

Bounded Perturbations of the Resolvent Operators Associated to the \mathcal{F} -Spectrum

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Abstract. Recently, we have introduced the \mathcal{F} -functional calculus and the \mathcal{SC} -functional calculus. Our theory can be developed for operators of the form $T = T_0 + e_1 T_1 + \dots + e_n T_n$ where (T_0, T_1, \dots, T_n) is an $(n + 1)$ -tuple of linear commuting operators. The \mathcal{SC} -functional calculus, which is defined for bounded but also for unbounded operators, associates to a suitable slice monogenic function f with values in the Clifford algebra \mathbb{R}_n the operator $f(T)$. The \mathcal{F} -functional calculus has been defined, for bounded operators T , by an integral transform. Such an integral transform comes from the Fueter's mapping theorem and it associates to a suitable slice monogenic function f the operator $\check{f}(T)$, where $\check{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ and Δ is the Laplace operator. Both functional calculi are based on the notion of \mathcal{F} -spectrum that plays the role that the classical spectrum plays for the Riesz-Dunford functional calculus. The aim of this paper is to study the bounded perturbations of the \mathcal{SC} -resolvent operator and of the \mathcal{F} -resolvent operator. Moreover we will show some examples of equations that lead to the \mathcal{F} -spectrum.

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Keywords. Functional calculus for n -tuples of commuting operators, \mathcal{F} -spectrum, perturbation of the \mathcal{SC} -resolvent operator, perturbation of the \mathcal{F} -resolvent operator, examples of equations that lead to the \mathcal{F} -spectrum.

1. Introduction

The recent theory of slice monogenic functions, mainly developed in the papers [2], [3], [8], [9], [10], [12], turned out to be very important because of its applications to the so-called \mathcal{S} -functional calculus for n -tuples of non-necessarily commuting operators (bounded or unbounded), see [3] and [11]. The theory admits a quaternionic version of the \mathcal{S} -functional calculus for quaternionic linear operators which can be found in [4]. It is crucial to note that slice monogenic functions have a Cauchy formula with slice monogenic kernel that admits two expressions. These

two expressions of the Cauchy kernel are not equivalent when we want to define a functional calculus for non-necessarily commuting operators. In this paper, we will consider the case of commuting operators and the expression of the Cauchy kernel which gives rise to the definition of \mathcal{F} -spectrum which is the natural tool to treat the case of commuting operators.

Let $x = x_0 + e_1x_1 + \dots + e_nx_n$ and $s = s_0 + e_1s_1 + \dots + e_ns_n$ be paravectors in \mathbb{R}^{n+1} . We consider the Cauchy kernel written in the form

$$\mathcal{S}_C^{-1}(s, x) = (s - \bar{x})(s^2 - 2\text{Re}[x]s + |x|^2)^{-1}$$

which is defined for $x^2 - 2xs_0 + |s|^2 \neq 0$. Let $f : U \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$, where \mathbb{R}_n is the real Clifford algebra with n imaginary units, U is a suitable open set that contains the singularities of $\mathcal{S}_C^{-1}(s, x)$. Let I be a 1-vector such that $I^2 = -1$ and let \mathbb{C}_I be the complex plane that contains 1 and I . Then we have the Cauchy formula

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, x) ds_I f(s), \quad ds_I = -I ds, \quad (1.1)$$

where the integral does not depend on the open set U and on the imaginary unit I . In the paper [5] we have introduced the so-called \mathcal{SC} -functional calculus (\mathcal{SC} stands for *slice-commuting*), which is defined for bounded but also for unbounded commuting operators, starting from the above Cauchy formula

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, T) ds_I f(s), \quad ds_I = -I ds. \quad (1.2)$$

The definition is well posed because the the integral in (1.2) does not depend on the open set U and on the imaginary unit I . The \mathcal{SC} -resolvent operator is defined by

$$\mathcal{S}_C^{-1}(s, T) := (sI - \overline{T})(s^2I - s(T + \overline{T}) + T\overline{T})^{-1}$$

whose associated spectrum is the \mathcal{F} -spectrum of T defined as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} : s^2I - s(T + \overline{T}) + T\overline{T} \text{ is not invertible}\}.$$

It is important to point out the meaning of the symbols. We have that, by definition, $T = T_0 + e_1T_1 + \dots + e_nT_n$, $\overline{T} = T_0 - e_1T_1 - \dots - e_nT_n$, so that $T + \overline{T} = 2T_0$, and since the components of T commute, we have $T\overline{T} = T_0^2 + T_1^2 + \dots + T_n^2$.

In the paper [6] we have proved the Fueter mapping theorem in integral form using the Cauchy formula (1.2). Precisely we have proved that, when n is an odd number, given the slice monogenic function f , we can associate to it the monogenic function $\check{f}(x)$ by the integral transform

$$\check{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \gamma_n(s - \bar{x})(s^2 - s(x + \bar{x}) + |x|^2)^{-\frac{n+1}{2}} ds_I f(s), \quad ds_I = -I ds, \quad (1.3)$$

where γ_n is a given constant. We recall that the Fueter mapping theorem in differential form is given in [19] for n odd and in Qian's paper [16] in the general case. Later on, Fueter's theorem has been generalized to the case in which a function f as above is multiplied by a monogenic homogeneous polynomial of degree k , see

[15], [20] and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane, see [17].

We point out that formula (1.3) allows us to define the \mathcal{F} -functional calculus by replacing $x = x_0 + x_1 e_1 + \dots + x_n e_n$ by $T = T_0 + T_1 e_1 + \dots + T_n e_n$. More precisely, in [6] we have defined the following version of the monogenic functional calculus by setting

$$\check{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T) ds_I f(s), \quad (1.4)$$

where

$$\mathcal{F}_n^{-1}(s, T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}. \quad (1.5)$$

The functional calculus in (1.4) is well defined since the integral does not depend on the open set U and on $I \in \mathbb{S}$. The natural notion of spectrum in this case is again the notion of \mathcal{F} -spectrum of T as it is suggested by the definition of the \mathcal{F} -resolvent operator defined in (1.5).

The goal of this paper is to prove that bounded perturbations of the \mathcal{SC} -resolvent operator and of the \mathcal{F} -resolvent operator produce bounded variations of the respective functional calculi.

We conclude by recalling that the well known theory of monogenic functions, see [1], [7], is the natural tool to define the monogenic functional calculus which has been well studied and developed by several authors, see the book of B. Jefferies [14] and the literature therein. For the analogies with the Riesz-Dunford functional calculus see for example the classical books [13] and [18].

2. Preliminary material

The setting in which we will work is the real Clifford algebra \mathbb{R}_n over n imaginary units e_1, \dots, e_n satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_A x_A$ where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index, $e_A = e_{i_1} e_{i_2} \dots e_{i_r}$ and $e_\emptyset = 1$. In the Clifford algebra \mathbb{R}_n , we can identify some specific elements with the vectors in the Euclidean space \mathbb{R}^n : an element $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1, x_2, \dots, x_n) \mapsto \underline{x} = x_1 e_1 + \dots + x_n e_n$.

An element $(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j$ called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The real part x_0 of x will be also denoted by $\text{Re}[x]$. A function $f : U \subseteq \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ is seen as a function $f(x)$ of x (and similarly for a function $f(\underline{x})$ of $\underline{x} \in U \subset \mathbb{R}^{n+1}$).

Definition 2.1. We will denote by \mathbb{S} the sphere of unit 1-vectors in \mathbb{R}^n , i.e.,

$$\mathbb{S} = \{\underline{x} = e_1 x_1 + \dots + e_n x_n : x_1^2 + \dots + x_n^2 = 1\}.$$

Note that \mathbb{S} is an $(n-1)$ -dimensional sphere in \mathbb{R}^{n+1} . The vector space $\mathbb{R} + I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by \mathbb{C}_I , while an element belonging

to \mathbb{C}_I will be denoted by $u + Iv$, for $u, v \in \mathbb{R}$. Observe that \mathbb{C}_I , for every $I \in \mathbb{S}$, is a 2-dimensional real subspace of \mathbb{R}^{n+1} isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.

Given a paravector $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_x = \begin{cases} \frac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}$$

By definition we have that a paravector x , with $\underline{x} \neq 0$, belongs to \mathbb{C}_{I_x} .

Definition 2.2. Given an element $x \in \mathbb{R}^{n+1}$, we define $[x] = \{y \in \mathbb{R}^{n+1} : y = \text{Re}[x] + I|\underline{x}|\}$, where $I \in \mathbb{S}$.

Remark 2.3. The set $[x]$ is an $(n-1)$ -dimensional sphere in \mathbb{R}^{n+1} . When $x \in \mathbb{R}$, then $[x]$ contains x only. In this case, the $(n-1)$ -dimensional sphere has radius equal to zero.

Definition 2.4 (Slice monogenic functions). Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \rightarrow \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane \mathbb{C}_I . We say that f is a (left) slice monogenic function, or s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2} \left(\frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0.$$

We denote by $\mathcal{SM}(U)$ the set of s-monogenic functions on U .

The natural class of domains in which we can develop the theory of s-monogenic functions are the so-called slice domains and axially symmetric domains.

Definition 2.5 (Slice domains). Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non-empty and if $\mathbb{C}_I \cap U$ is a domain in \mathbb{C}_I for all $I \in \mathbb{S}$.

Definition 2.6 (Axially symmetric domains). Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if, for every $u + Iv \in U$, the whole $(n-1)$ -sphere $[u + Iv]$ is contained in U .

Let us now introduce the notations necessary to deal with linear operators. By V and by V_n we denote a Banach space over \mathbb{R} with norm $\|\cdot\|$ and $V \otimes \mathbb{R}_n$, respectively. We recall that V_n is a two-sided Banach module over \mathbb{R}_n and its elements are of the type $\sum_A v_A \otimes e_A$ (where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index). The multiplications (right and left) of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined as $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. For short, in the sequel we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Moreover, we define $\|v\|_{V_n}^2 = \sum_A \|v_A\|_V^2$.

Let $\mathcal{B}(V)$ be the space of bounded \mathbb{R} -homomorphisms of the Banach space V into itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$. If $T_A \in \mathcal{B}(V)$, we

can define the operator $T = \sum_A T_A e_A$ and its action on $v = \sum_B v_B e_B$ as $T(v) = \sum_{A,B} T_A(v_B) e_A e_B$. The set of all such bounded operators is denoted by $\mathcal{B}_n(V_n)$ and the norm is defined by $\|T\|_{\mathcal{B}_n(V_n)} = \sum_A \|T_A\|_{\mathcal{B}(V)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}_n(V_n)$ in the norm of an operator and note also that $\|TS\| \leq \|T\|\|S\|$. A bounded operator $T = T_0 + \sum_{j=1}^n e_j T_j$, where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, \dots, n$, will be called, with an abuse of notation, an operator in paravector form. The set of such operators will be denoted by $\mathcal{B}_n^{0,1}(V_n)$. The set of bounded operators of the type $T = \sum_{j=1}^n e_j T_j$, where $T_\mu \in \mathcal{B}(V)$ for $\mu = 1, \dots, n$, will be denoted by $\mathcal{B}_n^1(V_n)$ and T will be said operator in vector form. We will consider operators of the form $T = T_0 + \sum_{j=1}^n e_j T_j$ where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, \dots, n$ for the sake of generality, but when dealing with n -tuples of operators, we will embed them into $\mathcal{B}_n(V_n)$ as operators in vector form, by setting $T_0 = 0$. The subset of those operators in $\mathcal{B}_n(V_n)$ whose components commute among themselves will be denoted by $\mathcal{BC}_n(V_n)$. In the same spirit we denote by $\mathcal{BC}_n^{0,1}(V_n)$ the set of paravector operators with commuting components.

We now recall some definitions and results from [5], [6].

Definition 2.7 (The \mathcal{F} -spectrum and the \mathcal{F} -resolvent sets). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$. We define the \mathcal{F} -spectrum of T as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} : s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T} \text{ is not invertible}\}.$$

The \mathcal{F} -resolvent set of T is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

Theorem 2.8 (Structure of the \mathcal{F} -spectrum). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and let $p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R}$, such that $p \in \sigma_{\mathcal{F}}(T)$. Then all the elements of the $(n-1)$ -sphere $[p_0 + p_1 I]$ belong to $\sigma_{\mathcal{F}}(T)$. Thus the \mathcal{F} -spectrum consists of real points and/or $(n-1)$ -spheres.

Theorem 2.9 (Compactness of \mathcal{F} -spectrum). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$. Then the \mathcal{F} -spectrum $\sigma_{\mathcal{F}}(T)$ is a compact non-empty set. Moreover $\sigma_{\mathcal{F}}(T)$ is contained in $\{s \in \mathbb{R}^{n+1} : |s| \leq \|T\|\}$.

Definition 2.10. Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s -domain containing the \mathcal{F} -spectrum $\sigma_{\mathcal{F}}(T)$, and such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of continuously differential Jordan curves for every $I \in \mathbb{S}$. Let W be an open set in \mathbb{R}^{n+1} . A function $f \in \mathcal{SM}(W)$ is said to be locally s -monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists a domain $U \subset \mathbb{R}^{n+1}$ as above such that $\overline{U} \subset W$. We will denote by $\mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ the set of locally s -monogenic functions on $\sigma_{\mathcal{F}}(T)$.

Definition 2.11 (The \mathcal{SC} -resolvent operator). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and $s \in \rho_{\mathcal{F}}(T)$. We define the \mathcal{SC} -resolvent operator as

$$S_{\mathcal{C}}^{-1}(s, T) := (s\mathcal{I} - \overline{T})(s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}. \quad (2.1)$$

Definition 2.12 (The \mathcal{SC} -functional calculus). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be a domain as in Definition 2.10 and set $ds_I = ds/I$ for $I \in \mathbb{S}$. We define the \mathcal{SC} -functional calculus as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, T) ds_I f(s). \quad (2.2)$$

Definition 2.13 (\mathcal{F} -resolvent operator). Let n be an odd number and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the \mathcal{F} -resolvent operator as

$$\mathcal{F}_n^{-1}(s, T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \left(\frac{n-1}{2} \right)!.$$

Next we define $\check{f}(T)$ when \check{f} is a monogenic function which comes from an s -monogenic function f via Fueter's theorem. The \mathcal{F} -functional calculus will be defined for those monogenic functions that are of the form $\check{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$, where f is an s -monogenic function. For the functional calculus associated to standard monogenic functions we mention the book [14].

Definition 2.14 (The \mathcal{F} -functional calculus). Let n be an odd number and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. Let U be an open set as in Definition 2.10. Suppose that $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\check{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the \mathcal{F} -functional calculus as

$$\check{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T) ds_I f(s). \quad (2.3)$$

Remark 2.15. The definitions of the \mathcal{SC} -functional calculus and of the \mathcal{F} -functional calculus are well posed since the integrals in (2.2) and in (2.3) are independent of $I \in \mathbb{S}$ and of the open set U .

3. Examples of equations for the \mathcal{F} -spectrum

Example (The case of Dirac operator). Let us consider the n -tuple of operators $(\partial_{x_1}, \dots, \partial_{x_n})$, each of them acting on the vector space of functions of class \mathcal{C}^2 over an open set $U \subseteq \mathbb{R}^{n+1}$. The vector operator associated to them is the Dirac operator

$$T = \partial_{x_1} e_1 + \dots + \partial_{x_n} e_n.$$

Let us determine the equation which gives its \mathcal{F} -spectrum. We have $\overline{T} = -\partial_{x_1} e_1 - \dots - \partial_{x_n} e_n$, and, since $\partial_{x_i} \partial_{x_j} = \partial_{x_j} \partial_{x_i}$, for all $i, j = 1, \dots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial_{x_1}^2 + \dots + \partial_{x_n}^2 = \Delta$. The \mathcal{F} -spectrum is associated to the equation

$$(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})v = 0 \quad \text{for } v \neq 0$$

which, in this case, becomes

$$(s^2\mathcal{I} + \Delta)v = 0 \quad \text{for } v \neq 0. \quad (3.1)$$

The paravector s can be considered as an element belonging to a complex plane $s \in \mathbb{C}_{I_0}$, so we can assume that $s = s_0 + s_1 I_0$ is a solution of (3.1) for some I_0 . Then the \mathcal{F} -spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.1)}} \{s = s_0 + s_1 I, \text{ for all } I \in \mathbb{S}\}.$$

Example (The case of second derivatives). Let us consider the second-order operators $(\partial_{x_1}^2, \dots, \partial_{x_n}^2)$ each of them acting on the vector space of functions of class \mathcal{C}^4 over an open set $U \subseteq \mathbb{R}^{n+1}$, and let us write

$$T = \partial_{x_1}^2 e_1 + \dots + \partial_{x_n}^2 e_n.$$

Determine the equation which gives its \mathcal{F} -spectrum. We have $\overline{T} = -\partial_{x_1}^2 e_1 - \dots - \partial_{x_n}^2 e_n$, and, since $\partial_{x_i}^2 \partial_{x_j}^2 = \partial_{x_j}^2 \partial_{x_i}^2$, for all $i, j = 1, \dots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial_{x_1}^4 + \dots + \partial_{x_n}^4$. The \mathcal{F} -spectrum is associated to the equation

$$(s^2 \mathcal{I} + \partial_{x_1}^4 + \dots + \partial_{x_n}^4)v = 0 \quad \text{for } v \neq 0. \quad (3.2)$$

We solve the equation (3.2) on the complex plane $s \in \mathbb{C}_{I_0}$ for some I_0 . Then the \mathcal{F} -spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.2)}} \{s = s_0 + s_1 I, \text{ for all } I \in \mathbb{S}\}.$$

Example (The case of powers of a real matrix). Let A be a matrix $n \times n$ with real entries and consider the operators $T_j := A^j$ for $j = 1, \dots, n$. Determine the equation associated to the \mathcal{F} -spectrum. It is well known that $T_j T_k = T_k T_j$, for $j, k = 1, \dots, n$. So we consider the operator

$$T = A e_1 + \dots + A^n e_n.$$

We have $\overline{T} = -A e_1 - \dots - A^n e_n$, $T + \overline{T} = 0$ and also $T\overline{T} = \sum_{j=1}^n A^{2j}$. The \mathcal{F} -spectrum is associated to the equation

$$(s^2 \mathcal{I} + \sum_{j=1}^n A^{2j})v = 0 \quad \text{for } v \neq 0. \quad (3.3)$$

Let us conclude this short list of examples with an explicit computation of the \mathcal{F} -spectrum.

Example (The case of two triangular commuting matrices). Let us consider $a, b, \alpha, \beta \in \mathbb{R}$ and the two matrices:

$$T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}.$$

It is easy to verify that $T_1 T_2 = T_2 T_1$. So we associate to T_1 and T_2 the operator

$$T = T_1 e_1 + T_2 e_2 = \begin{bmatrix} a e_1 + \alpha e_2 & b e_1 + \beta e_2 \\ 0 & a e_1 + \alpha e_2 \end{bmatrix}.$$

We have

$$\overline{T} = - \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

$T + \overline{T} = 0$ and

$$T\overline{T} = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}.$$

The \mathcal{F} -spectrum is associated to the equation

$$\left(s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix} \right) v = 0 \quad \text{for } v \neq 0$$

which becomes

$$\begin{bmatrix} s^2 + a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & s^2 + a^2 + \alpha^2 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \quad \text{for } \begin{bmatrix} u \\ w \end{bmatrix} \neq 0.$$

Consider a paravector s on the complex plane \mathbb{C}_{I_0} : with some calculation we obtain $s = \pm I_0 \sqrt{a^2 + \alpha^2}$. Thus the \mathcal{F} -spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \{\pm I \sqrt{a^2 + \alpha^2} \text{ for all } I \in \mathbb{S}\}.$$

4. Bounded perturbations of the \mathcal{SC} -resolvent

Lemma 4.1. *The set $\mathcal{U}(V_n)$ of elements in $\mathcal{B}_n(V_n)$ which have inverse in $\mathcal{B}_n(V_n)$ is an open set in the uniform topology of $\mathcal{B}_n(V_n)$. If $\mathcal{U}(V_n)$ contains an element A , then it contains the ball*

$$\Sigma = \{B \in \mathcal{B}_n(V_n) : \|A - B\| < \|A^{-1}\|^{-1}\}.$$

If $B \in \Sigma$, its inverse is given by the series

$$B^{-1} = A^{-1} \sum_{m \geq 0} [(A - B)A^{-1}]^m. \quad (4.1)$$

Furthermore, the map $A \mapsto A^{-1}$ from $\mathcal{U}(V_n)$ onto $\mathcal{U}(V_n)$ is a homeomorphism in the uniform operator topology.

Proof. See Lemma 7.1 in [3]. □

In order to state our results, we need the following definitions:

Definition 4.2. Let $T \in \mathcal{BC}_n^{0,1}(V_n)$. We denote by $\sigma_L(T)$ the so-called left spectrum of T related to the resolvent operator $(s\mathcal{I} - T)^{-1}$ that is defined as

$$\sigma_L(T) = \{s \in \mathbb{R}^{n+1} : s\mathcal{I} - T \text{ is not invertible in } \mathcal{BC}_n^{0,1}(V_n)\},$$

where the notation $s\mathcal{I}$ in $\mathcal{B}^R(V)$ means that $(s\mathcal{I})(v) = sv$.

Definition 4.3. Let \mathcal{W} be a subset of \mathbb{R}^{n+1} . We denote by $B(\mathcal{W}, \varepsilon)$, for $\varepsilon > 0$, the ε -neighborhood of \mathcal{W} defined as

$$B(\mathcal{W}, \varepsilon) := \{x \in \mathbb{R}^{n+1} : \inf_{s \in \mathcal{W}} |s - x| < \varepsilon\}.$$

Lemma 4.4. *Let $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and consider*

$$\mathcal{S}_C(s, T) = s\mathcal{I} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1}, \quad (4.2)$$

$$\mathcal{S}_C(s, Z) = s\mathcal{I} - (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1}. \quad (4.3)$$

Then there exists a strictly positive constant $K(s)$, depending on s and also on the operators T and Z , such that

$$\|\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z)\| \leq K(s)\|T - Z\|. \quad (4.4)$$

Proof. Consider the chain of equalities

$$\begin{aligned} & \mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z) \\ &= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1} \\ &= (s\mathcal{I} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1} \\ &\quad + (s\mathcal{I} - \overline{T})Z(s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})T(s\mathcal{I} - \overline{T})^{-1} \\ &= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})[Z(s\mathcal{I} - \overline{Z})^{-1} - T(s\mathcal{I} - \overline{T})^{-1}] \\ &= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} \\ &\quad + (s\mathcal{I} - \overline{T})[(Z - T)(s\mathcal{I} - \overline{Z})^{-1} + T((s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})^{-1})] \\ &= (\overline{T} - \overline{Z})Z(s\mathcal{I} - \overline{Z})^{-1} \\ &\quad + (s\mathcal{I} - \overline{T})[(Z - T)(s\mathcal{I} - \overline{Z})^{-1} + T(s\mathcal{I} - \overline{Z})^{-1}(\overline{Z} - \overline{T})(s\mathcal{I} - \overline{T})^{-1}]. \end{aligned}$$

By taking the norm and observing that $\|T - Z\| = \|\overline{T} - \overline{Z}\|$, we have

$$\begin{aligned} \|\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z)\| &\leq \|T - Z\| \left(\|Z\| \|(s\mathcal{I} - \overline{Z})^{-1}\| \right. \\ &\quad \left. + \|s\mathcal{I} - \overline{T}\| \left[\|(s\mathcal{I} - \overline{Z})^{-1}\| + \|T\| \|(s\mathcal{I} - \overline{Z})^{-1}\| \|(s\mathcal{I} - \overline{T})^{-1}\| \right] \right). \end{aligned}$$

If we now set

$$K(s) := \|(s\mathcal{I} - \overline{Z})^{-1}\| \left(\|Z\| + \|s\mathcal{I} - \overline{T}\| \left[1 + \|T\| \|(s\mathcal{I} - \overline{T})^{-1}\| \right] \right), \quad (4.5)$$

we have that $K(s) > 0$ and we get the statement. \square

Lemma 4.5. *Let $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and suppose that*

$$\|T - Z\| < \frac{1}{K(s)} \|\mathcal{S}_C^{-1}(s, T)\|^{-1},$$

where $K(s)$ is defined in (4.5). Then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\mathcal{S}_C^{-1}(s, Z) - \mathcal{S}_C^{-1}(s, T) = \mathcal{S}_C^{-1}(s, T) \sum_{m \geq 1} [(\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z))\mathcal{S}_C^{-1}(s, T)]^m. \quad (4.6)$$

Proof. Let us recall (4.2) and (4.3) and set

$$A := \mathcal{S}_C(s, T), \quad B := \mathcal{S}_C(s, Z), \quad A^{-1} = \mathcal{S}_C^{-1}(s, T). \quad (4.7)$$

By formula (4.1) in Lemma 4.1 with $B^{-1} := \mathcal{S}_C^{-1}(s, Z)$, we get

$$\mathcal{S}_C^{-1}(s, Z) = \mathcal{S}_C^{-1}(s, T) \sum_{m \geq 0} [(\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z))\mathcal{S}_C^{-1}(s, T)]^m. \quad (4.8)$$

The series in (4.8) converges since

$$\|(\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z))\mathcal{S}_C^{-1}(s, T)\| \leq K(s)\|T - Z\|\|\mathcal{S}_C^{-1}(s, T)\| < 1,$$

so we get the statement. \square

Theorem 4.6. *Let $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Let $\varepsilon > 0$ and consider the ε -neighborhood $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$. Then there exists $\delta > 0$ such that, for $\|T - Z\| < \delta$, we have*

$$\sigma_{\mathcal{F}}(Z) \subseteq B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$$

and

$$\|\mathcal{S}_C^{-1}(s, Z) - \mathcal{S}_C^{-1}(s, T)\| < \varepsilon, \quad \text{for } s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon).$$

Proof. Let $\overline{T}, \overline{Z} \in \mathcal{BC}_n^{0,1}(V_n)$ and let $\varepsilon > 0$. Thanks to Lemma 4.1 there exists a $\eta > 0$ such that if

$$\|T - Z\| < \eta$$

then $\sigma_L(\overline{Z}) \subset B(\sigma_L(\overline{T}), \varepsilon)$, where $B(\sigma_L(\overline{T}), \varepsilon)$ is the ε -neighborhood of $\sigma_L(\overline{T})$. So we can always choose η such that $\sigma_L(\overline{Z}) \subset B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$. Consider the function $K(s)$ defined in (4.5) and observe that the constant K_ε defined by

$$K_\varepsilon = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)} K(s)$$

is finite since $s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$, for the above observation $\sigma_L(\overline{Z}) \subset B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ and because

$$\lim_{s \rightarrow \infty} \|(s\mathcal{I} - \overline{Z})^{-1}\| = \lim_{s \rightarrow \infty} \|(s\mathcal{I} - \overline{T})^{-1}\| = 0.$$

Observe that since $s \in \rho_{\mathcal{F}}(T)$ the map $s \mapsto \|\mathcal{S}_C^{-1}(s, T)\|$ is continuous and

$$\lim_{s \rightarrow \infty} \|\mathcal{S}_C^{-1}(s, T)\| = 0,$$

for s in the complement set of $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ we have that there exists a positive constant N_ε such that

$$\|\mathcal{S}_C^{-1}(s, T)\| \leq N_\varepsilon.$$

From Lemma 4.5, if $\delta_1 > 0$ is such that $\|Z - T\| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1$, then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\begin{aligned} \|\mathcal{S}_C^{-1}(s, Z) - \mathcal{S}_C^{-1}(s, T)\| &\leq \frac{\|\mathcal{S}_C^{-1}(s, T)\|^2 \|\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z)\|}{1 - \|\mathcal{S}_C^{-1}(s, T)\| \|\mathcal{S}_C(s, T) - \mathcal{S}_C(s, Z)\|} \\ &\leq \frac{N_\varepsilon^2 K_\varepsilon \|Z - T\|}{1 - N_\varepsilon K_\varepsilon \|Z - T\|} < \varepsilon \end{aligned}$$

if we take

$$\|Z - T\| < \delta_2 := \frac{\varepsilon}{K_\varepsilon(N_\varepsilon^2 + \varepsilon N_\varepsilon)}.$$

To get the statement it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$. \square

Theorem 4.7. *Let $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ and*

$$\|f(Z) - f(T)\| < \varepsilon.$$

Proof. We recall that operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, T) ds_I f(s)$$

where $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Suppose that U is an ε -neighborhood of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$ and it is contained in the domain in which f is s -monogenic. By Lemma 4.6 there is a $\delta_1 > 0$ such that $\sigma_{\mathcal{F}}(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 4.6 $\mathcal{S}_C^{-1}(s, T)$ is uniformly near to $\mathcal{S}_C^{-1}(s, Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|f(T) - f(Z)\| = \frac{1}{2\pi} \left\| \int_{\partial(U \cap \mathbb{C}_I)} [\mathcal{S}_C^{-1}(s, T) - \mathcal{S}_C^{-1}(s, Z)] ds_I f(s) \right\| < \varepsilon. \quad \square$$

5. Bounded perturbations of the \mathcal{F} -resolvent

Let n be an odd number. For $s \in \rho_{\mathcal{F}}(T)$ the \mathcal{F} -resolvent operator associated to T is

$$\mathcal{F}_n^{-1}(s, T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}}, \quad (5.1)$$

while its inverse is

$$\mathcal{F}_n(s, T) := \frac{1}{\gamma_n}(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{\frac{n+1}{2}}(s\mathcal{I} - \overline{T})^{-1}, \quad (5.2)$$

for $s \notin \sigma_L(\overline{T})$. Analogously for $s \in \rho_{\mathcal{F}}(Z)$ the \mathcal{F} -resolvent operator associated to Z is

$$\mathcal{F}_n^{-1}(s, Z) := \gamma_n(s\mathcal{I} - \overline{Z})(s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z})^{-\frac{n+1}{2}}, \quad (5.3)$$

and it has the inverse

$$\mathcal{F}_n(s, Z) := \frac{1}{\gamma_n}(s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z})^{\frac{n+1}{2}}(s\mathcal{I} - \overline{Z})^{-1}, \quad (5.4)$$

for $s \notin \sigma_L(\overline{Z})$.

Lemma 5.1. *Let n be an odd number, $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$ and let $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Then there exists a positive constant $C_n(s)$ depending on s and also on the operators T and Z such that*

$$\|\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)\| \leq C_n(s)(|s| + \vartheta)^{n-1} \|T - Z\|, \quad (5.5)$$

where $\vartheta := \max\{\|T\|, \|Z\|\}$.

Proof. For simplicity let us set the positions $\frac{n+1}{2} := k+1$, for $k \in \mathbb{N}$, so that $k = \frac{n-1}{2}$, for $n = 1, 3, 5, \dots$. The case $k = 0$ has been studied in the previous section. Here we consider $k \geq 1$. We set $\beta_k := \gamma_{2k+1}$ and we define, for $s \in \rho_{\mathcal{F}}(T)$,

$$\tilde{\mathcal{F}}_k^{-1}(s, T) := \beta_k \mathcal{S}_{\mathcal{C}}^{-1}(s, T) (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-k}. \quad (5.6)$$

The inverse of operator $\tilde{\mathcal{F}}_k^{-1}(s, T)$ exists for $s \notin \sigma_L(\overline{T})$ and is given by

$$\tilde{\mathcal{F}}_k(s, T) = \frac{1}{\beta_k} (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T})^k \mathcal{S}_{\mathcal{C}}(s, T), \quad (5.7)$$

while the inverse of operator $\tilde{\mathcal{F}}_k^{-1}(s, Z)$ exists for $s \notin \sigma_L(\overline{Z})$ and is given by

$$\tilde{\mathcal{F}}_k(s, Z) = \frac{1}{\beta_k} (s^2 \mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z})^k \mathcal{S}_{\mathcal{C}}(s, Z). \quad (5.8)$$

Consider (5.7) and (5.8) for $k = 1$; we have

$$\begin{aligned} & \beta_1 [\tilde{\mathcal{F}}_1(s, T) - \tilde{\mathcal{F}}_1(s, Z)] \\ &= (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) \mathcal{S}_{\mathcal{C}}(s, T) - (s^2 \mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z}) \mathcal{S}_{\mathcal{C}}(s, Z) \\ &= (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) \mathcal{S}_{\mathcal{C}}(s, T) - (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) \mathcal{S}_{\mathcal{C}}(s, Z) \\ &\quad + (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) \mathcal{S}_{\mathcal{C}}(s, Z) - (s^2 \mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z}) \mathcal{S}_{\mathcal{C}}(s, Z) \\ &= (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) [\mathcal{S}_{\mathcal{C}}(s, T) - \mathcal{S}_{\mathcal{C}}(s, Z)] \\ &\quad + [-s(T + \overline{T}) + T\overline{T} + s(Z + \overline{Z}) - Z\overline{Z}] \mathcal{S}_{\mathcal{C}}(s, Z) \\ &= (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) [\mathcal{S}_{\mathcal{C}}(s, T) - \mathcal{S}_{\mathcal{C}}(s, Z)] \\ &\quad + [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})] \mathcal{S}_{\mathcal{C}}(s, Z) \end{aligned}$$

and taking the norm we get

$$\begin{aligned} & |\beta_1| \|\tilde{\mathcal{F}}_1(s, T) - \tilde{\mathcal{F}}_1(s, Z)\| \\ &\leq (|s|^2 + 2|s| \|T\| + \|T\overline{T}\|) \|\mathcal{S}_{\mathcal{C}}(s, T) - \mathcal{S}_{\mathcal{C}}(s, Z)\| \\ &\quad + [2|s| \|Z - T\| + \|T - Z\| (\|\overline{T}\| + \|Z\|)] \|\mathcal{S}_{\mathcal{C}}(s, Z)\|. \\ &\leq (|s| + \vartheta)^2 \|\mathcal{S}_{\mathcal{C}}(s, T) - \mathcal{S}_{\mathcal{C}}(s, Z)\| + [2(|s| + \vartheta) \|Z - T\|] \|\mathcal{S}_{\mathcal{C}}(s, Z)\|. \end{aligned}$$

Now observe that

$$(|s| + \vartheta)^{-1} \|\mathcal{S}_{\mathcal{C}}(s, Z)\| \leq (|s| + \vartheta)^{-1} [|s| + \|(s\mathcal{I} - \overline{Z})\| \|Z\| \|(s\mathcal{I} - \overline{Z})^{-1}\|] =: M(s) \quad (5.9)$$

where $M(s)$ is a continuous function since $s \notin \sigma_L(\overline{Z})$. Using Lemma 4.4 we get

$$\|\tilde{\mathcal{F}}_1(s, T) - \tilde{\mathcal{F}}_1(s, Z)\| \leq \frac{1}{|\beta_1|} [K(s) + 2M(s)] (|s| + \vartheta)^2 \|Z - T\|. \quad (5.10)$$

We now use the induction principle. We assume that the estimate

$$\|\tilde{\mathcal{F}}_k(s, T) - \tilde{\mathcal{F}}_k(s, Z)\| \leq \frac{1}{|\beta_k|} (|s| + \vartheta)^{2k} \left[K(s) + 2kM(s) \right] \|Z - T\| \quad (5.11)$$

holds for $k \geq 1$. Observe that (5.8) implies that the estimate

$$\|\mathcal{F}_k(s, Z)\| \leq \frac{1}{|\beta_k|} (|s| + \vartheta)^{2k} \|\mathcal{S}_C(s, Z)\| \quad (5.12)$$

holds. We prove that

$$\|\tilde{\mathcal{F}}_{k+1}(s, T) - \tilde{\mathcal{F}}_{k+1}(s, Z)\| \leq \frac{1}{|\beta_{k+1}|} (|s| + \vartheta)^{2(k+1)} \left[K(s) + 2(k+1)M(s) \right] \|Z - T\|.$$

In fact, we have that

$$\begin{aligned} & \beta_{k+1} (\tilde{\mathcal{F}}_{k+1}(s, T) - \tilde{\mathcal{F}}_{k+1}(s, Z)) \\ &= \beta_k (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) \tilde{\mathcal{F}}_k(s, T) - \beta_k (s^2 \mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z}) \tilde{\mathcal{F}}_k(s, Z) \\ &= \beta_k (s^2 \mathcal{I} - s(T + \overline{T}) + T\overline{T}) [\tilde{\mathcal{F}}_k(s, T) - \tilde{\mathcal{F}}_k(s, Z)] \\ &\quad - \beta_k [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})] \tilde{\mathcal{F}}_k(s, Z) \end{aligned}$$

and taking the norms we have

$$\begin{aligned} & |\beta_{k+1}| \|\tilde{\mathcal{F}}_{k+1}(s, T) - \tilde{\mathcal{F}}_{k+1}(s, Z)\| \\ & \leq |\beta_k| (|s| + \vartheta)^2 \|\tilde{\mathcal{F}}_k(s, T) - \tilde{\mathcal{F}}_k(s, Z)\| + 2|\beta_k| (|s| + \vartheta) \|\mathcal{F}_k(s, Z)\| \|Z - T\|. \end{aligned}$$

Using (5.11) and (5.12) we obtain

$$\begin{aligned} & |\beta_{k+1}| \|\tilde{\mathcal{F}}_{k+1}(s, T) - \tilde{\mathcal{F}}_{k+1}(s, Z)\| \\ & \leq (|s| + \vartheta)^{2k+2} \left[K(s) + 2kM(s) \right] \|Z - T\| + 2(|s| + \vartheta)^{2k+1} \|\mathcal{S}_C(s, Z)\| \|Z - T\| \\ & \leq (|s| + \vartheta)^{2k+2} \left[K(s) + 2kM(s) + 2(|s| + \vartheta)^{-1} \|\mathcal{S}_C(s, Z)\| \right] \|Z - T\| \\ & \leq (|s| + \vartheta)^{2k+2} \left[K(s) + 2(k+1)M(s) \right] \|Z - T\|. \end{aligned}$$

Setting $\tilde{C}_k(s) := \frac{1}{|\beta_k|} [K(s) + 2kM(s)]$ the constant $C_n(s)$ in estimate (5.5) is given by

$$C_n(s) := \frac{1}{|\gamma_n|} [K(s) + (n-1)M(s)]. \quad (5.13)$$

This concludes the proof. \square

Lemma 5.2. *Let n be an odd number, $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, let $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and suppose that*

$$\|T - Z\| < \frac{1}{C_n(s)} (|s| + \vartheta)^{-(n-1)} \|\mathcal{F}_n^{-1}(s, T)\|^{-1},$$

where $C_n(s)$ is defined in (5.13). Then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\mathcal{F}_n^{-1}(s, Z) - \mathcal{F}_n^{-1}(s, T) = \mathcal{F}_n^{-1}(s, T) \sum_{m \geq 1} [(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)) \mathcal{F}_n^{-1}(s, T)]^m. \quad (5.14)$$

Proof. Let us recall (5.2), (5.4) and set

$$A := \mathcal{F}_n(s, T), \quad B := \mathcal{F}_n(s, Z), \quad A^{-1} = \mathcal{F}_n^{-1}(s, T)(s, T). \quad (5.15)$$

By Lemma 4.1, formula (4.1), for $B^{-1} := \mathcal{F}_n^{-1}(s, Z)$ we get

$$\mathcal{F}_n^{-1}(s, Z) = \mathcal{F}_n^{-1}(s, T) \sum_{m \geq 0} [(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)) \mathcal{F}_n^{-1}(s, T)]^m. \quad (5.16)$$

Using the hypothesis, we have that the series converges since

$$\begin{aligned} & \|(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)) \mathcal{F}_n^{-1}(s, T)\| \\ & \leq \|(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z))\| \|\mathcal{F}_n^{-1}(s, T)\| \\ & \leq C_n(s)(|s| + \vartheta)^{n-1} \|Z - T\| \|\mathcal{F}_n^{-1}(s, T)\| < 1. \end{aligned} \quad \square$$

Theorem 5.3. *Let n be an odd number, $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Let $\varepsilon > 0$ and consider the ε -neighborhood $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$. Then there exists $\delta > 0$ such that, for $\|T - Z\| < \delta$, we have*

$$\sigma_{\mathcal{F}}(Z) \subseteq B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$$

and

$$\|\mathcal{F}_n^{-1}(s, Z) - \mathcal{F}_n^{-1}(s, T)\| < \varepsilon, \quad \text{for } s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon).$$

Proof. Let $\overline{T}, \overline{Z} \in \mathcal{BC}_n^{0,1}(V_n)$ and let $\varepsilon > 0$. Thanks to Lemma 4.1 there exists a $\eta > 0$ such that if

$$\|T - Z\| < \eta,$$

then $\sigma_L(\overline{Z}) \subset B(\sigma_L(\overline{T}), \varepsilon)$, where $B(\sigma_L(\overline{T}), \varepsilon)$ is the ε -neighborhood of $\sigma_L(\overline{T})$. So we can always choose η such that $\sigma_L(\overline{Z}) \subset B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$. Consider the function $C_n(s)$ defined in (5.13). The constant $C_{n,\varepsilon}$ defined as

$$C_{n,\varepsilon} = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)} C_n(s)$$

is finite because $s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ and

$$\lim_{s \rightarrow \infty} \|(s\mathcal{I} - \overline{Z})^{-1}\| = \lim_{s \rightarrow \infty} \|(s\mathcal{I} - \overline{T})^{-1}\| = 0.$$

Observe that since $s \in \rho_{\mathcal{F}}(T)$ map $s \mapsto \|\mathcal{F}_n^{-1}(s, T)\|$ is continuous and

$$\lim_{s \rightarrow \infty} \|\mathcal{F}_n^{-1}(s, T)\| = 0,$$

and so for s in the complement set of $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ we have that there exists a positive constant M_ε such that

$$\|\mathcal{F}_n^{-1}(s, T)\| \leq M_\varepsilon.$$

From Lemma 5.2 if $\delta_1 > 0$ is such that

$$\|Z - T\| < \frac{1}{C_{n,\varepsilon}M_\varepsilon} := \delta_3,$$

then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\begin{aligned} & \|\mathcal{F}_n^{-1}(s, Z) - \mathcal{F}_n^{-1}(s, T)\| \\ & \leq \frac{\|\mathcal{F}_n^{-1}(s, T)\|^2 \|\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)\|}{1 - \|\mathcal{F}_n^{-1}(s, T)\| \|\mathcal{F}_n(s, T) - \mathcal{F}(s, Z)\|} \\ & \leq \frac{M_\varepsilon^2 C_{n,\varepsilon} \|Z - T\|}{1 - M_\varepsilon C_{n,\varepsilon} \|Z - T\|} < \varepsilon \end{aligned}$$

if we take

$$\|Z - T\| < \delta_4 := \frac{\varepsilon}{C_{n,\varepsilon}(M_\varepsilon^2 + \varepsilon M_\varepsilon)}.$$

To get the statement it suffices to set $\delta = \min\{\eta, \delta_3, \delta_4\}$. \square

Theorem 5.4. *Let n be an odd number, $T, Z \in \mathcal{BC}_{n,n}^{0,1}(V_n)$, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ and*

$$\|\check{f}(Z) - \check{f}(T)\| < \varepsilon.$$

Proof. We recall that

$$\check{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T) ds_I f(s)$$

and $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Let U be an ε -neighborhood of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$ contained in the domain in which f is s -monogenic. By Lemma 5.3 there is a $\delta_1 > 0$ such that $\sigma_{\mathcal{F}}(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 5.3, $\mathcal{F}_n^{-1}(s, T)$ is uniformly near to $\mathcal{F}_n^{-1}(s, Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|\check{f}(T) - \check{f}(Z)\| = \frac{1}{2\pi} \left\| \int_{\partial(U \cap \mathbb{C}_I)} [\mathcal{F}_n^{-1}(s, T) - \mathcal{F}_n^{-1}(s, Z)] ds_I f(s) \right\| < \varepsilon. \quad \square$$

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