that converges to ι. □

Lemma 2.3 *Let A be a closed absolutely convex set having span F in a docile locally complete space E. Then* $\dim(E/F)$ *is either finite or at least* $2^{\aleph_0}$.

Proof Assume $\dim(E/F)$ is infinite. Then there is an infinite-dimensional subspace G of E that is transverse to F. By docility there is an infinite-dimensional bounded absolutely convex set B in G, and, trivially, its span is transverse to F. Let x_1 be an arbitrary nonzero member of B. The bipolar theorem provides $f_1 \in E'$ such that

$$f_1 \in A^\circ \text{ and } f_1(x_1) = 1 \cdot 2^1.$$

Now the span of $B_1 := f_1^\perp \cap B$ is infinite-dimensional and transverse to the span of $A_1 := A + \operatorname{span}\{x_1\}$, and A_1 is absolutely convex and closed. Let x_2 be an arbitrary nonzero member of B_1, and use the bipolar theorem as before to obtain $f_2 \in E'$ such that

$$f_2 \in A_1^\circ \text{ and } f_2(x_2) = 2 \cdot 2^2.$$

By setting $B_2 = f_2^\perp \cap B_1$ and $A_2 = A_1 + sp\{x_2\}$ and continuing inductively, we obtain $(x_n)_n \subset B$ and $(f_n)_n \subset E'$ such that

$$(f_n)_n \subset A^\circ;\ f_i\left(x_j\right) = 0 \text{ for } i \neq j;\ \text{and } f_n(x_n) = n \cdot 2^n \text{ for all } n \in \mathbb{N}.$$

Since $(x_n)_n$ is bounded and E is locally convex, the series $\sum_n 2^{-n} x_n$ is absolutely, and hence subseries, convergent. Let $\{\mathbb{N}_\iota\}_{\iota \in I}$ be a collection of $\mathfrak{c}$ denumerable subsets of $\mathbb{N}$ that are almost disjoint (Lemma 2.2), and for every $\iota \in I$ set

$$y_\iota = \sum_{n \in \mathbb{N}_\iota} 2^{-n} x_n.$$

We claim that the set $\{y_\iota\}_{\iota \in I}$ is linearly independent and has span transverse to F. Indeed, if y is any (finite) linear combination in which the coefficient of some y_ι is a nonzero scalar a, then we have

$$f_n(y) = n \cdot a \text{ for all but finitely many } n \in \mathbb{N}_\iota.$$

Therefore $(f_n)_n$ is unbounded at y, whereas $(f_n)_n \subset A^\circ$ implies that $(f_n)_n$ is bounded at all points of F since they are absorbed by A. We have thus shown that $\{y_\iota\}_{\iota \in I}$ consists of $\mathfrak{c}$ linearly independent vectors whose span is transverse to F, as desired. This implies that every infinite-dimensional locally complete docile space E has dimension at least $2^{\aleph_0}$. Indeed, take $A = F = \{0\}$. In particular, every infinite-dimensional Fréchet space has dimension at least $2^{\aleph_0}$; see also [277]. □

We are ready to prove Theorem 2.6.

Proof Let B be a barrel in F, let $(E_n)_n$ be a defining sequence of Fréchet spaces for E, let A be the closure of B in E and let G be the span of A. In each Fréchet space E_n, the intersection $A \cap E_n$ is closed and has a span of codimension less than $2^{\aleph_0}$

in E_n; by Lemma 2.3, the codimension is finite. Since the countable union of finite sets is countable, the codimension of G is countable in E. Therefore G is barrelled by the Saxon–Levin–Valdivia theorem, and A is a neighborhood of zero in G. It follows that $B = A \cap F$ is a neighborhood of zero in F. □

The following observation will be used below; see [328, Proposition 4.1.6].

Proposition 2.14 *If E is a barrelled lcs covered by an increasing sequence $(E_n)_n$ of vector subspaces, then a linear map $f : E \to F$ from E into an lcs F is continuous if and only if there exists $m \in \mathbb{N}$ such that the restrictions $f|E_n : E_n \to F$ are continuous for all $n \geq m$, where the topology of each E_n is that induced by E.*

There is an interesting application of Baire-like spaces for closed graph theorems; see [354]. It is known that the class of barrelled spaces is the largest class of lcs for which the closed graph (open mapping) theorem holds vis-á-vis Fréchet spaces. Bourbaki [66, Theorem III.2.1] observed that the class of barrelled spaces is also the largest one for which the uniform boundedness theorem holds.

Saxon [354] showed that the Grothendieck factorization theorem for linear maps from a Baire lcs into an (LF)-space with closed graph remains true for linear maps from a Baire-like space into an (LB)-space. We have the following theorem.

Theorem 2.7 (Saxon) *Let F be a Baire-like space. Let E be an (LB)-space with a defining sequence $(E_n)_n$ of Banach spaces. Then every linear map $f : F \to E$ with a closed graph is continuous.*

Proof Via equivalent norms, we may assume that the unit ball B_n for E_n is contained in B_{n+1} and that $(B_n)_n$ covers E. Thus $(f^{-1}(B_n))_n$ covers F, and there is some $m \in \mathbb{N}$ such that $\overline{f^{-1}(B_n)}$ is a neighborhood of zero in F for each $n \geq m$. Hence the closure of $f^{-1}(V_n)$ in $f^{-1}(E_n)$ is a neighborhood of zero whenever V_n is a neighborhood of zero in the Banach space E_n and $n \geq m$ (i.e., the restriction of f to $f^{-1}(E_n)$ is almost continuous as a mapping into the Banach space E_n). Since the Banach topology is finer than that induced by E, the graph of this restriction is also closed. By Ptak's closed graph theorem [328], this mapping is continuous, and all the more so when the range space E_n is given the coarser topology induced by E. Since F is barrelled, Proposition 2.14 ensures that the unrestricted f is also continuous. □

We complete this section with a characterization of the Bairelikeness of spaces $C_c(X)$ for some concrete spaces X.

In what follows, X is a completely regular Hausdorff topological space. If $\mathscr{F} := \{f \in C(X) : f(X) \subset [0, 1]\}$, the subspace

$$\{(f(x) : f \in \mathscr{F}) : x \in X\}$$

is homeomorphic to X; we identify X with this subspace, and the closure of X in $[0, 1]^{\mathscr{F}}$ is the Stone–Čech compactification of X, denoted by βX. Taking into account the restrictions to βX of the coordinate projections of $[0,1]^{\mathscr{F}}$, we have $f \in \mathscr{F}$, and therefore each uniformly bounded $f \in C(X)$ has a unique continuous extension to βX.

By the *realcompactification* υX of X, we mean the subset of βX such that $x \in \upsilon X$ if and only if each $f \in C(X)$ admits a continuous extension to $X \cup \{x\}$. From the regularity of X, it follows that each $f \in C(X)$ admits a continuous extension to υX. Therefore the closure in $\mathbb{R}^{C(X)}$ of $\{(f(x) : f \in C(X)) : x \in X\}$ is homeomorphic to υX.

By definition, X is called *realcompact* if $X = \upsilon X$. From the continuity of the coordinate projections, it follows that X is realcompact if and only if X is homeomorphic to a closed subspace of a product of the real lines. Clearly, closed subspaces of a realcompact space are realcompact, and any product of realcompact spaces is realcompact. The intersection of a family of realcompact subspaces of a space is realcompact because this intersection is homeomorphic to the diagonal of a product.

Recall that a subset $A \subset X$ of a topological space X is *topologically bounded* if the restricted map $f|A$ is bounded for each $f \in C(X)$; otherwise we will say that A is *topologically unbounded.*

If X is completely regular and Hausdorff, then $A \subset X$ is topologically bounded if and only if for each locally finite family $\mathscr{F}$ the family $\{F \in \mathscr{F} : F \cap A \neq \emptyset\}$ is finite. If X is a topologically bounded set of itself, then X is called *pseudocompact.*

Recall here that an lcs E is said to be *bornological* if every bornivorous absolutely convex subset of E is a neighborhood of zero.

The link between properties of $C_c(X)$ and $C_p(X)$ and topological properties of X is illustrated by Nachbin [307], Shirota [377], De Wilde and Schmets [113] and Buchwalter and Schmets [69].

Proposition 2.15 (Nachbin–Shirota) *$C_c(X)$ is barrelled if and only if X is a μ-space* (*i.e.*, *every topologically bounded subset of X has compact closure*).

Proposition 2.16 (Nachbin–Shirota, De Wilde–Schmets) *The space $C_c(X)$ is bornological if and only if $C_c(X)$ is the inductive limit of Banach spaces if and only if X is realcompact.*

Proposition 2.17 (Buchwalter–Schmets) *$C_p(X)$ is barrelled if and only if every topologically bounded subset of X is finite.*

A good sufficient condition for $C_c(X)$ to be a Baire space is hard to locate. Let us mention the following one: If X is a locally compact and paracompact space, $C_c(X)$ is Baire. The argument uses the well-known fact that X can be written as the topological direct sum of locally compact, σ-compact (hence hemicompact) disjoint subspaces $\{X_t : t \in T\}$ of X. Since $C_c(X)$ is isomorphic to the product $\prod_t C_c(X_t)$ of Fréchet spaces, the space $C_c(X)$ is a Baire space; see, for example, [328]. Gruenhage and Ma [194] defined and studied the *moving off property* and proved that if X is locally compact or first-countable, $C_c(X)$ is Baire if and only if X has the moving off property.

Combining results of Lehner [266, Theorem III.2.2, Theorem III.3.1] and Proposition 2.15, we note the following proposition.

Proposition 2.18 (Lehner) *$C_c(X)$ is Baire-like if and only if for each decreasing sequence $(A_n)_n$ of closed noncompact subsets of X there exists a continuous function $f \in C(X)$ that is unbounded on each A_n.*

This yields the useful Proposition 2.19; see [266] and also [225]. Recall that a topological space X is of *pointwise countable type* if each $x \in X$ is contained in a compact set $K \subset X$ of *countable character* in X (i.e., having a countable basis of open neighborhoods). All first-countable spaces, as well as Čech-complete spaces (hence locally compact spaces), are of pointwise countable type; see [146], [27].

Proposition 2.19 (Lehner) (*i*) *If X is a locally compact space and $C_c(X)$ is barrelled, then $C_c(X)$ is a Baire-like space.*

(*ii*) *If X is a space of pointwise countable type and $C_c(X)$ is Baire-like, X is a locally compact space.*

Proof (i) Assume $C_c(X)$ is a barrelled space. Then, by Proposition 2.15, the space X is a μ-space. Having in mind Proposition 2.18, we need only to find for a decreasing sequence $(A_n)_n$ of closed, not topologically bounded sets in X a continuous function $f \in C(X)$ that is unbounded on each A_n. By the assumption, on each A_n there exists $f_n \in C(X)$ unbounded on A_n. Note that the proof will be completed if we find a number $m \in \mathbb{N}$ such that f_m is unbounded on each A_n. Assume that for each $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N}$ such that f_n is bounded on A_{k_n}. Since $(A_n)_n$ is decreasing, we may assume that f_n is bounded on A_{n+1} for each $n \in \mathbb{N}$.

Two cases are possible:

(a) $A := \bigcap_n A_n$ is noncompact. Since X is a μ-space, there exists a continuous function $f \in C(X)$ that is unbounded on A, and the proof is finished.

(b) A is compact. Since X is locally compact, there exists an open set V_0 such that $A \subset V_0$ whose closure W_0 is compact. But f_1 is bounded on $A_2 \cup W_0$ and unbounded on A_1, so $A_1 \not\subset A_2 \cup W_0$. Select

$$x_1 \in A_1 \setminus (A_2 \cup W_0).$$

Then there exists an open neighborhood V_1, $x_1 \in V_1$, whose closure W_1 is compact, and

$$V_1 \subset X \setminus (A_2 \cup W_0).$$

By a simple induction, we select a sequence $(x_n)_n$ in X and a pairwise disjoint sequence $(V_n)_n$ of open neighborhoods of x_n whose closure W_n is compact for each $n \in \mathbb{N}$. Since $(V_n)_n$ is an open cover of

$$\{x_n : n \in \mathbb{N}\} \cup A,$$

we deduce that this cover does not admit a finite subcover. We conclude that $\{x_n : n \in \mathbb{N}\} \cup A$ is not compact. Set $L := \{x_n : n \in \mathbb{N}\}$.

Claim 2.5 $L \cup A$ is closed in X. Indeed, if $x \in \overline{L} \setminus A$, then $x \in \overline{\{x_k : k > n\}}$ for all $n \in \mathbb{N}$. Since $\{x_k : k > n\} \subset A_n$ for each $n \in \mathbb{N}$, $x \in A$. This yields that $L \cup A$ is closed. As every topologically bounded set in X is relatively compact, there exists a continuous function $f \in C(X)$ unbounded on $L \cup A$. Therefore f is unbounded on each A_n.

(ii) Fix $x \in X$. By the assumption, there exists a compact set K, $x \in K$, and a decreasing basis $(U_n)_n$ of open neighborhoods of K. Set

$$H_n := \left\{ f \in C(X) : \sup_{h \in U_n} |f(h)| \leq n \right\}.$$

Then, as is easily seen, the family $\{H_n : n \in \mathbb{N}\}$ covers $C(X)$. Since $C_c(X)$ is Baire-like, there exist a compact set $D \subset X$, $\varepsilon > 0$, and $n \in \mathbb{N}$ such that

$$\left\{ f \in C(X) : \sup_{d \in D} |f(d)| < \varepsilon \right\} \subset H_n.$$

Observe that $U_n \subset D$. Indeed, if $z \in U_n \setminus D$, we can find a continuous function $f \in C(X)$ such that $f(z) = n + 1$ and $f(d) = 0$ for all $d \in D$. Hence $f \in H_n$, so $f(z) \leq n$, which provides a contradiction. □

We already mentioned that the product of two normed Baire spaces need not be a Baire space. Nevertheless, particular products of Baire spaces are Baire (see the proof of Proposition 2.2 and text below Proposition 2.7). On the other hand, any product of metrizable and separable Baire spaces is a Baire space; see [324]. This fact can be applied to get the following interesting result concerning the products of Baire spaces $C_p(X)$; see [398, Theorem 4.8], [278].

Theorem 2.8 (Tkachuk) *Let $\{C_p(X_t) : t \in A\}$ be a family of Baire spaces. Then the product $\prod_{t \in A} C_p(X_t)$ is a Baire space.*

Proof Note that $\prod_{t \in A} C_p(X_t)$ is isomorphic to $C_p(X)$, where $X = \bigoplus_{t \in A} X_t$. We will need here the following useful fact stating that $C_p(X)$ is a Baire space if and only if $T_D(C_p(X))$ is a Baire space for each countable set $D \subset X$, where as usual $T_D(f) := f|D$ for $f \in C(X)$ means the restriction map; see [278, Theorem 3.6]. Let $D \subset X$ be a countable set, and set $D_t := D \cap X_t$ for each $t \in A$. Clearly, $W_D := \{t \in A : D_t \neq \emptyset\}$ is countable. Since by the assumption each $C_p(X_t)$ is a Baire space, $T_{D_t}(C_p(X_t))$ is a Baire space. On the other hand, as is easily seen,

$$T_D(C_p(X)) = \prod_{t \in W_D} T_{D_t}(C_p(X_t)).$$

As each space $T_{D_t}(C_p(X_t))$ is a metrizable and separable Baire space, the product $T_D(C_p(X))$ is a Baire space. Hence $C_p(X)$ is Baire. □

2.5 Strongly realcompact spaces X and spaces $C_c(X)$

This section deals with the class of strongly realcompact spaces, introduced and studied in [228]; see also [404]. The following well-known characterization of realcompact spaces will be used in the sequel; see [181] or [146].

Proposition 2.20 *A completely regular Hausdorff space X is realcompact if and only if for every element $x \in X^* := \beta X \setminus X$ there exists $h_x \in C(\beta X)$, $h_x(X) \subset]0, 1]$, which is positive on X, and $h_x(x) = 0$.*

Proof Assume the condition holds. Then $X = \bigcap\{h_y^{-1}]0, 1] : y \in \beta X\setminus X\}$. As each $h_y^{-1}]0, 1]$ is a realcompact subspace of βX (since $h_y^{-1}]0, 1]$ is homeomorphic to $(\beta X \times]0, 1]) \cap G(h_y)$, where $G(h_y)$ means the graph of h_y), then X is also realcompact. Conversely, if X is realcompact and $x_0 \in \beta X\setminus X = \beta X\setminus \upsilon X$, there exists a continuous function $f : X \to \mathbb{R}$ that cannot be extended continuously to $X \cup \{x_0\}$. From

$$f(x) = \max(f(x), 0) + \min(f(x), 0) =$$

$$1 + \max(f(x), 0) - (1 - \min(f(x), 0))$$

we know that one of the functions $g_1(x) = 1 + \max(f(x), 0)$, or $g_2(x) = 1 - \min(f(x), 0)$ cannot be extended continuously to $X \cup \{x_0\}$. Hence there exists a continuous function $g : X \to [1, \infty[$ that cannot be extended continuously to $X \cup \{x_0\}$. Let $\widehat{h}$ be a continuous extension of the bounded function $h := 1/g$ to βX. If $\widehat{h}(x_0) \neq 0$, we reach a contradiction. Hence $\widehat{h}(x_0) = 0$. □

We shall say that X is *strongly realcompact* [228] if for every sequence $(x_n)_n$ of elements in X^* there exists $f \in C(\beta X)$ that is positive on X and vanishes on some subsequence of $(x_n)_n$. Clearly, every strongly realcompact space is realcompact. It is known [411, Exercise 1B. 4] that if X is locally compact and σ-compact, then X^* is a zero set in βX, so X is strongly realcompact.

A subset $A \subset X$ is said to be C-embedded (C^*-embedded) if every real-valued continuous (bounded and continuous) function on A can be extended to a continuous function on the whole space X.

For strongly realcompact spaces, we note the following property. The proof presented below (see [228]) uses some arguments (due to Negrepontis [311]) from [180, Theorem 2.7].

Proposition 2.21 *If X is strongly realcompact, every infinite subset D of X^* contains an infinite subset S that is relatively compact in X^* and C^*-embedded in βX.*

Proof Let $(x_n)_n$ be an injective sequence in D (i.e., $x_n \neq x_m$ if $n \neq m$), and let $f : \beta X \to [0, 1]$ be a continuous function that is positive on X and vanishes on a subsequence of $(x_n)_n$. Set

$$S = \{x_n : n \in \mathbb{N}\} \cap f^{-1}\{0\}, \quad Y_n = \{x \in \beta X : |f(x)| \geqslant n^{-1}\}, n \in \mathbb{N},$$

$$X_1 = S \cup \bigcup_n Y_n.$$

Note that the space X_1 is regular and σ-compact; hence it is normal (see Lemma 6.1). Since S is closed in X_1, it is C^*-embedded in X_1. Therefore S is C^*-embedded in βX_1. As $X \subset X_1 \subset \beta X$, we conclude that $\beta X_1 = \beta X$. □

This implies that, if X is a strongly realcompact space, every infinite closed subset of X^* contains a copy of the space $\beta\mathbb{N}$. On the other hand, Baumgartner and van Douwen [48, Example 1.11] provided a separable first-countable locally compact realcompact space X (hence strongly realcompact by Theorem 2.9 below) for which X^* contains a discrete countable subset that is not C^*-embedded in βX. This result with [48, Theorem 1.2] can be used to distinguish an example of a locally compact realcompact space X such that X^* contains a sequence $(x_n)_n$ for which there does not exist $f \in C(\beta X)$ that is positive on X and vanishes on $(x_n)_n$.

The space $\mathbb{Q}$ of the rational numbers is not strongly realcompact, but by [139] one gets that $\mathbb{Q}^*$ is a $\beta\omega$-space (i.e., if D is a countable discrete subset of $\mathbb{Q}^*$, and $\overline{D}$ (the closure in $\mathbb{Q}^*$) is compact, then $\overline{D} = \beta D$). Hence D is C^*-embedded in $\beta\mathbb{Q}$. It is known (see [181]) that $\mathbb{Q}^*$ contains a countable subset that is not C^*-embedded in $\beta\mathbb{Q}$.

A filter (filterbasis) $\mathscr{F}$ on a topological space X is said to be *unbounded* if there exists a continuous real-valued function f on X that is unbounded on each element of $\mathscr{F}$. We call f unbounded on $\mathscr{F}$.

In order to prove Theorem 2.9, we need the following two lemmas.

Lemma 2.4 *A filter $\mathscr{F}$ on a topological space X is unbounded if and only if there exists $x \in \bigcap_{F\in\mathscr{F}} \overline{F} \setminus \upsilon X$, where the closure is taken in βX.*

Proof Set $K := \bigcap_{F\in\mathscr{F}} \overline{F}$, and assume by contradiction that $K \subset \upsilon X$. Then, for each continuous real-valued function f on X there exists an open set $U_f \subset \beta X$ such that $K \subset U_f$ and the restriction $f|U_f \cap X$ is bounded. Note that there exists $F \in \mathscr{F}$ contained in U_f. Indeed, otherwise the family of sets $\{\overline{F} \setminus U_f : F \in \mathscr{F}\}$ satisfies the finite intersection property, which leads to a point in $K \setminus U_f$. This is a contradiction. We proved that there exists $F \in \mathscr{F}$ contained in U_f. This shows that $\mathscr{F}$ is not unbounded.

To prove the converse, assume that there exists $x \in K \setminus \upsilon X$. It is known (see [181]) that $\upsilon X = \bigcap_{f\in C(X)} \upsilon_f(X)$, where

$$\upsilon_f(X) := \{x \in \beta X : f^\beta(x) \neq \infty\}.$$

Then there exists $f \in C(X)$ for which the extension $f^\beta : \beta X \to \mathbb{R}_\infty$ has property $f^\infty(x) = \infty$, where $\mathbb{R}_\infty := \mathbb{R} \cup \{\infty\}$ (the Alexandrov one-point compactification). Since $x \in \overline{F}$ for each $F \in \mathscr{F}$, the map f is unbounded on $\mathscr{F}$. □

Lemma 2.5 *Each unbounded filterbasis $\mathscr{F}$ on a topological space X is contained in an unbounded ultrafilter $\mathscr{U}$ on X.*

Proof If $\mathscr{M} := \{M \subset X : \exists F \in \mathscr{F}; F \subset M\}$, then $\mathscr{M}$ is an unbounded filter on X. By Lemma 2.4, there exists $x \in \bigcap_{F \in \mathscr{M}} \overline{F} \setminus \upsilon X$. Let $\mathscr{A}$ be the family of all filters $\mathscr{G}$ on X containing $\mathscr{M}$ and such that $x \in \bigcap_{F \in \mathscr{G}} \overline{F}$. Order $\mathscr{A}$ by inclusion. Since there exists a maximal chain in $\mathscr{A}$, its union $\mathscr{U}$ is an ultrafilter on X containing $\mathscr{F}$ such that $x \in \bigcap_{F \in \mathscr{U}} \overline{F}$, and we use Lemma 2.4 to conclude that $\mathscr{U}$ is unbounded on X. □

We are ready to prove the following characterization of strongly realcompact spaces. Parts (i) and (ii) were proved in [228]; part (iii) is from [404].

Theorem 2.9 (*i*) *A topological space X is strongly realcompact if and only if X is realcompact and X^* is countably compact. Hence every locally compact realcompact space is strongly realcompact.*

(*ii*) *Every strongly realcompact space of pointwise countable type is locally compact.*

(*iii*) *A realcompact space X is strongly realcompact if and only if for each sequence $(\mathscr{F}_n)_n$ of unbounded filters (filterbases) on X there exists a continuous real-valued function on X and a subsequence $(\mathscr{F}_{n_k})_k$ such that f is unbounded on each $\mathscr{F}_{n_k}$.*

Proof (i) Suppose X is strongly realcompact. Let $P \subset X^*$ be an infinite set, and let $(x_n)_n$ be an injective sequence in P. There exists a continuous function $f : \beta X \to [0, 1]$ that is positive on X and zero on some subsequence $(x_{k_n})_n$ of $(x_n)_n$. Then

$$\{x_{k_n} : n \in \mathbb{N}\} \subset f^{-1}(0) \subset X^*.$$

Hence

$$\{x_{k_n} : n \in \mathbb{N}\}^d \subset f^{-1}(0).$$

Note that $\{x_{k_n} : n \in \mathbb{N}\}^d$ is nonempty, where A^d is the set of all accumulation points of a set A. This proves that P has an accumulation point in X^*.

To prove the converse, assume that X is realcompact and every infinite subset of X^* has an accumulation point in X^*.

Let $(x_n)_n$ be a sequence in X^*. If $P = \{x_n : n \in \mathbb{N}\}$ is finite, then (since X is realcompact) there exists a continuous function $f : \beta X \to [0, 1]$ that is positive on X and zero on a subsequence of $(x_n)_n$. If $P = \{x_n : n \in \mathbb{N}\}$ is infinite, take $p \in P^d \setminus X$. Then there exists a continuous function $f : \beta X \to [0, 1]$ that is positive on X and vanishes on p. Note that for every $r > 0$ the set $P \cap f^{-1}([0, r))$ is infinite since $f^{-1}([0, r))$ is a neighborhood of the point $p \in P^d$.

We consider two possible cases:

Case 1. The set $P \cap f^{-1}(0)$ is infinite. Then f is positive on X and zero on some subsequence of the sequence $(x_n)_n$.

Case 2. The set $P \cap f^{-1}(0)$ is finite. As for every $r > 0$ the set $P \cap f^{-1}([0, r))$ is infinite, there exists an injective sequence $(t_n)_n$ in P such that the sequence $(f(t_n))_n$

is strictly decreasing and converges to zero. Set $P_0 = \{t_n : n \in \mathbb{N}\}$, $s_0 = 1$ and $s_k \in (f(t_{k+1}), f(t_k))$ for all $k \in \mathbb{N}$. Then $(s_k)_k$ is decreasing and converges to zero. Set

$$F_k = f^{-1}([s_k, s_{k-1}])$$

for $k \in \mathbb{N}$. Then F_k are compact and $t_k \in F_n$ if and only if $k = n$. Moreover,

$$X \subset f^{-1}((0, 1]) = \bigcup_k F_k$$

and

$$P_0 \cap F_k = \{t_k\}, \quad k \in \mathbb{N}.$$

If $f(x) = c > 0$, then $x \in f^{-1}((2^{-1}c, 1])$. Since $f(t_k) \to 0$, we have that $x \notin P_0^d$. Hence, if $x \in P_0^d$, then $x \in f^{-1}(0)$. Hence $x \notin \bigcup_k F_k$. We showed that

$$P_0^d \cap \left(\bigcup_k F_k\right) = \emptyset.$$

Since X is realcompact, for every $k \in \mathbb{N}$ there exists a continuous function $f_k : \beta X \to [0, 1]$ that is positive on X and zero on t_k. Next set

$$T_n^k = f_k^{-1}([n^{-1}, 1]), \;\; n, k \in \mathbb{N}.$$

Then

$$X \subset f_k^{-1}((0, 1]) = \bigcup_n T_n^k$$

and $t_k \notin T_n^k$ for all $k, n \in \mathbb{N}$. Moreover,

$$X \subset \bigcup_k F_k \cap X \subset \bigcup_k \bigcup_n F_k \cap T_n^k, \quad P_0 \cap (F_k \cap T_n^k) \subset P_0 \cap F_k = \{t_k\}.$$

As $t_k \notin F_k \cap T_n^k$,

$$P_0 \cap (F_k \cap T_n^k) = \emptyset$$

for all $n, k \in \mathbb{N}$. Hence $P_0 \cap W = \emptyset$, and

$$P_0^d \cap W \subset P_0^d \cap \bigcup_k F_k = \emptyset,$$

where $W = \bigcup_k \bigcup_n F_k \cap T_n^k$. Therefore $\overline{P_0} \cap W = \emptyset$. We showed that there exist an infinite subset P_0 of P and an infinite sequence of compact sets $(K_n)_n$ such that

$$X \subset \bigcup_n K_n \subset \beta X, \;\; \left(\bigcup_n K_n\right) \cap \overline{P_0} = \emptyset.$$

For every $n \in \mathbb{N}$, let $g_n : \beta X \to [0,1]$ be a continuous function such that

$$g_n | K_n = 1, \quad g_n | \overline{P_0} = 0.$$

Put $g = \sum_n 2^{-n} g_n$. The function $g : \beta X \to [0,1]$ is continuous, positive on X, and zero on some subsequence of the sequence $(x_n)_n$. This shows that for every sequence $(x_n)_n$ in X^* there exists a continuous function on βX that is positive on X and vanishes on some subsequence of $(x_n)_n$.

(ii) Assume X is a strongly realcompact space of pointwise countable type and is not locally compact. Then there exist $x_0 \in X$ for which there does not exist a relatively compact open neighborhood, and a compact set K with $x_0 \in K$ that admits a countable (decreasing) basis $(U_n)_n$ of neighborhoods of K. For every $n \in \mathbb{N}$, choose $x_n \in (\overline{U_n} \setminus X)$, where the closure is taken in βX. Note that

$$(\beta X \setminus K) \cap \{x_n\}^d = \emptyset.$$

Indeed, let $x \in (\beta X \setminus K)$. Let $V \subset \beta X$ be an open neighborhood of K such that $x \in (\beta X \setminus \overline{V})$. Then there exists $n_0 \in \mathbb{N}$ such that $U_{n_0} \subset V \cap X$, so $\overline{U_{n_0}} \subset \overline{V}$. Since

$$\{x_n\}^d \subset \overline{U_{n_0}} \subset \overline{V},$$

$x \in \beta X \setminus \{x_n\}^d$. Hence $\{x_n\}^d \subset K$. This shows that X is not strongly realcompact, a contradiction.

(iii) Assume X is strongly realcompact and each $\mathscr{F}_n$ is an unbounded filterbasis on X. For each $n \in \mathbb{N}$, there exists an accumulation point of $\mathscr{F}_n$, say $x_n \in \beta X \setminus \upsilon X$. Since X is a realcompact space, $\upsilon X = X$. As X is strongly realcompact, there exists a subsequence $(x_{n_k})_k$ of $(x_n)_n$, and a positive continuous function $g \in C(X)$ such that $g(x) \le 1$ and $g^\beta(x_{n_k}) = 0$ for all $k \in \mathbb{N}$. Then $f := g^{-1} \in C(X)$. Hence f is unbounded on each $\mathscr{F}_{n_k}$ since

$$x_{n_k} \in \bigcap_{F \in \mathscr{F}_{n_k}} \overline{F} \setminus \upsilon_f(X).$$

This proves one direction of the claim (iii).

To prove the converse, assume that $(x_n)_n$ is a sequence in $\beta X \setminus X$. Then, for each $n \in \mathbb{N}$ there exists a filter $\mathscr{F}_n$ on X that converges to x_n in the space βX. But

$$x_n \in \bigcap_{F \in \mathscr{F}_n} \overline{F}, \;\; x_n \notin X = \upsilon X.$$

This shows that each $\mathscr{F}_n$ is unbounded on X. By the assumption, there exists a subsequence $(\mathscr{F}_{n_k})_k$ of $(\mathscr{F})_n$ and $f \in C(X)$ that is unbounded on each $\mathscr{F}_{n_k}$. Set

$$g(x) := (1 + |f(x)|)^{-1}$$

for each $x \in X$. Clearly, the function g is positive on X and is continuous and bounded. Therefore there exists a continuous extension g^β of g to βX, and clearly $g^\beta(x_{n_k}) = 0$. This proves that X is strongly realcompact. $\square$

Example 2.1 There is a strongly realcompact space that is not locally compact. The space $\mathbb{R}^{\mathbb{N}}$ is realcompact and not strongly realcompact.

Proof Let P be a countable and nonempty subset of $\mathbb{N}^*$. Note that the subspace $X := \mathbb{N} \cup P$ of $\beta\mathbb{N}$ is a Lindelöf space. Hence it is a realcompact space. On the other hand, since every countable and closed subset of $\beta\mathbb{N}$ is finite (see [411, p. 71]) one gets that the space $X^* = \mathbb{N}^* \setminus P$ is countably compact. Now Theorem 2.9 is applied to deduce that X is strongly realcompact. Note that X is not locally compact. The second statement follows directly from Theorem 2.9. □

If D is an absolutely convex subset of $C_c(X)$, a *hold* K of D is a compact subset of βX such that $f \in C(X)$ belongs to D if its continuous extension $f^{\beta} : \beta X \to \beta\mathbb{R}$ is identically zero on a neighborhood of K. The intersection $k(D)$ of all holds of an absolutely convex set D in $C_c(X)$ is again a hold. $k(D)$ is called a *support* of D. If, moreover, D is bornivorous, $k(D)$ is contained in υX; see [328, Lemma 10.1.9].

We also need the following fact due to Valdivia [419]; see also [328].

Lemma 2.6 (Valdivia) *Let E be a barrelled space. Let $(A_n)_n$ be an increasing sequence of absolutely convex closed subsets of E covering E. Then, for every bounded set $B \subset E$ there exists $m \in \mathbb{N}$ such that $B \subset mA_m$.*

Proof Assume that for each $n \in \mathbb{N}$ there exists $x_n \in B \setminus nA_n$. Then the sequence $(y_n)_n$, $y_n := n^{-1}x_n$ converges to zero in E, and $y_n \notin A_n$ for all $n \in \mathbb{N}$. Let $(U_n)_n$ be a decreasing sequence of absolutely convex neighborhoods of zero in E such that $U_{n+1} + U_{n+1} \subset U_n$ and $y_n \notin A_n + U_n$ for all $n \in \mathbb{N}$. Then

$$y_n \notin \overline{A_n + U_{n+1}}$$

for all $n \in \mathbb{N}$. Set

$$U := \bigcap_n \overline{(A_n + U_n)}.$$

The set U is closed, absolutely convex and absorbing in E and $y_n \notin U$ for all $n \in \mathbb{N}$. Hence U is a barrel in E. Since E is barrelled, U is a neighborhood of zero in E. This proves that U contains almost all elements of the sequence $(y_n)_n$, a contradiction. □

We are prepared to prove the following result from [228].

Theorem 2.10 (Kąkol–Śliwa) (*i*) *If X is a strongly realcompact space, $C_c(X)$ is Baire-like and bornological.*

(*ii*) *If X is locally compact, $C_c(X)$ is Baire-like and bornological if and only if X is realcompact.*

(*iii*) *If X is a space of pointwise countable type, $C_c(X)$ is bornological and Baire-like if and only if X is strongly realcompact.*

Proof (i) Let $(D_n)_n$ be a bornivorous sequence in $C_c(X)$ (i.e., every bounded set in $C_c(X)$ is absorbed by some D_n). We may assume that every bounded set in $C_c(X)$ is contained in some D_n. Note that $k(D_{m_0}) \subset X$ for some $m_0 \in \mathbb{N}$. Indeed, otherwise for every $n \in \mathbb{N}$ there exists $x_n \in k(D_n) \setminus X$. Since X is strongly realcompact, there exists a continuous function $f : \beta X \to [0, 1]$ that is positive on X and zero on some subsequence of $(x_n)_n$. Since $(D_n)_n$ is increasing, we may assume that $f(x_n) = 0$, $n \in \mathbb{N}$. The sets

$$A_m = \{y \in \beta X : f(y) > m^{-1}\}$$

are open in βX and compose an increasing sequence covering X. Since $x_n \notin \overline{A_n}$ for $n \in \mathbb{N}$, where the closure is taken in βX, we have $k(D_n) \not\subset \overline{A_n}$ for every $n \in \mathbb{N}$. This implies that $\overline{A_n}$ is not a hold of D_n for any $n \in \mathbb{N}$. Hence there exists a sequence

$$f_n \in C_c(X) \setminus D_n$$

such that the extension $f_n^{\beta} = 0$ on some neighborhood of $\overline{A_n}$. As the sequence $(f_n)_n$ converges to zero in $C_c(X)$, there exists $p \in \mathbb{N}$ such that $f_n \in D_p$ for all $n \in \mathbb{N}$, a contradiction.

We proved that there exists $m_0 \in \mathbb{N}$ such that $k(D_{m_0}) \subset X$. Next we show that there exist $m \geqslant m_0$ and $r > 0$ such that

$$\left\{f \in C(X) : \sup_{x \in k(D_m)} |f(x)| < r\right\} \subset D_m, \tag{2.6}$$

which will show that $C_c(X)$ is b-Baire-like.

To show (2.6), it is enough to prove that there exist $r > 0$ and $n \geqslant m_0$ such that

$$\left\{f \in C(X) : \sup_{x \in X} |f(x)| < r\right\} \subset D_n;$$

see [370, Theorem II.1.4]. Assume this fails. Then there exists a sequence

$$f_n \in C(X) \setminus D_n$$

such that $|f_n(x)| < n^{-1}$ for every $x \in X$ and $n \in \mathbb{N}$. Since $(f_n)_n$ converges to zero in $C_c(X)$ and $(D_n)_n$ is bornivorous, we reach a contradiction.

On the other hand, since X is realcompact, by Proposition 2.16 the space $C_c(X)$ is the inductive limit of Banach spaces and therefore $C_c(X)$ is both barrelled and bornological. By Proposition 2.12, any barrelled b-Baire-like space is Baire-like. Hence $C_c(X)$ is Baire-like. An alternative (shorter) proof of (i) uses Theorem 2.9(iii). Indeed, since a sequence $(A_n)_n$ of unbounded subsets of X provides a sequence $((A_n))_n$ of unbounded filterbases, we apply (iii) of Theorem 2.9.

(ii) Assume X is locally compact and $C_c(X)$ is Baire-like and bornological. Then X is realcompact. The rest follows from Theorem 2.9.

(iii) Assume X is of pointwise countable type and the space $C_c(X)$ is bornological and Baire-like. Then X is realcompact by Proposition 2.16. We prove that X is locally compact. Let $x \in X$. Since X is of pointwise countable type, there exist a compact set K in X containing x and a decreasing basis $(U_n)_n$ of open neighborhoods of the set K. Then the absolutely convex and closed sets

$$W_n = \left\{ f \in C(X) : \sup_{x \in U_n} |f(x)| \leqslant n \right\}$$

cover $C_c(X)$. By assumption, there exist $n \in \mathbb{N}$, $\varepsilon > 0$, and a compact subset S of X such that

$$\left\{ f \in C(X) : \sup_{y \in S} |f(y)| < \varepsilon \right\} \subset W_n.$$

Hence $U_n \subset S$. We proved that X is locally compact. Theorem 2.9 is applied to deduce that X is strongly realcompact. For the converse, we again apply Theorem 2.9 and the previous case. □

From Theorem 2.9, we know that a realcompact space X for which $\beta X \setminus X$ is countably compact is strongly realcompact. As concerns the converse to Theorem 2.10, we note only the following fact.

Proposition 2.22 *If $C_c(X)$ is a Baire space and X is realcompact, then $\beta X \setminus X$ is pseudocompact* (*i.e., its image under any real-valued continuous function is bounded*).

Proof Assume that $X^* := \beta X \setminus X$ is not pseudocompact. Then there exists a locally finite sequence $(U_n)_n$ of open disjoint subsets in X^*. Then, by regularity, we obtain a sequence $(V_n)_n$ of open nonempty sets in βX such that

$$\emptyset \neq V_n \cap X^* \subset \overline{V_n} \cap X^* \subset U_n,$$

where the closure is taken in βX. Since X is realcompact, $\overline{A_n} = \overline{V_n}$, where $A_n := V_n \setminus X^* = V_n \cap X$. As every topologically bounded set in X is relatively compact by Proposition 2.15 (since $C_c(X)$ is barrelled), A_n is not topologically bounded in X for $n \in \mathbb{N}$. Since $C_p(X)$ is a Baire space, by [266] there exists a continuous function $f \in C(X)$ and a subsequence $(A_{n_k})_k$ such that $f|A_{n_k}$ is unbounded for each $k \in \mathbb{N}$.

Let $\mathbb{R}_\infty$ be the Alexandrov one-point compactification of $\mathbb{R}$, and let $f^\infty : \beta X \to \mathbb{R}_\infty$ be the continuous extension of f. As each $f|A_{n_k}$ is unbounded, there exists a sequence $(x_k)_k$ such that $f^\infty(x_k) = \infty$ for each $k \in \mathbb{N}$ and

$$x_k \in \overline{A_{n_k}} \setminus X \subset U_{n_k}.$$

Since the sequence $(U_{n_k})_k$ is locally finite in X^*, we deduce that the sequence $(x_k)_k$ has an adherent point $x \in X$. But $f^\infty(x_k) = \infty$ for each $k \in \mathbb{N}$, so $f(x) = f^\infty(x) = \infty$. This provides a contradiction since $f(X) \subset \mathbb{R}$. We proved that $\beta X \setminus X$ is pseudocompact. □

We need the following fact following from [285, Theorem 5.3.5].

Lemma 2.7 *If there exists an infinite family $\mathscr{K}$ of nonempty compact subsets of X such that* (*i*) *for every compact set L of X there exists $K \in \mathscr{K}$ with $K \cap L = \emptyset$ and* (*ii*) *any infinite subfamily of $\mathscr{K}$ is not discrete, then $C_c(X)$ is not a Baire space.*

There are several examples of barrelled spaces $C_c(X)$ that are not Baire; see, for example, [266], [398], [278]. The next example, from [228], is motivated by [48, Example 1.11].

Example 2.2 (*Kąkol–Śliwa*) There exists a locally compact and strongly realcompact space X such that the bornological Baire-like space $C_c(X)$ is not Baire.

Proof Let X be the set $\mathbb{R}$ of reals endowed with a topology defined as follows:

(a) For every $t \in \mathbb{Q}$, the set $\{t\}$ is open in X.

(b) For every $t \in \mathbb{R} \setminus \mathbb{Q}$, there exists a sequence $(t_n)_n \subset \mathbb{Q}$ that converges to t such that the sets

$$V_n(t) = \{t\} \cup \{t_m : m \geqslant n\},$$

$n, m \in \mathbb{N}$, form a base of neighborhoods of t in X.

(c) For all dense sets $A, B \subset \mathbb{Q}$ in the natural topology of $\mathbb{R}$, the set $\overline{A} \cap \overline{B}$ is non-empty, where the closure is taken in X.

As the space X is a locally compact and realcompact space, it is strongly realcompact by Theorem 2.9.

We prove that X satisfies conditions (i) and (ii) from Lemma 2.7. Let $(P_n)_n$ be a sequence of pairwise disjoint finite subsets of $\mathbb{R} \setminus \mathbb{Q}$ such that for any subsequence $(P_{n_k})_k$ the set $\bigcup_k P_{n_k}$ is dense in $\mathbb{R}$. Set

$$P_n = \{t_k^n : 1 \leqslant k \leqslant m_n\}$$

for all $n \in \mathbb{N}$. For all $n, m \in \mathbb{N}$, the set

$$K_{n,m} = \bigcup_{1 \leqslant k \leqslant m_n} V_m(t_k^n)$$

is nonempty and compact in X.

Claim 2.6 $\mathscr{K} = \{K_{n,m} : n, m \in \mathbb{N}\}$ satisfies (i).

Indeed, any compact subset L of X is contained in a set of the form

$$V_1(t^1) \cup \cdots \cup V_1(t^m) \cup \{p_1, \ldots, p_k\},$$

where $m, k \in \mathbb{N}$ and $t^1, t^2, \ldots, t^m \in \mathbb{R} \setminus \mathbb{Q}$, $p_1, \ldots, p_k \in \mathbb{Q}$. In fact, the set $L \cap (\mathbb{R} \setminus \mathbb{Q})$ is finite since the open cover

$$\{V_1(t) : t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\} \cup \{\{q\} : q \in L \cap \mathbb{Q}\}$$

of L has a finite subcover. On the other hand,

$$L \setminus \left(\bigcup\{V_1(t) : t \in L \cap (\mathbb{R} \setminus \mathbb{Q})\}\right)$$

is finite since $X^d \subset (\mathbb{R} \setminus \mathbb{Q})$. Hence $\mathscr{K}$ indeed satisfies (i).

Claim 2.7 $\mathscr{K}$ satisfies (ii).

Indeed, otherwise some infinite subfamily $\{K^n : n \in \mathbb{N}\}$ of $\mathscr{K}$ is discrete. But then sets $A = \bigcup_n K^{2n}$ and $B = \bigcup_n K^{2n+1}$ are disjoint and closed in X. Applying the property of $(P_n)_n$, one gets that $A \cap \mathbb{Q}$ and $B \cap \mathbb{Q}$ are dense in $\mathbb{R}$. Hence, by (c), the set $\overline{(A \cap \mathbb{Q})} \cap \overline{(B \cap \mathbb{Q})}$ is nonempty, a contradiction. We proved that $C_c(X)$ is not Baire. □

Tkachuk [398] provided an example of a countable space X that has exactly one nonisolated point, has no infinite topologically bounded sets (hence $C_p(X)$ is barrelled by Proposition 2.17) and $C_p(X)$ is not Baire. We show another example of this type.

Set $X := \mathbb{N}^2 \cup \{x\}$ for $x \notin \mathbb{N}^2$, where all points of $\mathbb{N}^2$ are isolated in X, and set $X_n := \{(m, n) : m \in \mathbb{N}\}$ for each $n \in \mathbb{N}$. The basis $\mathscr{B}(x)$ at x is formed by the sets

$$\{U \subset X : x \in U,\ |\{n : |(X \setminus U) \cap X_n| = \infty\}| < \infty\}.$$

The space X (originally defined by Arens) is completely regular and Hausdorff. Since X is countable, it is Lindelöf.

Example 2.3 The space $C_p(X)$ for the Arens space X is barrelled and not Baire.

Proof If $K \subset X$ is compact, K is finite (so X is hemicompact). Indeed, note that $|K \cap X_n| < \infty$ for each $n \in \mathbb{N}$ and

$$|\{n \in \mathbb{N} : K \cap X_n \neq \emptyset\}| < \infty.$$

Clearly, $C_p(X)$ is metrizable since X is countable. By Proposition 2.17 and Proposition 2.15, the space $C_c(X) = C_p(X)$ is barrelled.

We show that $C_p(X)$ is not Baire. First observe that for each $f \in C(X)$ there exists $n \in \mathbb{N}$ such that the restriction $f|X_n$ is bounded. Indeed, otherwise, if there exists $f \in C(X)$ that is unbounded on each X_n, then f is unbounded on each open neighborhood of the point x. This implies that f is discontinuous at x, a contradiction. Now set

$$B_{m,n} := \left\{ f \in C(X) : \sup_{y \in X_m} |f(y)| \leq n \right\}$$

for each $n, m \in \mathbb{N}$. The sets $B_{m,n}$ are absolutely convex and closed. Clearly, $C(X) = \bigcup_n B_{m,n}$. Assume that $C_p(X)$ is a Baire space. Then there exist $m, n \in \mathbb{N}$ such that $B_{m,n}$ is a neighborhood of zero in $C_p(X)$. Hence there exists a compact set $K \subset X$ and $\varepsilon > 0$ such that

$$\left\{ f \in C(X) : \sup_{y \in K} |f(y)| < \varepsilon \right\} \subset B_{m,n}. \tag{2.7}$$

We claim that $X_m \subset K$ (which will provide a contradiction). If there exists $y \in X_m \setminus K$, then there exists $f \in C(X)$ such that $f(z) = 0$ for each $z \in K$ and $f(y) > n$. This yields a contradiction with (2.7). $\square$

The following problem is motivated by the previous results.

Problem 2.1 *Characterize a strongly realcompact space X in terms of topological properties of $C_c(X)$ (or $C_p(X)$).*

2.6 Pseudocompact spaces, Warner boundedness and spaces $C_c(X)$

In this section, we characterize pseudocompact spaces. For example, we show that X is pseudocompact, a Warner-bounded set or $C_p(X)$ is a (df)-space if and only if for each sequence $(\mu_n)_n$ in the dual $C_c(X)'$ of $C_c(X)$ there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is weakly bounded, strongly bounded, or equicontinuous, respectively.

This result will be used to provide an example of a (df)-space $C_c(X)$ that is not a (DF)-space. This solves an open question; see [231]. Parts of this section will be used to present concrete examples of quasi-Suslin spaces that are not K-analytic.

Buchwalter [68] called a topological space X *Warner bounded* if for every sequence $(U_n)_n$ of nonempty open subsets of X there exists a compact set $K \subset X$ such that $U_n \cap K \neq \emptyset$ for infinitely many $n \in \mathbb{N}$. In fact, Buchwalter required that $(U_n)_n$ be disjoint, but due to regularity of X this condition could be omitted. Warner has already observed [413] that any Warner-bounded space is pseudocompact.

First we provide the following useful analytic characterization of Warner boundedness; see [232]. In this section, X always means a completely regular Hausdorff topological space.

Theorem 2.11 (Kąkol–Saxon–Todd) *$C_c(X)$ does not contain a dense subspace $\mathbb{R}^{\mathbb{N}}$ if and only if X is Warner bounded.*

For the proof, we need the following lemma.

Lemma 2.8 *(a) An lcs E contains a dense subspace G of $\mathbb{R}^{\mathbb{N}}$ if and only if there exists a sequence $(w_n)_n$ of nonzero elements in E such that every continuous seminorm in E vanishes at w_n for almost all $n \in \mathbb{N}$.*

(b) If an lcs E contains a dense subspace G of $\mathbb{R}^{\mathbb{N}}$, then the strong dual $(E', \beta(E', E))$ contains the space φ.

Proof (a) Assume that E contains a sequence as mentioned. For $(w_n)_n$ there exists a biorthogonal sequence $(v_n, u_n)_n$ in $F \times F'$, where $(v_n)_n$ is a subsequence of

$(w_n)_n$, F is a linear span of v_n, $n \in \mathbb{N}$, and F' is spanned by u_n, $n \in \mathbb{N}$. Note that F is isomorphic to the linear span G of the unit vectors of $\mathbb{R}^{\mathbb{N}}$. Now assume that $\mathbb{R}^{\mathbb{N}}$ contains a dense subspace G. Let $(p_n)_n$ be a fundamental sequence of continuous seminorms on G. Then each $G_n := \bigcap_{i=1}^{n} p_i^{-1}(0)$ is an infinite-dimensional subspace of G. Therefore we can find $w_n \in G_n \setminus \{0\}$ for each $n \in \mathbb{N}$, as required.

(b) It is well known that the strong dual of $\mathbb{R}^{\mathbb{N}}$ is the space φ; see, for example, [328]. On the other hand, it is also well known that every bounded set in the completion of $\hat{G} = \mathbb{R}^{\mathbb{N}}$ is contained in the $\hat{G}$-completion of a bounded set in G; see also [328, Observation 8.3.23]. This completes the proof of (b). $\square$

Now we are ready to prove Theorem 2.11.

Proof Assume that $C_c(X)$ contains a dense subspace of $\mathbb{R}^{\mathbb{N}}$. Then there exists a sequence $(f_n)_n$ of nonzero elements of $C(X)$ that vanishes for almost all $n \in \mathbb{N}$ on any compact subset of X. Then, for each $n \in \mathbb{N}$ there exists an open nonzero set U_n in X such that $f_n(y) \neq 0$ for all $y \in U_n$. Therefore every compact set K in X misses U_n for almost all $n \in \mathbb{N}$. This shows that X is not a Warner-bounded set.

To prove the converse, assume that $(U_n)_n$ is a sequence of nonempty open sets in X such that almost all of them miss each compact set in X. Then we can select a sequence $(f_n)_n$ of nonzero continuous functions on X such that each $f_n(X \setminus U_n) = \{0\}$. Hence each continuous seminorm on $C_c(X)$ vanishes on almost all elements of the sequence $(f_n)_n$. Now we apply Lemma 2.8. $\square$

Theorem 2.12 (see [231]) looks much more interesting if we have already in mind Theorem 2.11.

Theorem 2.12 (Kąkol–Saxon–Todd) *The following assertions are equivalent for X:*

(*i*) *X is pseudocompact.*

(*ii*) *For each sequence $(\mu_n)_n$ in the weak* dual F of $C_c(X)$, there exists a sequence $(t_n)_n \subset (0, 1]$ such that $(t_n\mu_n)_n$ is bounded in F.*

(*iii*) *The weak* dual F of $C_c(X)$ is docile* (*i.e., every infinite-dimensional subspace of F contains an infinite-dimensional bounded set*).

(*iv*) *$C_c(X)$ does not contain a copy of $\mathbb{R}^{\mathbb{N}}$.*

Proof (i) $\Rightarrow$ (ii): Take a sequence $(\mu_n)_n$ in F. Set

$$t_n := (\|\mu_n\| + 1)^{-1}$$

for each $n \in \mathbb{N}$. Since X is pseudocompact, $(t_n\mu_n(f))_n$ is bounded for each $f \in C(X)$.

(ii) $\Rightarrow$ (iii) holds for any lcs.

(iii) $\Rightarrow$ (i) is clear.

(iii) $\Rightarrow$ (iv): Assume G is a subspace of $C_c(X)$ isomorphic to $\mathbb{R}^{\mathbb{N}}$. It is clear that the *weak** dual of $\mathbb{R}^{\mathbb{N}}$, and hence $(G', \sigma(G', G))$, is not docile. Using the Hahn–Banach theorem, we extend elements of G' to the whole space $C(X)$ that generate a nondocile subspace of the *weak** dual of $C_c(X)$. This contradicts (iii).

(iv) $\Rightarrow$ (i): Suppose that X is not pseudocompact. Then there exists a sequence $(U_n)_n$ of disjoint open nonempty sets in X that is locally finite. Choose $x_n \in U_n$ and $f_n \in C(X)$ such that $f_n(x_n) = 1$ and $f_n(X \setminus U_n) = \{0\}$ for each $n \in \mathbb{N}$. Note that, because the sequence $(U_n)_n$ is locally finite, the series $\sum_n a_n f_n$ converges in $C_c(X)$ for any scalar sequence $(a_n)_n$ in $\mathbb{R}^{\mathbb{N}}$. Define a map $T : \mathbb{R}^{\mathbb{N}} \to C_c(X)$ by

$$T((a_n)_n) := \sum_n a_n f_n.$$

The map T is injective. Indeed, since for each evaluation map δ_{x_m} we have

$$\delta_{x_m} \in (C_c'(X), \sigma(C_c(X)', C_c(X))),$$

and

$$\delta_{x_m}\left(\sum_n a_n f_n\right) = a_m,$$

then indeed T is an injective open map. We show that T is continuous. Each partial sum map T_N defined by

$$T_N((a_n)_n) := \sum_{n=1}^{N} a_n f_n$$

is continuous, and T is the pointwise limit of the sequence $(T_N)_N$. Since the space $\mathbb{R}^{\mathbb{N}}$ is Baire, the sequence $(T_N)_N$ is equicontinuous, and T is continuous by the classical Banach–Steinhaus theorem. Consequently, the space $C_c(X)$ contains a copy of $\mathbb{R}^{\mathbb{N}}$. $\square$

Many barrelled spaces $C_c(X)$ are not Baire-like; for example, $C_c(\mathbb{Q})$ is not Baire-like (although barrelled by Proposition 2.15) since $\mathbb{Q}$ is not locally compact, and we apply Proposition 2.19. By Theorem 2.5, the space $C_c(\mathbb{Q})$ contains φ. It is interesting that owing to Theorem 2.12 each space $C_c(X)$ that contains the nondocile space φ contains also the docile space $\mathbb{R}^{\mathbb{N}}$. So we deduce that every barrelled non-Baire-like space $C_c(X)$ contains both spaces φ and $\mathbb{R}^{\mathbb{N}}$. Note that $C_c(\mathbb{R})$ contains $\mathbb{R}^{\mathbb{N}}$ but not φ.

Theorem 2.11 and Lemma 2.8 apply to simplify essentially Warner's fundamental theorem [413, Theorem 11].

Theorem 2.13 *The following assertions are equivalent*:

(*i*) *X is Warner bounded.*

(*ii*) $[X, 1] := \{f \in C(X) : \sup_{x \in X} |f(x)| \leq 1\}$ *absorbs bounded sets in $C_c(X)$.*

(*iii*) *$C_c(X)$ has a fundamental sequence of bounded sets.*

(*iv*) *Every Cauchy sequence in $C_c(X)$ is a Cauchy sequence in the space $C_b(X)$ of the continuous bounded functions on X with the uniform Banach topology.*

(*v*) *X is pseudocompact*, *and $C_c(X)$ is sequentially complete.*

(*vi*) *X is pseudocompact*, *and $C_c(X)$ is locally complete.*

Proof (i) ⇒ (ii): Let $B \subset C_c(X)$ be a bounded set. Hence B is uniformly bounded on each compact set in X. Assume B is not absorbed by $[X, 1]$. Then for each $n \in \mathbb{N}$ there exist $f_n \in B$, and $x_n \in X$ such that $|f_n(x_n)| > n$. Consequently, for each $n \in \mathbb{N}$ there exists an open neighborhood U_n of x_n such that $|f_n(x)| > n$ for all $x \in U_n$. Since X is Warner bounded, there exists a compact set $K \subset X$ such that $K \cap U_n \neq \emptyset$ for almost all $n \in \mathbb{N}$, a contradiction since $\{f_n : n \in \mathbb{N}\} \subset B$ must be uniformly bounded on K.

(ii) ⇒ (iii): The sets $n[X, 1]$ for $n \in \mathbb{N}$ form a fundamental sequence of bounded sets.

(iii) ⇒ (i): By the assumption, the strong dual of $C_c(X)$ is metrizable. Now we apply Lemma 2.8 (b) and Theorem 2.11 to complete the proof.

(i) ⇒ (iv): By (i) ⇒ (ii), the space X is pseudocompact. To prove the second part, let $(f_n)_n$ be a null sequence in $C_c(X)$.

We show that $(f_n)_n$ is a null sequence in the uniform Banach topology of $C_b(X)$. Assume this fails. Then there exist a subsequence $(h_n)_n$ of $(f_n)_n$, $\varepsilon > 0$, and a sequence $(U_n)_n$ of nonzero open sets in X such that $|h_n(x)| > \varepsilon$ for all $x \in U_n$. Since $(h_n)_n$ converges to zero uniformly on compact sets of X, they miss U_n for almost all $n \in \mathbb{N}$. This contradicts (i).

(iv) ⇒ (v): By (iv), X is pseudocompact and $C_b(X)$ is complete.

(v) ⇒ (vi): Any sequentially complete lcs is locally complete.

(vi) ⇒ (ii): In a locally complete lcs, barrels absorb bounded sets; this fact is elementary; see, for example, [328, Corollary 5.1.10]. Therefore the set $[X, 1]$ (which is clearly closed, absolutely convex and absorbing) absorbs bounded sets. □

It is worth noticing here another interesting characterization of a completely regular Hausdorff space X to be pseudocompact (see [416]): X is pseudocompact if and only if every uniformly bounded pointwise compact set H in the space $C_b(X)$ is weakly compact.

In order to prove the main result of this section, we shall need two additional lemmas.

Lemma 2.9 *If every countable subset of X is relatively compact*, $[X, 1]$ *is bornivorous in $C_c(X)$.*

Proof If a bounded set $A \subset C_c(X)$ is not absorbed by $[X, 1]$, there exist two sequences, $(x_n)_n$ in X and $(f_n)_n$ in A, such that $(f_n(x_n))_n$ is not bounded. Since the closure of the set $\{x_n : n \in \mathbb{N}\}$ is compact in X, $[K, 1]$ is a neighborhood of zero in $C_c(X)$ that does not absorb A, a contradiction. □

We also need the following useful fact; see [328] and [360].

Proposition 2.23 *An lcs E is locally complete if and only if it is ℓ^1-complete* (*i.e., for each $(t_n)_n \in \ell^1$ and each bounded sequence $(x_n)_n$ in E, the series $\sum_n t_n x_n$ converges in E*).

Proof Let ξ be the original topology of E. Assume that E is locally complete, and fix arbitrary $(t_n)_n$ in ℓ^1 and a bounded sequence $(x_n)_n$ in E. Then the closed, absolutely convex hull B of $(x_n)_n$ is a Banach disc. Since for $r > s$ we have

$$\sum_{n\le r} t_n x_n - \sum_{n<s} t_n x_n \in (|t_s| + \cdots + |t_r|)B,$$

the series $\sum_n t_n x_n$ converges in the Banach space $(E_B, \|.\|_B)$. Therefore $\sum_n t_n x_n$ converges in E since $\xi|E_B \le \xi_B$, where ξ_B is a locally convex topology generated by the norm $\|.\|_B$.

To prove the converse, assume that E is ℓ^1-complete, and let B be a closed, bounded, absolutely convex subset of E.

Let $(y_n)_n$ be a Cauchy sequence in E_B. There exists an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that $\|y_r - y_s\| \le 2^{-k}$ if $r, s, k \in \mathbb{N}$ and $r, s \ge n_k$. Set

$$t_k := 2^{-k},\ \ x_k := 2^k(y_{n_{k+1}} - y_{n_k}).$$

Then $(x_n)_n \subset B$ is bounded and the series $\sum_n t_n x_n$ converges in E to some $x \in E$. For $r > k$, we have

$$\sum_{n\le r} t_n x_n - \sum_{n<k} t_n x_n \in (2^{-k} + \cdots + 2^{-r}) \subset 2^{-k+1}B.$$

Then $x - \sum_{n<k} t_n x_n$ is the limit of the sequence

$$\left(\sum_{n\le r} t_n x_n - \sum_{n<k} t_n x_n\right)_r$$

whose members belong to the set $2^{-k+1}B$. As the last set is closed,

$$x - \sum_{n<k} t_n x_n \in 2^{-k+1}B$$

for each $k \in \mathbb{N}$. This implies that

$$y_{n_k} = y_{n_1} + \sum_{n<k} t_n x_n \to y_{n_1} + x$$

in the space E_B. Hence $(y_n)_n$ converges in the space E_B. We proved that E is locally complete. □

An lcs E is called *dual locally complete* [424] if $(E', \sigma(E', E))$ is locally complete. The space E is called *dual ℓ^1-complete* if $(E', \sigma(E', E))$ is ℓ^1-complete. Note also a dual version of Proposition 2.23; see [360].

Proposition 2.24 *For an lcs E, the following assertions are equivalent*:

(*i*) *E is dual locally complete.*

(ii) E is dual ℓ^1-complete.

(iii) If A is an absorbing and absolutely convex subset of E, f is a linear functional over E and $f|A$ is continuous on A, then f is continuous on E.

Proof Because of Proposition 2.23, it is enough to prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii).

(ii) $\Rightarrow$ (iii): Let A be an absolutely convex and absorbing closed subset of E, and let f be a linear functional on E that is continuous on A. We may assume that there exists $x \in E$ such that $f(x) = 1$. For each $n \in \mathbb{N}$, set $A_n := nA$ and

$$C_n := nA \cap \{y \in E : f(y) = 0\}.$$

Then, for each $n \in \mathbb{N}$ there exists a continuous linear functional f_n with $f_n(x) = 1$ and $|f_n(y)| < 2^{-n}$ for each $y \in C_n$. Clearly, the sequence $(2^n(f_{n+1} - f_n))_n$ is $\sigma(E', E)$-bounded, so by assumption we deduce that

$$f = f_1 + \sum_n 2^{-n}[2^n(f_{n+1} - f_n)]$$

is continuous.

(iii) $\Rightarrow$ (ii): Choose an arbitrary sequence $(t_n)_n$ in ℓ^1, and a sequence $(f_n)_n$ in E' that is $\sigma(E', E)$-bounded. Set

$$f(x) = \sum_n t_n f_n(x)$$

for each $x \in E$. We need to show that f is continuous. Because of (iii), it is enough to show that $f|A$ is continuous at zero, where $A := \{f_n : n \in \mathbb{N}\}^\circ$. Fix $\varepsilon > 0$, and choose $k \in \mathbb{N}$ with $\sum_{n>k} |t_n| < 2^{-1}\varepsilon$. Select a neighborhood of zero U in E such that

$$\left|\left(\sum_{n \le k} t_n f_n\right)(x)\right| < 2^{-1}\varepsilon$$

for all $x \in U$. Finally, let $x \in A \cap U$. Then

$$|f(x)| \le \left|\left(\sum_{n \le k} t_n f_n\right)(x)\right| + \sum_{n>k} |t_n| < \varepsilon.$$

The proof is completed. □

A subset K of X is the *support* of some continuous linear functional λ on $C_c(X)$ (denoted by $\operatorname{supp}\lambda$) if it is the intersection of closed sets $M \subset X$ having the property that $\lambda(f) = 0$ if $f \in C(X)$ and $f(M) = \{0\}$. We need the following lemma; see [231].

Lemma 2.10 *With $E := C_c(X)$, if one of $(E', \beta(E', E))$, $(E', \sigma(E', E))$, or $(E', n(E', E))$ is both docile and locally complete, then $[X, 1]$ is bornivorous, where $(E', n(E', E))$ denotes the dual of E' endowed with the topology generated by the norm $\|\lambda\| := \sup\{|\lambda(f)| : f \in [X.1]\}$.*

Proof Assume $[X, 1]$ is not bornivorous. Then there exist a sequence $(x_n)_n$ in X and a bounded sequence $(f_n)_n$ in $C_c(X)$ such that $f_n(x_n) > n$ for each $n \in \mathbb{N}$. Since the sequence $(f_n)_n$ is bounded, the set $\{x_n : n \in \mathbb{N}\}$ is infinite. We may assume that the sequence $(x_n)_n$ is injective (i.e., $x_n \neq x_m$ for $n \neq m$). Hence the evaluation functionals δ_{x_n} form a Hamel basis for their linear span $D \subset E'$. Clearly, $\gamma = \sum_n b_n \delta_{x_n}$ for $\gamma \in D$, where all but finite numbers b_n are zero. Then $\operatorname{supp} \gamma := \{x_n : b_n \neq 0\}$ is finite, so there exists

$$\min \gamma := \min\{|b_n| : x_n \in \operatorname{supp} \gamma\}.$$

By the assumption about docility, we deduce that there exists a linearly independent sequence $(\gamma_n)_n$ in D that is bounded in each dual mentioned above. By the local completeness, the series $\sum_n a_n \gamma_n$ converges for each absolutely summable sequence $(a_n)_n$ (by Proposition 2.23), and $\sum_n a_n \gamma_n$ converges pointwisely for each $f \in [X, 1]$.

Choose inductively an absolutely summable sequence $(a_n)_n$ of nonzero elements such that $a_n \to 0$ and

$$\sum_{k>n} |a_k \gamma_k| < |a_n| \min \gamma_n$$

for each $n \in \mathbb{N}$. Hence $\gamma := \sum_n a_n \gamma_n \in E'$. This implies that there exists a compact set $K \subset X$ such that γ is bounded on the neighborhood of zero $[K, 1]$ in $C_c(X)$. On the other hand, linear independence of $(\gamma_n)_n$ yields that $\bigcup_n \operatorname{supp} \gamma_n$ is an infinite subset of $(x_n)_n$. Then, since $(f_n)_n$ is uniformly bounded, we deduce that $\bigcup_n \operatorname{supp} \gamma_n$ is not a subset of K. Let $p \in \mathbb{N}$ such that $\operatorname{supp} \gamma_p$ is not a subset of K. Fix $y \in \operatorname{supp} \gamma_p \setminus K$. The set

$$A := K \cup \{x \neq y : x \in \operatorname{supp} \gamma_p\}$$

is closed and misses y, and there exists $g \in [X, 1]$ with $g(y) = 1$ and $g(A) = \{0\}$. Since $cg \in [K, 1]$ for each scalar c, we deduce that linear functional γ vanishes on g.

On the other hand, we note that

$$|\gamma(g)| = \left|\sum_n a_n \gamma_n(g)\right| = \left|\sum_{n \leq p} a_n \gamma_n(g)\right|$$

$$\geq |a_p \gamma_p(g)| - \sum_{k>p} |a_k \gamma_k(g)| \geq |a_p| \min \gamma_p - \sum_{k>p} |a_k| \|\gamma_k\| > 0.$$

This provides a contradiction. □

Buchwalter and Schmets [69, Theorem 4.1] proved the following.

Proposition 2.25 *$C_c(X)$ is ℓ^∞-barrelled if and only if the weak dual of $C_c(X)$ is locally complete if and only if a countable union of support sets in X is relatively compact provided it is topologically bounded.*

An lcs E is called a (DF)*-space* if E admits a fundamental sequence of bounded sets and is $\aleph_0$*-quasibarrelled* (i.e., any countable union of equicontinuous sets that is $\beta(E', E)$-bounded is equicontinuous). An lcs E is $\aleph_0$*-barrelled* if any countable union of equicontinuous sets in E' that is $\sigma(E', E)$-bounded is equicontinuous.

By Warner ([413]; see also [328, Theorem 10.1.22]) we have the following proposition.

Proposition 2.26 *$C_c(X)$ is a (DF)-space if and only if each countable union of compact sets in X is relatively compact if and only if $C_c(X)$ has a fundamental sequence of bounded sets and is $\aleph_0$-barrelled.*

Recall that an lcs E is called a (df)*-space* if it admits a fundamental sequence of bounded sets and is c_0*-quasibarrelled* (i.e., every null sequence in $(E', \beta(E', E))$ is equicontinuous). The classes of (DF)-spaces and (df)-spaces will be discussed in the following chapters. Clearly, every (DF)-space is a (df)-space. It was for a long time unknown whether every (df)-space is (DF). Kąkol, Saxon and Todd [231] provided an example of a (df)-space $C_c(X)$ that is not a (DF)-space; see Example 2.4 below. To present Example 2.4, first we recall some additional concepts and facts. An lcs E satisfies the *countable neighborhood property* (cnp) if for every sequence $(U_n)_n$ of neighborhoods of zero in E there exists a sequence $(a_n)_n$ of positive scalars such that $\bigcap_n a_n U_n$ is a neighborhood of zero. From Warner [413], the space $C_c(X)$ is a (DF)-space if and only if $C_c(X)$ satisfies the cnp. It is known that all support sets satisfy the cnp; see [213], [286]. We also need a couple of definitions concerning the *Borel measures*; see [173]. A Borel measure μ on a space X is a σ-additive real-valued finite function on all Borel subsets of X. It is called *regular* if its negative and positive parts satisfy

$$\mu(A) = \sup\{\mu(K) : K \subset A \text{ is compact}\}.$$

It is known that for each regular Borel measure there exists a smallest closed set M in X such that μ vanishes on each Borel set that misses M, and M is called the support of μ (again we use the notation $\operatorname{supp}\mu$), possibly noncompact. Nevertheless, one has $\operatorname{supp}\lambda = \operatorname{supp}\mu$ if either μ, a nonnegative regular Borel measure, or $\lambda \in C_c(X)'$ is given and the other is appropriately chosen.

We are prepared to formulate and prove the main result of this section; see [231]. Note that the equivalence (ii) $\Leftrightarrow$ (xi) has already been proved by Mazon [283]. Also, (iv) $\Leftrightarrow$ (viii) was known to McCoy and Todd [286]. By $C_d(X)$ we denote the space $C(X)$ endowed with the topology of the uniform convergence on support sets of X.

Theorem 2.14 (Kąkol–Saxon–Todd) *The following assertions are equivalent for $E := C_c(X)$:*

(*i*) *E is a (df)-space.*

(*ii*) *$(E', \beta(E', E))$ is a Fréchet space.*

(*iii*) *$(E', \beta(E', E))$ is a Banach space and equals $(E', n(E', E))$.*

(*iv*) *$(E', n(E', E))$ is a Banach space.*

(*v*) *$(E', \beta(E', E))$ is docile and locally complete.*

(*vi*) *$(E', \sigma(E', E))$ is docile and locally complete.*

(*vii*) *X is pseudocompact, and $(E', \sigma(E', E))$ is locally complete.*

(*viii*) *Every countable union of support sets in X is relatively compact.*

(*ix*) *For each sequence $(\mu_n)_n$ in E', there exists a sequence $(t_n)_n \subset (0, 1]$ such that $\{t_n \mu_n)_n$ is equicontinuous.*

(*x*) *$C_d(X)$ satisfies the cnp.*

(*xi*) *Each regular Borel measure on X has compact support.*

Proof (i) $\Rightarrow$ (ii): It is clear that $(E', \beta(E', E))$ is locally complete (since the strong dual of a c_0-quasibarrelled space is locally complete [328, Proposition 8.2.23(b)]). But then $(E', \beta(E', E))$ is a Fréchet space since E has a fundamental sequence of bounded sets.

Conditions from (ii) to (vi) are equivalent since any of them implies that $[X, 1]$ is bornivorous owing to Lemma 2.10. Then $(E', \beta(E', E)) = (E', n(E', E))$, so applying the Banach–Steinhaus theorem, we deduce that $\sigma(E', E)$-bounded sets are $\beta(E', E)$-bounded.

(vi) $\Leftrightarrow$ (vii): This follows from Theorem 2.12.

(vii) $\Rightarrow$ (viii): Since X is pseudocompact, every subset of X is topologically bounded. Since by assumption $(E', \sigma(E', E))$ is locally complete, applying Proposition 2.25 we have that (viii) holds.

(viii) $\Rightarrow$ (i): Clearly, every singleton subset of X is a support set, so by Lemma 2.9 we note that $\{[X, n] : n \in \mathbb{N}\}$ is a fundamental sequence of bounded sets in E. Hence, by Proposition 2.25, the space E is ℓ^∞-barrelled and hence c_0-quasibarrelled.

(vii) $\Rightarrow$ (ix): Choose $(\mu_n)_n$ in E', and set $t_n := (\|\mu_n\| + 1)^{-1}$ for each $n \in \mathbb{N}$. The sequence $(t_n \mu_n)_n$ is uniformly bounded on the barrel $[X, 1]$, so it is $\sigma(E', E)$-bounded. Now, again applying Proposition 2.25, we have that $(t_n \mu_n)_n$ is equicontinuous.

(ix) $\Rightarrow$ (x): Let $(U_n)_n$ be a sequence of neighborhoods of zero in $C_d(X)$. Let $(K_n)_n$ be a sequence of support sets in X, and let $(a_n)_n$ be a sequence of positive scalars such that

$$\left\{ f \in C(X) : \sup_{x \in K_n} |f(x)| \le a_n \right\} := [K_n, a_n] \subset U_n$$

for each $n \in \mathbb{N}$. By μ_n we denote a positive continuous linear functional on $C_c(X)$ whose support is the set K_n. By the assumption, there exists a sequence $(t_n)_n$ of positive scalars such that $(t_n \mu_n)_n$ is equicontinuous. This implies that the series $\sum_n 2^{-n} t_n \mu_n$ has a limit $\mu \in E'$ in the topology $\sigma(E', E)$ with support K. If $f(K) = \{0\}$ for $f \in C(X)$, the positive part, f^+, of f and the negative one, f^-, vanish on K. Therefore

$$\mu(f) = \mu(f^+) = \mu(f^-) = 0.$$

Consequently, we have

$$\mu_n(f) = \mu_n(f^+) = \mu_n(f^-) = 0$$

for each $n \in \mathbb{N}$, so $K_n \subset K$. This means that

$$[K, 1] \subset [K_n, 1] \subset a_n^{-1} U_n$$

and hence $\bigcap_n a_n^{-1} U_n$ is a neighborhood of zero in $C_d(X)$. We proved (x).

(x) $\Rightarrow$ (viii): Let $(K_n)_n$ be a sequence of support sets in X. By the assumption from (x), it follows that there exists a sequence $(a_n)_n$ of positive scalars such that

$$U := \bigcap_n a_n [K_n, 1]$$

is a neighborhood of zero in $C_d(X)$. Then there exists a compact set K in X such that $[K, \varepsilon] \subset U$ for some $\varepsilon > 0$. Note that $K_n \subset K$ for each $n \in \mathbb{N}$. Indeed, assume that $x \in X \setminus K_n$. Then there exists $f \in [X, 1]$ such that $f(x) = 0$ and $f(K) = \{0\}$. Hence

$$(a_n + 1) f \in [K, \varepsilon] \subset U \subset [K_n, a_n]$$

for each $n \in \mathbb{N}$. This implies $x \notin K_n$. Since K is compact, the closure of the set $\bigcup_n K_n$ is compact in X. This proves (viii).

To complete the proof, we need to show the equivalence (viii) $\Leftrightarrow$ (xi).

(viii) $\Rightarrow$ (xi): Note that for a regular Borel measure μ on X there exist sequences $(K_n^+)_n$, $(K_n^-)_n$ of compact subsets of $\operatorname{supp}\mu^+$ and $\operatorname{supp}\mu^-$, respectively, such that

$$\mu^+(X) = \sup_n \mu^+(K_n^+), \quad \mu^-(X) = \sup_n \mu^-(K_n^-).$$

By (viii) we deduce that the closure K^+ of $\bigcup_n K_n^+$ and K^- of $\bigcup_n K_n^-$ are compact sets. This shows that μ^+ and μ^- vanish on the Borel sets that miss K^+ and K^-, respectively. Hence $\mu = \mu^+ - \mu^-$ vanishes on the Borel sets that miss $K^+ \cup K^-$. Consequently, the set $\operatorname{supp}\mu$ is a closed set of the compact set $K^+ \cup K^-$. This proves that $\operatorname{supp}\mu$ is compact.

(xi) $\Rightarrow$ (viii): Assume that $(K_n)_n$ is a sequence of nonempty support sets in X. For each $n \in \mathbb{N}$, choose a nonnegative regular Borel measure μ_n on X such that $\mu_n(X) = \{1\}$ and $\operatorname{supp}\mu_n = K_n$. Set

$$\mu(A) := \sum_n 2^{-n} \mu_n(A).$$

It is easy to see that μ is a nonnegative regular Borel measure on X whose support contains each K_n. Applying (xi), we deduce that the closure of $\bigcup_n K_n$ is compact. $\square$

Using Theorem 2.14, we provide a version of Proposition 2.26 about (df)-spaces $C_c(X)$.

Corollary 2.10 *The following assertions are equivalent*:

(*i*) $C_c(X)$ *is a* (df)*-space.*

(*ii*) $C_c(X)$ *has a fundamental sequence of bounded sets and is* ℓ^∞*-barrelled.*

Proof (i) $\Rightarrow$ (ii): By Theorem 2.14, we know that the *weak** dual of $C_c(X)$ is locally complete. Then $C_c(X)$ is ℓ^∞-barrelled.

(ii) $\Rightarrow$ (i): In general, ℓ^∞-barrelled $\Rightarrow$ c_0-barrelled $\Rightarrow$ c_0-quasibarrelled. $\square$

Recall that a topological space X satisfies the *countable chain condition* if every pairwise disjoint collection of nonempty open subsets of X is countable. We are ready to show the following interesting example.

Example 2.4 (Kąkol–Saxon–Todd) There exists a (df)-space $C_c(X)$ that is not a (DF)-space.

Proof Recall that a point p in a topological space X is a P-point if every G_δ-set containing p is a neighborhood of p; see [181, Problems 4L]. From van Mill [297], there exists a non–P-point x_0 in the closed subspace $\beta\mathbb{N}\setminus\mathbb{N}$ such that for a sequence $(K_n)_n$ of closed subsets in $\beta\mathbb{N}\setminus\mathbb{N}$ not containing x_0 (and satisfying the countable chain condition) the closure of $\bigcup_n K_n$ does not contain x_0. Support sets satisfy the countable chain condition. Then a countable union of support sets of $X := (\beta\mathbb{N}\setminus\mathbb{N})\setminus\{x_0\}$ has the same closure in $\beta\mathbb{N}$ as in X and hence is compact. Applying (viii) of Theorem 2.14, we note that $C_c(X)$ is a (df)-space. We prove that $C_c(X)$ is not a (DF)-space. Indeed, since x_0 is not a P-point of $\beta\mathbb{N}\setminus\mathbb{N}$, there exists a sequence $(U_n)_n$ of open neighborhoods of x_0 in $\beta\mathbb{N}\setminus\mathbb{N}$ such that $\bigcap_n U_n$ is not a neighborhood of x_0. This implies that x_0 belongs to the closure of the set $X\setminus\bigcap_n U_n = \bigcup_n(X\setminus U_n)$. Since each set

$$K_n := X\setminus U_n = (\beta\mathbb{N}\setminus\mathbb{N})\setminus U_n$$

is compact but the union $\bigcup_n K_n$ has noncompact closure, by Proposition 2.26 the space $C_c(X)$ is not a (DF)-space. $\square$

This concrete space $C_c(X)$ provides an example of an ℓ^∞-barrelled space $C_c(X)$ (by Corollary 2.10) that is not $\aleph_0$-quasibarrelled. This space $C_c(X)$ is not a $\aleph_0$-barrelled space. This answers an old question posed by Buchwalter and Schmets [69].

2.7 Sequential conditions for locally convex Baire-type spaces

In this section, we collect a few results (mostly from [120] and [223]) about sequential conditions for an lcs with some Baire-type assumptions. It is well known from a classical theorem of Mahowald [280] that an lcs E is barrelled if and only if every closed linear map from E into a Banach space is continuous.

It turns out that for many concrete spaces E (for example, locally complete bornological spaces E) every linear map with a sequentially closed graph from E into a Banach space is continuous. Snipes [383] characterized an lcs E (called C-*sequential*; see also [317]) for which every sequentially continuous linear map of E into a Banach space is continuous. De Wilde [110] showed the following.

Proposition 2.27 (De Wilde) *If E is the inductive limit of a family of metrizable Baire lcs spaces and F is a webbed space, then every sequentially closed linear map of E into F is continuous.*

In particular, Proposition 2.27 applies if F is an (LF)-space. Recall again for convenience that an lcs E is called *bornological* if every absolutely convex bornivorous set in E is a neighborhood of zero if and only if every linear map from E into a Banach space that transforms bounded sets into bounded sets is continuous if and only if E is the inductive limit of a family of normed spaces.

In this section, we study an lcs E having the following property (s): Every sequentially closed linear map of E into an arbitrary Fréchet space F is continuous.

We shall say that an lcs E is *s-barrelled* [120] if it satisfies the property (s) above. Clearly, for a metrizable lcs the barrelledness and s-barrelledness conditions coincide. Inductive limits and Hausdorff quotients of s-barrelled spaces and are s-barrelled; in particular inductive limits of metrizable barrelled spaces and hence ultrabornological (or locally complete bornological) spaces are s-barrelled.

In [223], we introduced and characterized a class of lcs called *CS-barrelled*, for which the condition (s) holds with *Fréchet space F* replaced by *Banach space F*. We shall say that an lcs E is CS-barrelled if every sequentially closed linear map of E into a Banach space is continuous. Clearly, the class of s-barrelled spaces is included in the class of CS-barrelled spaces. It seems to be unknown whether both classes coincide.

A sequence $(U_n)_n$ of absolutely convex subsets of an lcs E such that

$$U_{n+1} + U_{n+1} \subset U_n,$$

$n \in \mathbb{N}$, is said to be *CS-closed* if whenever $n \in \mathbb{N}$, $x_m \in U_m$, $m > n$ and $x = \sum_{m=n+1}^{\infty} x_m$ exists in E, then $x \in U_n$. The sequence $(U_n)_n$ will be called *absorbing* if every U_n is absorbing in E. Set

$$Q = \left\{(t_n)\ t_n \geq 0, \sum_n t_n = 1\right\}.$$

By a *convex series* of elements of a subset $A \subset E$ we mean a series of the form $\sum_n t_n x_n$, where $x_n \in A$ and $(t_n) \in Q$, $n \in \mathbb{N}$.

Following Jameson [211], [212], a subset $A \subset E$ is called *CS-closed* if it contains the sum of every convergent convex series of its elements. Clearly, CS-closed sets are convex. Every open (or sequentially closed) convex set is CS-closed. Every convex G_δ-subset of a Fréchet space is CS-closed; see [174]. Moreover, if $U \subset E$ is CS-closed, the sets $2^{-n}U$ for $n \in \mathbb{N}$ form a CS-closed sequence.

Let $\mathscr{U} = (U_n)_n$ be an absorbing CS-closed sequence in an lcs E. Set $N(\mathscr{U}) = \bigcap_n U_n$, and let

$$Q_{\mathscr{U}} : E \to E/N(\mathscr{U})$$

be the quotient map. Set $E_{\mathscr{U}} = E/N(\mathscr{U})$. Then the sets $Q_{\mathscr{U}}(U_n)$ form a basis of neighborhoods of zero of a metrizable locally convex topology on $E/N(\mathscr{U})$. Let

$\tilde{E}_{\mathscr{U}}$ be the completion of $E_{\mathscr{U}}$. Clearly, $Q_{\mathscr{U}} : E \to E_{\mathscr{U}}$ is continuous if and only if $Q_{\mathscr{U}} : E \to \tilde{E}_{\mathscr{U}}$ is continuous. We start with the following characterization of an lcs as s-barrelled that is due to S. Dierolf and Kąkol; see [120].

Proposition 2.28 (*i*) *For every absorbing CS-closed sequence $\mathscr{U} = (U_n)_n$ in E, the map $Q_{\mathscr{U}}$ as a map from E into $\tilde{E}_{\mathscr{U}}$ is sequentially closed and $U_{n+1} \subset Q_{\mathscr{U}}^{-1}(Q_{\mathscr{U}}(U_{n+1})) \subset U_n$, $n \in \mathbb{N}$. Hence each U_n absorbs Banach discs of E.*

(*ii*) *For every Fréchet space F and every basis $(V_n)_n$ of absolutely convex closed neighborhoods of zero in F such that $V_{n+1} + V_{n+1} \subset V_n$, and every sequentially closed linear map $T : E \to F$, the sequence $(T^{-1}(V_n))_n$ is a CS-closed sequence in E.*

(*iii*) *E is s-barrelled if and only if every absorbing CS-closed sequence $(U_n)_n$ of E is topological* (*i.e., each U_n is a neighborhood of zero*).

Proof (i) Let $(x_n)_n$ be a nullsequence in E such that $Q_{\mathscr{U}}(x_n) \to y$ in $\tilde{E}_{\mathscr{U}}$. Then there exists a sequence $(n(k))_k$ in $\mathbb{N}$ such that $Q_{\mathscr{U}}(x_{n(k+1)} - x_{n(k+2)}) \in Q_{\mathscr{U}}(U_{k+1})$ for $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, set

$$y_k = x_{n(k+1)} - x_{n(k+2)}.$$

Then $y_k \in U_{k+1} + N(\mathscr{U}) \subset U_k$ for each $k \in \mathbb{N}$. Since $(U_n)_n$ is CS-closed and

$$x_{n(k+1)} = \sum_{m=k}^{\infty} y_m = \sum_{m=1}^{\infty} y_{m+k-1},$$

$x_{n(k+1)} \in U_{k-1}$ for each $k \in \mathbb{N}$. This implies that $Q_{\mathscr{U}}(x_{n(k)}) \to 0$, so $y = 0$.

If B is a Banach disc in E, the map

$$Q_{\mathscr{U}} \circ J_B : [B] \to \tilde{E}_{\mathscr{U}}$$

is closed, where $[B]$ denotes the Banach space E_B. Then $Q_{\mathscr{U}} \circ J_B$ is continuous by Proposition 2.27. Hence $Q_{\mathscr{U}}(B)$ is bounded in $E_{\mathscr{U}}$. This shows that B is absorbed by

$$Q_{\mathscr{U}}^{-1}(Q_{\mathscr{U}}(U_{n+1})) \subset U_n.$$

(ii) Fix $m \in \mathbb{N}$, and for all $n \geqslant m + 1$ choose $y_n \in T^{-1}(V_n)$ such that $y := \sum_{n=m+1}^{\infty} y_n$ exists in E. Then $(\sum_{n=m+1}^{k} T(y_n))_{k \in \mathbb{N}}$ is a Cauchy sequence in F and hence converges to some $z \in V_m$. Since T is sequentially closed, $z = T(y)$. Consequently, $y \in T^{-1}(V_m)$.

(iii) This is a direct consequence of the previous cases (i) and (ii). □

By Proposition 2.28, we have the following corollary.

Corollary 2.11 *An lcs E is CS-barrelled if and only if every absorbing CS-closed set in E is a neighborhood of zero.*

It turns out that the class of s-barrelled spaces is quite far from the class of barrelled spaces. We show that the space $\mathbb{R}^{\mathbb{R}}$ contains a dense Baire and bornological subspace that is not s-barrelled (see Example 2.5 below).

We will need the following technical facts. Let $G = (G, \tau)$ be an lcs, let L be a sequentially closed linear subspace, and let B be a Banach disc in G. Let $E = L + [B]$, and fix a linear subspace $M \subset [B]$ such that $L \cap M = \{0\}$ and $L + M = E$. Let $q : E \to E/L$ be the quotient map, and set $r = (q|M)^{-1}$. By $j : M \to [B]$ and

$$p : [B] \to [B]/([B] \cap L),$$

we denote the inclusion map and the quotient map, respectively. Then $[B] \cap L$ is closed in the Banach space $[B]$, so $[B]/([B] \cap L)$ is a Banach space.

We need the following lemma.

Lemma 2.11 *The linear map*

$$f := p \circ j \circ r \circ q : (E, \tau|E) \to [B]/([B] \cap L)$$

is sequentially closed. If L is not closed in $(E, \tau|E)$, *then* f *is not closed.*

Proof First we show that the map f is sequentially closed. Indeed, let $(x_n)_n$ be a null sequence in $(E, \tau|E)$, and let $y \in [B]/([B] \cap L)$ be such that $f(x_n) \to y$ in $[B]/([B] \cap L)$. Then there exist a sequence $(z_n)_n$ in $[B]$ and an element $z \in [B]$ such that $p(z_n) = f(x_n)$, $p(z) = y$ and $z_n \to z$ in the space $[B]$. This implies that $z_n \to z$ in the space $(E, \tau|E)$.

Moreover, for each $n \in \mathbb{N}$, one has

$$z_n - r(q(x_n)) \in [B] \cap L.$$

On the other hand, $r(q(x_n)) - x_n \in L$ for each $n \in \mathbb{N}$. Consequently,

$$z_n - x_n = (z_n - r(q(x_n))) + (r(q(x_n)) - x_n) \in L.$$

Since $z_n - x_n \to z$ in $(E, \tau|E)$, and L is sequentially closed, $z \in L$. Also $z \in [B]$, so $y = p(z) = 0$.

Note also that the map f is not closed. It is enough to show that the kernel $\ker f$ of f is not closed in E; this is a simple consequence of the fact that $L = \ker f$. $\square$

Now we are ready to present the promised example. Recall that a subspace F of an lcs E is *locally dense* if for every $x \in E$ there exists a Banach disc B in E such that E_B admits a sequence in F that converges to x in E_B (or, equivalently, if for every $x \in E$ there exists a sequence $(x_n)_n$ in F and an increasing scalar sequence $(a_n)_n$ with $a_n \to \infty$ such that $a_n(x_n - x) \to 0$). Note that every lcs that contains a locally dense bornological subspace is bornological; see [328, Proposition 6.2.7].

For the following example, we refer to [120].

Example 2.5 The space $\mathbb{R}^{\mathbb{R}}$ is s-barrelled and contains a dense subspace that is Baire bornological but is not CS-barrelled.

Proof Set $G := \mathbb{R}^{\mathbb{R}}$, and let

$$L := \{(x_t)_{t\in\mathbb{R}} \in \mathbb{R}^{\mathbb{R}} : |\{t \in \mathbb{R} : x_t \neq 0\}| \leq \aleph_0\}$$

be endowed with the product topology. Then L is a sequentially closed dense Baire subspace of G. Let C be the absolutely convex hull of the set $\{\varphi_{[a,b)} : \; a, b \in \mathbb{Q}, \; a < b\}$, where $[a, b)$ is the half-closed interval and $\varphi_{[a,b)}$ is the characteristic function. Then the closure A of C is a Banach disc in G. Next, by B we denote the closure of C in the Banach space $[A]$. Then B is a Banach disc in $[A]$ and hence in G. Let $E = L + [B]$ with the topology of G. Now Lemma 2.11 is applied to deduce that there exists a sequentially closed discontinuous linear map of E into a Banach space $[B]/([B] \cap L)$. This shows that E is not CS-barrelled. The space E contains L as a dense subspace. Hence E is a Baire space. We prove that E is a bornological space. Indeed, by [118] the space $L + [C]$ is a bornological subspace of G. Since $[C]$ is locally dense in $[B]$ with respect to the norm topology of $[A]$, $[C]$ is locally dense in $[B]$. Therefore $L + [C]$ is locally dense in $L + [B]$. As every lcs that contains a locally dense bornological subspace is bornological, E is bornological. Finally, since G is the inductive limit of Banach spaces (that means G is ultrabornological), G is s-barrelled by Proposition 2.27. □

Note that the space E from Example 2.5 is bornological and Baire, which is not the inductive limit of metrizable barrelled spaces. In fact, otherwise E would be CS-barrelled, which is impossible.

It is well known that an lcs E is barrelled if and only if every pointwise bounded family of continuous linear maps of E into a Fréchet space is equicontinuous; see [328, Proposition 4.1.3]. One can ask if a similar result for the s-barrelledness holds if the assumption *continuous linear maps* is replaced by *sequentially continuous linear maps*. The answer is negative. In fact, if E is a bornological and barrelled space, then every pointwise bounded family of sequentially continuous linear maps of E into a Fréchet space is equicontinuous. On the other hand, the space E from Example 2.5 is bornological and barrelled and is not s-barrelled.

The next example uses the following simple fact for which the proof is obvious.

Proposition 2.29 *If F is an s-barrelled locally dense subspace of an lcs E, then E is s-barrelled.*

Example 2.6 (Dierolf–Kąkol) There exists an bornological s-barrelled lcs that is not the inductive limit of metrizable barrelled spaces.

Proof Let F be an (LB)-space of Banach spaces $(F_n)_n$ that admits a bounded set not bounded in any step F_n (i.e., the space F is not *regular*); for such examples, we refer the reader for example to [328, Example 6.4.5]). Let E be the completion of F, and choose $x \in E \setminus F$ for which there exists a sequence $(x_n)_n$ in F that locally converges to x. Since F is s-barrelled, F is locally dense in $G := F + [x]$ and Proposition 2.29 applies to deduce that G is s-barrelled. As F is bornological and F is locally dense in G, then G is bornological. Hence the quotient space $G/[x]$

is s-barrelled and bornological. Let $q : G \to G/[x]$ be the quotient map. Then the restriction map

$$q|F : F \to G/[x]$$

is continuous and bijective. We prove that $G/[x]$ is not the inductive limit of metrizable barrelled spaces. Indeed, assume that H is a metrizable and barrrelled space, and let $j : H \hookrightarrow G/[x]$ be a continuous inclusion. The map

$$(q|F)^{-1} \circ j : H \to F$$

has the closed graph from a metrizable Baire-like space into an (LB)-space, so we apply Theorem 2.7 to deduce that this map is continuous. Since $q|F$ is not open, $G/[x]$ is not the inductive limit of metrizable and barrelled spaces. □

We also refer the reader to [120, Example 2.7] to find the proof of the following interesting fact.

Example 2.7 There exists an s-barrelled space that is not bornological.

Note that s-barrelled spaces enjoy some properties typical for bornological spaces. For example, we prove a Mackey–Ulam theorem for s-barrelled spaces. Let $\{E_t : t \in T\}$ be a family of lcs. Let E be the product $\prod_t E_t$. For $x \in E$, set $E_x = \prod_t [x_t]$. If $x_t \neq 0$ for every $t \in T$, then E_x is called a *simple subspace* of E; see [114]. Clearly, E_x is isomorphic to $\mathbb{R}^T$. De Wilde [114, Theorem 1] observed that a linear map of E into an lcs F is continuous if and only if its restrictions to all factor subspaces and to simple subspaces of E are continuous.

Recall (see [328, Definition 6.2.21]) that a set T satisfies the *Mackey–Ulam condition* if no Ulam measure can be defined on it (i.e., no (0, 1)-valued measure $\mathfrak{m}$ on the set 2^T of all the subsets of T with $\mathfrak{m}(T) = 1$ and $\mathfrak{m}(\{t\}) = 0$ for all $t \in T$ can be defined). It is known (see [328, Theorem 6.2.23]) that T satisfies the Mackey–Ulam condition if and only if $\mathbb{R}^T$ is bornological.

We shall need the following fact due to López-Pellicer [275]; see also [276], [328].

Lemma 2.12 *For a completely regular Hausdorff space X, the space $C_c(X)$ is bornological if and only if every weakly sequentially continuous functional on $C_c(X)$ is continuous.*

Proof Assume that every linear weakly sequentially continuous functional on $C_c(X)$ is continuous. We prove that $C_c(X)$ is bornological. It is enough to show that X is realcompact and apply Proposition 2.16.

Fix $x \in \upsilon X$. Define $\xi : C_c(X) \to \mathbb{R}$ by $\xi(f) := f^{\upsilon}(x)$ for each $f \in C_c(X)$. Note that ξ transforms bounded sets into bounded sets. Indeed, let $(f_n)_n$ be a bounded sequence in $C_c(X)$ such that $|\xi(f_n)| \geq n$ for each $n \in \mathbb{N}$. Since there exists $y \in X$ such that $f_n^{\upsilon}(x) = f_n(y)$ for each $n \in \mathbb{N}$, we have $f_n(y) \geq n$ for each $n \in \mathbb{N}$, yielding a contradiction. By the assumption, the map ξ is continuous. Hence there exists $\varepsilon > 0$ and a compact set $K \subset X$ such that

$$|\xi(f)| \leq \varepsilon \sup_{z \in K} |f(z)|$$

for each $f \in C(X)$. Then $x \in K$; otherwise there exists $g \in C(X)$ such that $g|K$ vanishes, and $g^{\upsilon}(x) = \xi(g) = 1$, a contradiction. Hence $\upsilon X = X$, so X is realcompact. □

Now we prove the following theorem from [120].

Theorem 2.15 *Let $\{E_t : t \in T\}$ be a family of s-barrelled spaces.*
(*i*) *The product $E = \prod_t E_t$ is s-barrelled if and only if $\mathbb{R}^T$ is s-barrelled.*
(*ii*) $\mathbb{R}^T$ *is s-barrelled iff T satisfies the Mackey–Ulam condition.*

Proof (i) Since $\mathbb{R}^T$ is a minimal space (i.e., $\mathbb{R}^T$ does not admit a weaker Hausdorff vector topology), $\mathbb{R}^T$ is complemented in the product space E; see [328, Corollary 2.6.4]. Hence, since E is s-barrelled, the space $\mathbb{R}^T$ is s-barrelled. The converse follows from De Wilde's theorem [114, Theorem 1] mentioned above.

(ii) Assume T satisfies the Mackey–Ulam condition. Then $\mathbb{R}^T$ is bornological by [328, Theorem 6.2.23]. The space $\mathbb{R}^T$ is complete. Hence $\mathbb{R}^T$ is the inductive limit of Banach spaces. The space $\mathbb{R}^T$ is s-barrelled by Proposition 2.27. If T does not satisfy the Mackey–Ulam condition, $\mathbb{R}^T$ is not bornological. Endow the set T with the discrete topology. Then $\mathbb{R}^T = C_p(T) = C_c(T)$. By Lemma 2.12, there exists on $\mathbb{R}^T$ a discontinuous sequentially continuous linear functional ξ. Hence $\mathbb{R}^T$ is not s-barrelled. □

Lemma 2.12 also provides the following corollary.

Corollary 2.12 *$C_c(X)$ is s-barrelled if and only if X is realcompact.*

It is known that the completion of a bornological space need not be bornological; see [328, Notes and Remarks, p. 196]. If Z is a realcompact space whose associated k_r-space Z_{k_r} is not realcompact (in [58] Blasco provided such examples), then the completion of $C_c(Z)$ is $C_c(Z_{k_r})$. By Corollary 2.12, the space $C_c(Z)$ is s-barrelled but its completion is not. Since $C_c(X)$ is barrelled if and only if X is a μ-space (Proposition 2.15), and there exist μ-spaces that are not realcompact, Corollary 2.12 also provides barrelled spaces $C_c(X)$ that are not s-barrelled.

http://www.springer.com/978-1-4614-0528-3

Descriptive Topology in Selected Topics of Functional Analysis
Kąkol, J.; Kubiś, W.; López-Pellicer, M.
2011, XII, 496 p., Hardcover
ISBN: 978-1-4614-0528-3