
Contents

Preface	v
----------------------	---

Part I Introduction and Survey

1 The Kepler Conjecture and Its Proof, by J. C. Lagarias	3
1. The Kepler Problem for Sphere Packing	5
2. Why the Kepler Problem Is Difficult	9
3. Local Density Approach: History	14
4. Hales Program and Hales-Ferguson Papers	16
5. Peer-Reviewing of the Hales-Ferguson Papers	17
6. Reliability of the Hales-Ferguson Proof	18
7. Formal Proof of the Kepler Conjecture	20
8. Applications of the Hales-Ferguson Proof Methodology	21
9. Contents of This Volume	22
2 Bounds for Local Density of Sphere Packings and the Kepler Conjecture, by J. C. Lagarias	27
1. Introduction	29
2. Local Density Inequalities	31
3. History	36
4. Hales-Ferguson Partition Rule and Score Function	40
5. Kepler Conjecture	47
6. Concluding Remarks	52

Part II Proof of the Kepler Conjecture

Guest Editor's Foreword	62
--------------------------------------	----

3	Historical Overview of the Kepler Conjecture, by T. C. Hales	65
1.	Introduction	67
1.1.	The face-centered cubic packing	67
1.2.	Early history, Hariot, and Kepler	69
1.3.	History	70
1.4.	The Literature.	71
2.	Overview of the Proof	74
2.1.	Experiments with other decompositions	74
2.2.	Contents of the papers	77
2.3.	Complexity	78
2.4.	Computers	78
4	A Formulation of the Kepler Conjecture, by T. C. Hales and S. P. Ferguson	83
3.	The Top-Level Structure of the Proof.	87
3.1.	Statement of theorems	87
3.2.	Basic concepts in the proof.	91
3.3.	Logical skeleton of the proof	92
3.4.	Proofs of the central claims	94
4.	Construction of the Q -System	94
4.1.	Description of the Q -system	95
4.2.	Geometric considerations	97
4.3.	Incidence relations	99
4.4.	Overlap of simplices	103
5.	V -Cells	106
5.1.	V -cells	107
5.2.	Orientation	110
5.3.	Interaction of V -cells with the Q -system	111
6.	Decomposition Stars	116
6.1.	Indexing sets	116
6.2.	Cells attached to decomposition stars	118
6.3.	Colored spaces	120
7.	Scoring	121
7.1.	Definitions	122
7.2.	Negligibility	128
7.3.	Fcc-compatibility	128
7.4.	Scores of standard clusters	129
7.5.	Scores of simplices and cones	131
7.6.	The example of a dodecahedron	132
5	Sphere Packings III. Extremal Cases, by T. C. Hales	135
8.	Local Optimality	138
8.1.	Results	138
8.2.	Rogers simplices	139
8.3.	Bounds on simplices	141
8.4.	Breaking clusters into pieces	143

8.5. Proofs	148
9. The \mathcal{S} -System	151
9.1. Overview	151
9.2. The set $\delta(v)$	152
9.3. Overlap	159
9.4. The \mathcal{S} -system defined	160
9.5. Disjointness	161
9.6. Separation of simplices of type A	162
9.7. Separation of simplices of type B	162
9.8. Separation of simplices of type C	163
9.9. Simplices of type C'	163
9.10. Scoring	164
10. Bounds on the Score in Triangular and Quadrilateral Regions	165
10.1. The function τ	165
10.2. Types	166
10.3. Limitations on types	169
10.4. Bounds on the score in quadrilateral regions	170
10.5. A volume formula	173
6 Sphere Packings IV. Detailed Bounds, by T. C. Hales	177
11. Upright Quarters	180
11.1. Erasing upright quarters	180
11.2. Contexts	181
11.3. Three anchors	182
11.4. Six anchors	183
11.5. Anchored simplices	183
11.6. Anchored simplices do not overlap	184
11.7. Five anchors	186
11.8. Four anchors	188
11.9. Summary	190
11.10. Some flat quarters	192
12. Bounds in Exceptional Regions	193
12.1. The main theorem	193
12.2. Nonagons	195
12.3. Distinguished edge conditions	196
12.4. Scoring subclusters	196
12.5. Proof	197
12.6. Preparation of the standard cluster	198
12.7. Reduction to polygons	199
12.8. Some deformations	200
12.9. Truncated corner cells	202
12.10. Formulas for truncated corner cells	203
12.11. Containment of truncated corner cells	204
12.12. Convexity	207
12.13. Proof that distances remain at least 2	208

13. Convex Polygons	211
13.1. Deformations	211
13.2. Truncated corner cells	211
13.3. Analytic continuation	212
13.4. Penalties	213
13.5. Penalties and bounds	214
13.6. Penalties	216
13.7. Constants	217
13.8. Triangles	219
13.9. Quadrilaterals	220
13.10. Pentagons	221
13.11. Hexagons and heptagons	222
13.12. Loops	222
14. Further Bounds in Exceptional Regions	225
14.1. Small dihedral angles	225
14.2. A particular 4-circuit	226
14.3. A particular 5-circuit	228
7 Sphere Packings V. Pentahedral Prisms, by S. P. Ferguson	235
15. Pentahedral Prisms	237
15.1. The main theorem	238
15.2. Propositions	238
16. The Main Propositions	243
16.1. Scoring	243
16.2. Dimension reduction	244
16.3. Proof of Proposition 15.4	246
16.4. Proof of Proposition 15.5: Top level	247
16.5. Proof of Proposition 15.5: Flat quad clusters	247
16.6. Proof of Proposition 15.5: Octahedra	248
16.7. Proof of Propositions 15.5: Pure Voronoi quad clusters	251
16.8. Pure Voronoi quad clusters: Acute case	252
16.9. Pure Voronoi quad clusters: Obtuse case	253
17. Calculations	264
17.1. Interval arithmetic	264
17.2. The method of subdivision	265
17.3. Numerical considerations	265
17.4. Calculations	266
8 Sphere Packings VI. Tame Graphs and Linear Programs, by	
 T. C. Hales	275
18. Tame Graphs	279
18.1. Basic definitions	279
18.2. Weight assignments	280
18.3. Plane graph properties	282
19. Classification of Tame Plane Graphs	283

19.1. Statement of the theorem	283
19.2. Basic definitions	283
19.3. A finite state machine	284
19.4. Pruning strategies	285
20. Contravening Graphs	288
20.1. A review of earlier results	288
20.2. Contravening plane graphs defined	292
21. Contravention is Tame	294
21.1. First properties	294
21.2. Computer calculations and their consequences	295
21.3. Linear programs	296
21.4. A noncontravening 4-circuit	299
21.5. Possible 4-circuits	300
22. Weight Assignments	300
22.1. Admissibility	301
22.2. Proof that $\text{tri}(v) > 2$	302
22.3. Bounds when $\text{tri}(v) \in \{3, 4\}$	304
22.4. Weight assignments for aggregates	306
23. Linear Program Estimates	308
23.1. Relaxation	308
23.2. The linear programs	309
23.3. Basic linear programs	310
23.4. Error analysis	312
24. Elimination of Aggregates	313
24.1. Triangle and quad branching	313
24.2. A pentagonal hull with $n = 8$	314
24.3. $n = 8$, Hexagonal hull	314
24.4. $n = 7$, Pentagonal hull	314
24.5. Type $(p, q, r) = (5, 0, 1)$	316
24.6. Summary	316
25. Branch and Bound Strategies	316
25.1. Review of internal structures	316
25.2. 3-crowded and 4-crowded upright diagonals	318
25.3. Five anchors	319
25.4. Penalties	319
25.5. Pent and Hex branching	321
25.6. Hept and Oct branching	322
25.7. Branching on upright diagonals	324
25.8. Branching on flat quarters	325
25.9. Branching on simplices that are not quarters	326
25.10. Branching on quadrilateral subregions	327
25.11. Implementation details for branching	327
25.12. Variables related to score	327
25.13. Appendix: Hexagonal inequalities	329
25.14. Conclusion	337

Part III A Revision to the Proof of the Kepler Conjecture

9	A Revision of the Proof of the Kepler Conjecture, by T. C. Hales, J. Harrison, S. McLaughlin, T. Nipkow, S. Obua, and R. Zumkeller . .	341
	Part 1. Formal Proof Initiative	344
	1. The Flyspeck project	344
	2. Blueprint edition of the Kepler Conjecture	346
	3. Formalizing the ordinary mathematics	348
	4. Standard ML reimplementations of code	350
	5. Proving nonlinear inequalities with Bernstein bases	352
	6. Tame graph enumeration	356
	7. Verifying linear programs	359
	Part 2. Addendum to and Errata in the Original Proof	361
	8. Biconnected graphs	361
	9. Errata listing	372

Part IV Initial Papers of the Hales Program

10	Sphere Packings I, by T. C. Hales	379
	1. Introduction	381
	2. The Program	384
	3. Quasi-Regular Tetrahedra	388
	4. Quadrilaterals	393
	5. Restrictions	397
	6. Combinatorics	402
	7. The Method of Subdivision	405
	8. Explicit Formulas for Compression, Volume, and Angle	406
	9. Floating-Point Calculations	417
	Appendix. Proof of Theorem 6.1	428
11	Sphere Packings II, by T. C. Hales	433
	1. Introduction	435
	2. Some Polyhedra	438
	3. The Score Attached to a Delaunay Star	441
	4. The Main Theorem	444
	Appendix	448
	Index of Symbols	451
	Index of Subjects	453

The Kepler Conjecture

The Hales-Ferguson Proof

Lagarias, J.C. (Ed.)

2011, XIV, 456 p. 11 illus., Softcover

ISBN: 978-1-4614-1128-4