Chapter 2
Univariate Right Caputo Fractional Ostrowski Inequalities

Here we present general univariate right Caputo fractional Ostrowski inequalities. One of them is proved sharp and attained. Estimates are with respect to $\|\cdot\|_p$, $1 \leq p \leq \infty$. This chapter is based on [4].

2.1 Introduction

In 1938, A. Ostrowski [8] proved the following important inequality:

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \cdot (b-a) \left\| f' \right\|_\infty, \tag{2.1}$$

for any $x \in [a,b]$. The constant $\frac{1}{4}$ is the best possible.

Since then there has been a lot of activity around these inequalities with important applications to numerical analysis and probability.

This chapter is greatly motivated and inspired also by the following result.

Theorem 2.2 (see [1]). *Let $f \in C^{n+1}([a,b])$, $n \in \mathbb{N}$ and $x \in [a,b]$ be fixed, such that $f^{(k)}(x) = 0$, $k = 1, \ldots, n$. Then it holds*

$$\left| \frac{1}{b-a} \int_a^b f(y)\,\mathrm{d}y - f(x) \right| \leq \frac{\left\| f^{(n+1)} \right\|_\infty}{(n+2)!} \cdot \left(\frac{(x-a)^{n+2} + (b-x)^{n+2}}{b-a} \right). \tag{2.2}$$

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Inequality (2.2) is sharp. In particular, when n is odd is attained by $f^{*}(y) := (y-x)^{n+1} \cdot (b-a)$, while when n is even the optimal function is

$$\overline{f}(y) := |y-x|^{n+\alpha} \cdot (b-a), \quad \alpha > 1.$$

Clearly inequality (2.2) generalizes inequality (2.1) for higher order derivatives of f.

Also in [2], see Chaps. 24–26, we gave a complete theory of left fractional Ostrowski inequalities.

2.2 Main Results

We need

Definition 2.3 ([3,5–7,9]). Let $f \in L_1([a,b])$, $\alpha > 0$. The right Riemann–Liouville fractional operator of order α by

$$I_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} f(J)\,\mathrm{d}J, \tag{2.3}$$

$\forall\ x \in [a,b]$, where Γ is the gamma function. We set $I_{b-}^{0} := I$ (the identity operator).

Definition 2.4 ([3,5–7,9]). Let $f \in AC^m([a,b])$ ($f^{(m-1)}$ is in $AC([a,b])$), $m \in \mathbb{N}$, $m = \lceil \alpha \rceil$, $\alpha > 0$ ($\lceil \cdot \rceil$ the ceiling of the number). We define the right Caputo fractional derivative of order $\alpha > 0$ by

$$D_{b-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (J-x)^{m-\alpha-1} f^{(m)}(J)\,\mathrm{d}J, \quad \forall\ x \le b. \tag{2.4}$$

If $\alpha = m \in \mathbb{N}$, then

$$D_{b-}^{m} f(x) = (-1)^m f^{(m)}(x), \quad \forall\ x \in [a,b].$$

If $x > b$ we define $D_{b-}^{\alpha} f(x) = 0$.

We also need

Theorem 2.5 ([3]). *Let $f \in AC^m([a,b])$, $x \in [a,b]$, $\alpha > 0$, $m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{k=0}^{m-1} \frac{f^{(k)}(b)}{k!}(x-b)^k + \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J)\,\mathrm{d}J, \tag{2.5}$$

the right Caputo fractional Taylor formula with integral remainder.

We give

Theorem 2.6. *Let $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. Assume $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{\Gamma(\alpha+2)} (b-a)^{\alpha}. \tag{2.6}$$

Proof. Let $x \in [a,b]$. We have

$$f(x) - f(b) = \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} D_{b-}^{\alpha} f(J)\, \mathrm{d}J.$$

Then

$$\begin{aligned}
|f(x) - f(b)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| \mathrm{d}J \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (J-x)^{\alpha-1}\, \mathrm{d}J \right) \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \\
&= \frac{1}{\Gamma(\alpha)} \left(\left. \frac{(J-x)^{\alpha}}{\alpha} \right|_x^b \right) \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \\
&= \frac{1}{\Gamma(\alpha+1)} (b-x)^{\alpha} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}.
\end{aligned}$$

Therefore

$$|f(x) - f(b)| \leq \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}, \quad \forall\, x \in [a,b].$$

Hence it holds

$$\begin{aligned}
&\left| \frac{1}{b-a} \int_a^b f(x)\, \mathrm{d}x - f(b) \right| = \left| \frac{1}{b-a} \int_a^b (f(x) - f(b))\, \mathrm{d}x \right| \\
&\leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\, \mathrm{d}x \leq \frac{1}{b-a} \int_a^b \frac{(b-x)^{\alpha}}{\Gamma(\alpha+1)} \left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \mathrm{d}x \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} \int_a^b (b-x)^{\alpha}\, \mathrm{d}x \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} \left(- \left(\left. \frac{(b-x)^{\alpha+1}}{\alpha+1} \right|_a^b \right) \right)
\end{aligned}$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+1)} (-1) \left(0 - \frac{(b-a)^{\alpha+1}}{\alpha+1}\right)$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]}}{(b-a)\,\Gamma(\alpha+2)} \cdot (b-a)^{\alpha+1} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{\infty,[a,b]} \cdot (b-a)^{\alpha}}{\Gamma(\alpha+2)},$$

proving the claim. □

We present

Theorem 2.7. *Let* $\alpha \geq 1$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_1([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-a)^{\alpha-1}. \tag{2.7}$$

Proof. We have again

$$|f(x) - f(b)| \leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| dJ$$

$$\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \int_x^b \left| D_{b-}^{\alpha} f(J) \right| dJ$$

$$\leq \frac{1}{\Gamma(\alpha)} (b-x)^{\alpha-1} \left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}.$$

Hence

$$|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}, \quad \forall\, x \in [a,b].$$

Therefore

$$\left| \frac{1}{b-a} \int_a^b f(x)\,dx - f(b) \right| \leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\,dx$$

$$\leq \frac{1}{b-a} \int_a^b \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha)} (b-x)^{\alpha-1}\,dx$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \int_a^b (b-x)^{\alpha-1}\,dx$$

$$= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{(b-a)\,\Gamma(\alpha)} \frac{(b-x)^{\alpha}}{\alpha} = \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_1([a,b])}}{\Gamma(\alpha+1)} (b-x)^{\alpha-1},$$

proving the claim. □

We continue with

Theorem 2.8. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\alpha > 1 - \frac{1}{p}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Assume that* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, *and* $D_{b-}^{\alpha} f \in L_q([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)\left(p(\alpha-1)+1\right)^{\frac{1}{p}} \left(\alpha + \frac{1}{p}\right)} (b-a)^{\alpha-1+\frac{1}{p}}. \tag{2.8}$$

Proof. We have again

$$\begin{aligned}
|f(x) - f(b)| &\leq \frac{1}{\Gamma(\alpha)} \int_x^b (J-x)^{\alpha-1} \left| D_{b-}^{\alpha} f(J) \right| \mathrm{d}J \\
&\leq \frac{1}{\Gamma(\alpha)} \left(\int_x^b (J-x)^{p(\alpha-1)}\,\mathrm{d}J \right)^{\frac{1}{p}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q \mathrm{d}J \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left(\int_x^b \left| D_{b-}^{\alpha} f(J) \right|^q \mathrm{d}J \right)^{\frac{1}{q}} \\
&\leq \frac{1}{\Gamma(\alpha)} \frac{(b-x)^{(\alpha-1)+\frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}.
\end{aligned}$$

Therefore

$$|f(x) - f(b)| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} (b-x)^{\alpha-1+\frac{1}{p}}, \quad \forall\, x \in [a,b].$$

Hence

$$\begin{aligned}
\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| &\leq \frac{1}{b-a} \int_a^b |f(x) - f(b)|\,\mathrm{d}x \\
&\leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{(b-a)\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \int_a^b (b-x)^{\alpha-1+\frac{1}{p}}\,\mathrm{d}x \\
&= \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_q([a,b])}}{\Gamma(\alpha)(p(\alpha-1)+1)^{\frac{1}{p}}} \frac{(b-a)^{\alpha-1+\frac{1}{p}}}{\left(\alpha+\frac{1}{p}\right)}.
\end{aligned}$$

□

Corollary 2.9. *Let* $\alpha > \frac{1}{2}$, $m = \lceil \alpha \rceil$, $f \in AC^m([a,b])$. *Suppose* $f^{(k)}(b) = 0$, $k = 1, ..., m-1$, $D_{b-}^{\alpha} f \in L_2([a,b])$. *Then*

$$\left| \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x - f(b) \right| \leq \frac{\left\| D_{b-}^{\alpha} f \right\|_{L_2([a,b])}}{\Gamma(\alpha)\left(\sqrt{2\alpha-1}\right)\left(\alpha+\frac{1}{2}\right)} (b-a)^{\alpha-\frac{1}{2}}. \tag{2.9}$$

We finish the chapter with

Proposition 2.10. *Inequality (2.6) is sharp, namely it is attained by*

$$f(x) = (b-x)^{\alpha}, \ \alpha > 0, \alpha \notin \mathbb{N}, x \in [a,b].$$

Proof. Notice that $(b-x)^{\alpha} \in AC^{m}([a,b])$. We observe that

$$\begin{aligned} f'(x) &= -\alpha (b-x)^{\alpha-1}, \\ f''(x) &= (-1)^2 \alpha (\alpha-1)(b-x)^{\alpha-2}, \\ &\vdots \\ f^{(m-1)}(x) &= (-1)^{m-1} \alpha (\alpha-1)(\alpha-2)\cdots(\alpha-m+2)(b-x)^{\alpha-m+1}, \end{aligned}$$

and

$$f^{(m)}(x) = (-1)^{m} \alpha (\alpha-1)(\alpha-2)\cdots(\alpha-m+2)(\alpha-m+1)(b-x)^{\alpha-m}.$$

Thus

$$\begin{aligned} D_{b-}^{\alpha} f(x) &= \frac{(-1)^{2m}}{\Gamma(m-\alpha)} \alpha(\alpha-1)\cdots(\alpha-m+1) \int_x^b (J-x)^{m-\alpha-1}(b-J)^{\alpha-m}\,\mathrm{d}J \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{\Gamma(m-\alpha)} \int_x^b (b-J)^{(\alpha-m+1)-1}(J-x)^{(m-\alpha)-1}\,\mathrm{d}J \\ &= \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{\Gamma(m-\alpha)} \frac{\Gamma(\alpha-m+1)\Gamma(m-\alpha)}{\Gamma(1)} \\ &= \alpha(\alpha-1)\cdots(\alpha-m+1)\Gamma(\alpha-m+1) = \Gamma(\alpha+1). \end{aligned}$$

That is,

$$D_{b-}^{\alpha} f(x) = \Gamma(\alpha+1), \quad \forall\, x \in [a,b].$$

Also we see that $f^{(k)}(b) = 0$, $k = 0, 1, ..., m-1$, and $D_{b-}^{\alpha} f \in L_{\infty}([a,b])$. So f fulfills all assumptions. Next we see

$$\text{R.H.S. (2.6)} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+2)}(b-a)^{\alpha} = \frac{(b-a)^{\alpha}}{(\alpha+1)}.$$

$$\begin{aligned} \text{L.H.S. (2.6)} &= \frac{1}{b-a} \int_a^b (b-x)^{\alpha}\,\mathrm{d}x \\ &= \frac{1}{b-a} \frac{(b-a)^{\alpha+1}}{(\alpha+1)} = \frac{(b-a)^{\alpha}}{\alpha+1}, \end{aligned}$$

proving attainability and sharpness of (2.6). ∎

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