

# Chapter 2

## Relations

**Abstract** We examine the notion of “relation”, which is central to this book, at various progressive levels: relations on sets (Sect. 2.1), linear relations on vector spaces (Sect. 2.2), smooth relations and reductions (Sect. 2.3), linear symplectic relations (Sect. 3.1), symplectic relations on symplectic manifolds (Chap. 3), and symplectic relations on cotangent bundles (Chap. 4).

### 2.1 Relations on sets

A *relation*  $R$  between two sets  $A$  and  $B$  is a subset of their Cartesian product:

$$R \subseteq B \times A.$$

The sets  $A$  and  $B$  are the *domain* and the *codomain* of the relation, respectively. For a relation we use the notation<sup>1</sup>

$$R: B \leftarrow A, \text{ or } A \overset{R}{\leftarrow} B.$$

The composition of two relations  $R: B \leftarrow A$  and  $S: C \leftarrow B$  is the relation

$$S \circ R: C \leftarrow A$$

defined by

$$(c, a) \in S \circ R \iff \begin{cases} \text{There exists } b \in B \text{ such that} \\ (b, a) \in R \text{ and } (c, b) \in S. \end{cases}$$

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<sup>1</sup> This is not the most common convention, but the use of the backward arrow (i.e., from right to left) turns out to be more convenient in dealing with the composition of relations.

The composition of relations is associative:

$$(S \circ R) \circ Q = S \circ (R \circ Q).$$

Then “sets” and “relations” are objects and morphisms of a category.

### 2.1.1 The transposition functor

With a relation  $R \subseteq B \times A$  we associate the *transpose relation* or *inverse relation*

$$R^\top \subseteq A \times B,$$

made of the same pairs of  $R$ , but in reverse order. The *contravariant transposition rule*

$$(S \circ R)^\top = R^\top \circ S^\top$$

holds. Hence, if we put by definition  $A^\top = A$  for all sets, the transposition operator  $^\top$  is a contravariant functor in the category of set relations.

A relation  $R \subseteq B \times A$  is *symmetric* if  $A = B$  and  $R^\top = R$ .

A map  $\rho: A \rightarrow B$  can be interpreted as a relation by its graph,  $R = \text{graph}(\rho) \subset B \times A$ :

$$(b, a) \in R \iff b = \rho(a).$$

Hence, a relation  $R \subseteq B \times A$  is a map if and only if

$$\begin{cases} R^\top \circ B = A, \\ (b, a) \in R, (b', a) \in R \implies b' = b. \end{cases}$$

The *diagonal* of a product  $A \times A$  is denoted by  $\Delta_A$ ,

$$\Delta_A = \{(a, a') \in A \times A \mid a = a'\}.$$

It behaves as the *identity relation* over the set  $A$ ; if  $R: B \leftarrow A$  then  $R \circ \Delta_A = R$  and  $\Delta_B \circ R = R$ .

*Remark 2.1.* In the category of relations it is convenient to interpret a subset  $S \subseteq A$  as a relation  $S \subseteq A \times \{0\}$  where  $\{0\}$  is a *singleton*, an arbitrary set made of a single element. If  $R: B \leftarrow A$  then  $R \circ S$  is the *image of the subset*  $S$  by the relation  $R$ . In particular  $R \circ A \subseteq B$  is the *image* of the relation  $R$  and  $R^\top \circ B \subseteq A$  is the *inverse image* of  $R$ .  $\diamond$

## 2.2 Linear relations

**Definition 2.1.** A *linear relation*  $R: B \leftarrow A$  is a linear subspace of the direct sum  $B \oplus A$  of two vector spaces  $A$  and  $B$ .  $\heartsuit$

The direct sum is the Cartesian product endowed with the natural structure of a vector space. This definition is suggested by the fact that *a map  $f: A \rightarrow B$  is linear if and only if its graph  $R$  is a linear subspace of  $B \oplus A$* . It can be shown that *the composition of two linear relations is a linear relation*. Vector spaces and linear relations form a category (Benenti and Tulczyjew 1979).

## 2.3 Smooth relations and reductions

**Definition 2.2.** A *smooth relation* is a submanifold  $R \subseteq M_2 \times M_1$  of the product of two smooth manifolds  $M_1$  and  $M_2$ .  $\heartsuit$

The composition of two smooth relations may not be a smooth relation; that is  $S \circ R$  may not be a submanifold. The graph of a smooth map is a special case of smooth relation.

With a smooth relation  $R: M_2 \leftarrow M_1$  we associate the *tangent relation*

$$TR \subset TM_2 \times TM_1 \simeq T(M_2 \times M_1).$$

This is always a smooth relation. If  $R$  is the graph of a map  $\rho: M_1 \rightarrow M_2$ , then  $TR$  is the graph of the tangent map  $T\rho: TM_1 \rightarrow TM_2$ .

**Definition 2.3.** A *reduction* is a smooth relation  $R: M_0 \leftarrow M$  which is the graph of a surjective submersion  $\rho: C \rightarrow M_0$  from a submanifold  $C \subseteq M$  onto  $M_0$ . The transpose  $R^\top$  of a reduction  $R$  is called *coreduction*. A *fiber of a reduction*  $R: M_0 \leftarrow M$  is the inverse image of a point of  $M_0$ :  $R^\top\{p\}$ ,  $p \in M_0$ .  $\heartsuit$

*Remark 2.2.* A fiber of a reduction is a submanifold (Theorem 1.3). If all fibers are connected and  $S_0 \subseteq M_0$  is a submanifold, then  $S = R^\top \circ S_0 \subseteq M$  is a submanifold.  $\diamond$

*Remark 2.3.* (i) A surjective submersion is a reduction (case  $C = M$ ). (ii) The transpose of the injection of a submanifold  $C \subseteq M$  is a reduction (case  $C = M_0$ ). (iii) A reduction is always the compositions of reductions of type (ii) and (i).  $\diamond$

*Remark 2.4.* The composition of two reductions is a reduction. Reductions are morphisms of a category (Benenti 1983).  $\diamond$

### 2.3.1 Reduction of submanifolds

Let  $R: M_0 \leftarrow M$  be a reduction,  $\rho: C \rightarrow M_0$  the associated submersion, and  $A \subset M$  a submanifold. We call  $A_0 = R \circ A$  the *reduced set*. In general,  $A_0$  is not a submanifold of  $M_0$ . This depends on the way  $A$  intersects  $C$  and the fibers of  $R$ .

In order to give an answer to this question, let us consider, for each point  $x \in A \cap C$ , the subspace  $V_x \subset T_x C$  of the vectors tangent to the fiber of the submersion  $\rho$  passing through  $x$  (*vertical vectors*). Furthermore, let us denote by  $\rho': A \cap C \rightarrow M_0$  the restriction of  $\rho$  to the intersection  $A \cap C$ .

**Theorem 2.1.** *Assume that (i)  $A$  and  $C$  have a clean intersection,<sup>2</sup> and that (ii)  $\dim(V_x \cap T_x A)$  does not depend on the point  $x \in A \cap C$ . Then  $\rho'$  is a subimmersion and for each point  $x_0 \in N \cap C$  there exists a neighborhood  $U$  of  $x_0$  in  $N \cap C$  such that  $R \circ U$  is a submanifold of  $M_0$ .*

*Proof.* Let  $L: A \rightarrow B$  be a linear map. We define

$$\begin{cases} \text{rank}(L) \doteq \dim(\text{image of } L), \\ \ker(L) \doteq L^{-1}(0). \end{cases}$$

We know that  $\dim \ker(L) + \dim(\text{image of } L) = \dim A$ . Then,

$$\text{rank}(L) = \dim A - \dim(\ker(L)).$$

Apply this formula to  $L = T_x \rho': T_x(A \cap C) \rightarrow T_{\rho'(x)} M_0$ :

$$\text{rank}(T_x \rho') = \dim(T_x(A \cap C)) - \dim(\ker(T_x \rho')).$$

Observe that  $\ker(T_x \rho') = V_x \cap T_x(A \cap C)$ . Then,

$$\text{rank}(T_x \rho') = \dim(T_x(A \cap C)) - \dim(V_x \cap T_x(A \cap C))$$

Due to the clean intersection,

$$V_x \cap T_x(N \cap C) = V_x \cap T_x N \cap T_x C = V_x \cap T_x A,$$

inasmuch as  $V_x \subset T_x C$ . Then,

$$\text{rank}(T_x \rho') = \dim(T_x(A \cap C)) - \dim(V_x \cap T_x A). \quad (2.1)$$

Hence, due to assumption (ii),  $\text{rank}(T_x \rho')$  is constant. This proves that  $\rho'$  is a subimmersion. Then apply Theorem 1.2.  $\square$

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<sup>2</sup> Recall Definition 1.10 of a clean intersection:  $A \cap C$  is a submanifold and  $T_x(A \cap C) = T_x A \cap T_x C$  for all  $x \in A \cap C$ .



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