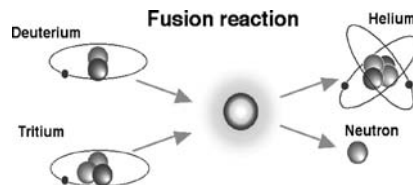


## Chapter 2

# Hydrogen Fusion: Light Nuclei and Theory of Fusion Reactions

Deuterium, tritium, neutron, and helium are the main players in the reaction ( $D + T \rightarrow He + n + 17.6 \text{ MeV}$ ; see Figure 2.1) to realize the Sun on Earth. In the microworld of atoms and nuclei, particles exhibit wave characteristics, and nuclear properties and reaction mechanics are governed by the Schrödinger wave equation.

These players have interesting properties. An encounter between deuterium and tritium results in the formation of a compound nucleus through the tunnel effect at a fractional energy of 500 keV Coulomb barrier potential. The compound nucleus has a high reaction probability near 80 keV due to the resonance phenomenon. In this way, nature gives humans a chance to use this reaction.

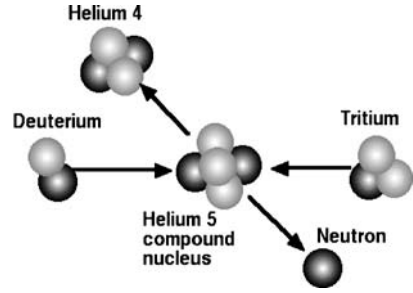


**Figure 2.1** The fusion reaction

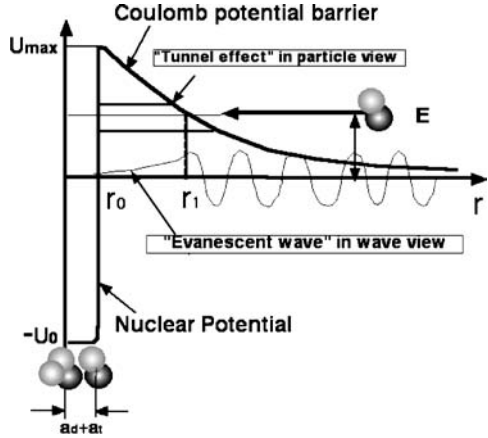
### 2.1 Fusion: Fusion of Little Nuts

The fusion that we are currently trying to achieve in ITER is the reaction of deuterium and tritium. This is different from the hydrogen fusion that takes place slowly in the Sun. “Nucleus” is a Latin word meaning “little nuts” [1]. Deuterium and tritium form a “compound nucleus” (a concept originating from N. Bohr),  $^5\text{He}$  (helium), when they get close enough to each other for the nuclear force to operate beyond the Coulomb barrier when the distance  $r$  is less than 3 fermi if we use nu-

**Figure 2.2** Schematic diagram of the DT fusion reaction via a compound nucleus

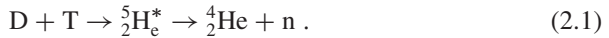


**Figure 2.3** Potential and wave function structure in the fusion reaction of deuterium and tritium



clear radius formula  $R_c = 1.1 A^{1/3}$  fermi (1 fermi =  $10^{-15}$  m;  $A$  is mass number) as seen in Figure 2.2.

The kinetic energy of the incident nuclei is distributed to nuclei in the compound nucleus. And, the neutron and helium, which have large energies, will, by chance, escape from compound nucleus.



Let  $U_{\max}$  be the barrier height for charged particles,  $U_{\max} = e^2/(4\pi\epsilon_0 r) = 0.48$  MeV for  $r = 3$  fermi as seen in Figure 2.3. Fusion will occur if the relative energy of deuterium and tritium is larger than 0.48 MeV, but it is difficult to raise the temperature to this level (3.7 billion degrees).

However, thanks to the wave nature of particles, fusion can occur at low energy (several 10 keV) by penetrating the Coulomb barrier. This is called the tunnel effect. Scattering and penetration of the particle beam can be investigated by solving the Schrödinger equation under the Coulomb field (the potential  $V = e^2/(4\pi\epsilon_0 r)$ ) (see note). This Schrödinger equation was first derived by Austrian physicist Erwin Schrödinger (1887–1961) who was awarded the 1933 Nobel Prize in physics.

In the Sections 2.2–2.5, deuterium, tritium, helium, and neutron will be described as the fuel and products of the fusion reaction.

**Note: Schrödinger's Wave Equation [2]**

In the world of atoms (radius of  $10^{-10} \text{ m} = 1 \text{ Ångstrom (Å)}$ ) and nuclei (radius of  $10^{-15} \text{ m} = 1 \text{ fermi}$ ), the motion of matter exhibits a “particle nature” and “wave nature.” In 1905, Einstein, explained the photoelectric effect through the concept of “photons” (light quantization) with energy in proportion to frequency.

$$E = \hbar \omega . \quad (2.2)$$

Here,  $\hbar$  is Planck's constant, which was introduced by German physicist Max Planck (1858–1947) to describe black-body radiation. The wave nature of the particle (“matter-wave”) is given by the French physicist de Broglie (1892–1987) with the following matter-wave relation:

$$p = \hbar k . \quad (2.3)$$

From the relation of light quantization and matter-wave, the Austrian physicist E. Schrödinger (Figure 2.4) derived a wave equation to describe the dynamics of the atomic world in 1926. In general, “wave” amplitude  $\psi$  is expressed as  $\psi = \exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ , and relationships  $(\partial/\partial \mathbf{x})\psi = i\mathbf{k}\psi$  and  $(\partial/\partial t)\psi = -i\omega\psi$  hold. In other words, the wave number  $\mathbf{k}$  and angular frequency  $\omega$  correspond to differential operators  $(\partial/\partial \mathbf{x})$  and  $(\partial/\partial t)$ , respectively.

$$i\mathbf{k} = \partial/\partial \mathbf{x} \quad (2.4)$$

$$-i\omega = \partial/\partial t . \quad (2.5)$$

Let the matter of the atomic world have energy  $E$  and momentum  $\mathbf{p}$ . Assuming the energy conservation law of Newtonian mechanics based on macroscopic world  $\mathbf{p}^2/2m + V(\mathbf{x}) = E$  applies similarly to the microscopic world of matter-wave, the microscopic energy conservation law can be obtained by using the Equations 2.2 and 2.3 as follows

$$\hbar^2 \mathbf{k}^2/2m + V(\mathbf{x}) = \hbar \omega . \quad (2.6)$$

Substituting wave number and angular frequency from Equations 2.4 and 2.5 into Equation 2.6 results in the following equation of the differential operator consistent with the energy conservation law,

$$-(\hbar^2/2m)\partial^2/\partial \mathbf{x}^2 + V(\mathbf{x}) = i\hbar \partial/\partial t . \quad (2.7)$$

Application of differential operator relation (2.7) to the wave amplitude  $\psi$  from left leads to the following Schrödinger's wave equation.

$$[-(\hbar^2/2m)\partial^2/\partial \mathbf{x}^2 + V(\mathbf{x})]\psi = i\hbar(\partial/\partial t)\psi . \quad (2.8)$$

Stationary solution of the wave amplitude is obtained by solving the eigen value problem by setting  $\psi = u \exp(-iEt/\hbar)$ .

$$[-(\hbar^2/2m)\partial^2/\partial \mathbf{x}^2 + V(\mathbf{x})]u = Eu . \quad (2.9)$$

In classical mechanics including the theory of relativity, the variables that appear in physics, represents numerical values, and there is a one-to-one correspondence between the variable and the number. Physical quantity is, in principle, measurable and can be regarded as a numerical value. However, this relationship is broken in quantum theory, physical quantity is not the physical number but becomes an “operator”, which itself does not correspond to the measurement. Following this idea, momentum  $\mathbf{p}$  becomes a differential operator  $-i\hbar \partial/\partial \mathbf{x}$ , energy  $E$  becomes the differential operator  $i\hbar \partial/\partial t$ .

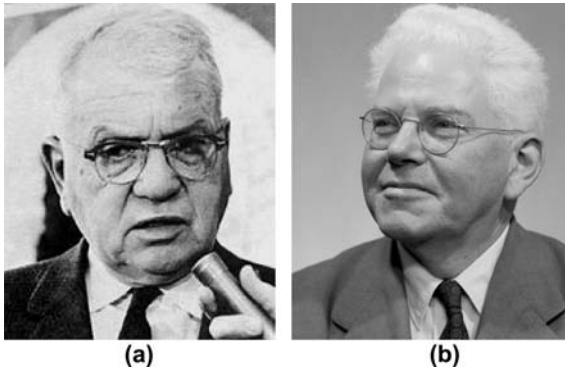


**Figure 2.4** Erwin Schrödinger who created the Schrödinger equation governing the dynamics of microscopic worlds

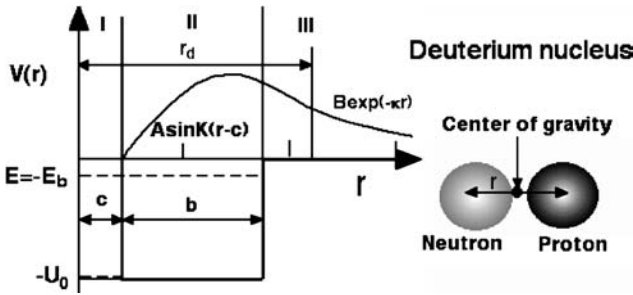
## 2.2 Deuterium: Nucleus Loosely Bound by a Neutron and Proton

The deuterium nucleus consists of one proton and one neutron. Among the combinations of the two nuclei, p-p, n-n and p-n, the only possible bound state is p-n, which is deuterium [3]. Deuterium was discovered in 1932 by American chemist H. C. Urey (1893–1981; Figure 2.5(a)) [4], who showed that 1 of every 7,000 hydrogen atoms is deuterium. Urey was awarded the 1934 Nobel Prize in Chemistry.

The bound state of a proton and a neutron to form deuterium can be treated as a two-body problem of the nuclear force between a neutron and proton. Then, its state can be solved using the Schrödinger wave equation (2.9) if the potential field  $V(r)$  is given. The nuclear force between the neutron and proton is the meson exchange force (see the note) and is represented by a central force given by the potential  $V$  as a function of the distance between nucleons  $r$  only. This nuclear force is termed a “strong interaction” and has a range of quite a short distance. While long range interaction is explained by the exchange of massless photons, strong interaction is explained by the exchange of mesons with an intermediate mass between that of protons and electrons. Here, the potential is approximated by the graph given in Figure 2.6 for simplicity.



**Figure 2.5** (a) H. Urey and (b) M. Oliphant who discovered deuterium and tritium, respectively (Courtesy of the Australian National University)



**Figure 2.6** Potential model and wave function as a function of distance  $r$  from the center of gravity of the neutron and proton in the deuterium nucleus. For deuterium,  $c = 0.4$  fermi,  $b = 1.337$  fermi,  $r_d = 2.1$  fermi,  $U_0 = 73$  MeV,  $E_b = 2.225$  MeV [3].

Scattering experiments suggest a minimum distance  $c \sim 0.4$  fermi between a proton and neutron. Therefore, the wave function is zero at  $r < c$  (region I in Figure 2.6) (existing probability is zero). The potential outside (region II) is negative since the attractive nuclear force works. The wave function ( $ru$ ) in region II is a sine function  $A \sin K(r - c)$  ( $A$  and  $K$  are constants). The wave function in region III (where the nuclear force does not work) is  $B e^{-\kappa r}$  ( $B, \kappa$  are constants) considering the boundary condition of  $ru = 0$  at  $r = \infty$ . The width  $b$  of region II, and potential  $U_0$  can be determined by the condition of the smooth connection of wave function at the boundary of region II and III then matching the condition to the measured binding energy and nucleus radius  $r_d$  of deuterium, leading to  $b = 1.337$  fermi,  $U_0 = 73$  MeV, respectively. Here, the physical relation between wave function in region III and the binding energy  $E_b$  is interesting. The decay rate  $\kappa$  of the wave function is related to the binding energy as  $\kappa \sim E_b^{0.5}$ , which implies radial decay of the wave function is weak with a probability that two nucleons will stay in region III ( $\sim |ru_0|^2 \sim e^{-2\kappa r}$  from Born relation) and becomes larger if  $E_b$  is small. In other words, the two nucleons are not strongly bound and are easily separated. Conversely, if the binding energy  $E_b$  is large, the wave function decays rapidly in

region III, resulting in a low probability that the nucleons will separate. The binding energy of deuterium  $E_b = 2.225$  MeV is small and the wave function decays slowly in the  $r$  direction, while the wave function in III decays rapidly for helium-4 since  $E_b = 28.3$  MeV.

Now, let us consider  $U$  caused by the attractive nuclear force. A nucleus confined in a small space (a few fermi) via Yukawa's meson nuclear exchange force, will have a very large wave number  $K = (m_r(U_0 - E_b))^{0.5}/\hbar = 1/1.09$  fermi), which implies large kinetic energy of the nucleon ( $E = p^2/2m_r \sim K^2\hbar^2/2m_r \sim 50$  MeV:  $m_r$  is reduced mass). This means that the nucleon is moving around at breakneck speed in the nuclear potential. To keep this nucleon confined,  $U_0$  must be several tens of MeV.

Heavy water (chemical symbol  $D_2O$ , chemically bonded two deuterium and one oxygen) exists at about 158 ppm (ppm: part per million) in seawater, and  $\sim 140$  ppm in the freshwater. There is approximately about  $5 \times 10^{13}$  ton of deuterium on the planet since the volume of seawater is  $1.8 \times 10^{18} \text{ m}^3$ . Since the amount of combustion of deuterium in a 1 GW level fusion reactor is about 73 kg per year, there exists 70 billion years of resource even if one assumes 10,000 nuclear fusion reactors. Heavy water can be manufactured by electrolysis. Commercially, the GS [GS: Girdler–Spevack] method is used to produce heavy water from fresh water using an isotopic exchange method. Specifically, hydrogen is exchanged in a countercurrent between  $H_2S$  (hydrogen sulfide) and  $H_2O$  (water). At room temperature, deuterium is exchanged with hydrogen in the water ( $H_2O + HDS \rightarrow HDO + H_2S$ ) and at high temperature ( $100^\circ\text{C}$  or more), deuterium is exchanged with hydrogen in the hydrogen sulfide ( $HDO + H_2S \rightarrow H_2O + HDS$ ). Thus heavy water production is mediated by hydrogen sulfide. The energy consumed in the manufacturing process is quite small.

### Note: Yukawa's Meson Theory [3,5]

To form a “little nut” by binding neutrons and protons, a new force, the “nuclear force,” was needed; this is different from gravity or electromagnetic forces, which were known in the nineteenth century. The first answer to this question was given by Japanese physicist, Hideki Yukawa (1907–1981; Figure 2.7). This force is called a “strong interaction.” In the Coulomb interaction, the particle produces a “field” in the space and the force is formed by the interaction of the field with another particle in the space. From a quantum mechanical perspective, it seemed to Yukawa that emission and absorption of massless photons between particles produces this long-range Coulomb force, the so-called “exchange force.” To explain the short-range nuclear force, Yukawa considered it to be an exchange force associated with a finite mass particle or “field with mass.”

The wave equation of the electric field  $[\partial^2/\partial x^2 - \partial^2/(c^2\partial t^2)]U = 0$  can be obtained by applying quantization rules for the momentum  $\mathbf{p} \rightarrow -i\hbar \partial/\partial \mathbf{x}$  and energy  $E \rightarrow i\hbar \partial/\partial t$  to the photon momentum equation  $-\mathbf{p}^2 + (E/c)^2 = 0$ . Then, the “field equation” with rest mass  $m$  has to be obtained by applying the quantization rules to the relativistic momentum balance equation of the particle,  $-\mathbf{p}^2 - m^2c^2 + (E/c)^2 = 0$  as follows,

$$\left[ \hbar^2 \frac{\partial^2}{\partial \mathbf{x}^2} - m^2 c^2 - \hbar^2 \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] U = 0 . \quad (2.10)$$

The static solution of this equation is called the Yukawa potential  $U = ge^{-\kappa r}/r$ . The fundamental difference to the long range Coulomb field by photon exchange force is that this nuclear force decays quickly with radius at  $r > r_0 \sim 1/\kappa = \hbar/mc$  due to finite mass. Yukawa predicted the mass of the exchange particle to be 200 times the electron mass considering a 2 fermi range of interaction of nuclear force. This is  $\pi$  meson. The  $\pi$  meson was discovered by the British physicist C. Powell (1903–1969) in cosmic rays, and was found to be 273 times the mass of the electron. For this achievement, Yukawa and Powell received the 1949 and 1950 Nobel Prize in Physics, respectively.



**Figure 2.7** Hideki Yukawa who explained strong force by the exchange force of meson (Courtesy of Yukawa Archival Library, Yukawa Hall, Yukawa Institute for Theoretical Physics, Kyoto University)

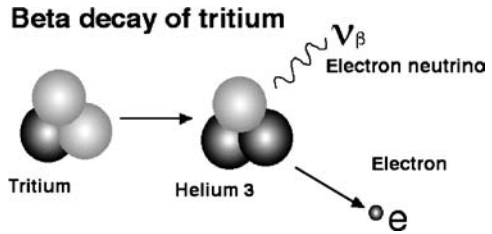
## 2.3 Tritium: Nucleus Emitting an Electron and Neutrino

Hydrogen with mass number 3 is called tritium. The word “tritium” comes from the Greek word meaning “third.” The nucleus of tritium consists of one proton and two neutrons. Tritium rarely exists in nature, and is made only in the atmosphere by cosmic rays. This element decays into helium-3 by emitting a high-energy electron beam (see Figure 2.8). This is called beta decay and has a half-life of 12.26 years. Tritium was first produced in the laboratory by Australian physicist M. Oliphant (1901–2000; Figure 2.5 (b)) in 1934 by colliding deuterium [6]

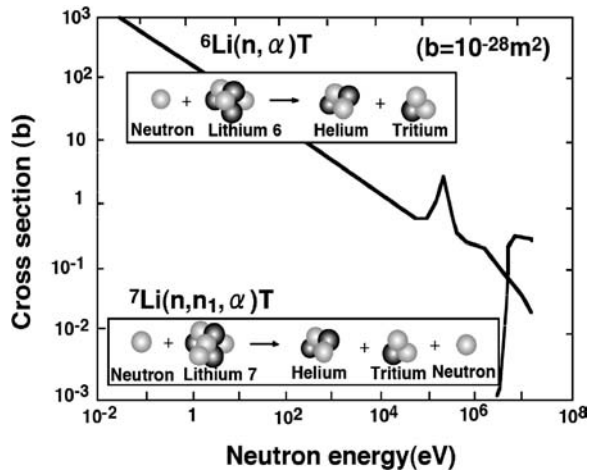
Tritium as fuel of DT fusion is generated by the nuclear reaction of neutron with lithium. The DT fusion produces neutrons and these neutrons can be used to generate tritium by the reaction with lithium (Figure 2.9). Therefore, high temperature plasma is covered with a device called a “blanket” containing lithium to generate tritium.

Lithium has two isotopes ( ${}^6\text{Li}$  and  ${}^7\text{Li}$ ) and the abundance of  ${}^6\text{Li}$  and  ${}^7\text{Li}$  in natural Li is 7.4 and 92.4%, respectively. Tritium is created by neutron absorption. The

**Figure 2.8** Beta decay of tritium into  $^3\text{He}$ , an electron, and a neutrino



**Figure 2.9** Reaction cross-section versus neutron energy for  $^6\text{Li}(n, \alpha)\text{T}$  and  $^7\text{Li}(n, n', \alpha)\text{T}$  reactions



$^6\text{Li}$  reaction  $^6\text{Li} + n \rightarrow ^3\text{T} + ^4\text{He} + 4.8\text{ MeV}$  is an exothermic reaction, while the  $^7\text{Li}$  reaction  $^7\text{Li} + n \rightarrow ^3\text{T} + ^4\text{He} + n' - 2.5\text{ MeV}$  is an endothermic reaction. The cross-section of the  $^6\text{Li}$  reaction is of the  $1/v$  type and it readily reacts at low energy. Meanwhile, the  $^7\text{Li}$  reaction is called the “threshold reaction” whose cross-section becomes nonzero above a critical energy. The reaction rate for  $^6\text{Li}$  is much larger than that from  $^7\text{Li}$ . Therefore, a net energy production occurs in the blanket. There is 233 billion tons of lithium in seawater and the resources can be thought of as infinite if low-cost technologies for the recovery of lithium from seawater are established.

- Lithium resources (reserve base): 9.4 million tons (3.7 million tons of reserves)
- Resources in the sea: 233 billion tons
- Annual production: 21 thousand tons/year (1996) (low demand)

### Salon: Beta Decay and Neutrinos

A strange thing happens in the beta decay. While the sum of the mass and kinetic energy is conserved in other nuclear reactions, it seems that the sum of the energy and mass are not conserved for beta decay. In other words, the beta-rays do not have energy equivalent to the mass lost by the beta decay. It is a famous story that Prof. Bohr, one of founders of the quantum mechanics, almost gave up the energy conservation law. This issue troubled physicists for a long time, but



in 1931, Swiss physicist W. Pauli (1900–1958) proposed the theory that an electrically neutral particle is emitted by the beta decay. It was named “neutrino” (meaning “little neutron” in Italian) by E. Fermi. Based on this theory, Fermi constructed a theory of beta decay, considering that during beta decay, a neutron is converted to a proton, electron, and neutrino in the nucleus. By assuming the existence of the neutrino, energy, momentum, and angular momentum are conserved. The existence of this neutrino was confirmed, 25 years after Pauli proposed it, by the proton–neutrino reaction in a water tank by F. Reines and C. L. Cowan in 1956. Given the existence of the neutrino, the beta decay of tritium is given as  ${}^3\text{H} \rightarrow {}^3\text{He} + e^- + \nu + 18.6\text{ keV}$ . The mass lost by the beta decay of tritium  $\Delta M_N$  can be calculated using the fact that mass of a beta-ray  $M(\beta)$  is the electron mass  $M(e)$

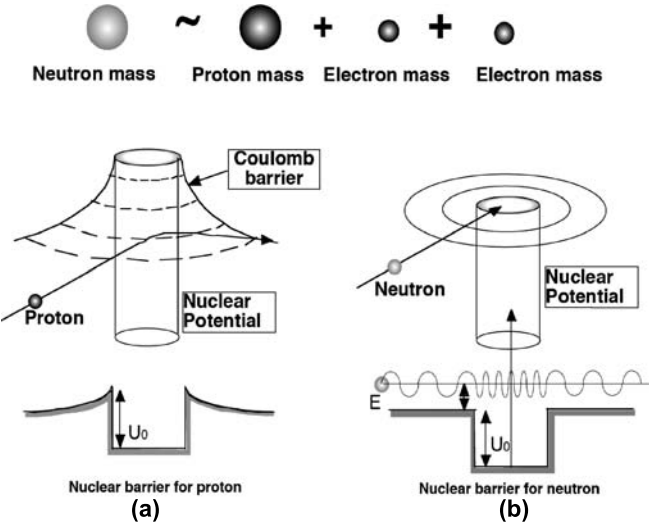
$$\begin{aligned}\Delta M_N &= M_N({}^3\text{H}) - (M_N({}^3\text{He}) + M(\beta)) \\ &= M_a({}^3\text{H}) - M_a({}^3\text{He}) = 2 \times 10^{-5}u = 18.6\text{ keV}(= E_0) .\end{aligned}\tag{2.11}$$

Here,  $M_N$  and  $M_a$  are the masses of the nucleus and atom, respectively. The maximum and average energy of beta-rays emitted from tritium are 18.6 keV and 5.7 keV, respectively. The difference between lost mass (mass defect) and beta-ray energy is the energy of the neutrino. Whether a neutrino has a mass is a long-discussed question in physics. Beta decay of tritium, has been used to determine the maximum value of the neutrino mass due to its low electron kinetic energy. In 1995, A. I. Belesev [7] showed that neutrino mass to be  $m_\nu < 4.35\text{ eV}$ . In 1998, the existence of neutrino mass was confirmed by the observation of neutrino oscillation between  $\mu$  neutrino and  $\tau$  neutrino [8]. Decay phenomena such as beta decay are generally very long compared to the duration of a nuclear reaction, which implies the interaction to be a weak force rather than a strong force binding the protons and neutrons in the nucleus. Recently, studies of neutrinos are expanding as the field Neutrino Astrophysics.

## 2.4 Neutron: Elementary Particles with No Charge

The neutron was discovered by the British physicist J. Chadwick (1891–1974) in 1932. A particle with a mass similar to a proton and no charge was considered to explain the nuclear spin by Rutherford around 1920, and was named “neutron” by the American chemist W. D. Harkins in 1921. Neutrons have no net charge, but their center has a slightly positive charge distribution, which is cancelled by a slightly negative charge distribution in the periphery. The mass is distributed within a radius of about 0.8 fermi.

A neutron is slightly heavier than a proton and the difference is about twice the electron mass (1.29 MeV). Neutron alone cannot exist stably, and decay to a proton emitting an electron and neutrino with a half-life of about 12 min ( $n \rightarrow p + e + \nu$ ). The mass of the neutron is greater than the sum of the masses of a proton and



**Figure 2.10** A neutron is heavier than a proton by two electron masses. (a) It is difficult for a proton to react due to the Coulomb potential. (b) A neutron can easily get into the nuclear potential inside the nucleus at low energy. In the potential well, particle energy becomes higher due to the attractive strong force and its wave number becomes much larger than the wave number in free space

an electron, and the mass difference leads to the energy release. This reaction was confirmed in 1948 by observations of the electrical bending of protons and electrons from the beta decay of a strong neutron beam in a large cylindrical tank.

Natural decay of isolated particles always seems to end with a decrease in mass. Since the mass of neutron is larger than that of a proton by about double the electron mass, a neutron easily decays to a proton while it is difficult for a proton to decay to a neutron. Whether proton decay occurs has become an important research subject in physics, which led to the construction of “KAMIOKANDE” in the Kamioka mine to detect the proton decay.

Tritium, which has two neutrons and one proton, is subject to beta decay, while a nucleus having nearly the same number of protons and neutrons can exist stably without beta decay. Why is that?

If the state after beta decay is a higher energy state than the initial state, beta decay will not occur naturally. Since the proton has a charge, conversion of a neutron to proton in the nucleus may produce extra Coulomb energy and beta decay will not occur in such a case. Conversely, Coulomb energy will not increase in beta decay of a free neutron and beta decay will occur. The neutron in the nucleus can thus exist stably and becomes a proton or neutron by exchanging  $\pi$  meson.

Figure 2.10 (a) shows schematics of charged particle collision in the Coulomb potential. A low energy particle is repelled and does not reach the nucleus. Fusion is a reaction between charged particles, and the charged particles must be close enough to be affected by the nuclear force beyond the Coulomb barrier. This requires the

charged particles to have high energy, which leads to the requirement of production and control of plasma at a few hundred million degrees.

Figure 2.10(b) shows schematics of neutron collision without Coulomb potential. In this case, neutrons at room temperature can enter into the nucleus. Fission utilizes this property to produce energy from uranium by the absorption of neutrons. In this sense, fission is much easier than fusion. The ease of the fission reaction led to the success of Fermi in producing the first chain reaction in 1942. Problem associated with fission energy exists in different aspect as discussed in Chapter 10.

### **Salon: Interaction of Fusion Neutron with Material**

Neutrons at almost room temperature ( $\sim 300$  K) cause a fission reaction in a thermal fission reactor, while about 1 MeV ( $3 \times 10^7$  times thermal neutron) per neutron is produced in the fast reactor. When high-energy neutrons dive into the material, its atoms may be dislocated due to collision with neutrons. This is called the neutron irradiation damage. In the past, swelling and blistering happened in fast reactors, but we now have radiation resistant materials improved by adjusting the composition of materials. A fusion neutron has an energy of 14 MeV, ten times that of a fast reactor, which may lead to the greater complexity of neutron irradiation damage

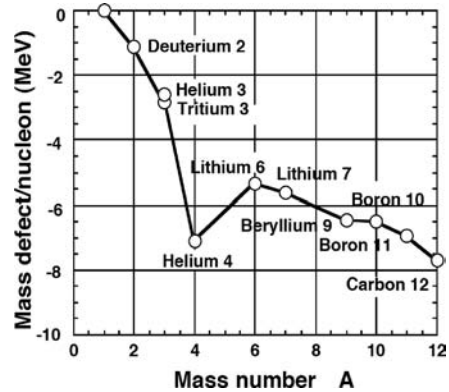
Thanks to the experiences with materials in fission, reduced activation ferritic steel (RAF), SiC/SiC composite materials, and vanadium alloys are being developed as materials to withstand a powerful fusion neutron. These materials are being developed with a target, 150–200 dpa (150–200 times dislocation of material atoms). Among them, RAF is considered to be a primary candidate material for DEMO (DEMONstration Power Plant) reactors. Research to lower the ductile to brittle transition temperature (DBTT), reduce creep deformation at high temperature, and study transmutation He effects by 14 MeV neutrons is continuing.

## **2.5 Helium: Stabilized Element with a Magic Number**

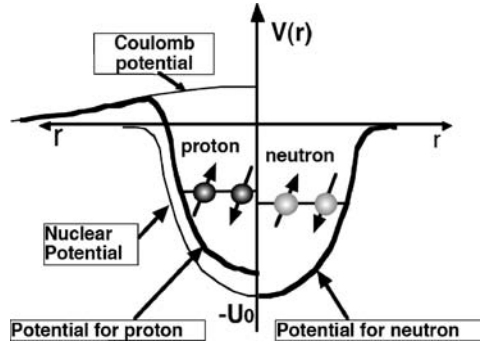
Helium is an element with two protons and two neutrons, and its mass number is four. The origin of the name of helium is the Greek word “helios” meaning the Sun. When a solar eclipse was observed in India on August 18, 1868, British astronomer J. N. Lockyer (1836–1920), who launched the prominent scientific journal *Nature*, observed the solar corona and discovered a new emission spectrum. He thought that the emission came from unknown element, which he called as “helium.” As shown in Figure 2.11, the binding energy (mass defect) of  ${}^4\text{He}$  is extremely large compared to that of hydrogen and lithium.

This characteristic cannot be explained by Weizsaecker–Bethe’s nuclear “liquid drop model” in which energies from surface tension and Coulomb repulsion, *etc.*, are taken into account. Such a large binding energy for a particular nucleus is ex-

**Figure 2.11** Mass defect per nucleon for low mass number elements



**Figure 2.12** Potential structure and energy states of nucleons (neutron and proton) for  $^4\text{He}$ . A closed-shell structure is formed by the proton and neutron occupying the same two orbits (spin angular momentum of  $+1/2$  and  $-1/2$ ) in the ground state S orbit (orbital angular momentum = 0)



plained by the nuclear “shell model,” where neutrons and protons form energy levels under the average nuclear potential and become stable when a particular energy level is filled up. Helium is composed of two protons and two neutrons, forming spherically symmetric potential. The potential shape for a neutron is different from that for a proton as shown in Figure 2.12 since only nuclear potential operates on neutrons while Coulomb potential is superimposed for protons.

Under these potentials, protons and neutrons can take discrete energy states, independently. This energy level can be obtained by solving the Schrödinger’s wave equation. In the case where the potential is proportional to the square of the distance (harmonic oscillator:  $V(r) = cr^2$ ), the solution for such a potential is well-known as  $u = R(r)Y(\theta, \phi)$ , where  $Y(\theta, \phi)$  represents the spherical harmonics and  $R(r)$  is the function including Laguerre polynomials. The energy level is expressed by  $E_n = -U_0 + (n_1 + (3/2)\hbar\omega)$  ( $\hbar\omega = (2U_0\hbar^2/mR_0^2)^{1/2}$ ), where  $n_1 = 2(n-1) + l$  where  $n$  is the principal quantum number and  $l$  the azimuthal quantum number, respectively. Ground state (lowest energy level) is  $E_0 = -U_0 + (3/2)\hbar\omega$  for  $n_1 = 0$  ( $n = 1, l = 0$ ). For the non-zero azimuthal quantum number  $l$ , the nucleon has its angular momentum around the center of the nucleus (orbital angular momentum:  $L = rmv = l\hbar$ ). The number of nucleons to the  $l$  state is the number of degenerate states of the level ( $2l + 1$ ) and is 1 for the ground state.

In the nucleus, there is a spin of the nucleon itself in addition to their angular momentum to the center of the nucleus. The value of the spin for neutrons and protons is also quantized and the spin angular momentum associated with rotation of the nucleon itself is  $1/2$ . The states of the spin are clockwise and counterclockwise rotations with respective values of  $+1/2$  and  $-1/2$ .

In helium, two nucleons sit in the ground state (principal quantum number  $n = 0$ ), defined for neutrons and protons independently. In the ground state of  ${}^4\text{He}$ , each nucleon takes the orbit with the orbital angular momentum 0 (called the 1S orbit). It is mysterious in some sense that protons and neutrons form an independent energy state. Although protons and neutrons form the nucleus via meson exchange, they behave as if there is no relationship between them. The recent nuclear experiment shows that they are not completely independent.

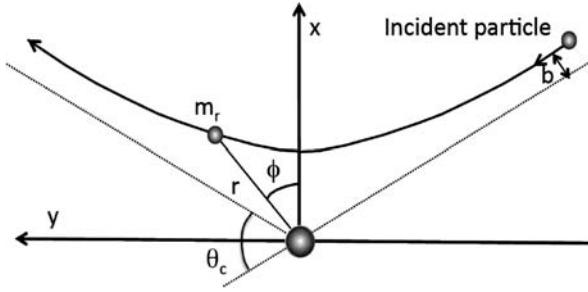
${}^4\text{He}$  is the first one in the closed-shell state and “2” is the smallest “magic number (2, 8, 20, ...)”. This leads to large binding energy for  ${}^4\text{He}$ . Thus, the  ${}^4\text{He}$  nucleus is particularly stable and abundant in the universe created by the Big Bang.

### Salon: Production and Uses of Helium

Commercially available helium gas is refined from natural gas from wells, while you might get the helium gas from the atmosphere. Natural gas fields in Algeria and the United States contain high concentrations of helium (1–7%). Lunar rock contains a large amount of helium blown in from the Sun as the solar wind and  ${}^3\text{He}$  is expected to be a raw material for fusion for the use of bases on the Moon. Helium is a light ( $1/7$  of air or less) and stable (not burning) gas and is used for balloons and airships. However, helium atoms are small, and permeate glass and rubber. This is the reason why balloons inflated with helium gas deflate. In addition, the cryogenic liquid helium is used as a coolant to achieve ultra low temperatures (below  $-269^\circ\text{C}$ ) for magnetic fusion devices and linear motor vehicles using superconducting magnets. In fusion reactors, such as the Sun on Earth, hydrogen isotopes are fused to produce this stable helium and the lost mass is converted to energy. In future magnetic fusion reactors, energetic helium will be created in the hot plasma and the superconducting magnet of the magnetic fusion reactor will be cooled by cold helium.

## 2.6 Fusion Cross-section: Tunneling Effect and Resonance

In previous sections, we reached a certain degree of understanding about the nature of deuterium, tritium, neutron, and helium, which are fuel and products for the fusion reactions we are currently investigating. Now, we will talk about the wave dynamics and the cross-section of fusion reactions following Mott and Massey [9]. The classical orbit in Coulomb scattering is given by the hyperbolic curve,  $r = r_0/(1 - \alpha \cos \phi)$  (in the center of mass coordinate) shown in Figure 2.13.



**Figure 2.13** Particle orbit in Coulomb potential ( $r = r_0/(1 - \alpha \cos \phi)$ )  $r_0 = b^2/b_0$ , and  $\alpha = (1 + (b/b_0)^2)^{1/2}$

### Note: Particle Motion in Central Field [10]

If the potential energy between two particles depends only on distance, the force field is called the **central force field** and is given as  $\mathbf{F} = -(dU/dr)\mathbf{r}/r$ . In the center of the mass system, angular momentum  $\mathbf{M} = \mathbf{r} \times \mathbf{p}$  is conserved. Since  $\mathbf{M}$  is constant,  $r$  stays on a plane perpendicular to  $\mathbf{M}$ . In the polar coordinates  $(r, \phi)$ , the Lagrangian is given as,

$$L = m(\dot{r}^2 + r^2\dot{\phi}^2) - U(r) . \quad (2.12)$$

Since  $L$  is independent of  $\phi$ , canonical momentum conjugates to  $\phi$ ,  $M = p_\phi = m r^2 \dot{\phi}$  is conserved. Substituting this relation into the energy conservation relation  $E = m(\dot{r}^2 + r^2\dot{\phi}^2) + U(r)$ , we obtain

$$\frac{dr}{dt} = \sqrt{2(E - U(r))/m - M^2/(m^2 r^2)} . \quad (2.13)$$

In the case of repulsive Coulomb potential  $U(r) = e_i e_j / (4\pi \epsilon_0 r)$ , the particle orbit is given by following hyperbolic curve.

$$r = \frac{r_0}{1 - \alpha \cos \phi} \quad (2.14)$$

where

$$r_0 = \frac{M^2}{m_r(e_i e_j / 4\pi \epsilon_0)} , \quad \alpha = \sqrt{1 + \frac{2EM^2}{m_r(e_i e_j / 4\pi \epsilon_0)^2}} . \quad (2.15)$$

Since  $M = b m_r u$  and  $E = m_r u^2 / 2$  ( $b$  is impact parameter in Figure 2.13,  $u$  is particle speed at infinity), we can convert these parameters as follows,

$$r_0 = \frac{b^2}{b_0} , \quad \alpha = \sqrt{1 + \frac{b^2}{b_0^2}} \quad (2.16)$$

where,

$$b_0 = \frac{e_i e_j}{4\pi\epsilon_0 m_r u^2} . \quad (2.17)$$

The wave front of the incident wave should be perpendicular to this hyperbolic curve. Let  $k$  be the de Broglie wave number, the wave front satisfying this criteria is,

$$z + b_0 \ln k(r - z) = \text{const.} \quad (2.18)$$

where  $b_0 = e_i e_j / (4\pi\epsilon_0 m_r u^2) = 7.2 \times 10^{-10} Z_i Z_j / E_r$  (eV) (m), called the Landau parameter. The incident wave is already distorted at infinite distance before encountering the nucleus since the Coulomb field operates to infinity. So, the incident wave is given by  $\exp[ik\{z + b_0 \ln k(r - z)\}]$ .

For a non-relativistic collision  $E_r = p^2/2m_r = \hbar^2 k^2/2m_r$ , and the Schrödinger's wave equation (2.9) becomes  $[\partial^2/\partial x^2 + (k^2 - \beta/r)]\psi = 0$ , where  $\beta = m_r e_i e_j / 2\pi\epsilon_0 \hbar^2$ . Substituting  $\psi = \exp(ikz)F(x)$ ,  $F$  should have the form  $F = F(r - z)$ . Then, the wave equation becomes  $\zeta^2 d^2 F/d\zeta^2 + (1 - ik\zeta)dF/d\zeta - (\beta/2)F = 0$  for  $\zeta = r - z$ . Taylor expanding  $F$  ( $F = \sum a_n \zeta^n$  ( $a_0 = 1$ )) and substituting into the wave equation gives  $a_n$  and  $F$  is found to be the hyper geometric function  $F$  as follows,

$$\psi = \exp(-\pi\alpha/2)\Gamma(1 + i\alpha)e^{ikz}F(-i\alpha, 1; ik\zeta) . \quad (2.19)$$

Here  $\alpha = \beta/2k$ ,  $F(a, b; z) \equiv \sum \Gamma(a + n)\Gamma(b)z^n / \Gamma(a)\Gamma(b)/\Gamma(n + 1)$  is a hypergeometric function,  $\Gamma$  is the gamma function. Permeability of the Coulomb barrier  $P$  is then given from wave function at the origin as,

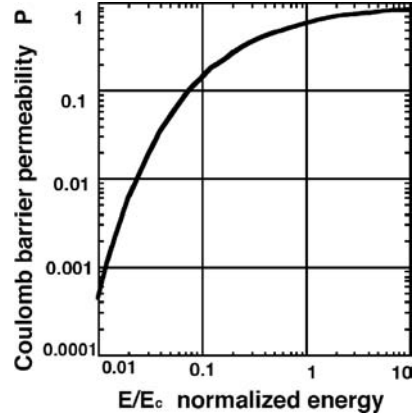
$$P(E/E_c) = \frac{\sqrt{E_c/E}}{\exp \sqrt{E_c/E} - 1} \quad (2.20)$$

$$E_c = \frac{m_r e^4}{8\epsilon_0^2 \hbar^2} = 0.98 A_r (\text{MeV}) . \quad (2.21)$$

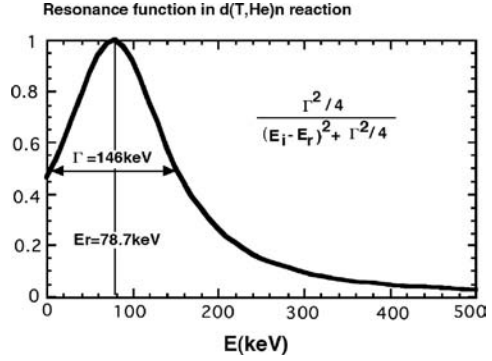
Here,  $m_r$  is the reduced mass ( $m_a m_b / (m_a + m_b)$ ) between particles a and b,  $A_r$  is a mass number of the reduced mass,  $\epsilon_0$  is the vacuum permittivity ( $= 8.854 \times 10^{12}$  F/m), and  $\hbar$  is Planck's constant. Figure 2.13 shows permeability of the Coulomb barrier  $P$  against  $E/E_c$ . The critical energy  $E_c$  given by Equation 2.21. For the  $T(d,n)^4\text{He}$  reaction, it is 1.18 MeV, while numerically fitting  $E_c$  to reproduce the measured fusion cross-section is 1.27 MeV showing good agreement.

A particle with reduced mass jumping into the nuclear potential, will have higher kinetic energy than the original one (de Broglie wavelength shorter than the original de Broglie wavelength). Such big changes in wave number will cause a resonance interaction within the nucleus. For the DT fusion reaction, resonance energy and resonance width are  $E_r = 78.65$  keV and  $\Gamma = 146$  keV, respectively (Figure 2.15). The probability amplitude of a material wave is large since the compound nucleus  $^5\text{He}$  has an energy level corresponding to a certain boundary condition.

**Figure 2.14** Energy dependence of the Coulomb barrier permeability. The Coulomb barrier permeability  $P$  is well-behaved ( $P \rightarrow 1$ ) in the high energy limit ( $E/E_c \gg 1$ ) while the formula given by Gamov  $P(E/E_c) = (E/E_c)^{-1/2} \exp(-(E/E_c)^{-1/2})$  [11] is valid only at low energy



**Figure 2.15** Normalized resonance function for DT reaction in laboratory frame



The compound nucleus is unstable and will decay at a decay time constant. The decay time constant  $\tau$  and the resonance width  $\Gamma$  have the relationship  $\Gamma\tau = \hbar$ , the decay time is calculated as  $\tau = 4.5 \times 10^{-21}$  s since  $\Gamma = 146$  keV. This time is much longer (100 times) than the transit time of a nucleon with Fermi energy,  $\tau_F = 2R/v_F = 4.4 \times 10^{-23}$  s.

The fusion cross-section, considering the above, is given as,

$$\sigma_r = \pi\lambda^2 P(E/E_c) \frac{\Gamma_i \Gamma_f}{(E - E_r)^2 + \Gamma^2/4}. \quad (2.22)$$

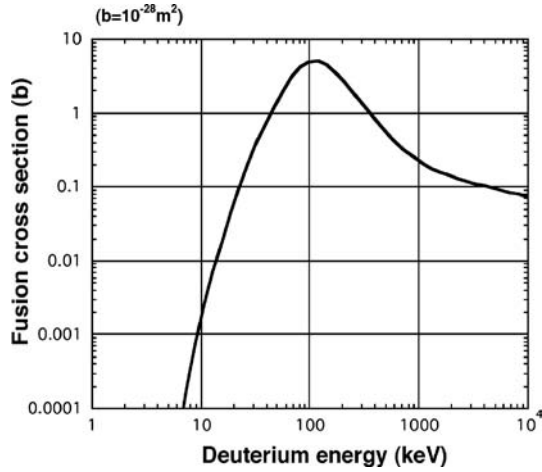
Here,  $\lambda^2 = \hbar^2(2ME)^{-1}$ , and the second factor comes from the Breit-Wigner formula of resonance cross-section [12], here  $\Gamma_i \sim k \sim 1/E^{0.5}$ . In fact, the measured values of the fusion cross-section are given in a form of Equation 2.22 [13].

$$\sigma_r = \sigma_0 \frac{E_{cl}}{E_l [\exp \sqrt{E_{cl}/E_l} - 1]} \left[ \frac{1}{1 + 4(E_l - E_{tl})^2/\Gamma_l^2} + \alpha \right]. \quad (2.23)$$

Here,  $\sigma_0 = 23.79$  b,  $E_{cl} = 2.11$  MeV,  $E_{tl} = 78.65$  keV,  $\Gamma_l = 146$  keV,  $\alpha = 0.0081$  for the T(d,n)<sup>4</sup>H reaction [12]. Figure 2.16 shows DT cross-section



**Figure 2.16** DT-fusion cross-sections in laboratory frame (in b (barn =  $10^{-28} \text{ m}^2$ ))



of nuclear fusion reactions (in the laboratory system) illustrating the dependence on deuterium energy. Deuterium energy in the laboratory frame  $E_l$  is related to the energy in the center of mass frame  $E$  as  $E = m_t/(m_d + m_t)E_l$  and thus  $E_c = 0.6E_{cl} = 1.27 \text{ MeV}$ .

Recently Li et al. [14] gave more clear analysis of fusion cross section. Using the Landau's analysis of resonant scattering of charged particles [12], they derived different form of fusion cross section using the optical potential ( $= U_r + iU_i$ ) for D + T reaction as follows,

$$\sigma_r = \frac{\pi}{k^2 \theta^2} \frac{-4w_i}{w_r^2 + (w_i - \theta^{-2})^2} \quad (2.24)$$

where  $\theta^2 = (\exp((E_c/E)^{1/2}) - 1)/2\pi$  and  $w = w_r + iw_i = \cot(\delta_0)/\theta^2$  where  $\delta_0$  is phase shift due to nuclear potential. Using approximate expression  $w = C_1 + C_2 E_\ell + iC_3$ , fusion cross-section in laboratory frame is given as follows,

$$\sigma_r(E_\ell) = \frac{\pi \hbar^2}{2m_d E_\ell \theta^2} \frac{-4C_3}{(C_1 + C_2 E_\ell)^2 + (C_4 - \theta^{-2})^2} \quad (2.25)$$

For DT fusion,  $C_1 = -0.5405$ ,  $C_2 = 0.005546$ ,  $C_3 = -0.3909$  gives good agreement with measured fusion cross-section.

## References

1. Asimov I (1991) Atom. Nightfall Inc.
2. Schiff LI (1974) Quantum Mechanics. 3rd edn. McGraw-Hill Book Company Inc.
3. Enge H (1966) Introduction to Nuclear Physics. Addison-Wesley Pub. Co.
4. Urey HC et al. (1932) Phys. Rev. 39, 164L.

5. Yukawa H (1935) Proc. Phys. Math. Soc., Japan, 17, 48.
6. Oliphant ML et al. (1934) Nature, 133, 413.
7. Beisev AI et al. (1995) Phys. Rev. Lett., B350, 263.
8. Fukuda Y et al. (1998) Phys. Rev. Lett., 81, 1562.
9. Mott NF and Massey HSW (1965) The Theory of Atomic Collisions. 3rd edn. Clarendon Press, p. 57, eq. 17.
10. Landau LD, Lifschitz EM (1969) Mechanics. Pergamon Press.
11. Gamov G, Critchfield CL (1951) The Theory Atomic Nucleus and Nuclear Energy Sources. Clarendon Press, Ch. X, eq. 2.
12. Landau LD, Lifschitz EM (1987) Quantum Mechanics. 3rd edn., Pergamon Press.
13. Duane BH (1972) Fusion Cross-section Theory. Brookhaven National Laboratory Report BNWL-1685, pp. 75–92.
14. Li ZX, Wie MQ, Liu B (2008) Nucl. Fusion 48, 125003.



<http://www.springer.com/978-1-84996-410-4>

Frontiers in Fusion Research

Physics and Fusion

Kikuchi, M.

2011, XIII, 270 p., Hardcover

ISBN: 978-1-84996-410-4