Preface

With some justification, measure theory has a bad reputation. It is regarded by most students as a subject that has little æsthetic appeal and lots of fussy details. In order to make the subject more palatable, many authors have chosen to add spice by embedding measure theory inside one of the many topics in which measure theory plays a central role. In the past, Fourier analysis was usually the topic chosen, but in recent years Fourier analysis has been frequently displaced by probability theory. There is a lot to be said for the idea of introducing a running metaphor with which to motivate the technical definitions and minutiae with which measure theory is riddled. However, I[1] have not adopted this pedagogic device. Instead, I have attempted to present measure theory as an essential branch of analysis, one that has merit of its own. Thus, although I often digress to demonstrate how measure theory answers questions whose origins are in other branches of analysis, this book is about measure theory, unadorned.

In the first chapter I give a résumé of Riemann's theory of integration, including Stieltjes's extension of that theory. My reason for including Riemann's theory is twofold. In the first place, when I turn to Lebesgue's theory, I want Riemann's theory available for comparison purposes. Secondly, and perhaps more important, I believe that Riemann's theory provides many of the basic tools with which one does actual computations. Lebesgue's theory enables one to prove equalities between abstract quantities, but evaluation of those quantities usually requires Riemann's theory. The final section of Chapter 1 contains an analysis of the rate at which Riemann sums approximate his integral. In no sense is this section serious numerical analysis. On the other hand, it gives an amusing introduction to the Euler–Maclaurin formula.

Modern (i.e., *après* Lebesgue) measure theory is introduced in Chapter 2. I begin by trying to explain why countable additivity is the *sine qua non* in Lebesgue's theory of integration. This explanation is followed by the derivation of a few elementary properties possessed by countably additive measures. In the second section of the chapter, I develop a somewhat primitive procedure for constructing measures on metric spaces and then apply this procedure to the construction of Lebesgue measure $\lambda_{\mathbb{R}^N}$ on $\mathbb{R}^N$, the measure

[1]Contrary to the convention in most modern mathematical exposition and the wishes of the GTM editors, I often use the first person singular rather than the "royal we" when I expect the reader to be playing a passive role. I restrict the use of "we" to places, like proofs, where I expect the active participation of my readers.

μ_F on $\mathbb{R}$ determined by a distribution function F, and the Bernoulli measures β_p on $\{0,1\}^{\mathbb{Z}^+}$. Included here are a proof of the way in which Lebesgue measure transforms under linear maps and of the relationship between $\lambda_{\mathbb{R}^N}$ and $\beta_{\frac{1}{2}}$.

Lebesgue integration theory is taken up in Chapter 3. The basic theory is covered in the first section, and its miraculous stability (i.e., the Monotone Convergence and Lebesgue's Dominated Convergence Theorems as well as Fatou's Lemma) is demonstrated in next section. The third section is a bit of a digression. There I give a proof, based on Riesz's Sunrise Lemma, of Lebesgue's Differentiation Theorem for increasing functions.

The first section of Chapter 4 is devoted to the construction of product measures and the proof of Fubini's Theorem. As an application, in the second section I describe Steiner's symmetrization procedure and use it to prove the isodiametric inequality, which I then apply to show that N-dimensional Hausdorff measure in $\mathbb{R}^N$ is Lebesgue measure there.

In Chapter 5 I discuss several topics that are tied together by the fact that they all involve changes of variables. The first of these is the application of distribution functions to show that Lebesgue integrals can be represented as Riemann integrals, and the second topic is polar coordinates. Both of these are in § 5.1. The second section contains a proof of Jacobi's transformation formula and an application of his formula to the construction of surface measure for hypersurfaces in $\mathbb{R}^N$. My treatment of these is, from a differential geometric perspective, extremely inelegant: there are no differential forms here. In particular, my construction of surface measure is concertedly non-intrinsic. Instead, I have adopted a more geometric measure-theoretic point of view and constructed surface measure by "differentiating" Lebesgue measure. Similarly, my derivation in § 5.3 of the Divergence Theorem is devoutly extrinsic and devoid of differential form technology.

Some of the bread and butter inequalities (specifically, Jensen's, Hölder's, and Minkowski's) of integration theory are derived in the first section of Chapter 6. In the second section, these inequalities are used to study some elementary geometric facts about the Lebesgue spaces L^p as well as the mixed Lebesgue spaces $L^{(p,q)}$. The results obtained in § 6.2 are applied in § 6.3 to the analysis of boundedness properties for transformations defined by kernels on the Lebesgue space. Particular emphasis is placed on transformations given by convolution, for which Young's inequality is proved. The chapter ends with a brief discussion of Friedrichs mollifiers.

In preparation for Fourier analysis, Chapter 7 begins with a cursory introduction to Hilbert spaces. The basic L^2-theory of Fourier series is given in § 7.2 and is applied there to complete the program, started in § 1.3 of Chapter 1, of understanding the Euler–Maclaurin formula. The elementary theory of the Fourier transform is developed in § 7.3, where I first give the L^1-theory and then the L^2-theory. My approach to the latter is via Hermite functions.

The concluding chapter contains several vital topics that were either given short shrift or entirely neglected earlier. The first of these is the Radon–Nikodym Theorem, which I, following von Neumann, prove as an application of Riesz's Representation Theorem for Hilbert space. The second topic is Daniell's theory of integration, which I use first to derive the standard criterion that says when a finite, finitely additive measure admits a countably additive extension and second to derive the Riesz Representation Theorem for non-negative linear functionals on continuous functions. The final topic is Carathéodory's method for constructing measures from subadditive functions and its application to the construction of Hausdorff measures on $\mathbb{R}^N$. Although my treatment of Hausdorff measures barely touches on the many beautiful and deep aspects of this subject, I do show that the restriction of $(N-1)$-dimensional Hausdorff measure to a hypersurface in $\mathbb{R}^N$ coincides with the surface measure constructed in § 5.2.2.

It is my hope that this book will be useful both as a resource for students trying to learn measure theory on their own and as text for a course. I have used it at M.I.T. as the text for a one semester course. However, M.I.T. students are accustomed to abuse,[2] and it is likely that as a text for a one semester course elsewhere some picking and choosing will be necessary. My suggestion is that one be sure to cover the first four chapters and the first sections of Chapters 7 and 8, perhaps skipping § 1.3, § 3.3, and § 4.2. Depending on the interests of the students, one can supplement this basic material with selections from Chapters 5, 6, as well as from material that one skipped earlier.

There are exercises at the end of each section. Some of these are quite trivial and others are quite challenging. Especially for those attempting to learn the subject on their own, I strongly recommend that, at the very least, my readers look at all the exercises and solve enough of them to become facile with the techniques they wish to master. At least for me, it is not possible to learn mathematics as a spectator.

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[2]It has been said that getting an education at M.I.T. is like taking a drink from a fire hydrant.

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