Chapter 2

Function Spaces and Operators

Sobolev spaces $W^{s,p}$ proved to be very convenient in the study of elliptic problems in bounded domains. For unbounded domains, it is also useful to introduce some generalizations of these spaces in such a way that they coincide with $W^{s,p}$ in bounded domains and have a prescribed behavior at infinity in unbounded domains. In this chapter we introduce spaces of functions in unbounded domains and study their properties.

It turns out that such spaces can be constructed for arbitrary Banach spaces of distributions, not only Sobolev spaces, as follows. Consider first functions defined on $\mathbb{R}^n$. As usual we denote by D the space of infinitely differentiable functions with compact support and by D' its dual. Let $E \subset D'$ be a Banach space, the inclusion is understood both in an algebraic and a topological sense. Denote by E_{loc} the collection of all $u \in D'$ such that $fu \in E$ for all $f \in D$. Let $\omega(x) \in D,\ \ 0 \le \omega(x) \le 1,\ \ \omega(x) = 1$ for $|x| \le 1/2,\ \ \omega(x) = 0$ for $|x| \ge 1$.

Definition. $E_q \ \ (1 \le q \le \infty)$ is the space of all $u \in E_{\text{loc}}$ such that

$$\|u\|_{E_q} := \left(\int_{R^n} \|u(.)\omega(. - y)\|_E^q dy \right)^{1/q} < \infty, \quad 1 \le q < \infty,$$

$$\|u\|_{E_\infty} := \sup_{y \in R^n} \|u(.)\omega(. - y)\|_E < \infty.$$

In what follows we will also use an equivalent definition based on a partition of unity. It will be proved that E_q is a Banach space. If Ω is a domain in $\mathbb{R}^n$, then by definition $E_q(\Omega)$ is the space of restrictions of E_q to Ω with the usual norm of restrictions. It is easy to see that if Ω is a bounded domain, then

$$E_q(\Omega) = E(\Omega), \quad 1 \le q \le \infty.$$

In particular, if $E = W^{s,p}$, then we set $W_q^{s,p} = E_q \quad (1 \le q \le \infty)$. We will show that

$$W_p^{s,p} = W^{s,p} \quad (s \ge 0,\ 1 < p < \infty).$$

V. Volpert, *Elliptic Partial Differential Equations: Volume 1: Fredholm Theory of Elliptic Problems in Unbounded Domains*, Monographs in Mathematics 101,
DOI 10.1007/978-3-0346-0537-3_2,

Hence the spaces $W_q^{s,p}$ generalize Sobolev spaces ($q < \infty$) and Stepanov spaces ($q = \infty$) (see [309], [310]).

1 The space E

Everywhere below we denote by $D(\mathbb{R}^n)$ the space of infinitely differentiable functions with finite supports, and by $D'(\mathbb{R}^n)$ the space of generalized functions, i.e., linear continuous functionals on $D(\mathbb{R}^n)$. In this section we consider only the whole $\mathbb{R}^n$, and we will use the notations D and D'.

Consider a Banach space E with the elements from D'. The inclusion $E \subset D'$ is understood both in the algebraic and topological sense.

Definition 1.1. The space of multipliers $M(E)$ on E is a set of infinitely differentiable functions $f(x)$, $x \in \mathbb{R}^n$ such that the operator of multiplication by f is a bounded operator in E. All functions defined in $\mathbb{R}^n$, which are infinitely differentiable and have all bounded derivatives, are multipliers in E.

We denote by $\|\cdot\|_E$ the norm in E and by $\|\cdot\|_M$ the norm in $M(E)$. By definition

$$\|fu\|_E \leq \|f\|_M \|u\|_E, \quad \forall f \in M(E), \ \forall u \in E.$$

Proposition 1.2. *Let E be invariant with respect to translation in $\mathbb{R}^n$ and*

$$\|\tau_h u\|_E = \|u\|_E, \quad \forall u \in E,$$

where τ_h is an operator of translation. Let further $f \in M(E)$, $\tau_h f(x) = f(x+h)$, $h \in \mathbb{R}^n$. Then

$$\|\tau_h f\|_M = \|f\|_M.$$

Proof. We first prove that

$$\tau_h(fu) = \tau_h f(\tau_h u). \tag{1.1}$$

Indeed, by definition for any $\phi \in D$ we have

$$\langle \tau_h(fu), \phi \rangle = \langle fu, \tau_{-h}\phi \rangle,$$

$$\langle \tau_h f(\tau_h u), \phi \rangle = \langle \tau_h u, (\tau_h f) \cdot \phi \rangle = \langle u, \tau_{-h}((\tau_h f) \cdot \phi) \rangle = \langle u, f \cdot \tau_{-h}\phi \rangle = \langle fu, \tau_{-h}\phi \rangle,$$

and (1.1) is proved. Further,

$$\|fu\|_E = \|\tau_h(fu)\|_E = \|\tau_h f(\tau_h u)\|_E \leq \|\tau_h f\|_M \|\tau_h u\|_E = \|\tau_h f\|_M \|u\|_E.$$

Hence

$$\|f\|_M \leq \|\tau_h f\|_M. \tag{1.2}$$

Therefore

$$\|\tau_h f\|_M \leq \|\tau_{-h}(\tau_h f)\|_M = \|f\|_M.$$

Together with (1.2) this estimate proves the proposition. $\square$

In what follows we suppose that for any $f \in D$,

$$\sup_h \|\tau_h f\|_M < \infty. \tag{1.3}$$

Example 1.3. If $E = H^{s,p}$ or $E = W^{s,p}$, where $-\infty < s < \infty$, $1 < p < \infty$, then any infinitely differentiable function f from $C^{[|s|]+1}(\mathbb{R}^n)$ belongs to $M(E)$ and

$$\|f\|_M \le K \|f\|_{C^{[|s|]+1}},$$

where K is a positive constant.

Definition 1.4. E_{loc} is a space of all $u \in D'$ such that $fu \in E$ for all $f \in D$.

2 Systems of functions

Definition 2.1. Partition of unity is a sequence $\{\phi_i\}$, $i = 1, 2, \ldots$ of functions $\phi_i \in D$, $\phi_i(x) \ge 0$ such that

$$\sum_{i=1}^{\infty} \phi_i(x) = 1, \quad x \in \mathbb{R}^n.$$

Condition 2.2. Let $\{\phi_i\}$, $i = 1, 2, \ldots$ be a sequence of functions $\phi_i \in D$. For some given N and any i there exists no more than N functions ϕ_j such that $\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_i \ne \oslash$.

Everywhere below we consider partitions of unity for which Condition 2.2 is satisfied.

Definition 2.3. Two systems of functions $\{\phi_i\}$, $\{\psi_j\}$, $i = 1, 2, \ldots$, $j = 1, 2, \ldots$, $\phi_i \in D$, $\psi_j \in D$ are called *equivalent* if there exists a number N such that:

- for any i there exists no more than N functions ψ_j such that $\operatorname{supp}\psi_j \cap \operatorname{supp}\phi_i \ne \oslash$,
- for any j there exists no more than N functions ϕ_i such that $\operatorname{supp}\phi_i \cap \operatorname{supp}\psi_j \ne \oslash$.

Proposition 2.4. *The equivalence relation introduced by Definition* 2.3 *is reflexive, symmetric, and transitive.*

We will also use systems of functions satisfying the following condition.

Condition 2.5. System of functions ϕ_i satisfies the following conditions:

1. $\phi_i(x) \ge 0$, $\phi_i \in D$,
2. Condition 2.2 is satisfied,
3. $\sup_i \|\phi_i\|_M < \infty$,

4. $\phi(x) = \sum_{i=1}^{\infty} \phi_i(x) \geq m > 0$ for some constant m,
5. the following estimate holds:

$$\sup_x |D^\alpha \phi(x)| \leq M_\alpha,$$

where D^α denotes the operator of differentiation, and M_α are positive constants.

3 The space E_p

Definition 3.1. Let $\{\phi_i\},\ i = 1, 2, \ldots$ be a partition of unity. E_p is the space of all $u \in E_{\text{loc}}$ such that

$$\sum_{i=1}^{\infty} \|\phi_i u\|_E^p < \infty,$$

where $1 \leq p < \infty$, with the norm

$$\|u\|_{E_p} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^p \right)^{1/p}.$$

Proposition 3.2. *Let $\{\phi_i^1\}$ and $\{\phi_i^2\}$ be two partitions of unity such that*

$$\sup_i \|\phi_i^1\|_M < \infty, \quad \sup_i \|\phi_i^2\|_M < \infty.$$

Suppose that E_p^1 and E_p^2 are the spaces E_p corresponding to $\{\phi_i^1\}$ and $\{\phi_i^2\}$, respectively. If the partitions of unity are equivalent, then $E_p^1 = E_p^2$, and their norms are equivalent.

Proof. Let $u \in E_p^2$. We have

$$\phi_i^1 u = \phi_i^1 \sum_{j=1}^{\infty} \phi_j^2 u = \sum_{j'} \phi_i^1 \phi_{j'}^2 u,$$

where j' are all the numbers j such that $\operatorname{supp} \phi_i^1 \cap \operatorname{supp} \phi_j^2 \neq \oslash$. By Definition 2.3 the number of such j' is no more than N. We have the estimate

$$\|\phi_i^1 u\|_E^p \leq \left(\sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E \right)^p.$$

Let $a_j \geq 0,\ j = 1, \ldots, m$. Then from convexity of the function s^p we obtain the estimate

$$\left(\sum_{j=1}^m a_j \right)^p = m^p \left(\sum_{j=1}^m \frac{1}{m} a_j \right)^p \leq m^{p-1} \sum_{j=1}^m a_j^p.$$

Therefore

$$\left(\sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E\right)^p \le m^{p-1} \sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E^p,$$

where m is the number of j'. Since $m \le N$, then

$$\|\phi_i^1 u\|_E^p \le N^{p-1} \sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E^p = N^{p-1} \sum_{j=1}^{\infty} \|\phi_i^1 \phi_j^2 u\|_E^p.$$

Let k be a positive integer. We have

$$\sum_{i=1}^{k} \|\phi_i^1 u\|_E^p = N^{p-1} \sum_{i=1}^{k} \sum_{j=1}^{\infty} \|\phi_i^1 \phi_j^2 u\|_E^p = N^{p-1} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^p, \tag{3.1}$$

$$\sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^p = \sum_{i'} \|\phi_{i'}^1 \phi_j^2 u\|_E^p \le \sum_{i'} \|\phi_{i'}^1\|_M^p \|\phi_j^2 u\|_E^p,$$

where i' are those of i for which $\operatorname{supp} \phi_i^1 \cap \operatorname{supp} \phi_j^2 \neq \oslash$. The number of such i' is less than or equal to N. Let

$$K_j = \sup_i \|\phi_i^j\|_M, \quad j = 1, 2.$$

Then

$$\sum_{i=1}^{k} \|\phi_i^1 \phi_j^2 u\|_E^p \le N K_1 \|\phi_j^2 u\|_E^p.$$

It follows from (3.1) that

$$\sum_{i=1}^{k} \|\phi_i^1 u\|_E^p \le N^p K_1 \sum_{j=1}^{\infty} \|\phi_j^2 u\|_E^p = N^p K_1 \|u\|_{E_p^2}^p.$$

From this we obtain

$$\sum_{i=1}^{\infty} \|\phi_i^1 u\|_E^p \le N^p K_1 \|u\|_{E_p^2}^p.$$

Hence $u \in E_p^1$ and

$$\|u\|_{E_p^1} \le N K_1^{1/p} \|u\|_{E_p^2}, \quad E_p^2 \subset E_p^1.$$

Similarly we get

$$\|u\|_{E_p^2} \le N K_2^{1/p} \|u\|_{E_p^1}, \quad E_p^1 \subset E_p^2.$$

The proposition is proved. □

Proposition 3.3. *The space E_p is complete.*

Proof. Consider a fundamental sequence u_m in the space E_p. Then for any $\epsilon > 0$ there exists $N(\epsilon)$ such that

$$\sum_{i=1}^{\infty} \|(u_k - u_m)\phi_i\|_E^p \le \epsilon \tag{3.2}$$

for any $k, m \ge N(\epsilon)$. Denote $\Phi_n = \sum_{i=1}^n \phi_i$. Let Ψ_n be an infinitely differentiable function with a finite support such that $\Psi_n = 1$ in the support of Φ_n. Since E is a Banach space and the sequence $\Psi_n u_m$ is fundamental with respect to m for any n fixed, then $\Psi_n u_m \to v_n$ in E as $m \to \infty$. Obviously, $\Phi_n u_m \to \Phi_n v_n$ in E as $m \to \infty$.

Consider a sequence n_j, $n_j \to \infty$ as $j \to \infty$. We construct the sequence of limiting functions v_{n_j} such that

$$\|\Phi_{n_j}(u_m - v_{n_j})\|_E \to 0 \ \text{ as } \ m \to \infty,$$

and for any $j_2 > j_1$,

$$\Phi_{n_{j_1}} v_{n_{j_1}} = \Phi_{n_{j_1}} v_{n_{j_2}}.$$

Therefore we have constructed the limiting function v defined in R^n. It coincides with v_j in the support of Φ_j. We have

$$\|\Phi_{n_j}(u_m - v)\|_E \to 0, \ \text{ as } \ m \to \infty. \tag{3.3}$$

We note that for any $\delta > 0$ there exists $N(\delta)$ and $i_0(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{\infty} \|u_k \phi_i\|_E^p \le \delta \tag{3.4}$$

for any $k \ge N(\delta)$. Indeed, we choose $N(\delta)$ such that

$$\sum_{i=1}^{\infty} \|(u_k - u_m)\phi_i\|_E^p \le C_p \delta \tag{3.5}$$

for any $k, m \ge N(\delta)$. Here $C_p = 2^{-p}$. On the other hand, for a fixed m we can choose $i_0(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{\infty} \|u_m \phi_i\|_E^p \le C_p \delta \tag{3.6}$$

since the corresponding series converges. From (3.5) it follows that for m fixed and any $k \ge N(\delta)$,

$$\sum_{i=i_0(\delta)}^{\infty} \|(u_k - u_m)\phi_i\|_E^p \le C_p \delta. \tag{3.7}$$

From (3.6) and (3.7) we obtain (3.4).

We prove next that

$$\sum_{i=i_0(\delta)}^{\infty} \|v\phi_i\|_E^p \le \delta, \tag{3.8}$$

where $i_0(\delta)$ is the same as in (3.4). Suppose that this estimate is not true. Then there exists $i_1(\delta)$ such that

$$\sum_{i=i_0(\delta)}^{i_1(\delta)} \|v\phi_i\|_E^p > \delta. \tag{3.9}$$

On the other hand from (3.3) we have

$$\sum_{i=i_0(\delta)}^{i_1(\delta)} \|(u_m - v)\phi_i\|_E^p \to 0 \ \text{ as } \ m \to \infty.$$

This convergence and (3.9) contradict (3.4).

From (3.3), (3.4), and (3.8) we conclude that u_m converges to v in E_p. The proposition is proved. □

Proposition 3.4. *Let $u_k = \sum_{i=1}^k u\phi_i$. Then $u_k \to u$ in E_q for $1 \le q < \infty$.*

Proof. We have

$$\|u - u_k\|_{E_q}^q = \sum_{i=1}^{\infty} \|\phi_i(u-u_k)\|_E^q = \sum_{i=1}^{\infty} \|\phi_i \sum_{j=k+1}^{\infty} u\phi_j\|_E^q = \sum_{i=k'}^{\infty} \|\phi_i \sum_{j=k+1}^{\infty} u\phi_j\|_E^q \equiv S,$$

where the external sum is taken over all i such that $\operatorname{supp}\phi_i \cap \operatorname{supp}\phi_j \ne \oslash$ for all $j \ge k+1$. The value k' depends on k, and $k' \to \infty$ as $k \to \infty$,

$$S = \sum_{i=k'}^{\infty} \|\phi_i \sum_{j'} u\phi_{j'}\|_E^q,$$

where j' denotes all j such that $\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_i \ne \oslash$ for a given i. Since the number of such j is uniformly bounded, we have the estimate

$$S \le C_1 \sum_{i=k'}^{\infty} \|\phi_i u\phi_{j'}\|_E^q \le C_2 \sum_{i=k'}^{\infty} \|u\phi_{j'}\|_E^q.$$

The last sum converges to zero as $k \to \infty$. The proposition is proved. □

Corollary 3.5. *Infinitely differentiable functions with bounded supports are dense in E_q, $1 \le q < \infty$.*

Proof. It is sufficient to note that D is dense in E, and $u_k \in E$. □

Definition 3.6. Let $\{\phi_i\},\ i = 1, 2, \dots$ be a system of functions satisfying Condition 2.5. E_p is the space of all $u \in E_{\text{loc}}$ such that

$$\sum_{i=1}^{\infty} \|\phi_i u\|_E^p < \infty,$$

where $1 \le p < \infty$, with the norm

$$\|u\|_{E_p} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^p \right)^{1/p}.$$

Proposition 3.7. *The spaces in Definitions* 3.1 *and* 3.6 *coincide and their norms are equivalent.*

The proof is similar to the proof of Proposition 4.4 below.

We introduce now one more definition of the norm in the space E_q. Let the norm be given by the equality

$$\|u\|_{E_q} = \left(\int_{\mathbb{R}^n} \|u(\cdot)\phi(\cdot - y)\|_E^q dy \right)^{1/q}. \tag{3.10}$$

We show that this norm is equivalent to the norm defined through a partition of unity. We note first of all that the function

$$s(y) = \|u(\cdot)\phi(\cdot - y)\|_E^q$$

is continuous. Indeed,

$$|s^{1/q}(y) - s^{1/q}(y_0)| \le \|u(\cdot)(\phi(\cdot - y) - \phi(\cdot - y_0))\|_E \to 0 \ \text{ as } \ y \to y_0$$

by the properties of multipliers.

We have

$$\|u\|_{E_q}^q = \int_{\mathbb{R}^n} s(y)dy = \sum_{i=1}^{\infty} \int_{Q_i} s(y)dy,$$

where Q_i are unit cubes of the square lattice in $\mathbb{R}^n$,

$$\int_{Q_i} s(y)dy = s(y_i)$$

for some $y_i \in Q_i$ since $s(y)$ is continuous. Hence

$$\|u\|_{E_q}^q = \sum_{i=1}^{\infty} s(y_i). \tag{3.11}$$

This equality is obtained without specific assumptions on the function $\phi(x)$. Suppose now that it equals 1 in the ball of the radius $r = \sqrt{n}$, and 0 outside of the ball with the radius $2r$. Then for any $y_i \in Q_i$,

$$\phi(x - y_i) = 1, \quad x \in Q_i.$$

Therefore the system of functions $\phi_i(x) = \phi(x - y_i)$ satisfies the following conditions:

(1) $m \le \sum_{i=1}^{\infty} \phi_i(x) \le M$ for all $x \in R^n$ and some positive constants m and M,
(2) for each $x \in \mathbb{R}^n$ there exists a finite number of functions ϕ_i different from zero at this point. The estimate of this number is independent of x.

Hence the norm (3.11) is equivalent to the norm defined with any other system of functions equivalent to ϕ_i We have proved the following proposition.

Proposition 3.8. *The norm* (3.10) *is equivalent to the norm in Definition* 3.1.

4 The space E_∞

Definition 4.1. Let $\{\phi_i\}$ be a system of functions from D, $\phi_i(x) \ge 0$. E_∞ is the space of all functions $u \in E_{\text{loc}}$ such that

$$\sup_i \|\phi_i u\|_E < \infty,$$

with the norm

$$\|u\|_{E_\infty} = \sup_i \|\phi_i u\|_E.$$

Proposition 4.2. *Let* $\{\phi_i^1\}$ *and* $\{\phi_i^2\}$ *be two partitions of unity satisfying Condition* 2.2*,*

$$\sup_i \|\phi_i^1\|_M < \infty, \quad \sup_i \|\phi_i^2\|_M < \infty.$$

Suppose that E_∞^1 *and* E_∞^2 *are the spaces* E_∞ *corresponding to* $\{\phi_i^1\}$ *and* $\{\phi_i^2\}$*, respectively. If* $\{\phi_i^1\}$ *and* $\{\phi_i^2\}$ *are equivalent, then* $E_\infty^1 = E_\infty^2$*, and their norms are equivalent.*

Proof. We have

$$\phi_i^1 u = \phi_i^1 \sum_{j=1}^{\infty} \phi_j^2 u = \sum_{j'} \phi_i^1 \phi_{j'}^2 u,$$

where j' are all the numbers j such that $\operatorname{supp} \phi_i^1 \cap \operatorname{supp} \phi_j^2 \ne \oslash$. By Definition 2.3 the number of j' is less than or equal to N. Hence

$$\|\phi_i^1 u\|_E \le \sum_{j'} \|\phi_i^1 \phi_{j'}^2 u\|_E \le \|\phi_i^1\|_M \sum_{j'} \|\phi_{j'}^2 u\|_E \le N \|\phi_i^1\|_M \|u\|_{E_\infty^2},$$

$$\sup_i \|\phi_i^1 u\|_E \le N \sup_i \|\phi_i^1\|_M \|u\|_{E_\infty^2}.$$

Therefore $u \in E^1_\infty$ and

$$\|u\|_{E^1_\infty} \le N \sup_i \|\phi^1_i\|_M \|u\|_{E^2_\infty}.$$

Similarly it can be proved that $E^1_\infty \subset E^2_\infty$ with the corresponding inequality between their norms. The proposition is proved. □

Example 4.3. If $E = H^{s,p}$ or $W^{s,p}$, $-\infty < s < \infty$, $1 < p < \infty$, then instead of 3 in Condition 2.5 we can require

$$\sup_i \|\phi_i\|_{C^{[|s|]+1}} < \infty.$$

Proposition 4.4. *Let $\{\phi^1_i\}$ and $\{\phi^2_i\}$ be two systems of functions satisfying Condition 2.5, E^1_∞ and E^2_∞ be two spaces E_∞ corresponding to $\{\phi^1_i\}$ and $\{\phi^2_i\}$, respectively. If $\{\phi^1_i\}$ and $\{\phi^2_i\}$ are equivalent, then $E^1_\infty = E^2_\infty$, and their norms are equivalent.*

Proof. We introduce the system of functions $\theta^1_i(x) = \phi^1_i(x)/\phi^1(x)$, where

$$\phi^1(x) = \sum_{i=1}^{\infty} \phi^1_i(x).$$

Obviously θ^1_i is a partition of unity. Denote by E^3_∞ the space which is constructed with the functions θ^1_i according to Definition 4.1. We will prove that $E^1_\infty = E^3_\infty$ and that their norms are equivalent. Indeed, let $u \in E^3_\infty$. Then

$$\sup_i \|\theta^1_i u\|_E < \infty, \quad \|u\|_{E^3_\infty} = \sup_i \|\theta^1_i u\|_E.$$

We have

$$\|\phi^1_i u\|_E = \|\phi^1 \theta^1_i u\|_E \le \|\phi^1\|_M \|\theta^1_i u\|_E \le \|\phi^1\|_M \|u\|_{E^3_\infty}.$$

Hence $u \in E^1_\infty$ and

$$\|u\|_{E^1_\infty} \le \|\phi^1\|_M \|u\|_{E^3_\infty}.$$

We have proved that $E^3_\infty \subset E^1_\infty$. Conversely, let $u \in E^1_\infty$. We have

$$\|\theta^1_i u\|_E = \|\frac{\phi^1_i}{\phi^1} u\|_E \le \|\frac{1}{\phi^1}\|_M \|\phi^1_i u\|_E \le \|\frac{1}{\phi^1}\|_M \|u\|_{E^1_\infty}.$$

Hence $u \in E^3_\infty$ and

$$\|u\|_{E^3_\infty} \le \|\frac{1}{\phi^1}\|_M \|u\|_{E^1_\infty}.$$

Therefore $E^1_\infty \subset E^3_\infty$.

We can repeat the same construction for the second system of functions. Let $\theta_i^2(x) = \phi_i^2(x)/\phi^2(x)$, where

$$\phi^2(x) = \sum_{i=1}^{\infty} \phi_i^2(x).$$

Denote by E_∞^4 the space constructed with θ_i^2. Then we obtain $E_\infty^4 = E_\infty^2$ and the corresponding equivalence of the norms. It remains to apply the previous proposition to the spaces E_∞^3 and E_∞^4. The proposition is proved. □

5 Completeness of the space E_∞

Theorem 5.1. *The space E_∞ is complete.*

Proof. Let $u_k \in E_\infty$, $k = 1, 2, \dots$ be a fundamental sequence. This means that for any $\epsilon > 0$ there exists $N = N(\epsilon)$ such that

$$\|u_k - u_l\|_{E_\infty} < \epsilon, \quad k, l > N. \tag{5.1}$$

Let $\{\phi_i\}$, $i = 1, 2, \dots$ be a partition of unity in $\mathbb{R}^n$ such that supports of ϕ_i belong to the cubes of a lattice in R^n and $\sup_i \|\phi_i\|_M < \infty$. By Definition 4.1 and (5.1) we have

$$\sup_i \|\phi_i u_k - \phi_i u_l\|_{E_\infty} < \epsilon, \quad k, l > N. \tag{5.2}$$

It follows that for all i,

$$\|\phi_i u_k - \phi_i u_l\|_{E_\infty} < \epsilon, \quad k, l > N. \tag{5.3}$$

This implies that for any i the sequence $\phi_i u_k$, $k = 1, 2, \dots$ is fundamental in the space E. Since E is a complete space, we conclude that there exists $u^i \in E$ such that

$$\phi_i u_k \to u^i, \quad k \to \infty \tag{5.4}$$

in E. Passing to the limit in (5.3) we get

$$\|u^i - \phi_i u_l\|_{E_\infty} \le \epsilon, \quad \forall i, \ l > N. \tag{5.5}$$

For any i we can construct a function $\psi_i \in D$ such that $\psi_i(x) = 1, \quad x \in \operatorname{supp} \phi_i$. Then $\phi_i(x)\psi_i(x) = \phi_i(x)$ for $x \in \mathbb{R}^n$.

Consider the formal sum $u = \sum_i \psi_i u^i$. We introduce the following functional: for any $\phi \in D$,

$$\langle u, \phi \rangle = \sum_{i'} \langle \psi_{i'} u^{i'}, \phi \rangle.$$

Here i' are those of i for which $\operatorname{supp} \psi_i \cap \operatorname{supp} \phi \neq \oslash$. We note that

$$\langle u, \phi \rangle = \sum_i \langle \psi_i u^i, \phi \rangle$$

for any finite set of i which contains i' since $\langle \psi_i u^i, \phi\rangle = \langle u^i, \psi_i \phi\rangle = 0$ if $\operatorname{supp}\psi_i \cap \operatorname{supp}\phi = \oslash$. Obviously u is a linear functional on D since for any $\phi_1, \phi_2 \in D$ we can take those i which contain i' for ϕ_1, ϕ_2, and $\phi_1 + \phi_2$.

We now prove that $u \in D'$, i.e., that the functional is continuous. Indeed, let $\phi_k \to 0$ in D. This means that $\operatorname{supp}\phi_k \subset B$ for all k and for some ball $B \subset \mathbb{R}^n$, and $\phi_k \to 0$ uniformly with all their derivatives. We take $u = \sum_{i'} \psi_{i'} u^{i'}$, where i' are those of i for which $\operatorname{supp}\psi_i \cap B \neq \oslash$. Since $\psi_i u^i \in D'$, then $u \in D'$.

Moreover, $u \in E_{\text{loc}}$. Indeed, let $f \in D$. We have for any $\phi \in D$:

$$\langle fu, \phi\rangle = \langle u, f\phi\rangle = \sum_{i'} \langle \psi_{i'} u^{i'}, f\phi\rangle = \langle f \sum_{i'} \psi_{i'} u^{i'}, \phi\rangle.$$

Hence $fu = f\sum_{i'} \psi_{i'} u^{i'}$. Here i' are those of i for which $\operatorname{supp}\psi_i \cap \operatorname{supp} f = \oslash$. Since $u_i \in E$ and f and ψ_i are multipliers, we get $fu \in E$. Therefore, $u \in E_{\text{loc}}$.

It remains to prove that $u \in E_\infty$ and $\lim_{k\to\infty} u_k = u$ in E_∞. We have

$$\phi_i u = \phi_i \sum_j \psi_j u^j, \tag{5.6}$$

where j are all of the subscripts for which

$$\operatorname{supp}\psi_j \cap \operatorname{supp}\phi_i \neq \oslash. \tag{5.7}$$

Further, from (5.4)

$$\phi_i \sum_j \psi_j u^j = \phi_i \sum_j \psi_j \lim_{k\to\infty} \phi_j u_k. \tag{5.8}$$

Since ψ_j and ϕ_i are multipliers in E, we obtain

$$\sum_j \phi_i \psi_j \lim_{k\to\infty} \phi_j u_k = \sum_j \lim_{k\to\infty} \phi_i \psi_j \phi_j u_k = \lim_{k\to\infty} \sum_j \phi_i \phi_j u_k. \tag{5.9}$$

The subscripts j are defined by (5.7). For all other j we have $\operatorname{supp}\psi_j \cap \operatorname{supp}\phi_i = \oslash$. Hence $\operatorname{supp}\phi_j \cap \operatorname{supp}\phi_i = \oslash$. Therefore

$$\sum_j \phi_i \phi_j = \phi_i \sum_{j=1}^{\infty} \phi_j = \phi_i.$$

From (5.8), (5.9)

$$\phi_i \sum_j \psi_j u^j = \lim_{k\to\infty} \phi_i u_k = u^i.$$

From (5.6), $\phi_i u = u^i$. We get now from (5.5)

$$\|\phi_i u - \phi_i u_l\|_E = \|u^i - \phi_i u_l\|_E \le \epsilon, \ \ \forall i, \ l > N. \tag{5.10}$$

It follows that

$$\|\phi_i u\|_E \le \epsilon + \|\phi_i u_l\|_E \le \epsilon + \|u_l\|_{E_\infty}.$$

Hence $u \in E_\infty$.

We obtain from (5.10)

$$\sup_i \|\phi_i(u - u_l)\|_E \le \epsilon, \;\; l > N.$$

Therefore

$$\|u - u_l\|_{E_\infty} \le \epsilon, \;\; l > N.$$

Hence $\lim_{l\to\infty} u_l = u$ in E_∞. The theorem is proved. □

6 Other definitions of the space E_∞

Definition 6.1. Let $\eta(x) \in D$ satisfy the following conditions:

1. $0 \le \eta(x) \le 1, \;\; x \in R^n$,
2. $\eta(x) = 1$ in the cube $|x_i| \le a_1$, $i = 1, 2, \dots, n$,
3. $\eta(x) = 0$ outside the cube $|x_i| \le a_2$, $i = 1, 2, \dots, n$, where a_1 and a_2 are given numbers, $a_1 < a_2$.

Denote $\eta_y(x) = \eta(x - y), \;\; y \in R^n$. The space E_∞ is the set of all $u \in E_{\mathrm{loc}}$ such that

$$\sup_{y \in R^n} \|\eta_y u\|_E < \infty.$$

The norm in this space is given by the relation $\|u\|_{E_\infty} = \sup_{y\in R^n} \|\eta_y u\|_E$.

Proposition 6.2. *Let $\{\phi_i\}$ be a partition of unity in $\mathbb{R}^n$ with supports in lattice cubes,* $\sup_i \|\phi_i\|_M < \infty$*. Then the spaces in Definitions* 4.1 *and* 6.1 *coincide.*

Proof. Denote by E^1_∞ and E^2_∞ the spaces in Definitions 6.1 and 4.1, respectively. We will use the function $\eta(x)$ constructed in the following way. Let Q_a be the cube $|x_i| \le a, \; i = 1, \dots, n$, $\chi(x)$ be the characteristic function of the cube Q_{2a}. Set

$$\eta(x) = \int \omega_\epsilon(x - \xi)\chi(\xi)d\xi = \int \omega_\epsilon(\tau)\chi(x + \tau)d\tau,$$

where $\omega_\epsilon(x)$ is a symmetric averaging kernel,

$$\omega_\epsilon(x) = 0 \;\text{ for }\; |x| > \epsilon, \quad \int \omega_\epsilon(x)dx = 1.$$

For $\epsilon > 0$ sufficiently small we obtain

$$\eta(x) = 1, \; x \in Q_a, \;\; \eta(x) = 0, \; x \notin Q_{3a}.$$

Set $\eta_y(x) = \eta(x-y), \;\; \chi_y(x) = \chi(x-y)$. Obviously,

$$\eta_y(x) = \int \omega_\epsilon(x-\xi)\chi_y(\xi)d\xi = \int \omega_\epsilon(\tau)\chi_y(x+\tau)d\tau.$$

We cover R^n with the cubes obtained by translation of the cube Q_{2a} such that they intersect each other only by their sides. Let $\chi_i(x)$ be the characteristic functions of these cubes. Then

$$\sum_i \chi_i(x) = 1 \;\text{ almost everywhere in }\; \mathbb{R}^n. \tag{6.1}$$

We have $\chi_i(x) = \chi(x-h_i)$ for some h_i. Hence

$$\eta_{h_i}(x) = \int \omega_\epsilon(\tau)\chi_i(x+\tau)d\tau.$$

From (6.1) it follows that $\sum_i \eta_{h_i}(x) = 1$. Therefore $\eta_{h_i}(x)$ is a partition of unity, $\operatorname{supp}\eta_{h_i}$ belong to some cubes. Moreover, $\eta_{h_i}(x) = \eta(x-h_i)$, $\sup_i \|\eta_{h_i}\|_M < \infty$.

Let $u \in E^1_\infty$. We have

$$\sup_i \|\eta_{h_i}u\|_M \le \sup_y \|\eta_y u\|_E \le \|u\|_{E^1_\infty}.$$

Hence $u \in E^2_\infty$ and $\|u\|_{E^2_\infty} \le \|u\|_{E^1_\infty}$. We have proved that

$$E^1_\infty \subset E^2_\infty \;\text{ for this choice of }\; \eta(x). \tag{6.2}$$

Now let $u \in E^2_\infty$ and $\{\phi_i\}$ be the partition of unity in the formulation of the proposition. We have

$$\eta_y u = \sum_{i=1}^{\infty} \eta_y\phi_i u = \sum_{i'} \eta_y \phi_{i'} u, \tag{6.3}$$

where i' are all the numbers i for which $\operatorname{supp}\phi_i$ has a nonempty intersection with $\operatorname{supp}\eta_y$. The number of such i' is less than or equal to N, where N does not depend on y. It follows from (6.3) that

$$\|\eta_y u\|_E \le \sum_{i'} \|\eta_y\phi_{i'}u\|_E \le \|\eta_y\|_M \|\phi_{i'}u\|_E \le N\|\eta_y\|_M\|u\|_{E^2_\infty}.$$

Hence

$$\sup_y \|\eta_y u\|_E \le N \sup_y \|\eta_y\|_M \|u\|_{E^2_\infty} \le K\|u\|_{E^2_\infty}.$$

From this estimate it follows that $u \in E^1_\infty$ and

$$\|u\|_{E^1_\infty} \le K\|u\|_{E^2_\infty}, \;\; E^2_\infty \subset E^1_\infty. \tag{6.4}$$

The proposition is proved for the special choice of η_y. Below we will prove that E^1_∞ does not depend on the choice of η_y. $\square$

In what follows we use the space $E(G)$, where G is a domain in $\mathbb{R}^n$. The space $E(G)$ is defined as the set of all generalized functions from D'_G which are restrictions to G of generalized functions from E. The norm in this space is

$$\|u\|_{E(G)} = \inf \|v\|_E,$$

where the infimum is taken over all those generalized functions $v \in E$ whose restriction to G coincides with u.

Definition 6.3. The space E_∞ is the set of all $u \in E_{\mathrm{loc}}$ such that

$$\sup_{y \in R^n} \|u_y\|_{E(G_y)} < \infty, \tag{6.5}$$

where u_y is a restriction of u to G_y, $G \subset R^n$ is a bounded domain containing the origin, G_y is a shifted domain: the characteristic function of G_y is $\chi(x-y)$, where $\chi(x)$ is the characteristic function of G. The norm in E_∞ is given by

$$\|u\|_{E_\infty} = \sup_{y \in R^n} \|u_y\|_{E(G_y)}.$$

Proposition 6.4. *The spaces in Definitions* 6.1 *and* 6.3 *coincide.*

Proof. Denote by E^1_∞ and E^3_∞ the spaces given by Definitions 6.1 and 6.3, respectively. Let $u \in E^1_\infty$. We take a_1 sufficiently large such that $G \subset Q_{a_1}$, where Q_{a_1} is the cube $|x_i| < a_1$, $i = 1, \ldots, n$. Let u_y be the restriction of u to G_y. Then $\eta_y u$ is an extension of u_y to E. Hence

$$\|u_y\|_{E(G_y)} \le \|\eta_y u\|_E \le \|u\|_{E^1_\infty}.$$

Therefore, $u \in E^3_\infty$, and

$$\|u\|_{E^3_\infty} \le \|u\|_{E^1_\infty}, \quad E^1_\infty \subset E^3_\infty. \tag{6.6}$$

This inequality is proved only for such a_1 that $G \subset Q_{a_1}$. From (6.4) we have $E^2_\infty \subset E^1_\infty$ for any a_1. Hence $E^2_\infty \subset E^3_\infty$ for any choice of G.

Now let $u \in E^3_\infty$. By Definition 6.3 this means that $u \in E_{\mathrm{loc}}$ and (6.5) holds. By the definition of the norm $\|u_y\|_{E(G_y)}$, there exists a function $v \in E$ such that v is an extension of u_y and

$$\|v\|_E \le 2\|u_y\|_{E(G_y)} < \infty. \tag{6.7}$$

We take a function $\eta(x)$ in Definition 6.1 such that $Q_{a_2} \subset G$. We have

$$\|\eta_y v\|_E \le \|\eta_y\|_M \|v\|_E \le K\|v\|_E \tag{6.8}$$

since $\sup_y \|\eta_y\|_M < \infty$.

Since $Q_{a_2} \subset G$, then

$$\eta_y v = \eta_y u_y \;\text{ in }\; D'. \tag{6.9}$$

Indeed, for any $\phi \in D$ we have

$$\langle \eta_y v, \phi \rangle = \langle v, \eta_y \phi \rangle, \quad \langle \eta_y u_y, \phi \rangle = \langle u_y, \eta_y \phi \rangle.$$

From the inclusion $\operatorname{supp} \eta_y \phi \subset G_y$ follows the equality $\langle v, \eta_y \phi \rangle = \langle u_y, \eta_y \phi \rangle$ since v is an extension of u_y.

It follows from (6.9) that

$$\|\eta_y u_y\|_E = \|\eta_y v\|_E \le K\|v\|_E \le 2K\|u_y\|_{E(G_y)} \le 2K\|u\|_{E^3_\infty}.$$

Therefore

$$\|\eta_y u\|_E \le 2K\|u\|_{E^3_\infty} \tag{6.10}$$

since $\eta_y u = \eta_y u_y$ in D'. From (6.10) we conclude that $u \in E^1_\infty$ and

$$\|u\|_{E^1_\infty} \le 2K\|u\|_{E^3_\infty}, \quad E^3_\infty \subset E^1_\infty. \tag{6.11}$$

This result is obtained under the assumption that $Q_{a_2} \subset G$. We can take $\eta(x)$ as in the proof of (6.2). Since this result is true for any a, we obtain from (6.11) that $E^3_\infty \subset E^2_\infty$ for any choice of G. Therefore $E^3_\infty = E^2_\infty$ for any choice of G. We conclude that E^3_∞ does not depend on the choice of G.

Let us return to (6.6). We recall that it is obtained under the assumption that $G \subset Q_{a_1}$. But since E^3_∞ does not depend on the choice of G, a_1 can be taken arbitrary. Similarly, a_2 can be taken arbitrary in the assumption $Q_{a_2} \subset G$. Hence (6.11) is true for any a_2. From (6.6) and (6.11) we obtain $E^1_\infty = E^3_\infty$ for any choice of a_1 and a_2. The proposition is proved. □

Remark 6.5. It follows from the proposition that E_∞ in Definition 6.1 does not depend on the choice of $\eta(x)$. The same result is true if instead of cubes in Definition 6.1 we take balls. Indeed, since we have proved that E^3_∞ does not depend on the choice of G, we can repeat the same proof.

7 Bounded sequences in E_∞

Definition 7.1. A sequence $u_k \in E_{\mathrm{loc}}$ is called locally weakly convergent to $u \in E_{\mathrm{loc}}$ if for any $\phi \in D$,

$$\phi u_k \to \phi u \ \text{ weakly in } E.$$

Lemma 7.2. *If a sequence $u_k \in E_\infty$ is bounded in E_∞ and locally weakly convergent to u, then $u \in E_\infty$.*

Proof. We use Definition 4.1 of the space E_∞. Let $\{\phi_i\}$ be a partition of unity. Then $u \in E_\infty$ if $\sup_i \|\phi_i u\|_E < \infty$. Suppose that $u \notin E_\infty$. Then there is a subsequence i_k of i such that

$$\|\phi_i u\|_E \to \infty \ \text{ as } \ i_k \to \infty. \tag{7.1}$$

A set in a Banach space is bounded if and only if any functional from the dual space is bounded on it. Hence there exists a functional $F \in E^*$ such that $F(\phi_{i_k} u) \to \infty$ as $i_k \to \infty$. Since u_l is locally weakly convergent to u, then $F(\phi_{i_k} u_l) \to F(\phi_{i_k} u)$ as $l \to \infty$ for any i_k. Therefore we can choose l_k such that $|F(\phi_{i_k} u_{l_k}) - F(\phi_{i_k} u)| < 1$. It follows from (7.1) that

$$F(\phi_{i_k} u_{l_k}) \to \infty \ \text{ as } \ i_k \to \infty. \tag{7.2}$$

On the other hand, by assumption u_k is bounded in E_∞. Hence $\|u_k\|_{E_\infty} \le M$, $\|\phi_{i_k} u_k\|_E \le M$. This contradicts (7.2). The lemma is proved. □

Theorem 7.3. *Let E be a reflexive Banach space. If $\{u_k\}$, $k = 1, 2, \ldots$ is a bounded sequence in E_∞, then there exists a subsequence u_{k_i} of u_k and $u \in E_\infty$ such that*

$$u_{k_i} \to u \ \textit{ locally weakly and in } D'.$$

Proof. Denote by B_r a ball $|x| < r$ in R^n and consider the sequence B_j, $j = 1, 2, \ldots$. Suppose that f_j, $j = 1, 2, \ldots$ is a sequence of functions such that $f_j \in D$,

$$f_j(x) = 1, \ x \in B_j, \ \ f_j(x) = 0, \ x \notin B_{j+1}, \ \ j = 1, 2, \ldots$$

Let $\{\phi_i\}$, $i = 1, 2, \ldots$, $\phi_i \in D$ be a partition of unity for which E_∞ is defined. We suppose that $\operatorname{supp} \phi_i$ belong to unit cubes and $\sup_i \|\phi_i\|_M = K < \infty$. Since

$$\|\phi_i u_k\|_E \le K \|u_k\|_{E_\infty} \le M,$$

we get

$$\|f_j u_k\|_E \le M_j \tag{7.3}$$

with a constant M_j independent of k. Indeed,

$$\|f_j u_k\|_E = \|\sum_{i'} f_j \phi_{i'} u_k\|_E.$$

Here i' are those of i for which $\operatorname{supp} \phi_i \cap \operatorname{supp} f_j \neq \oslash$. The number of i' is less or equal to N_j, where N_j is a constant. Therefore

$$\|f_j u_k\|_E \le \sum_{i'} \|f_j\|_M \|\phi_{i'} u_k\|_E \le N_j M \|f_j\|_M,$$

and (7.3) is proved. Since E is a reflexive space, we conclude that there exists a subsequence u_i^j, $i = 1, 2, \ldots$ of u_k such that $f_j u_i^j \to v_j$ weakly in E, $v_j \in E$. This means that there exists a sequence $\tilde{u}_i$ such that

$$F(f_j \tilde{u}^i) \to F(v_j) \ \text{ as } \ i \to \infty \tag{7.4}$$

for any $F \in E^*$. Indeed, we choose u_i^j such that u_i^2 is a subsequence of u_i^1, u_i^3 is a subsequence of u_i^2 and so on. Denote by $\tilde{u}_i$ the diagonal subsequence. Then we obtain from (7.4)

$$F(f_j \tilde{u}_i) \to F(v_j) \ \text{ as } \ i \to \infty \tag{7.5}$$

for any j and any $F \in E^*$.

It follows from (7.5) that if $k > j$, then

$$f_j v_k = v_j. \tag{7.6}$$

Indeed, $F(f_j \cdot) \in E^*$. Hence $F(f_j f_k \tilde{u}_i) \to F(f_j v_k)$ as $i \to \infty$. But $f_j(x) f_k(x) = f_j(x)$. Therefore $F(f_j \tilde{u}_i) \to F(f_j v_k)$ as $i \to \infty$. From this and (7.5) we obtain (7.6).

From (7.6) it follows that

$$\langle v_j, \phi \rangle = \langle v_k, \phi \rangle \tag{7.7}$$

if $k > j$ and $\operatorname{supp} \phi \subset B_j$. Indeed,

$$\langle v_j, \phi \rangle = \langle f_j v_k, \phi \rangle = \langle v_k, f_j \phi \rangle = \langle v_k, \phi \rangle$$

since $f_j \phi = \phi$.

We introduce a generalized function $u \in D'$ such that for any $\phi \in D$,

$$\langle u, \phi \rangle = \langle v_j, \phi \rangle \tag{7.8}$$

if $\operatorname{supp} \phi \subset B_j$. The proof that u is a continuous linear functional on D is standard.

Obviously, $u \in E_{\text{loc}}$. Indeed, for any $\phi \in D$ and $f \in D$ we have

$$\langle f u, \phi \rangle = \langle u, f \phi \rangle = \langle v_j, f \phi \rangle = \langle f v_j, \phi \rangle,$$

where j is taken such that $\operatorname{supp} f \subset B_j$. Hence $fu = f v_j$ in D'. Since $v_j \in E$, we get $fu \in E$, and therefore $u \in E_{\text{loc}}$.

We prove now that

$$\tilde{u}_i \to u \text{ locally weakly as } i \to \infty.$$

We have to prove that

$$F(f \tilde{u}_i) \to F(f u) \text{ as } i \to \infty \tag{7.9}$$

for any $f \in D$ and $F \in E^*$. If $F \in E^*$, $f \in D$, then $fF \in E^*$. It follows from (7.5) that

$$f F(f_j \tilde{u}_i) \to f F(v_j) \text{ as } i \to \infty$$

or

$$F(f f_j \tilde{u}_i) \to F(f v_j) \text{ as } i \to \infty.$$

Since $f \in D$, we can take j so large that $\operatorname{supp} f \subset B_j$. Then $f(x) f_j(x) = f(x)$ for all $x \in \mathbb{R}^n$. Therefore $f f_j \tilde{u}_i = f \tilde{u}_i$ and

$$F(f \tilde{u}_i) \to F(f v_j) \text{ as } i \to \infty. \tag{7.10}$$

Further, for any $\phi \in D$ we have $\langle f v_j, \phi\rangle = \langle v_j, f\phi\rangle$. Since $\operatorname{supp} f\phi \subset B_j$, from (7.8) we obtain

$$\langle fu, \phi\rangle = \langle u, f\phi\rangle = \langle v_j, f\phi\rangle.$$

Hence

$$\langle fu, \phi\rangle = \langle f v_j, \phi\rangle, \quad \forall \phi \in D.$$

Therefore $fu = fv_j$ in D'. From (7.10) we obtain (7.9).

Since $\{\tilde{u}_i\}$ is a subsequence of $\{u_k\}$ and this latter is bounded in E_∞, we can conclude that $\{\tilde{u}_i\}$ is bounded in E_∞. From Lemma 7.2 it follows that $u \in E_\infty$.

It remains to prove that $\tilde{u}_i \to u$ in D'. Since $E \subset D'$ (inclusion with the topology), it follows that for any $\phi \in D$, $\phi(u) = \langle u, \phi\rangle \in E^*$. Let $f \in D$, $f(x) = 1$ in $\operatorname{supp}\phi$. Then

$$\langle \tilde{u}_i - u, \phi\rangle = \langle \tilde{u}_i - u, f\phi\rangle = \phi(f(u_i - u)) \to 0$$

as $i \to \infty$ because of the local weak convergence. The theorem is proved. □

8 The space $E_p(\Gamma)$

Let Γ be an m-dimensional manifold, $\Gamma \subset \mathbb{R}^n$, $(m < n)$. We first consider the case where it is C^∞ manifold. We recall the definition of $D'(\Gamma)$ (see [243]). We are given a family J of homeomorphisms ψ, called coordinate systems, of open sets $\Gamma_\psi \subset \Gamma$ on open sets $\tilde{\Gamma}_\psi \subset \mathbb{R}^m$ such that:

(i) If ψ and ψ' belong to J, then the mapping

$$\psi'\psi^{-1} : \psi(\Gamma_\psi \cap \Gamma_{\psi'}) \to \psi'(\Gamma_\psi \cap \Gamma_{\psi'}) \tag{8.1}$$

is infinitely differentiable,

(ii) $\cup_{\psi\in J}\Gamma_\psi = \Gamma$.

We define the space $D(\Gamma)$. If to every coordinate system ψ in Γ we are given a function $\theta_\psi \in D(\tilde{\Gamma}_\psi)$ such that

$$\theta_{\psi'} = \theta_\psi \circ (\psi(\psi')^{-1}) \ \text{ in } \ \psi(\Gamma_\psi \cap \Gamma_{\psi'}),$$

we say that $\theta \in D(\Gamma)$ and set $\theta_\psi = \theta \circ \psi^{-1}$.

If to every coordinate system ψ in Γ corresponds a distribution $u_\psi \in D'(\tilde{\Gamma}_\psi)$ such that

$$u_{\psi'} = u_\psi \circ (\psi(\psi')^{-1}) \ \text{ in } \ \psi'(\Gamma_\psi \cap \Gamma_{\psi'}), \tag{8.2}$$

we call the system u_ψ a distribution u in Γ. The set of all distributions in Γ is denoted by $D'(\Gamma)$. We write also $u_\psi = u \circ \psi^{-1}$, $u = u_\psi \circ \psi$.

As in Section 1, we consider the space $E = E(\mathbb{R}^m)$, $E \subset D'(\mathbb{R}^m)$. We denote by $M(E)$ the space of multipliers of E, and we suppose that $D(\mathbb{R}^m) \subset M(E)$.

Moreover we suppose that there exist numbers $\kappa > 0$ and $\nu > 0$ such that for any $\phi \in D(\mathbb{R}^m)$,

$$\|\phi\|_{M(E)} \leq \kappa \|\phi\|_{C^\nu}. \tag{8.3}$$

Definition 8.1. A function u belongs to the space $E_{\text{loc}}(\Gamma)$ if and only if $u \in D'(\Gamma)$ and for any $\theta \in D(\Gamma)$,

$$(\theta u) \circ \psi^{-1} \in E. \tag{8.4}$$

It is supposed that $(\theta u)_\psi$ is extended by zero from $\tilde{\Gamma}_\psi$ to $\mathbb{R}^m$. By definition $(\theta u)_\psi = \theta_\psi u_\psi$. We give the definition of the space $E_p(\Gamma)$. Let U_i, $i = 1, 2, \dots$ be a covering of Γ for which a coordinate system ψ_i is introduced. We suppose that there exists a number N such that for any i there is no more than N of j such that $U_i \cap U_j \neq \oslash$. Let $\theta_i \in D(\Gamma)$ be a partition of unity, $\operatorname{supp} \theta_i \subset U_i$, $\sum_i \theta_i(x) = 1$ for any $x \in \Gamma$.

Definition 8.2. A function u belongs to the space $E_p(\Gamma)$, $1 \leq p < \infty$ if and only if $u \in E_{\text{loc}}(\Gamma)$ and

$$\|u\|_{E_p(\Gamma)} = \left(\sum_{j=1}^{\infty} \|(\theta_j u) \circ \psi_j^{-1}\|_E^p \right)^{1/p} < \infty.$$

A function u belongs to the space $E_\infty(\Gamma)$ if and only if $u \in E_{\text{loc}}(\Gamma)$ and

$$\|u\|_{E_\infty(\Gamma)} = \sup_i \|(\theta_j u) \circ \psi_j^{-1}\|_E < \infty.$$

Definition 8.3. Two coverings U_i^1 and U_j^2 are called equivalent if there exists a number N such that for any i there is no more than N of j such that $U_i^1 \cap U_j^2 \neq \oslash$; for any j there is no more than N of i such that $U_i^1 \cap U_j^2 \neq \oslash$.

In the following theorem we prove the independence of the space $E^p(\Gamma)$ of the choice of equivalent coverings of Γ and of the choice of partition of unity.

Theorem 8.4. *Let U_i^1 and U_j^2 be two equivalent coverings of Γ, θ_i^1 and θ_j^2 be two corresponding partitions of unity. Suppose that the following conditions are satisfied:*

(α) *For any i, j such that $U_i^1 \cap U_j^2 \neq \oslash$ the norms of the operators of change of variables*

$$\psi_j^2(\psi_i^1)^{-1} : \psi_i^1(U_i^1 \cap U_j^2) \to \psi_j^2(U_i^1 \cap U_j^2),$$
$$\psi_i^1(\psi_j^2)^{-1} : \psi_j^2(U_i^1 \cap U_j^2) \to \psi_i^1(U_i^1 \cap U_j^2)$$

are uniformly bounded in E,

(β) *The estimates*

$$K_1 = \sup_i \|\theta_i^1 \circ (\psi_i^1)^{-1}\|_{C^\nu} < \infty, \quad K_2 = \sup_j \|\theta_j^2 \circ (\psi_j^2)^{-1}\|_{C^\nu} < \infty,$$

hold with the same ν as in (8.3).

Let $E_p^1(\Gamma)$ and $E_p^2(\Gamma)$ $(1 \leq p \leq \infty)$ be the spaces $E_p(\Gamma)$ that correspond to the coverings U_i^1 and U_j^2 and the partitions of unity θ_i^1 and θ_j^2, respectively. Then $E_p^1(\Gamma) = E_p^2(\Gamma)$, and their norms are equivalent.

Proof. Consider first the case $1 \leq p < \infty$. Let $u \in E_{\mathrm{loc}}(\Gamma)$ and $\|u\|_{E_p^2} < \infty$. We have

$$\theta_i^1 u = \theta_i^1 \sum_{j=1}^{\infty} \theta_j^2 u = \sum_{j'} \theta_i^1 \theta_{j'}^2 u, \tag{8.5}$$

where j' are all the numbers j for which $U_i^1 \cap U_j^2 \neq \oslash$. By assumption, the number of j' is less than or equal to N. It follows from (8.5) that

$$\|(\theta_i^1 u) \circ (\psi_i^1)^{-1}\|_E^p \leq \left(\sum_{j'} \|\theta_i^1 \theta_{j'}^2 u) \circ (\psi_i^1)^{-1}\|_E \right)^p .$$

From convexity of the function s^p we obtain

$$\begin{aligned} \|(\theta_i^1 u) \circ (\psi_i^1)^{-1}\|_E^p &\leq N^{p-1} \sum_{j'} \|(\theta_i^1 \theta_{j'}^2 u) \circ (\psi_i^1)^{-1}\|_E^p \\ &= N^{p-1} \sum_{j=1}^{\infty} \|(\theta_i^1 \theta_{j'}^2 u) \circ (\psi_i^1)^{-1}\|_E^p . \end{aligned}$$

Let k be a positive integer. We have

$$\sum_{i=1}^{k} \|(\theta_i^1 u) \circ (\psi_i^1)^{-1}\|_E^p \leq N^{p-1} \sum_{j=1}^{\infty} \sum_{i=1}^{k} \|(\theta_i^1 \theta_{j'}^2 u) \circ (\psi_i^1)^{-1}\|_E^p . \tag{8.6}$$

Further

$$\sum_{i=1}^{k} \|(\theta_i^1 \theta_{j'}^2 u) \circ (\psi_i^1)^{-1}\|_E^p \leq \sum_{i'} \|(\theta_{i'}^1 \theta_{j'}^2 u) \circ (\psi_{i'}^1)^{-1}\|_E^p , \tag{8.7}$$

where i' are those i for which $U_i^1 \cap U_j^2 \neq \oslash$. The number of such i' is less than or equal to N.

For any i and j such that $U_i^1 \cap U_j^2 \neq \oslash$ we have, according to (8.2),

$$(\theta_i^1 \theta_j^2 u) \circ (\psi_i^1)^{-1} = \left((\theta_i^1 \theta_j^2 u) \circ (\psi_j^2)^{-1}\right) \circ (\psi_j^2 (\psi_i^1)^{-1}). \tag{8.8}$$

By the condition of the theorem, the norms of the operators $\psi_j^2(\psi_i^1)^{-1}$ are uniformly bounded. Therefore we get from (8.8)

$$\|(\theta_i^1 \theta_j^2 u) \circ (\psi_i^1)^{-1}\|_E \leq M_1 \|(\theta_i^1 \theta_j^2 u) \circ (\psi_j^2)^{-1}\|_E .$$

Denote by $\tilde{\theta}_i^1(x)$ the restriction of $\theta_i^1(x)$ to $U_i^1 \cap U_j^2$. We have obviously $\theta_i^1\theta_j^2 = \tilde{\theta}_i^1\theta_j^2$. Hence

$$\begin{aligned}(\theta_i^1\theta_j^2 u)\circ(\psi_j^2)^{-1} = (\tilde{\theta}_i^1\theta_j^2 u)\circ(\psi_j^2)^{-1} &= \tilde{\theta}_i^1 \circ (\psi_j^2)^{-1} \cdot \theta_j^2 u \circ (\psi_j^2)^{-1} \\ &= \tilde{\theta}_i^2 \circ (\psi_i^1)^{-1} \cdot \theta_j^2 u \circ (\psi_j^2)^{-1}.\end{aligned}$$

It follows that

$$\begin{aligned}\|(\theta_i^1\theta_j^2 u)\circ(\psi_j^2)^{-1}\|_E &\le \|\tilde{\theta}_i^2 \circ (\psi_i^1)^{-1}\|_{M(E)} \|\theta_j^2 u \circ (\psi_j^2)^{-1}\|_E \\ &\le \kappa \|\tilde{\theta}_i^2 \circ (\psi_i^1)^{-1}\|_{C^\nu} \|\theta_j^2 u \circ (\psi_j^2)^{-1}\|_E\end{aligned}$$

by virtue of (8.3). Obviously

$$\|\tilde{\theta}_i^2 \circ (\psi_i^1)^{-1}\|_{C^\nu} \le \|\theta_i^2 \circ (\psi_i^1)^{-1}\|_{C^\nu} \le K_1$$

according to condition (β). Thus we have obtained

$$\|(\theta_i^1\theta_j^2 u)\circ(\psi_i^1)^{-1}\|_E \le \kappa K_1 M_1 \|(\theta_j^2 u)\circ(\psi_j^2)^{-1}\|_E. \tag{8.9}$$

Let us return to (8.7). From (8.9) we get

$$\sum_{i'} \|(\theta_{i'}^1\theta_j^2 u)\circ(\psi_{i'}^1)^{-1}\|_E^p \le N(\kappa K_1 M_1\|)^p \|(\theta_j^2 u)\circ(\psi_j^2)^{-1}\|_E^p. \tag{8.10}$$

Therefore (8.6), (8.7), and (8.10) imply

$$\sum_{i=1}^k \|(\theta_i^1 u)\circ(\psi_i^1)^{-1}\|_E^p \le c_1^p \sum_{j=1}^\infty \|(\theta_j^2 u)\circ(\psi_j^2)^{-1}\|_E^p,$$

where

$$c_1 = \kappa N K_1 M_1. \tag{8.11}$$

Passing to the limit as $k \to \infty$, we obtain

$$\|u\|_{E_p^1(\Gamma)} \le c_1 \|u\|_{E_p^2(\Gamma)}.$$

Hence $u \in E_p^1(\Gamma)$.

Similarly we get for $u \in E_p^1(\Gamma)$ that

$$\|u\|_{E_p^2(\Gamma)} \le c_2 \|u\|_{E_p^1(\Gamma)},$$

and therefore $u \in E_p^2(\Gamma)$.

We have proved that $E_p^1(\Gamma) = E_p^2(\Gamma)$ and that the norms in these spaces are equivalent. Thus the theorem is proved for $1 \le p < \infty$.

Consider the case $p = \infty$. Let $u \in E_{\mathrm{loc}}(\Gamma)$ and $\|u\|_{E_\infty^2(\Gamma)} < \infty$. From (8.5) we obtain

$$\|(\theta_i^1 u)\circ(\psi_i^1)^{-1}\|_E \le \sum_{j'} \|(\theta_i^1\theta_{j'}^2 u)\circ(\psi_i^1)^{-1}\|_E.$$

From (8.9) it follows that

$$\|(\theta_i^1\theta_{j'}^2 u)\circ(\psi_i^1)^{-1}\|_E \le \kappa K_1 M_1 \|(\theta_j^2 u)\circ(\psi_j^2)^{-1}\|_E \le \kappa K_1 M_1 \|u\|_{E^2_\infty(\Gamma)}.$$

Hence

$$\|(\theta_i^1 u)\circ(\psi_i^1)^{-1}\|_E \le c_1 \|u\|_{E^2_\infty(\Gamma)},$$

where c_1 is given by (8.11). Therefore

$$\|u\|_{E^1_\infty(\Gamma)} \le c_1 \|u\|_{E^2_\infty(\Gamma)}.$$

Similarly

$$\|u\|_{E^2_\infty(\Gamma)} \le c_2 \|u\|_{E^1_\infty(\Gamma)}.$$

The theorem is proved. □

Remark 8.5. In what follows $\Gamma = \partial\Omega$, where Ω is a domain satisfying Condition D (Section 2, Chapter 3), $E = W^{s,q}$, $(-\infty < s < \infty, 1 < q < \infty)$. In this case condition (α) of the theorem is satisfied if $|s| + 1 \le l$, where l is the number in Condition D. Indeed, from Condition D it follows that the functions $\psi_j^2(\psi_i^1)^{-1}$ and $\psi_i^1(\psi_j^2)^{-1}$ belong to C^l. It can be verified that the following proposition is true for $E = W^{s,q}$ and $|s| + 1 \le l$.

Proposition 8.6. *Let $g : \mathbb{R}^m \to \mathbb{R}^m$ be a homeomorphism, $g \in C^l(\mathbb{R}^m)$, $g^{-1} \in C^l(\mathbb{R}^m)$. If $u \in E$, then $u \circ g \in E$, and $\|u \circ g\|_E \le M\|u\|_E$, where M does not depend on u, and it depends continuously on $\|g\|_{C^l(\mathbb{R}^m)}$ and $\|g^{-1}\|_{C^l(\mathbb{R}^m)}$.*

Hence condition (α) of the theorem is satisfied. Moreover under the same assumptions on E and l, for any $f \in C^l(\mathbb{R}^m)$ and $u \in E$ we have $fu \in E$ and

$$\|fu\|_E \le M_1 \|f\|_{C^l(\mathbb{R}^m)} \|u\|_E,$$

where M_1 does not depend on f and u. Therefore (8.3) is satisfied with $\nu = l$.

Consider now the case where Γ is a C^l manifold, where $l \ge 1$ is an integer. In this case the space D' cannot be used since the multiplication of elements from D' by functions from C^l is not defined. We can consider instead the spaces D_l and D_l'. Here D_l is the space of all functions $\phi \in C^l$ with compact supports. The convergence in D_l is defined as follows: $\phi_i \to 0$ in D_l if $D^\alpha\phi_i \to 0$ uniformly, $|\alpha| \le l$, and there is a fixed compact set containing the supports of all ϕ_j. The space D_l' is defined as the space of all continuous functionals on D_l.

It is clear that the multiplication of elements of D_l' by functions $\phi \in C^l$ is defined. We can define the space $D_l'(\Gamma)$ similar to the definition of $D'(\Gamma)$ above. Using the space $D_l'(\Gamma)$ instead of $D'(\Gamma)$ we can give Definitions 8.1 and 8.2 and prove Theorem 8.3 for C^l manifolds exactly as it was done above. We consider the space $E = E(\mathbb{R}^m)$ such that $E \subset D_l'(\mathbb{R}^m)$, and we suppose that $D_l(\mathbb{R}^m) \subset M(E)$. In (8.3) we put $\nu = l$. In the definition of the space $E_p(\Gamma)$ we suppose that the partition of unity $\theta_i \in D_l(\Gamma)$.

The space $E_\infty(\Gamma)$ was defined in Definition 8.2. In what follows we need also other equivalent definitions. Suppose that $\Gamma = \partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is a domain satisfying Condition D (Section 2, Chapter 3). We assume that the space E satisfies the condition of the Proposition 8.6 in the Remark 8.5, and the number ν in (8.3) is equal to l.

Let δ be the number in Condition D and $0 < \rho \le \delta$. Consider a function $\eta(x) \ge 0$ such that $\eta \in C^\infty(\mathbb{R}^n)$, $\eta(x) = 1$ for $|x| < \rho/2$, $\eta(x) = 0$ for $|x| > \rho$. Let $\eta_z(x) = \eta(x - z)$, $z \in R^n$.

Definition 8.7. $E_\infty(\Gamma)$ is the space of functions $u \in E_{\mathrm{loc}}(\Gamma)$ such that

$$\|u\|_{E_\infty(\Gamma)} = \sup_{z\in\Gamma} \|(\eta_z u) \circ \psi_z^{-1}\|_E < \infty,$$

where ψ_z is the function in Condition D which corresponds to the point $z \in \Gamma$.

In Theorem 8.4 we have proved that the space $E_p(\Gamma)$ $(1 \le p \le \infty)$ does not depend on the choice of equivalent coverings of Γ. We now specify the equivalence class of coverings, which we shall use, by pointing out its representative: covering of Γ by cubes with the sides equal 2 and the centers at lattice points with the increment equal to 1.

Proposition 8.8. *Suppose that in Definition* 8.2*,*

$$\sup_i \|\theta_i \circ \psi_i^{-1}\|_{C^l(\mathbb{R}^{n-1})} < \infty. \tag{8.12}$$

Then the spaces $E_\infty(\Gamma)$ *in Definitions* 8.2 *and* 8.7 *coincide.*

Proof. Denote by $E^2_\infty(\Gamma)$ and $E^4_\infty(\Gamma)$ the spaces $E_\infty(\Gamma)$ in Definitions 8.2 and 8.7, respectively. Let $u \in E^4_\infty(\Gamma)$. Let us construct a covering of Γ and a partition of unity as follows. We cover Γ by balls B_j of the radii $\rho/2$ and the centers at the points $x_j \in \Gamma$ $(i = 1, 2, \dots)$. Denote by $\tilde{B}_j$ the balls of the radii ρ and the centers at x_j. We suppose that the points x_j are chosen such that the covering by the balls $\tilde{B}_j$ belongs to the equivalence class under consideration. Denote by N such number that for any i there is no more than N of j such that the intersection $\tilde{B}_i \bigcap \tilde{B}_j$ is not empty. Let

$$\omega(x) = \sum_j \eta_{x_j}(x). \tag{8.13}$$

Obviously, $\operatorname{supp} \eta_{x_j} \subset \tilde{B}_j$. Hence for any $x \in \Gamma$ we have no more than N nonzero terms in sum (8.13). Moreover, $\omega(x) \ge 1$, $x \in \Gamma$.

Set

$$\theta_j(x) = \frac{\eta_{x_j}(x)}{\omega(x)}. \tag{8.14}$$

Clearly, this is a partition of unity which corresponds to the covering by $\tilde{B}_j$. We have

$$\|(\theta_j u)\circ\psi_j^{-1}\|_E = \|((1/\omega)\circ\psi_j^{-1})((\eta_{x_j}u)\circ\psi_j^{-1})\|_E \\ \le \|(1/\omega)\circ\psi_j^{-1}\|_{M(E)}\|(\eta_{x_j}u)\circ\psi_j^{-1}\|_E,$$

where $M(E)$ is the space of multipliers. By Condition D and (8.3) we have $\|(1/\omega)\circ\psi_j^{-1}\|_{M(E)} \le K$, where the constant K does not depend on j. Hence

$$\|(\theta_i u)\circ\psi_j^{-1}\|_E \le K\|(\eta_{x_j}u)\circ\psi_j^{-1}\|_E \le K\|u\|_{E^4_\infty(\Gamma)}.$$

It follows that $\|u\|_{E^2_\infty(\Gamma)} \le K\|u\|_{E^4_\infty(\Gamma)}$ and $u \in E^2_\infty(\Gamma)$. We have proved that $E^4_\infty(\Gamma) \subset E^2_\infty(\Gamma)$.

Now, let $u \in E^2_\infty(\Gamma)$ and θ_i be the partition of unity in Definition 8.2. For any $z \in \Gamma$ we have

$$(\eta_z u)\circ\psi_z^{-1} = \sum_{i'}(\eta_z\theta_{i'}u)\circ\psi_z^{-1}, \tag{8.15}$$

where i' are all of i for which $\operatorname{supp}\theta_i$ has nonempty intersection with $\operatorname{supp}\eta_z$. By the assumption above on the choice of the equivalent class of coverings, it is obvious that there exists a number N independent of z such that the number of i' is less than N. It follows from (8.15) that

$$\|(\eta_z u)\circ\psi_z^{-1}\|_E \le \sum_{i'}\|(\eta_z\theta_{i'}u)\circ\psi_z^{-1}\|_E. \tag{8.16}$$

As in the proof of Theorem 8.4, we get

$$\|(\eta_z\theta_{i'}u)\circ\psi_z^{-1}\|_E \le M\|(\theta_{i'}u)\circ\psi_{i'}^{-1}\|_E,$$

where the constant M does not depend on z and i'. Since $\|(\theta_{i'}u)\circ\psi_{i'}^{-1}\|_E \le \|u\|_{E^2_\infty(\Gamma)}$, we obtain from (8.16) that

$$\|(\eta_z u)\circ\psi_z^{-1}\|_E \le NM\|u\|_{E^2_\infty(\Gamma)}.$$

Hence $\|u\|_{E^4_\infty(\Gamma)} \le NM\|u\|_{E^2_\infty(\Gamma)}$. Therefore $u \in E^4_\infty(\Gamma)$. It follows that $E^2_\infty(\Gamma) \subset E^4_\infty(\Gamma)$. The proposition is proved. □

It follows from the proof that this result remains true if instead of the partition of unity in Definition 8.2 we take an arbitrary system of functions $\theta_i \in D_l(\Gamma)$, $\operatorname{supp}\theta_i \subset U_i$ and such that (8.12) is satisfied. We now give another definition of the space $E_\infty(\Gamma)$ (Definition 8.9) and prove its equivalence to the previous ones. Let $G \subset R^n$ be a bounded domain, and the intersection $\Gamma\bigcap G$ be not empty. Denote by $D_l'(\Gamma\bigcap G)$ the restriction of $D_l'(\Gamma)$ to $\Gamma\bigcap G$.

Definition 8.9. The function $u \in E_\infty(\Gamma\bigcap G)$ if and only if $u \in D_l'(\Gamma\bigcap G)$ and there exists $v \in E_\infty(\Gamma)$ such that the restriction of v to $D_l'(\Gamma\bigcap G)$ coincides with u. The norm in $E_\infty(\Gamma\bigcap G)$ is given by

$$\|u\|_{E_\infty(\Gamma\bigcap G)} = \inf\|v\|_{E_\infty(\Gamma)},$$

where the infimum is taken over all v, for which u is the restriction of v to $D_l'(\Gamma \bigcap G)$.

Definition 8.10. The space $\tilde{E}_\infty(\Gamma)$ is the set of all $u \in E_{\mathrm{loc}}(\Gamma)$ such that

$$\|u\|_{\tilde{E}_\infty(\Gamma)} = \sup_{y\in\Gamma} \|u_y\|_{E_\infty(\Gamma \bigcap G_y)} < \infty, \tag{8.17}$$

where u_y is the restriction of u to $\Gamma \bigcap G_y$, G_y is the shifted domain: the characteristic function of G_y is $\chi(x-y)$, where $\chi(x)$ is the characteristic function of G.

In what follows it is supposed that G contains the origin.

Proposition 8.11.

$$\tilde{E}_\infty(\Gamma) = E_\infty(\Gamma).$$

Proof. Let $u \in E_\infty(\Gamma)$, u_y be the restriction of u to $\Gamma \bigcap G_y$ and θ_j be the partition of unity (8.14). Further, let j' be all of j for which $\operatorname{supp}\theta_j$ has a nonempty intersection with $\Gamma \bigcap G_y$. According to the choice of x_j in (8.14) it is clear that the number of j' is less than a number N which does not depend on y. The function $\sum_{j'} \theta_{j'} u$ is an extension of u_y. Hence

$$\|u_y\|_{E_\infty(\Gamma\cap G_y)} \le \|\sum_{j'} \theta_{j'} u\|_{E_\infty(\Gamma)} \le K\|u\|_{E_\infty(\Gamma)},$$

where K does not depend on y. Therefore $\|u\|_{\tilde{E}_\infty(\Gamma)} \le K\|u\|_{E_\infty(\Gamma)}$. We have proved that $E_\infty(\Gamma) \subset \tilde{E}_\infty(\Gamma)$.

Now let $u \in \tilde{E}_\infty(\Gamma)$. This means that

$$u \in E_{\mathrm{loc}}(\Gamma) \quad \text{and} \quad \sup_{y\in\Gamma} \|u_y\|_{E_\infty(\Gamma \bigcap G_y)} < \infty.$$

By Definition 8.9, there exists an extension $v \in E_\infty(\Gamma)$ of u_y such that

$$\|v\|_{E_\infty(\Gamma)} \le 2\|u_y\|_{E_\infty(\Gamma \bigcap G_y)}. \tag{8.18}$$

Since the space $E_\infty(\Gamma)$ does not depend on the choice of ρ, we can suppose that ρ is taken so small that $\operatorname{supp}\eta_y \in G_y$. We have

$$\|\eta_y v\|_{E_\infty(\Gamma)} \le K_1\|v\|_{E_\infty(\Gamma)}, \tag{8.19}$$

where K_1 does not depend on y.

Since $\operatorname{supp}\eta_y \in G_y$, we have

$$\eta_y v = \eta_y u_y, \tag{8.20}$$

where $\eta_y u_y$ is extended by zero outside $\Gamma \bigcap G_y$. From (8.18), (8.19) and (8.20),

$$\begin{aligned}\|\eta_y u_y\|_{E_\infty(\Gamma)} &= \|\eta_y v\|_{E_\infty(\Gamma)} \le K_1 \|v\|_{E_\infty(\Gamma)} \\ &\le 2K_1 \|u_y\|_{E_\infty(\Gamma \bigcap G_y)} \le 2K_1 \|u\|_{\tilde{E}_\infty(\Gamma)}.\end{aligned}$$

But $\eta_y u_y = \eta_y u$. Therefore

$$\|\eta_y u\|_{E_\infty(\Gamma)} \le 2K_1 \|u\|_{\tilde{E}_\infty(\Gamma)}. \tag{8.21}$$

By Definition 8.7, we have

$$\sup_{y\in\Gamma} \|\eta_y u\|_{E_\infty(\Gamma)} = \sup_{y\in\Gamma} \sup_{z\in\Gamma} \|(\eta_z \eta_y u) \circ \psi_z^{-1}\|_E. \tag{8.22}$$

On the other hand, since the space $E_\infty(\Gamma)$ in Definition 8.7 does not depend on the choice of η, we can take $\eta^2(x)$ instead of $\eta(x)$. Hence for some constant $\kappa > 0$ we have

$$\kappa \|u\|_{E_\infty(\Gamma)} \le \sup_{z\in\Gamma} \|\eta_z^2 u \circ \psi_z^{-1}\|_E \le \sup_{y\in\Gamma} \sup_{z\in\Gamma} \|(\eta_y \eta_z u) \circ \psi_z^{-1}\|_E = \sup_{y\in\Gamma} \|\eta_y u\|_{E_\infty(\Gamma)}$$

by (8.22). It follows from (8.21) that

$$\kappa \|u\|_{E_\infty(\Gamma)} \le 2K_1 \|u\|_{\tilde{E}_\infty(\Gamma)}.$$

Therefore $u \in E_\infty(\Gamma)$. We have proved that $\tilde{E}_\infty(\Gamma) \subset E_\infty(\Gamma)$. The proposition is proved. □

Remark 8.12. The formulation of Proposition 8.11 and its proof remain the same if instead of (8.17) we suppose that

$$\|u\|_{\tilde{E}_\infty(\Gamma)} = \sup \|u_y\|_{E_\infty(\Gamma \bigcap G_y)} < \infty,$$

where the supremum is taken over all $y \in \Omega$ such that the $\Gamma \bigcap G_y$ is not empty.

9 Spaces in unbounded domains

Let Ω be a domain in R^n, E be a space in Definition 1.1. Instead of the estimate

$$\|fu\|_E \le \|f\|_M \|u\|_E$$

for $f \in D$, $u \in E$, we have a similar estimate for the space $E(\Omega)$:

$$\|fu\|_{E(\Omega)} \le \|f\|_M \|u\|_{E(\Omega)}, \tag{9.1}$$

where $f \in D$, $u \in E(\Omega)$. Indeed, let u^c be an extension of u to E. Then fu^c is an extension of fu to E. Hence

$$\|fu\|_{E(\Omega)} \le \|fu^c\|_E \le \|f\|_M \|u\|_E.$$

If we take the infimum over all extensions u^c, we obtain (9.1).

Definition 9.1. The space $E(\Omega)$ is defined as the space of the generalized functions from D'_Ω that are restrictions to Ω of generalized functions from E. The norm in $E(\Omega)$ is defined as

$$\|u\|_{E(\Omega)} = \inf \|u^c\|_E,$$

where the infimum is taken over all $u^c \in E$ whose restriction to Ω coincide with u.

Definition 9.2. The space $E^1_\infty(\Omega)$ is defined as the set of the generalized functions from D'_Ω that are restrictions to Ω of generalized functions from E_∞. The norm in $E^1_\infty(\Omega)$ is defined as

$$\|u\|_{E^1_\infty(\Omega)} = \inf \|u^c\|_{E_\infty},$$

where the infimum is taken over all $u^c \in E_\infty$ whose restriction to Ω coincide with u.

We will also give another definition of the space $E_\infty(\Omega)$. Let $\eta_y(x)$ be the same as in Definition 6.1.

Definition 9.3. The space $E^2_\infty(\Omega)$ is defined as the set of such generalized functions from D'_Ω that $\eta_y u \in E(\Omega)$ for all $y \in \mathbb{R}^n$ and

$$\sup_{y \in \mathbb{R}^n} \|\eta_y u\|_{E(\Omega)} < \infty.$$

If $\operatorname{supp} \eta_y \cap \Omega = \oslash$, then $\eta_y u = 0$. The norm in $E^2_\infty(\Omega)$ is given by the equality

$$\|u\|_{E^2_\infty(\Omega)} = \sup_{y \in \mathbb{R}^n} \|\eta_y u\|_{E(\Omega)}.$$

Definition 9.4. Let $\{\phi_i\}$ be a system of functions satisfying Condition 2.2. The space $E^3_\infty(\Omega)$ is defined as the set of generalized functions $u \in D'_\Omega$ such that $\phi_i u \in E(\Omega)$ for all i and

$$\sup_i \|\phi_i u\|_{E(\Omega)} < \infty.$$

The norm in $E^3_\infty(\Omega)$ is given by the equality

$$\|u\|_{E^3_\infty(\Omega)} = \sup_i \|\phi_i u\|_{E(\Omega)}.$$

Definition 9.5. Denote by $B_{y,r}$ the ball $\{x : |x-y| < r\}$. The space $E^4_\infty(\Omega)$ is defined as the set of generalized functions $u \in D'_\Omega$ such that for any $y \in \Omega, \Omega \cap B_{y,r} \neq \oslash$ the restriction of u to $\Omega \cap B_{y,r}$ satisfies the condition

$$\|u\|_{E(\Omega \cap B_{y,r})} \leq M$$

with M independent of u. The norm in $E^4_\infty(\Omega)$ is defined by the equality

$$\|u\|_{E^3_\infty(\Omega)} = \sup \|u\|_{E(\Omega \cap B_{y,r})},$$

where the supremum is taken over all $y \in \Omega$ such that $\Omega \cap B_{y,r} \neq \oslash$.

Proposition 9.6. *Let $E^1_\infty(\Omega)$ and $E^3_\infty(\Omega)$ be the space defined above. Then $E^3_\infty(\Omega) \subset E^1_\infty(\Omega)$ and for any $u \in E^3_\infty(\Omega)$ we have*

$$\|u\|_{E^1_\infty(\Omega)} \le M\|u\|_{E^3_\infty(\Omega)},$$

where M is a constant independent of u.

Proof. Let $u \in E^3_\infty(\Omega)$. Then $\phi_i u \in E(\Omega)$ for any i. By the definition of $E(\Omega)$, there exists $u_i \in E$ such that

$$\langle u_i, \phi\rangle = \langle \phi_i u, \phi\rangle \ \text{ for any} \phi : \operatorname{supp}\phi \in \Omega \tag{9.2}$$

and

$$\|u_i\|_E \le 2\|\phi_i u\|_{E(\Omega)}. \tag{9.3}$$

We can suppose that $\{\phi_i\}$ is a partition of unity. Let $\{\psi_i\}$ be another system of functions satisfying Condition 2.2 such that $\psi_i(x) = 1$ for $x \in \operatorname{supp}\phi_i$. Then $\phi_i(x)\psi_i(x) = \phi_i(x)$ for any i and x. Set

$$u^c = \sum_{i=1}^{\infty} \psi_i u_i. \tag{9.4}$$

Obviously, for $\phi \in D$ the functional u^c is defined. Indeed, by definition

$$\langle u^c, \phi\rangle = \sum_{i=1}^{\infty}\langle \psi_i u_i, \phi\rangle = \sum_{i=1}^{\infty}\langle u_i, \psi_i\phi\rangle.$$

But this sum has only a finite number of nonzero terms. For definiteness we suppose that the supports of ϕ_i and ψ_i are cubes of a lattice.

We now prove that u^c is an extension of u. Indeed, if $\operatorname{supp}\phi \subset \Omega$, we have $\langle u^c, \phi\rangle = \sum_{i=1}^{\infty}\langle u_i, \psi_i\phi\rangle$. Since $\operatorname{supp}\psi_i\phi \subset \Omega$, we have, from (9.2),

$$\langle u_i, \psi_i\phi\rangle = \langle \phi_i u, \psi_i\phi\rangle = \langle u, \phi_i\psi_i\phi\rangle = \langle u, \phi_i\phi\rangle.$$

Hence

$$\langle u^c, \phi\rangle = \sum_{i=1}^{\infty}\langle u, \phi_i\phi\rangle = \langle u, \phi\rangle$$

since $\sum_{i=1}^{\infty}\phi_i(x) = 1$. It follows that u^c is an extension of u.

We will prove that $u^c \in E_\infty$. Indeed, we have

$$\phi_k u^c = \sum_i \phi_k\psi_i u_i = \sum_{i'} \phi_k\psi_{i'} u_{i'}, \tag{9.5}$$

where i' denotes all the subscripts for which $\operatorname{supp}\phi_k$ and $\operatorname{supp}\psi_i$ have a nonempty intersection. Let the number of such i' be no more than N, which does not depend on i and k. From (9.5) follows the estimate

$$\|\phi_k u^c\|_E \le \sum_{i'} \|\phi_k\psi_{i'} u_{i'}\|_E. \tag{9.6}$$

By Definition 1.1 of the space E we have $\|\phi_k\psi_i u_i\|_E \le M\|\phi_k\psi_i\|_M\|u_i\|_E$. By Condition 2.5 $\|\phi_k\psi_i\|_M \le K$, where K is independent of k and i. Hence $\|\phi_k\psi_i u_i\|_E \le MK\|u_i\|_E$. From (9.6) we obtain $\|\phi_k u^c\|_E \le MK\sum_{i'}\|u_{i'}\|_E$. The inequality (9.3) implies $\|\phi_k u^c\|_E \le 2MK\sum_{i'}\|\phi_{i'}u\|_{E(\Omega)}$. From Definition 9.4,

$$\|\phi_k u^c\|_E \le 2MKN\|u\|_{E^3_\infty(\Omega)}.$$

Since k is arbitrary, by the definition of E_∞ we get $u^c \in E_\infty$ and

$$\|u^c\|_{E_\infty} = \sup_k \|\phi_k u^c\|_E \le 2MKN\|u\|_{E^3_\infty(\Omega)}.$$

Since u is a restriction of u^c to Ω, we obtain that $u \in E^1_\infty(\Omega)$. Moreover,

$$\|u\|_{E^1_\infty(\Omega)} \le \|u^c\|_{E_\infty} \le 2MKN\|u\|_{E^3_\infty(\Omega)} = M_0\|u\|_{E^3_\infty(\Omega)}. \tag{9.7}$$

Thus the proposition is proved for the special choice of ϕ_i. We can write (9.7) in the form

$$\|u\|_{E^1_\infty(\Omega)} \le M_0 \sup_k \|\phi_k u\|_{E(\Omega)}. \tag{9.8}$$

We now prove the proposition for any system of functions $\{\omega_i\}$ satisfying Definition 2.3 and equivalent to the system of functions $\{\phi_i\}$, which is considered above. We have

$$\phi_k u = \phi_k \sum_i \frac{\omega_i}{\omega} u = \sum_{i'} \phi_k \frac{\omega_{i'}}{\omega} u,$$

where i' are those subscripts i for which $\operatorname{supp}\omega_i$ has a nonempty intersection with $\operatorname{supp}\phi_k$. The number of such i' is no more than N by the definition of equivalence of systems of functions.

We now have

$$\|\phi_k u\|_{E(\Omega)} = \sum_i \|\frac{\phi_k}{\omega}\omega_{i'}u\|_{E(\Omega)} \le K\sum_{i'}\|\omega_{i'}u\|_{E(\Omega)}.$$

Hence

$$\|\phi_k u\|_{E(\Omega)} \le KN\|u\|_{E^3_\infty(\Omega)}, \quad \sup_k \|\phi_k u\|_{E(\Omega)} \le KN\|u\|_{E^3_\infty(\Omega)}.$$

Therefore by the results of the first part of the proof we conclude that $u \in E^1_\infty(\Omega)$, and from (9.8) we obtain

$$\|u\|_{E^1_\infty(\Omega)} \le M_0 KN\|u\|_{E^3_\infty(\Omega)}.$$

The proposition is proved. □

Proposition 9.7. *Let* $E^1_\infty(\Omega)$ *and* $E^2_\infty(\Omega)$ *be the space in Definitions* 9.2 *and* 9.3*, respectively. Then* $E^1_\infty(\Omega) \subset E^2_\infty(\Omega)$ *and for any* $u \in E^1_\infty(\Omega)$ *we have*

$$\|u\|_{E^2_\infty(\Omega)} \le M\|u\|_{E^1_\infty(\Omega)},$$

where M *is a constant independent of* u.

Proof. Let $u \in E^1_\infty(\Omega)$. Then there exists $u^c \in E_\infty$ such that

$$\langle u, \phi \rangle = \langle u^c, \phi \rangle \text{ for any } \phi \in D, \ \operatorname{supp} \phi \subset \Omega. \tag{9.9}$$

By the definition of E_∞ we have $\eta_y u^c \in E$ for any $y \in \mathbb{R}^n$, and

$$\|u^c\|_{E_\infty} = \sup_{y \in \mathbb{R}^n} \|\eta_y u^c\|_E < \infty. \tag{9.10}$$

From (9.9) it follows that $\langle \eta_y u, \phi \rangle = \langle \eta_y u^c, \phi \rangle$ for any $\phi \in D$, $\operatorname{supp} \phi \subset \Omega$. Hence $\eta_y u^c$ is an extension of $\eta_y u$ to E_∞. By the definition of $E(\Omega)$ we have $\eta_y u \in E(\Omega)$ and $\|\eta_y u\|_{E(\Omega)} \le \|\eta_y u^c\|_E \le \|u^c\|_{E_\infty}$. Therefore $u \in E^2_\infty(\Omega)$ and

$$\|u\|_{E^2_\infty(\Omega)} = \sup_{y \in \Omega} \|\eta_y u\|_{E(\Omega)} \le \|u^c\|_{E_\infty}.$$

The left-hand side here does not depend on the extension u^c. Hence

$$\|u\|_{E^2_\infty(\Omega)} \le \inf_{u^c} \|u^c\|_{E_\infty} = \|u\|_{E^1_\infty(\Omega)}.$$

The proposition is proved. □

Proposition 9.8. *Let $E^2_\infty(\Omega)$ and $E^4_\infty(\Omega)$ be the space in Definitions* 9.3 *and* 9.5, *respectively. Then $E^2_\infty(\Omega) \subset E^4_\infty(\Omega)$ and for any $u \in E^2_\infty(\Omega)$ we have*

$$\|u\|_{E^4_\infty(\Omega)} \le M \|u\|_{E^2_\infty(\Omega)},$$

where M is a constant independent of u.

Proposition 9.9. *If $\{\phi^1_i\}$ and $\{\phi^2_i\}$ are two systems of functions equivalent in the sense of Definition* 2.3, *then the spaces $E^3_\infty(\Omega)$ corresponding to them coincide, and their norms are equivalent.*

Proposition 9.10. *Let $E^2_\infty(\Omega)$ and $E^3_\infty(\Omega)$ be the spaces in Definitions* 9.3 *and* 9.4, *respectively. Then $E^2_\infty(\Omega) = E^3_\infty(\Omega)$ and their norms are equivalent.*

Corollary 9.11. *The space $E^2_\infty(\Omega)$ does not depend on the choice of the numbers a_1 and a_2 in Definition* 6.1.

Proposition 9.12. *Let $E^2_\infty(\Omega)$ and $E^4_\infty(\Omega)$ be the spaces in Definitions* 9.3 *and* 9.5, *respectively. Then $E^4_\infty(\Omega) \subset E^2_\infty(\Omega)$ and for any $u \in E^4_\infty(\Omega)$ we have*

$$\|u\|_{E^2_\infty(\Omega)} \le M \|u\|_{E^4_\infty(\Omega)},$$

where M is a constant independent of u.

Theorem 9.13. *All definitions of the space $E_\infty(\Omega)$ are equivalent, i.e.,*

$$E^1_\infty(\Omega) = E^2_\infty(\Omega) = E^3_\infty(\Omega) = E^4_\infty(\Omega)$$

and their norms are equivalent.

10 Dual spaces

Let $E(\mathbb{R}^n)$ be a Banach space satisfying the conditions of Section 1, and $E^*(\mathbb{R}^n)$ be its dual. As above, we will denote them by E and E^*, respectively. We suppose that $D \subset E$, the inclusion being in the algebraic and topological sense, and that D is dense in E. Then $E^* \subset D'$, and this inclusion should be also understood in the algebraic and topological sense. We can define $(E^*)_q$ as it is done in Sections 3 and 4. For example the norm in the space $(E^*)_\infty$ is given by

$$\|v\|_{(E^*)_\infty} = \sup_i \|\phi_i v\|_{E^*}, \tag{10.1}$$

where ϕ_i is a partition of unity. The norm in the right-hand side of (10.1) is the norm of functionals from E^*.

Lemma 10.1. $(E_p)^* \subset E^*_{\text{loc}}$.

Proof. Let $v \in (E_p)^*$, $\phi \in D$. We should verify that $v\phi \in E^*$. For any $u \in E$, $\phi u \in E_p$. Therefore

$$|\langle \phi v, u\rangle| = |\langle v, \phi u\rangle| \le \|v\|_{(E_p)^*} \|\phi u\|_{E_p} \le M\|v\|_{(E_p)^*}\|u\|_E.$$

The lemma is proved. □

In what follows we say that two normed spaces are equal or coincide if they are linearly isomorphic and their norms are equivalent.

Theorem 10.2. *The spaces $(E^*)_\infty$ and $(E_1)^*$ coincide.*

Proof. Let $v \in (E_1)^*$. Then for any $u \in E_1$,

$$\langle v, u\rangle \le \|v\|_{(E_1)^*} \ \|u\|_{E_1}.$$

Since $v \in E^*_{\text{loc}}$ and $u \in E$, then $\langle \phi_i v, u\rangle$ is defined and

$$|\langle \phi_i v, u\rangle| = |\langle v, \phi_i u\rangle| \le \|v\|_{(E_1)^*} \ \|\phi_i u\|_{E_1} \le M\|v\|_{(E_1)^*} \ \|u\|_E.$$

Here $\{\phi_i\}$ is a partition of unity, $\sup_i \|\phi_i\|_M < \infty$.

Therefore

$$\|\phi_i v\|_{E^*} \le M\|v\|_{(E_1)^*}.$$

Consequently,

$$\|v\|_{(E^*)_\infty} \le M\|v\|_{(E_1)^*}.$$

Suppose that $v \in (E^*)_\infty$. Then $v \in E^*_{\text{loc}}$. Let $u \in E_1$, $u_k = \sum_{i=1}^k \phi_i u$. Then $u_k \in E$, and

$$|\langle v, u_k\rangle| = |\langle v, \sum_{i=1}^k \phi_i u\rangle| \le \sum_{i=1}^k |\langle v, \phi_i u\rangle| = \sum_{i=1}^k |\langle \phi_i v, \psi_i u\rangle| \le \sum_{i=1}^k \|\phi_i v\|_{E^*} \ \|\psi_i u\|_E$$
$$\le \|v\|_{(E^*)_\infty} \sum_{i=1}^\infty \|\psi_i u\|_E \le M\|v\|_{(E^*)_\infty}\|u\|_{E_1}.$$

Here $\psi_i \in D$, $\psi_i = 1$ in $\operatorname{supp}\phi_i$. We suppose that the system of functions ψ_i satisfies Condition 2.2. We can pass to the limit in the last estimate as $k \to \infty$. Therefore v can be considered as a functional on E_1, and

$$\|v\|_{(E_1)^*} \le M\|v\|_{(E^*)_\infty}.$$

The theorem is proved. $\square$

We note that functionals from both spaces $(E^*)_\infty$ and $(E_1)^*$ are considered in Theorem 10.2 on functions from E_1.

Theorem 10.3. *Let $1/p + 1/q = 1$, $1 < p, q < \infty$. Then $(E^*)_p \subset (E_q)^*$.*

Proof. Let $v \in (E^*)_p$. We show that $v \in (E_q)^*$. For any $u \in D$ we have

$$|\langle v, u\rangle| = |\langle v, \sum_{i=1}^{\infty} \phi_i u\rangle| \le \sum_{i=1}^{\infty} |\langle v, \phi_i u\rangle| = \sum_{i=1}^{\infty} |\langle \psi_i v, \phi_i u\rangle| \le \sum_{i=1}^{\infty} \|\psi_i v\|_{E^*} \|\phi_i u\|_E$$
$$\le \left(\sum_{i=1}^{\infty} \|\psi_i v\|_{E^*}^p\right)^{1/p} \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^q\right)^{1/q} \le M\|v\|_{(E^*)_p}\, \|u\|_{E_q}.$$

Here $\psi_i \in D$, $\psi_i(x) = 1$ in $\operatorname{supp}\phi_i(x)$. Since D is dense in E_q, this estimate remains valid for all $u \in E_q$. Therefore

$$\|v\|_{(E_q)^*} \le M\|v\|_{(E^*)_p}.$$

The theorem is proved. $\square$

Lemma 10.4. *Let $\phi \in (E_\infty)^*$, $u_n = \sum_{i=1}^n u\theta_i$, where $u \in E_\infty$, θ_i is a partition of unity. Then there exists the limit $\lim_{n\to\infty} \phi(u_n)$.*

Proof. We have

$$\|u_n\|_{E_\infty} = \sup_j \|u_n\theta_j\|_E = \sup_j \|\left(\sum_{i=1}^{n} u\theta_i\right)\theta_j\|_E$$
$$\le \sup_j \left(\sum_{i:\operatorname{supp}\theta_i \cap \operatorname{supp}\theta_j \ne \emptyset} \|u\theta_i\theta_j\|_E\right) \le MN \sup_j \|u\theta_j\| = MN\|u\|_{E_\infty}.$$

Suppose that the limit $\phi(u_n)$ does not exist. Then there exist two subsequences u_{nk} and u_{nm} such that

$$\phi(u_{nk}) \to C_1, \quad \phi(u_{nm}) \to C_2, \quad C_1 \ne C_2.$$

We will construct a bounded sequence in E_∞ such that the functional ϕ will be unbounded on it. This contradiction will prove the existence of the limit.

Without loss of generality we can assume that $C_1 > C_2$. For all k and m sufficiently large,

$$\phi(u_{nk}) \geq C_1 - \epsilon, \;\; \phi(u_{nm}) \leq C_2 + \epsilon.$$

For $\epsilon \leq (C_1 - C_2)/4$,

$$\phi(u_{nk} - u_{nm}) \geq \frac{C_1 - C_2}{2} (= a > 0).$$

We choose k and m such that this estimate is satisfied and write $v_1 = u_{nk} - u_{nm}$. We note that

$$u_{nk} - u_{nm} = \sum_{i=n_m}^{n_k} u\theta_i.$$

Therefore the support of the function v_1 is inside $\bigcup_{i=n_m}^{n_k} \operatorname{supp} \theta_i$.

Similarly, we choose other values of k and m and define the function v_2, $\phi(v_2) \geq a$. Moreover, if the new values k and m are sufficiently large, then $\operatorname{supp} v_1 \cap \operatorname{supp} v_2 = \oslash$. In the same way, we construct other functions v_l such that their supports do not intersect and $\phi(v_l) \geq a$. We finally put $w_j = \sum_{l=1}^{j} v_l$. Similar to the sequence u_n, the sequence w_j is uniformly bounded in E_∞. At the same time $\phi(w_j) \to \infty$. This contradicts the assumption that $\phi \in (E_\infty)^*$. The lemma is proved. $\square$

Consider a functional ϕ from $(E_\infty)^*$. We define a new functional $\tilde{\phi}$ as follows. For any function $u \in E_\infty$ with a bounded support we put

$$\tilde{\phi}(u) = \phi(u).$$

For any function $u \in E_\infty$, we put

$$\tilde{\phi}(u) = \lim_{n\to\infty} \phi(\sum_{i=1}^{n} u\theta_i).$$

Thus $\tilde{\phi}$ is a weak limit of $\sum_{i=1}^{n} \theta_i \phi$ in $(E_\infty)^*$. From Lemma 10.4 it follows that this limit exists. It is easy to verify that $\tilde{\phi}$ is a bounded linear functional on E_∞.

Let $\phi_0 = \phi - \tilde{\phi}$. Then $\phi_0(u) = 0$ for any function u with a bounded support. Thus we have the following result.

Lemma 10.5. *The space $(E_\infty)^*$ can be represented as a direct sum of two subspaces, $(E_\infty)^*_0$ and $(E_\infty)^*_\omega$, where $(E_\infty)^*_0$ consists of functionals equal to 0 on all functions with bounded supports, and $(E_\infty)^*_\omega$ consists of the functionals $\tilde{\phi}$ constructed above.*

Proof. It remains to prove that $(E_\infty)^*_\omega$ and $(E_\infty)^*_0$ are closed. Let $v_k \in (E_\infty)^*_\omega$, $v_k \to v$ in $(E_\infty)^*$. We have

$$\langle v_k,\, u\rangle = \lim_{n\to\infty} \langle v_k,\, u_n\rangle, \;\; \forall u \in E_\infty, \tag{10.2}$$

where $u_n = \sum_{i=1}^n \theta_i u$ (see Lemma 10.8 below). We prove that we can pass to the limit with respect to k in the right-hand side of (10.2). Indeed we have

$$|\langle v_k - v,\ u_n\rangle| \le \|v_k - v\|_{(E_\infty)^*}\|u_n\|_{E_\infty} \le M\|v_k - v\|_{(E_\infty)^*}$$

since $\|u_n\|_{E_\infty}$ is bounded. Hence

$$|\lim_{n\to\infty}\langle v_k - v,\ u_n\rangle| \le M\|v_k - v\|_{(E_\infty)^*} \to 0$$

as $k \to \infty$. Passing to the limit with respect to k in (10.2), we obtain

$$\langle v,\ u\rangle = \lim_{n\to\infty}\langle v,\ u_n\rangle, \quad \forall u \in E_\infty.$$

Therefore $v \in (E_\infty)^*_\omega$. The completeness of the space $(E_\infty)^*_\omega$ is proved. It can be easily verified that the second subspace is also closed. The lemma is proved. □

If $\phi_0 \in (E_\infty)^*_0$ and $\theta \in D$, then $\theta\phi_0 = 0$ is an element of $(E_\infty)^*$. Therefore, if $\phi = \phi_0 + \phi_1$, then $\theta\phi = \theta\phi_1$ (see the next lemma).

Lemma 10.6. *If $\phi \in (E_\infty)^*$, then $\phi \in (E^*)_1$ and*

$$\|\phi\|_{(E^*)_1} \le M\ \|\phi\|_{(E_\infty)^*}\ , \tag{10.3}$$

where M is a constant independent of ϕ.

Proof. We have $\theta_i\phi \in E^*$ for $\theta_i \in D$ and

$$\|\theta_i\phi\|_{E^*} = \sup_{u\in E,\ \|u\|_E=1} |\theta_i\phi(u)|.$$

Hence there exists $u_i \in E$ such that $\|\theta_i\phi\|_{E^*} \le 2\ |\theta_i\phi(u_i)| = 2\ \theta_i\phi(\sigma_i u_i)$, where $|\sigma_i| = 1$. Therefore

$$\sum_{i=1}^m \|\theta_i\phi\|_{E^*} \le 2\ \phi\left(\sum_{i=1}^m \theta_i\sigma_i u_i\right). \tag{10.4}$$

For any θ_k we have

$$\left\|\sum_{i=1}^m \theta_k\theta_i\sigma_i u_i\right\|_E \le \sum_{i=1}^m \|\theta_k\theta_i u_i\|_E \le \sum_{i'} \|\theta_k\theta_{i'}u_{i'}\|_E,$$

where i' are all those numbers i for which $\operatorname{supp}\theta_i \bigcap \operatorname{supp}\theta_k \ne \oslash$. It follows that

$$\left\|\sum_{i=1}^m \theta_k\theta_i\sigma_i u_i\right\|_E \le NK^2, \tag{10.5}$$

where N is the number from Condition 2.2 and $K = \sup_i \|\theta_i\|_{M(E)}$. Inequality (10.5) implies $\|\sum_{i=1}^m \theta_i\sigma_i u_i\|_{E_\infty} \le NK^2$. From (10.4) we obtain $\sum_{i=1}^m \|\theta_i\phi\|_{E^*} \le 2NK^2\|\phi\|_{(E_\infty)^*}$ and (10.3) follows. The lemma is proved. □

Lemma 10.7. *The following inclusions hold:*

$$E_1 \subset E, \ \ (E^*)_1 \subset E^*.$$

Proof. Suppose that $u \in E_1$. Then

$$\|u\|_E = \left\| \sum_{i=1}^{\infty} \phi_i u \right\|_E \le \sum_{i=1}^{\infty} \|\phi_i u\|_E = \|u\|_{E_1},$$

where ϕ_i is a partition of unity. Therefore $u \in E$. The second inclusion follows from the first one applied to the space E^*. The lemma is proved. $\square$

We will prove below that the spaces $(E_\infty)^*_\omega$ and $(E^*)_1$ coincide (Theorem 10.10). Let us introduce an operator $J : (E_\infty)^* \to (E^*)_1$ as follows. According to Lemma 10.6, to any $v \in (E_\infty)^*$ we can put in correspondence $w = Jv \in (E^*)_1$ such that

$$\langle w, \ u\rangle = \langle v, \ u\rangle, \ \ \forall u \in E. \tag{10.6}$$

The right-hand side in (10.6) has sense since $E \subset E_\infty$. The left-hand side in (10.6) is well defined since $(E^*)_1 \subset E^*$. There is only one w satisfying (10.6). Indeed, let w_1 be another one and $w_0 = w_1 - w$. Then $\langle w_0, \ u\rangle = 0, \ \ \forall u \in E$. This means that w_0 is the zero element in E^* and

$$\|w_0\|_{(E^*)_1} = \sum_{i=1}^{\infty} \|w_0 \ \theta_i\|_{E^*} = 0.$$

Hence $w_1 = w$. It is clear that J is a linear operator.

Lemma 10.8. *If* $v \in (E_\infty)^*_\omega$*, then*

$$\langle v, \ u\rangle = \lim_{n\to\infty} \left\langle v, \ \sum_{i=1}^{n} \ \theta_i u \right\rangle, \ \ \forall u \in E_\infty. \tag{10.7}$$

Proof. By definition of $(E_\infty)^*_\omega$, there exists $y \in (E_\infty)^*$ such that

$$\langle v, \ u\rangle = \lim_{n\to\infty} \left\langle y, \ \sum_{i=1}^{n} \ \theta_i u \right\rangle, \ \ \forall u \in E_\infty.$$

In order to prove (10.7), it is sufficient to verify the equality

$$\langle y, \ \theta_i u\rangle = \langle v, \ \theta_i u\rangle. \tag{10.8}$$

We have

$$\langle v, \ \theta_i u\rangle = \lim_{n\to\infty} \left\langle y, \ \sum_{j=1}^{n} \theta_j \theta_i u \right\rangle.$$

Since $\sum_{j=1}^{\infty} \theta_j \theta_i$ has only a finite number of terms, we can pass to the limit and obtain

$$\langle v,\ \theta_i u\rangle = \left\langle y,\ \sum_{j=1}^{\infty} \theta_j \theta_i u \right\rangle,$$

and (10.8) follows from this equality. The lemma is proved. $\square$

Lemma 10.9. *The space* $(E_\infty)^*$ *can be represented as a direct sum of linear subspaces* $(E_\infty)^*_\omega$ *and* Ker J*:*

$$(E_\infty)^* = (E_\infty)^*_\omega \ \oplus \ \mathrm{Ker}\, J. \tag{10.9}$$

Proof. Let $v \in (E_\infty)^*$ and $\tilde{v}$ be given by

$$\langle \tilde{v},\ u\rangle = \lim_{n\to\infty} \left\langle v,\ \sum_{i=1}^{n} \theta_i u \right\rangle, \quad \forall u \in E_\infty.$$

Then $\tilde{v} \in (E_\infty)^*_\omega$. Set $v_0 = v - \tilde{v}$. For any $\phi \in D$ we have

$$\langle \tilde{v},\ \phi\rangle = \left\langle v,\ \sum_{i=1}^{\infty} \phi\, \theta_i \right\rangle = \langle v,\ \phi\rangle$$

since the sum contains only a finite number of terms. It follows that $\langle v_0,\ \phi\rangle = 0,\ \ \forall \phi \in D$. Since D is dense in E and $E \subset E_\infty$ (inclusion with topology), it follows that $\langle v_0,\ u\rangle = 0, \ \ \forall u \in E$. From (10.6) it follows that $\langle Jv_0,\ u\rangle = 0, \ \ \forall u \in E$. Therefore $Jv_0 = 0$ in E^* and, hence, this equality also holds in $(E^*)_1$. This means that $v_0 \in \mathrm{Ker}\, J$. Thus we have proved that $v = \tilde{v} + v_0$, where $\tilde{v} \in (E_\infty)^*_\omega$ and $v_0 \in \mathrm{Ker}\, J$.

It remains to prove that (10.9) is a direct sum. Suppose that $v \in (E_\infty)^*_\omega$ and $v \in \mathrm{Ker}\, J$. We have to verify that $v = 0$. We have

$$\langle v,\ u\rangle = \lim_{n\to\infty} \left\langle v,\ \sum_{i=1}^{n} \theta_i u \right\rangle, \quad \forall u \in E_\infty \tag{10.10}$$

(see Lemma 10.8) because $v \in (E_\infty)^*_\omega$. Since $v \in \mathrm{Ker}\, J$ we conclude that $\langle v,\ \theta_i u\rangle = 0, \ \ \forall u \in E_\infty$. Indeed, if $u \in E_\infty$, then $\theta_i u \in E$. From (10.10) we obtain $\langle v,\ u\rangle = 0, \ \ \forall u \in E_\infty$. The lemma is proved. $\square$

Theorem 10.10. $(E_\infty)^*_\omega = (E^*)_1$.

Proof. The inclusion $(E_\infty)^*_\omega \subset (E^*)_1$ follows from Lemma 10.6. Suppose now that $\phi \in (E^*)_1$. Consider the functionals $\phi_k = \sum_{i=1}^{k} \theta_i \phi$, where θ_i is a partition of unity. By the definition of the space $(E^*)_1$, the series $\sum_{i=1}^{\infty} \|\theta_i \phi\|_{E^*}$ converges. We show that ϕ_k converges to ϕ in $(E^*)_1$. Indeed,

$$\|\phi - \phi_k\|_{(E^*)_1} = \left\| \phi - \sum_{i=1}^{k} \theta_i \phi \right\|_{(E^*)_1} = \sum_{j=1}^{\infty} \left\| \theta_j \left(\phi - \sum_{i=1}^{k} \theta_i \phi \right) \right\|_{E^*}$$

$$= \sum_{j=1}^{\infty} \left\| \theta_j \phi - \sum_{i=1}^{k} \theta_i (\theta_j \phi) \right\|_{E^*} \equiv S.$$

All terms of this sum, for which $\sum_{i=1}^{k} \theta_i$ equals 1 in the support of θ_j, disappear. The remaining terms begin with some k', where k' depends on k and tends to infinity together with it.

$$S = \sum_{j=k'}^{\infty} \left\| \theta_j \phi - \sum_{i=1}^{k} \theta_i (\theta_j \phi) \right\|_{E^*} \leq \sum_{j=k'}^{\infty} \|\theta_j \phi\|_{E^*} + \sum_{j=k'}^{\infty} \sum_{i=1}^{k} \|\theta_i \theta_j \phi\|_{E^*}$$

$$= \sum_{j=k'}^{\infty} \|\theta_j \phi\|_{E^*} + \sum_{j=k'}^{\infty} \sum_{i'} \|\theta_{i'} \theta_j \phi\|_{E^*}$$

$$\leq \sum_{j=k'}^{\infty} \|\theta_j \phi\|_{E^*} + NM \sum_{j=k'}^{\infty} \|\theta_j \phi\|_{E^*} \to 0 \text{ as } k \to \infty.$$

Here i' denotes all those i for which the support of θ_i intersects the support of θ_j for each j fixed. As usual, we use the fact that their number is limited by N.

Thus, the functional ϕ can be represented in the form $\phi = \sum_{i=1}^{\infty} \theta_i \phi$. Then it is also a continuous functional on E_∞. Indeed, for any $u \in E_\infty$,

$$|\langle \phi, u \rangle| \leq \sum_{i=1}^{\infty} |\langle \theta_i \phi, \psi_i u \rangle| \leq \sum_{i=1}^{\infty} \|\theta_i \phi\|_{E^*} \|\psi_i u\|_E$$

$$\leq C \|u\|_{E_\infty} \sum_{i=1}^{\infty} \|\theta_i \tilde{\phi}\|_{E^*} \leq C \|\phi\|_{(E^*)_1} \|u\|_{E_\infty}.$$

Here $\psi_i = 1$ in the support of θ_i. Therefore $\phi \in (E_\infty)^*$, and

$$\|\phi\|_{(E_\infty)^*} \leq C \|\phi\|_{(E^*)_1}.$$

Let $u \in E_\infty$. Put $u_k = \sum_{i=1}^{k} \theta_i u$. Then $\phi(u_k) = \phi_k(u)$. Hence

$$\phi(u) = \lim_{k \to \infty} \phi_k(u) = \lim_{k \to \infty} \phi(u_k).$$

This means that $\phi \in (E_\infty)^*_\omega$.

We prove that the spaces $(E_\infty)^*_\omega$ and $(E^*)_1$ are linearly isomorphic. Let $w \in (E^*)_1$. From the proof above it follows that there exists $v \in (E_\infty)^*_\omega$ such that (10.6) holds. This means that

$$Jv = w. \tag{10.11}$$

Denote by J_1 the restriction of the operator J to $(E_\infty)^*_\omega$. Then from (10.11) we get $J_1 v = w$. This means that the range of the operator J_1 coincides with $(E^*)_1$. Since according to Lemma 10.9 the operator J_1 is invertible, we conclude that the spaces under consideration are linearly isomorphic.

Since the operator J_1^{-1} is bounded, we can use the Banach theorem to conclude that J_1 is also bounded. The theorem is proved. □

Consider now the closure E_D of D in the norm E_∞. The proof of the following lemma is the same as the proof of Lemma 10.6.

Lemma 10.11. $(E_D)^* \subset (E^*)_1$.

Theorem 10.12. $(E_D)^* = (E^*)_1$.

Proof. Let $\phi \in (E^*)_1$, $u \in D$. Then $\phi(u)$ is well defined, and

$$\begin{aligned} |\phi(u)| &\le \sum_{i=1}^{\infty} |\phi(\theta_i u)| = \sum_{i=1}^{\infty} |\phi(\theta_i \psi_i u)| \\ &\le \sum_{i=1}^{\infty} \|\theta_i \phi\|_{E^*} \|\psi_i u\|_E \le M \|\phi\|_{(E^*)_1} \, \|u\|_{E_\infty}, \end{aligned}$$

where $\psi_i = 1$ in the support of θ_i.

This estimate remains valid for $u \in E_D$. Therefore, $\phi \in (E_D)^*$. The opposite inclusion follows from the previous lemma.

We now prove the isomorphism of the spaces. Let $v \in (E_D)^*$, then $v \in D'$. As in the proof of Lemma 10.6 we obtain that $v \in (E^*)_1$. We introduce the embedding operator

$$T : \ (E_D)^* \ \to \ (E^*)_1.$$

This means that to any $v \in (E_D)^*$ we put in correspondence $Tv \in (E^*)_1$ such that

$$\langle Tv, \ \phi \rangle = \langle v, \ \phi \rangle, \ \ \forall \phi \in D. \tag{10.12}$$

It is clear that T is a linear operator. It is easy to see that the range of the operator T coincides with $(E^*)_1$. Indeed let $w \in (E^*)_1$. Then, as in the proof above, we obtain

$$|\langle w, \ \phi \rangle| \le M \|w\|_{(E^*)_1} \, \|\phi\|_{E_\infty}. \tag{10.13}$$

Consider $v \in D'$ such that $\langle v, \ \phi \rangle = \langle w, \ \phi \rangle, \ \ \forall \phi \in D$. Then by (10.13) we have $v \in (E_D)^*$. Hence by (10.12) we get

$$\langle Tv, \ \phi \rangle = \langle w, \ \phi \rangle, \ \ \forall \phi \in D.$$

Since D is dense in E, we conclude that Tv and w coincide as elements of E^* and therefore as elements of $(E^*)_1$. We have proved that the equation $Tv = w$ has a solution $v \in (E_D)^*$ for any $w \in (E^*)_1$. Hence the range of the operator T coincides with $(E^*)_1$.

It remains to prove that the operator T is invertible. By definition of the operator T, the equality (10.12) holds. Hence if $Tv = 0$ in $(E^*)_1$, then $\langle v,\ \phi\rangle = 0,\ \ \forall\phi \in D$ and therefore $v = 0$ in $(E_D)^*$.

From (10.13) we have $|\langle v,\ \phi\rangle| \le M\|w\|_{(E^*)_1}\ \|\phi\|_{E_\infty}$, and $\|T^{-1}w\|_{(E_D)^*} \le M\|w\|_{(E^*)_1}$. Hence the operator T^{-1} is bounded. By the Banach theorem the operator T is also bounded. Therefore the norms in the spaces $(E_D)^*$ and $(E^*)_1$ are equivalent. The theorem is proved. □

The next theorem follows from Theorems 10.10 and 10.12. Nevertheless we give a direct proof of this theorem in order to obtain an explicit relation between the elements of these spaces.

Theorem 10.13. $(E_\infty)^*_\omega = (E_D)^*$.

Proof. We note that $E \subset E_D$. Indeed, let $u \in E$, then $u \in E_\infty$. Since D is dense in E, there exists a sequence $\{\phi_n\}$, $\phi_n \in D$ such that $\|\phi_n - u\|_E \to 0$. Hence

$$\|\phi_n - u\|_{E_\infty} \le M\|\phi_n - u\|_E \to 0.$$

This means that $u \in E_D$.

Let $v \in (E_D)^*$. We introduce a functional w as follows:

$$\langle w,\ u\rangle = \lim_{n\to\infty}\left\langle v,\ \sum_{i=1}^n \theta_i u\right\rangle,\ \forall u \in E_\infty. \tag{10.14}$$

We prove that the limit in (10.14) exists. Indeed, by the Hahn-Banach theorem we can extend v to a functional $\hat{v} \in (E_\infty)^*$. We have $\langle \hat{v},\ u\rangle = \langle v,\ u\rangle$, $\forall u \in E_D$. Since $\sum_{i=1}^n \theta_i u \in E \subset E_D$, $\forall u \in E_\infty$, we get

$$\left\langle \hat{v},\ \sum_{i=1}^n \theta_i u\right\rangle = \left\langle v,\ \sum_{i=1}^n \theta_i u\right\rangle,\ \forall u \in E_\infty.$$

From Lemma 10.4 it follows that the limit in (10.14) exists and $w \in (E_\infty)^*_\omega$.

Let us introduce an operator $S : (E_D)^* \to (E_\infty)^*_\omega$ by the formula $w = Sv$, where w is given by (10.14). It is clear that S is a linear operator. We will prove that S is invertible. Indeed, let $Sv = 0$. Then from (10.14) we obtain

$$\lim_{n\to\infty}\left\langle v,\ \sum_{i=1}^n \theta_i u\right\rangle = 0,\ \forall u \in E_\infty.$$

In particular, if $u \in D$, we get $\langle v,\ u\rangle = 0$, $\forall u \in D$. Since D is dense in E_D, it follows that $v = 0$.

To prove that $(E_\infty)^*_\omega$ is linearly isomorphic to $(E_D)^*$, it is sufficient to verify that the range of S coincides with $(E_\infty)^*_\omega$. Let $w \in (E_\infty)^*_\omega$. Then $w \in (E_\infty)^*$. From Lemma 10.8 it follows that

$$\langle w,\ u\rangle = \lim_{n\to\infty} \left\langle w,\ \sum_{i=1}^{n} \theta_i u \right\rangle,\ \forall u \in E_\infty. \tag{10.15}$$

Denote by v the restriction of w to E_D: $\langle v,\ u\rangle = \langle w,\ u\rangle$, $\forall u \in E_D$. From this equality it follows that

$$\left\langle w,\ \sum_{i=1}^{n} \theta_i u \right\rangle = \left\langle v,\ \sum_{i=1}^{n} \theta_i u \right\rangle,\ \forall u \in E_\infty$$

since $\theta_i u \in E_D$. From this and (10.15) we conclude that (10.14) is true. Therefore $w = Sv$. We have proved that the range of S coincides with $(E_\infty)^*_\omega$ and hence $(E_\infty)^*_\omega$ and $(E_D)^*$ are linearly isomorphic.

We now prove that the operator S is bounded. From (10.14) we get

$$|\langle w,\ u\rangle| = \lim_{n\to\infty} \left| \left\langle v,\ \sum_{i=1}^{n} \theta_i u \right\rangle \right|,\ \forall u \in E_\infty.$$

Further,

$$\left| \left\langle v,\ \sum_{i=1}^{n} \theta_i u \right\rangle \right| \le \sum_{i=1}^{n} |\langle \theta_i v,\ \psi_i u\rangle| \le \sum_{i=1}^{n} \|\theta_i v\|_{E^*} \|\psi_i u\|_E \le M \|v\|_{(E^*)_1} \|u\|_{E_\infty}.$$

As in Lemma 10.6, we prove that $\|v\|_{(E^*)_1} \le M_1 \|v\|_{(E_D)^*}$. Therefore

$$|\langle w,\ u\rangle| \le M_2 \|v\|_{(E_D)^*} \|u\|_{E_\infty}.$$

It follows that $\|w\|_{(E_\infty)^*} \le M_2 \|v\|_{(E_D)^*}$. Hence the operator S is bounded. The theorem is proved. □

Remark 10.14. The space E_D is a subspace of E_∞. Therefore we can expect that $(E_\infty)^* \subset (E_D)^*$. Nevertheless we obtain that $(E_D)^*$ coincides with a subspace $(E_\infty)^*_\omega$ of $(E_\infty)^*$. To explain this situation we note that E_D is not dense in E_∞. Therefore there exist different from zero functionals in $(E_\infty)^*$, equal zero at E_D. We call them "bad" functionals or functional with support at infinity. Each functional from $(E_\infty)^*$ can be formally considered as a functional from $(E_D)^*$. However, if we do not take into account zero functionals, then the inclusion $(E_\infty)^* \subset (E_D)^*$ does not hold. "Bad" functionals do not belong to D' and cannot be considered as generalized functions.

In the definition of the space E_p in Section 3 it is supposed that $E \subset D'$. Hence, in order to use this definition for the space $(E^*)_p$, we should assume that

$E^* \subset D'$. We will define the space $(E^*)_p$ without this assumption. We will give an intrinsic definition of the spaces E_1 and $(E^*)_p$ which coincides with the previous ones. We do not suppose now that $E \subset D'$ but assume, as before, that all $f \in D$ are multipliers in E according to Definition 1.1. Obviously, it follows that $D \subset M(E^*)$.

Definition 10.15. *E_1 is the space of all $u \in E$ such that*

$$\sum_{i=1}^{\infty} \|\phi_i u\|_E < \infty, \tag{10.16}$$

where $\{\phi_i\}$ is a partition of unity.

Proposition 10.16. *The spaces E_1 in Definition* 10.15 *and in Definition* 3.1 *coincide.*

Proof. The proof follows from the fact that any $u \in E_{\text{loc}}$ satisfying (10.16) belongs to E. □

Thus, the space E_1 is defined. Hence the space $(E_1)^*$ is also defined. We can now define the space $(E^*)_\infty$.

Definition 10.17. $u \in (E^*)_\infty$ if and only if $u \in (E_1)^*$ and

$$\|u\|_{(E^*)_\infty} = \sup_i \|\phi_i u\|_{E^*} < \infty.$$

Proposition 10.18. *The spaces $(E^*)_\infty$ and $(E_1)^*$ coincide and their norms are equivalent.*

The proof of this theorem is the same as in Theorem 10.2. We can now give the intrinsic definition of the space $(E^*)_p, \;\; 1 \le p < \infty$.

Definition 10.19. $u \in (E^*)_p$ if and only if $u \in (E_1)^*$ and

$$\|u\|_{(E^*)_p} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_{E^*}^p \right)^{1/p} < \infty.$$

If the space E is reflexive, we can also give an intrinsic definition of the spaces $E_p, \;\; 1 \le p \le \infty$. Indeed, as before we define $(E^*)_1$. Then $E_\infty = (E^{**})_\infty = ((E^*)^*)_\infty = ((E^*)_1)^*$ according to Proposition 10.18 applied to E^*. Therefore $(E^*)_1 \subset ((E^*)_1)^{**} = (E_\infty)^*$ (cf. Theorem 10.10).

Definition 10.20. $u \in E_p, \;\; 1 \le p < \infty$ if and only if $u \in E_\infty$ and

$$\|u\|_{E_p} = \left(\sum_{i=1}^{\infty} \|\phi_i u\|_E^p \right)^{1/p} < \infty.$$

11 Dual spaces in domains

Let $E \in D'$ be a reflexive Banach space. First of all, we will explain in what sense $\varphi \in D$ are understood as elements of the space E^*. We consider $\tilde{\varphi}(u)$ $(\varphi \in D,\ u \in E)$ as the functional

$$\tilde{\varphi}(u) = u(\varphi). \tag{11.1}$$

The right-hand side in (11.1) is defined since $E \subset D'$. The inclusion $E \subset D'$ is supposed to be with the topology. Hence if $u_n \to 0$ in E, then $u_n \to 0$ in D'. Therefore $u_n(\varphi) \to 0$ for any $\varphi \in D$. From (11.1) it follows that $\tilde{\varphi}(u_n) \to 0$ and $\tilde{\varphi} \in E^*$.

Moreover, the functions $\varphi \in C_0^\infty(\Omega)$ can be understood as elements of $[E(\Omega)]^*$. We consider $\hat{\varphi}(u)$ $(\varphi \in D,\ u \in E(\Omega))$ as the functional

$$\hat{\varphi}(u) = u(\varphi). \tag{11.2}$$

We can do this since if the convergence

$$u_n \longrightarrow 0 \tag{11.3}$$

holds in $E(\Omega)$, then

$$\hat{\varphi}(u_n) = u_n(\varphi) \to 0. \tag{11.4}$$

Indeed, let u_n^c be an extension of u_n such that

$$u_n^c(\varphi) = u_n(\varphi) \tag{11.5}$$

for $\varphi \in C_0^\infty(\Omega)$ and $\|u_n^c\|_E \le 2\|u_n\|_{E(\Omega)}$. Then from (11.3) it follows that $u_n^c \to 0$ in E. Since $E \subset D'$, then we get $u_n^c(\varphi) \to 0$ for $\varphi \in C_0^\infty(\Omega)$, and (11.4) follows from (11.5).

Denote by $\hat{C}_0^\infty(\Omega)$ the set of all $\varphi \in C_0^\infty(\Omega)$ considered as functionals on $E(\Omega)$ in the sense described by (11.2).

Theorem 11.1. *Let $E \in D'$ be a reflexive Banach space. Then*

$$[E(\Omega)]^* = \hat{E}_0^*(\Omega), \tag{11.6}$$

where $\hat{E}_0^*(\Omega)$ is the closure of $\hat{C}_0^\infty(\Omega)$ in $[E(\Omega)]^*$.

Proof. From the definition of $\hat{C}_0^\infty(\Omega)$ it follows that

$$\hat{C}_0^\infty(\Omega) \subset [E(\Omega)]^*. \tag{11.7}$$

Denote by $E_{\overline{\Omega}}$ the subspace of E consisting of such generalized functions that their supports are contained in $\overline{\Omega}$. Then it is known that

$$E(\Omega) = E/E_{C\Omega} \tag{11.8}$$

(for the proof see, for example, in [543], p. 22). It follows that $E(\Omega)$ is a reflexive Banach space.

From (11.7) we conclude that $\hat{E}_0^*(\Omega) \subset [E(\Omega)]^*$. Suppose now that (11.6) is not true. Then there exists a functional $f \in [E(\Omega)]^{**}$ such that

$$f \neq 0 \tag{11.9}$$

and

$$f(v) = 0 \tag{11.10}$$

for all $v \in \hat{E}_0^*(\Omega)$. Since $E(\Omega)$ is a reflexive Banach space, then $E(\Omega)$ coincides with $[E(\Omega)]^{**}$ by the natural embedding. This means that there exists $y \in E(\Omega)$ such that

$$f(v) = v(y) \tag{11.11}$$

for any $v \in [E(\Omega)]^*$. It follows from (11.10) that $f(\hat{\varphi}) = 0$ for all $\varphi \in C_0^\infty(\Omega)$. Equality (11.11) implies $\hat{\varphi}(y) = 0$ for all $\varphi \in C_0^\infty(\Omega)$. We conclude from (11.2) that $y(\varphi) = 0$ for all $\varphi \in C_0^\infty(\Omega)$. By definition of $E(\Omega)$, this means that $y = 0$ as an element of $E(\Omega)$, and from (11.11) $f(v) = 0$ for all $v \in [E(\Omega)]^*$. This contradicts (11.9). The theorem is proved. □

It follows from Theorem 11.1 that for any $v \in [E(\Omega)]^*$ there exists $v_k \in C_0^\infty(\Omega)$ such that

$$\|\hat{v}_k - v\|_{[E(\Omega)]^*} \to 0 \tag{11.12}$$

as $k \to \infty$.

Theorem 11.2. *Let $E \subset D'$ be a reflexive Banach space and $v \in [E(\Omega)]^*$. Then there exists a unique $\tilde{v} \in E^*$ such that for any $u \in E(\Omega)$ and any extension $\tilde{u}$ of u to E we have*

$$\tilde{v}(\tilde{u}) = v(u). \tag{11.13}$$

Moreover,

$$\|\tilde{v}\|_{E^*} = \|v\|_{[E(\Omega)]^*}. \tag{11.14}$$

Proof. Let $\varphi \in C_0^\infty(\Omega)$, $\tilde{\varphi}$ and $\hat{\varphi}$ be as in (11.1) and (11.2), respectively. Then for any $u \in E(\Omega)$ and any extension $\tilde{u}$ of u to E we have

$$\hat{\varphi}(u) = u(\varphi) = \tilde{u}(\varphi) = \tilde{\varphi}(\tilde{u}). \tag{11.15}$$

By definition of the norm in $E(\Omega)$ we have

$$\|u\|_{E(\Omega)} \leq \|\tilde{u}\|_E. \tag{11.16}$$

For any $\tilde{u} \in E$ and its restriction u to $E(\Omega)$, (11.15) and (11.16) hold. Therefore

$$|\tilde{\varphi}(\tilde{u})| = |\hat{\varphi}(u)| \leq \|\hat{\varphi}\|_{[E(\Omega]^*}\|u\|_{E(\Omega)} \leq \|\hat{\varphi}\|_{[E(\Omega)]^*}\|\tilde{u}\|_E.$$

Hence

$$\|\tilde{\varphi}\|_{E^*} \leq \|\hat{\varphi}\|_{[E(\Omega)]^*}. \tag{11.17}$$

Now, let $v_k \in C_0^\infty, k = 1, 2 \ldots$, be a sequence which satisfies (11.12). Then by (11.17) we obtain

$$\|\tilde{v}_k\|_{E^*} \leq \|\hat{v}_k\|_{[E(\Omega)]^*} \tag{11.18}$$

and

$$\|\tilde{v}_k - \tilde{v}_l\|_{E^*} \leq \|\hat{v}_k - \hat{v}_l\|_{[E(\Omega)]^*}. \tag{11.19}$$

It follows from (11.12) and (11.19) that the sequence $\{\tilde{v}_k\}$ is convergent in E^*. Denote by $\tilde{v}$ its limit. Then the inequality $\|\tilde{v}\|_{E^*} \leq \|v\|_{[E(\Omega)]^*}$ follows from (11.18). From (11.15) we get $\tilde{v}_k(\tilde{u}) = \hat{v}_k(u)$. Passing here to the limit we obtain (11.13).

We now show that the function $\tilde{v}$ with property (11.13) is unique. If we have two of them, $\tilde{v}_1$ and $\tilde{v}_2$, then for any $\tilde{u} \in E$ we take its restrictions u to $E(\Omega)$ and obtain $\tilde{v}_1(\tilde{u}) = v(u), \ \ \tilde{v}_2(\tilde{u}) = v(u)$. Hence $\tilde{v}_1 = \tilde{v}_2$.

It remains to prove that

$$\|v\|_{[E(\Omega)]^*} \leq \|\tilde{v}\|_{E^*}. \tag{11.20}$$

It follows from (11.13) that $|v(u)| = |\tilde{v}(\tilde{u})|$ for any $u \in E(\Omega)$ and any extension $\tilde{u}$ to E. Hence

$$|v(u)| \leq \|\tilde{v}\|_{E^*}\|\tilde{u}\|_E. \tag{11.21}$$

Let $\epsilon > 0$ be an arbitrary number. Since $\|u\|_{E(\Omega)} = \inf \|\tilde{u}\|_E$, where the infimum is taken over all extensions $\tilde{u}$ of u, we can take $\tilde{u}$ such that $\|\tilde{u}\|_E \leq (1+\epsilon)\|u\|_{E(\Omega)}$. Therefore from (11.21), $\|v\|_{[E(\Omega)]^*} \leq (1+\epsilon)\|\tilde{v}\|_{E^*}$. Since $\epsilon > 0$ is arbitrary, we get (11.20). The theorem is proved. □

Denote by $\tilde{C}_0^\infty(\Omega)$ the set of functionals (11.1) and by $E_0^*(\Omega)$ its closure in E^*.

Theorem 11.3. *Let $E \subset D'$ be a reflexive Banach space. Then $[E(\Omega)]^*$ is isometrically isomorphic to the subspace $E_0^*(\Omega)$ of E^*. The correspondence is described by Theorem* 11.2. *More exactly. Any functional $v \in [E(\Omega)]^*$ can be represented in the form*

$$v(u) = \tilde{v}(\tilde{u}), \ \ u \in E(\Omega), \tag{11.22}$$

where $\tilde{v}$ is the corresponding functional, which belongs to $E_0^*(\Omega)$, and $\tilde{u}$ is an arbitrary extension of u to E. Moreover,

$$\|v\|_{[E(\Omega)]^*} = \|\tilde{v}\|_{E*}.$$

Proof. The representation (11.22) follows from Theorem 11.2. It remains only to prove that $\tilde{v} \in E_0^*(\Omega)$. Let $v_k \in C_0^\infty(\Omega)$ be a sequence such that (11.12) holds. Then we have

$$\tilde{v}_k(\tilde{u}) = \hat{v}_k(u). \tag{11.23}$$

Moreover, (11.19) holds. Hence $\{\tilde{v}_k\}$ is convergent in E^*. Denote by $\tilde{v}$ its limit. Obviously $\tilde{v} \in E_0^*(\Omega)$, and from (11.23) we obtain (11.22). The theorem is proved. □

Let $\phi_i \in D$ be a partition of unity in $\mathbb{R}^n$, and $\Omega \subset \mathbb{R}^n$ be an unbounded domain. We consider the space $E(\Omega)$ and its dual $(E(\Omega))^*$. For each $u \in E(\Omega)$ the product $\phi_i u$ is defined, and $\phi_i u \in E(\Omega)$. Therefore we can define the product $\phi_i v$ for $v \in (E(\Omega))^*$:

$$\langle \phi_i v, u\rangle = \langle v, \phi_i u\rangle, \quad u \in E(\Omega). \tag{11.24}$$

It is a bounded functional on $E(\Omega)$:

$$|\langle \phi_i v, u\rangle| = |\langle v, \phi_i u\rangle| \le \|v\|_{E^*(\Omega)}\|\phi_i u\|_{E(\Omega)} \le M\|v\|_{E^*(\Omega)}\|u\|_{E(\Omega)}. \tag{11.25}$$

Thus $\phi_i v \in (E(\Omega))^*$, and $\|\phi_i v\|_{E^*(\Omega)} \le M\|v\|_{(E(\Omega))^*}$.

Let v be a functional on $E(\Omega)$. We do not assume a priori that it is bounded. We say that $v \in ((E(\Omega))^*)_{\mathrm{loc}}$ if $\phi_i v \in (E(\Omega))^*$ for any i.

Definition 11.4. The space $((E(\Omega))^*)_\infty$ is the set of all functionals $v \in ((E(\Omega))^*)_{\mathrm{loc}}$ such that

$$\|v\|_{((E(\Omega))^*)_\infty} = \sup_i \|\phi_i v\|_{(E(\Omega))^*} < \infty.$$

Theorem 11.5. *The spaces $((E(\Omega))^*)_\infty$ and $(E(\Omega))_1)^*$ coincide.*

The proof is the same as for Theorem 10.2

We have proved in Theorem 11.3 that a functional $v \in (E(\Omega))^*$ can be extended to $\tilde v \in E^* = (E(\mathbb{R}^n))^*$. We will use this result in order to show that a functional $v \in ((E(\Omega))^*)_\infty$ can be extended to $(E^*)_\infty$.

Theorem 11.6. *For any $v \in ((E(\Omega))^*)_\infty$ there exists an extension*

$$\tilde v \in (E^*)_\infty = ((E(\mathbb{R}^n))^*)_\infty$$

such that

$$\langle v, u\rangle = \langle \tilde v, \tilde u\rangle, \quad \forall u \in (E(\Omega))_1, \tag{11.26}$$

where $\tilde u \in E_1 = (E(\mathbb{R}^n))_1$ is an extension of u.

Proof. We can represent the functional $v \in (E(\Omega))^*$ in the form $v = \sum_{i=1}^\infty \phi_i v$ with the equality understood in the sense of equality of generalized functions. Let $\psi_i \in D$ equal 1 in the support of ϕ_i. Then $v = \sum_{i=1}^\infty \phi_i \psi_i v$. Denote by v_i the extension of $\psi_i v$ such that $\langle v_i, \tilde u\rangle = \langle \psi_i v, u\rangle$, where $u \in E(\Omega)$ and $\tilde u$ is its extension to E. We put $\tilde v = \sum_{i=1}^\infty \phi_i v_i$. We show that $\tilde v \in (E^*)_\infty$. Indeed,

$$\begin{aligned} |\langle \phi_i\phi_j v_i, \tilde u\rangle| &= |\langle v_i, \phi_i\phi_j \tilde u\rangle| = |\langle \psi_i v, \phi_i\phi_j u\rangle| = |\langle \phi_i\phi_j v, u\rangle| \\ &\le \|\phi_i\phi_j v\|_{(E(\Omega))^*}\|u\|_{E(\Omega)} \le \|\phi_i\phi_j v\|_{(E(\Omega))^*}\|\tilde u\|_E. \end{aligned}$$

Therefore

$$\|\phi_i\phi_j v_i\|_{E^*} \le \|\phi_i\phi_j v\|_{(E(\Omega))^*}.$$

We have

$$\|\phi_j \tilde{v}\|_{E^*} \le \sum_{i=1}^{\infty} \|\phi_i \phi_j v_i\|_{E^*} \le \sum_{i=1}^{\infty} \|\phi_i \phi_j v\|_{(E(\Omega))^*}$$
$$\le KM \|\phi_j v\|_{(E(\Omega))^*} \le KM \|v\|_{((E(\Omega))^*)_\infty}.$$

Thus

$$\|\tilde{v}\|_{(E^*)_\infty} \le KM \|v\|_{((E(\Omega))^*)_\infty}.$$

To finish the proof of the theorem we verify equality (11.26). It is sufficient to check it for functions u with a bounded support since they are dense in $(E(\Omega))_1$. We have

$$\langle \phi_i v_i, \tilde{u} \rangle = \langle \phi_i \psi_i v, u \rangle = \langle \phi_i v, u \rangle,$$

where an extension $\tilde{u}$ can also be chosen with a bounded support. Taking a sum with respect to those i for which the support of ϕ_i has a nonempty intersection with the supports of u and $\tilde{u}$, we obtain (11.26). The theorem is proved. □

12 Spaces $W_q^{s,p}(\mathbb{R}^n)$

In this section we consider the spaces E_q in the case where E is a Sobolev-Slobodetskii space $W^{s,p}(\mathbb{R}^n)$ with a real $s \ge 0$ and $1 \le p < \infty$. We denote them by $W_q^{s,p}(\mathbb{R}^n)$. If $s = 0$ we will use the notation $L_q^p(\mathbb{R}^n)$ and the conventional notation $L^p(\mathbb{R}^n)$. In what follows we do not specify the domain if it is the whole $\mathbb{R}^n$. Applying results of Section 10, we obtain the relations

$$(W_1^{s,p})^* = W_\infty^{-s,p'}, \ \ (W_D^{s,p})^* = W_1^{-s,p'},$$

(Theorems 10.2 and 10.10). We begin with a result that shows the relation of the spaces $W_q^{s,p}(\mathbb{R}^n)$ and usual Sobolev or Sobolev-Slobodetskii spaces.

Lemma 12.1. $L_p^p = L^p$.

Proof. We have

$$\|u\|_{L_p^p}^p = \sum_{i=1}^{\infty} \|\phi_i u\|_{L^p}^p = \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} |\phi_i u|^p dx,$$

where ϕ_i is a partition of unity. Denote by B_i the support of ϕ_i and recall that for any $x \in \mathbb{R}^n$ there exist no more than $N+1$ functions ϕ_i different from zero at this point. Therefore

$$\|u\|_{L_p^p}^p \le \sum_{i=1}^{\infty} \int_{B_i} |u|^p dx \le (N+1) \int_{\mathbb{R}^n} |u|^p dx.$$

On the other hand,

$$\int_{\mathbb{R}^n} |u|^p dx = \sum_{i=1}^{\infty} \int_{S_i} |u|^p dx,$$

where S_i are unit cubes forming a lattice in R^n,

$$\int_{S_i} |u|^p dx = \int_{S_i} |\sum_{i'} \phi_{i'} u|^p dx$$
$$\leq \int_{\mathbb{R}^n} |\sum_{i'} \phi_{i'} u|^p dx \leq C_1 \sum_{i'} \int_{\mathbb{R}^n} |\phi_{i'} u|^p dx$$

where $\sum_{i'} \phi_{i'}(x) = 1$ for $x \in S_i$, since the number of i' is bounded independently of i. Therefore

$$\int_{\mathbb{R}^n} |u|^p dx \leq C_2 \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} |\phi_i u|^p dx.$$

Thus the norms in L_p and L_p^p are equivalent. The lemma is proved. □

Lemma 12.2. *Let s be a positive integer. Then $W_p^{s,p} = W^{s,p}$.*

Proof. By the definition of the norm in $W_p^{s,p}$,

$$\|u\|_{W_p^{s,p}} = \sum_{i=1}^{\infty} \|\phi_i u\|_{W^{s,p}}^p = \sum_{i=1}^{\infty} \sum_{|\alpha| \leq s} \int_{\mathbb{R}^n} |D^\alpha (\phi_i u)|^p dx.$$

Taking into account that the derivatives of ϕ_i are uniformly bounded, we obtain the estimate

$$\sum_{i=1}^{\infty} \int_{\mathbb{R}^n} |D^\beta \phi_i D^\gamma u|^p dx \leq C_1 \int_{\mathbb{R}^n} |D^\gamma u|^p dx$$

in the same way as in the previous lemma. The opposite estimate

$$\int_{\mathbb{R}^n} |D^\gamma u|^p dx \leq C_2 \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} |\phi_i D^\gamma u|^p dx$$

can be also obtained as above. The lemma is proved. □

Theorem 12.3. *Let s be real and positive. Then $W_p^{s,p} = W^{s,p}$.*

Proof. Consider first the case where $0 < s < 1$. Then

$$\|u\|_{W_p^{s,p}}^p = \sum_{i=1}^{\infty} \|\phi_i u\|_{W^{s,p}}^p$$
$$= \sum_{i=1}^{\infty} \|\phi_i u\|_{L^p}^p + \sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi_i(x)u(x) - \phi_i(y)u(y)|^p}{|x-y|^{n+ps}} \, dxdy.$$

Denote by J_i the last integral in the right-hand side. Then

$$|J_i| \le C_1 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\phi_i(x)|^p \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy + C_1 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |u(y)|^p \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dxdy$$
$$\le C_1 \sup_x |\phi_i(x)|^p \int_{\mathbb{R}^n}\int_{B_i} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy + C_1 \int_{\mathbb{R}^n} |u(y)|^p \left(\int_{R^n} \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx\right) dy,$$

where B_i is the support of ϕ_i. To estimate $\sum_{i=1}^{\infty}|J_i|$ we first use the inequality

$$\sum_{i=1}^{\infty}\int_{\mathbb{R}^n}\int_{B_i} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \le C_2 \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy$$

which follows from the fact that at each $x \in \mathbb{R}^n$ the number of intersecting supports B_i is no more than $N+1$. We next estimate the function

$$\Phi(y) \equiv \sum_{i=1}^{\infty}\int_{\mathbb{R}^n} \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx = \Phi_1(y)+\Phi_2(y),$$

where $\Phi_1(y)$ contains all i such that $y \in B_i$ or $y \notin B_i$ but the distance from y to B_i is less than 1. The function $\Phi_2(y)$ contains the remaining terms. There is a finite number of terms in $\Phi_1(y)$. For each such i we have

$$\int_{\mathbb{R}^n} \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx = \int_{K_i} \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx + \int_{\mathbb{R}^n/K_i} \frac{|\phi_i(x)-\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx$$
$$\le C_3 \int_{K_i} \frac{|x-y|^p}{|x-y|^{n+ps}}\,dx + \int_{\mathbb{R}^n/K_i} \frac{|\phi_i(y)|^p}{|x-y|^{n+ps}}\,dx,$$

where K_i is the ball with the same center as B_i and the radius two times greater. Both integrals in the right-hand side are bounded. Therefore $\Phi_1(y)$ is also bounded.

Consider next $\Phi_2(y)$:

$$\Phi_2(y) = \sum_{i=1}^{\infty}\int_{B_i} \frac{|\phi_i(x)|^p}{|x-y|^{n+ps}}\,dx \le C_4 \int_{\mathbb{R}^n/S(y)} \frac{dx}{|x-y|^{n+ps}},$$

where $S(y)$ is the unit ball with the center at y. Hence $\Phi_2(y)$ is bounded. Thus, $\Phi(y)$ is bounded independently of y, and

$$\|u\|^p_{W_p^{s,p}} \le C_5\|u\|^p_{L_p^p} + C_5 \int_{R^n}\int_{R^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy = C_5\|u\|^p_{W^{s,p}}.$$

We now prove the opposite inequality. The $2n$-dimensional space $\mathbb{R}^n \times \mathbb{R}^n$ is represented as a sum of two sets,

$$\Pi = \{x \in \mathbb{R}^n, y \in \mathbb{R}^n, |x-y| \le \epsilon\},$$

and

$$\Lambda = \{x \in \mathbb{R}^n, y \in \mathbb{R}^n, |x-y| > \epsilon\}.$$

Consider a square lattice in $\mathbb{R}^n$ with distance d between its points, and the balls K_i with centers at the centers of the lattice and the radii $2d$. For $\epsilon > 0$ sufficiently small, $\Pi \subset \cup_{i=1}^{\infty} K_i \times K_i$. Let θ_i be a system of functions such that $\theta_i = 1$ in K_i. Then

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \\ &\quad \le \sum_{i=1}^{\infty}\int_{K_i}\int_{K_i} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy + \int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy,\end{aligned}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\int_{K_i}\int_{K_i} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy &= \sum_{i=1}^{\infty}\int_{K_i}\int_{K_i} \frac{|\theta_i(x)u(x)-\theta_i(y)u(y)|^p}{|x-y|^{n+ps}}\,dxdy \\ &\le \sum_{i=1}^{\infty}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\theta_i(x)u(x)-\theta_i(y)u(y)|^p}{|x-y|^{n+ps}}\,dxdy,\end{aligned}$$

$$\int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \le C_6 \int_{\Lambda} \frac{|u(x)|^p}{|x-y|^{n+ps}}\,dxdy + C_6 \int_{\Lambda} \frac{|u(y)|^p}{|x-y|^{n+ps}}\,dxdy,$$
$$\int_{\Lambda} \frac{|u(x)|^p}{|x-y|^{n+ps}}\,dxdy = \int_{\mathbb{R}^n} |u(x)|^p \left(\int_{|y-x|>\epsilon} \frac{dy}{|x-y|^{n+ps}}\right) dx.$$

Since the internal integral in the right-hand side of the last equality can be estimated by a constant independent of x, then

$$\int_{\Lambda} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dxdy \le C_7\|u\|_{L^p}^p \le C_8\|u\|_{L_p^p}^p.$$

Thus,

$$\|u\|_{W^{s,p}} \le C_9\|u\|_{W_p^{s,p}}.$$

This estimate is obtained for the system of functions θ_i. We recall that the norms in $W_p^{s,p}$ with equivalent systems of functions are equivalent.

We have proved that the norms in $W^{s,p}$ and $W_p^{s,p}$ are equivalent in the case of positive $s < 1$. For an integer $s \ge 1$, the assertion of the theorem follows from

Lemma 12.2. Consider now noninteger $s > 1$. We have

$$\begin{aligned}\|u\|^p_{W_p^{s,p}} &= \sum_{i=1}^{\infty} \|\phi_i u\|^p_{W^{s,p}} \\ &= \sum_{|\alpha|\le[s]} \sum_{i=1}^{\infty} \|D^\alpha(\phi_i u)\|^p_{L^p} \\ &\quad + \sum_{i=1}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^{[s]}(\phi_i(x)u(x)) - D^{[s]}(\phi_i(y)u(y))|^p}{|x-y|^{n+p\sigma}}\, dxdy,\end{aligned}$$

where $\sigma = s - [s]$. The estimate of the integral

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^\beta\phi_i(x)D^\gamma u(x) - D^\beta\phi_i(y)D^\gamma u(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy$$

can be done in the same way as above in the case $s < 1$. This allows us to obtain the estimate

$$\|u\|_{W_p^{s,p}} \le C_{10}\|u\|_{W^{s,p}}.$$

To prove the opposite inequality we use the estimate

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^\alpha u(x) - D^\alpha u(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy \\ &\quad\le \sum_{i=1}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\theta_i(x)D^\alpha u(x) - \theta_i(y)D^\alpha u(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy + \|u\|^p_{W^{[s],p}}\end{aligned} \tag{12.1}$$

similar to the estimate obtained for $s < 1$. Since

$$\|u\|_{W^{[s],p}} \le C_{11}\|u\|_{W_p^{[s],p}},$$

it remains to estimate the integral in the right-hand side of (12.1):

$$\begin{aligned}&\sum_{i=1}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|\theta_i(x)D^\alpha u(x) - \theta_i(y)D^\alpha u(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy \\ &\quad\le \sum_{i=1}^{\infty} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^\alpha(\theta_i(x)u(x)) - D^\alpha(\theta_i(y)u(y))|^p}{|x-y|^{n+p\sigma}}\, dxdy \\ &\qquad + \sum_{i=1}^{\infty} C_{12} \sum_{|\beta|+|\gamma|=|\alpha|, |\gamma|<|\alpha|} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D^\beta\theta_i(x)D^\gamma u(x) - D^\beta\theta_i(y)D^\gamma u(y)|^p}{|x-y|^{n+p\sigma}}\, dxdy \\ &\quad\le \|u\|^p_{W_p^{s,p}} + C_{13}\|u\|^p_{W^{s-1,p}}.\end{aligned} \tag{12.2}$$

The second term in (12.2) is estimated similar to the estimate of J_i above. Thus

$$\|u\|_{W^{s,p}} \le C_{14}(\|u\|_{W_p^{s,p}} + \|u\|_{W^{s-1,p}}) \le C_{15}(\|u\|_{W_p^{s,p}} + \|u\|_{W^{[s],p}}) \le C_{16}\|u\|_{W_p^{s,p}}.$$

The theorem is proved. □

To study further properties of the spaces $W^{s,p}_q$ we use the integral definition of the norm in $W^{s,p}_q$:

$$\|u\|_{W^{s,p}_q} = \left(\int_{\mathbb{R}^n} \|u(\cdot)\phi(\cdot - y)\|^q_{W^{s,p}} dy\right)^{1/q}$$

(see Section 3). If $s = 0$ it becomes

$$\|u\|_{L^p_q} = \left(\int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |u(x)\phi(x-y)|^p dx\right)^{q/p} dy\right)^{1/q}.$$

We can replace $\phi(x)$ by the characteristic function of the unit ball. Then

$$\|u\|_{L^p_q} = \left(\int_{\mathbb{R}^n} \left(\int_{|x-y|\le 1} |u(x)|^p dx\right)^{q/p} dy\right)^{1/q}.$$

We will determine conditions on $u(x)$ to belong to the space L^p_q. We have

$$\|u\|^q_{L^p_q} = \int_{\mathbb{R}^n} \left(\int_{|x-y|\le 1} |u(x)|^p dx\right)^{q/p} dy = I_1 + I_2,$$

where

$$I_1 = \int_{|y|\le 2} \left(\int_{|x-y|\le 1} |u(x)|^p dx\right)^{q/p} dy, \quad I_2 = \int_{|y|>2} \left(\int_{|x-y|\le 1} |u(x)|^p dx\right)^{q/p} dy.$$

Let $u(x) = |x|^{-\alpha}$, $\alpha > 0$. Then

$$I_1 = \int_{|y|\le 2} \left(\int_{|x-y|\le 1} \frac{dx}{|x|^{\alpha p}}\right)^{q/p} dy \le \int_{|y|\le 2} \left(\int_{|x|\le 3} \frac{dx}{|x|^{\alpha p}}\right)^{q/p} dy.$$

If $\alpha p < n$, then

$$I_1 \le 2^n \omega_n \left(\frac{3^{n-\alpha p}\kappa_n}{n - \alpha p}\right)^{q/p},$$

where ω_n and κ_n are the volume and the surface of the unit sphere, respectively.

Consider I_2. Since $|y| > 2$, then $|x| \ge |y| - 1 \ge \frac{1}{2}|y|$. If $n < \alpha q$, then

$$I_2 \le \int_{|y|>2} \left(\int_{|x-y|\le 1} \frac{2^{\alpha p} dx}{|y|^{\alpha p}}\right)^{q/p} dy = \frac{2^n \omega_n \kappa_n}{\alpha q - n}.$$

We have proved the following lemma.

Lemma 12.4. *If $\alpha p < n < \alpha q$, then $u(x) = 1/|x|^\alpha \in L_q^p$.*

From this lemma we easily obtain the following proposition.

Proposition 12.5. *If for some $R > 0$,*

$$\int_{|x|\le R} |u(x)|^p dx < \infty$$

and $|u(x)| \le K|x|^{-\alpha}$ for $|x| > R$, where K is a positive constant and $\alpha q > n$, then $u \in L_q^p$.

In the remaining part of this section we construct an example of "bad" functionals (Remark 10.12) in the space $L_\infty^2(\mathbb{R})$. Consider the subspace E_{lim} of this space that consists of functions $u(x)$ for which there exists the limit

$$\phi(u) = \lim_{x\to+\infty} \int_x^{x+1} u(s)ds.$$

We verify that E_{lim} is closed in the norm

$$\|u\|_{L_\infty^2} = \sup_x \left(\int_x^{x+1} u^2(s)ds\right)^{1/2}.$$

Let $u_n \in E_{lim}$, $u_n \to u_0$ in L_∞^2. Put

$$z_n(x) = \int_x^{x+1} u_n(s)ds, \quad a_n = \phi(u_n).$$

Then

$$|z_n(x)| \le \left(\int_x^{x+1} u_n^2(s)ds\right)^{1/2} \le \|u_n\|_{L_\infty^2} \le M$$

for some positive constant M. The last inequality follows from the assumption that the sequence is convergent. The sequence $\phi(u_n)$ is fundamental. Denote by a_0 its limit. We will show that $z_0(x) \to a_0$. Indeed,

$$|z_0(x) - a_0| \le |z_n(x) - a_n| + |a_n - a_0| + |z_0(x) - z_n(x)|.$$

For any $\epsilon > 0$ we can choose N such that for any $n \ge N$, $|a_n - a_0| < \epsilon/3$, and $|z_0(x) - z_n(x)| < \epsilon/3$ for all $x \in \mathbb{R}^1$. For a fixed $n \ge N$, we can choose x_0 such that $|z_n(x) - a_n| < \epsilon/3$ for $x \ge x_0$. Therefore $|z_0(x) - a| < \epsilon$ for $x \ge x_0$. This proves the convergence. Thus, $u_0 \in E_{lim}$.

By the Hahn-Banach theorem we can extend the functional $\phi(u)$ to the whole space L_∞^2. For any $u \in D$, $\phi(u) = 0$.

13 Local operators

1. Operators in $\mathbb{R}^n$. Let E and F be local spaces, that is the spaces of distributions introduced in Section 1. We suppose that $D \subset E$, $D \subset F$, and D is dense in F.

Definition 13.1. An operator $A : E \to F$ is called local if for any $u \in E$ with a compact support the inclusion $\operatorname{supp} Au \subset \operatorname{supp} u$ holds.

Theorem 13.2. *If $A : E \to F$ is a bounded local operator, then $A^* : E^* \to F^*$ is also a bounded local operator.*

Proof. Let $v \in F^*$ be a function with compact support. We have to prove that $\operatorname{supp} A^*v \subset \operatorname{supp} v$. Suppose that it is not the case. Then there exists a point $x_0 \in R^n$ such that $x_0 \in \operatorname{supp} A^*v$, $x_0 \notin \operatorname{supp} v$. Let B be a closed ball with the center at x_0 such that $B \cap \operatorname{supp} v = \oslash$ and $f \in D$ be such that $\operatorname{supp} f \subset D$,

$$\langle A^*v, f\rangle \neq 0. \tag{13.1}$$

On the other hand,

$$\langle A^*v, f\rangle = \langle v, Af\rangle \tag{13.2}$$

and $\langle v, Af\rangle = 0$ since the support of Af belongs to B and it does not intersect the support of v. Here we use the density of D in F in order to approximate Af by functions from D with supports in B. This contradiction proves the theorem. $\square$

Theorem 13.3. *Let $A : E \to F$ be a local operator. Then*

$$A_{\mathrm{loc}}u = \sum_{i,j=1}^{\infty} \theta_j A(\theta_i u), \quad \forall u \in E_{\mathrm{loc}} \tag{13.3}$$

is a linear operator acting from E_{loc} to F_{loc}. Convergence of the series in (13.3) *is understood in the sense of distributions, and it does not depend on the choice of the partition of unity θ_i.*

Proof. Let

$$A_{m,n}u = \sum_{i=1}^{m}\sum_{j=1}^{n} \theta_j A(\theta_i u).$$

Since $u \in E_{\mathrm{loc}}$, then $\theta_i u \in E$ and $A(\theta_i u) \in F$. Moreover $\operatorname{supp} A(\theta_i u) \subset \operatorname{supp} \theta_i u \subset \theta_i$. Let $\phi \in D$. We have

$$\langle A_{m,n}u, \phi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} \langle A(\theta_i u), \theta_j \phi\rangle. \tag{13.4}$$

Denote by N the number of functions θ_i for which $\operatorname{supp}\theta_i \cap \operatorname{supp}\phi \neq \oslash$. Then the right-hand side of (13.4) contains no more than N^2 terms. Therefore we can pass to the limit in (13.4) as $m, n \to \infty$, and

$$\langle A_{\mathrm{loc}} u, \phi \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle A(\theta_i u), \theta_j \phi \rangle \tag{13.5}$$

for m and n sufficiently large (depending on $\operatorname{supp}\phi$).

We show next that $A_{\mathrm{loc}} u \in F_{\mathrm{loc}}$, that is $\psi A_{\mathrm{loc}} u \in F$ for any $\psi \in D$. We have

$$\langle \psi A_{\mathrm{loc}} u, \phi \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} \langle A(\theta_i u), \theta_j \psi \phi \rangle,$$

where m and n depend on $\operatorname{supp}\psi$ but do not depend on $\operatorname{supp}\phi$. Hence for n sufficiently large,

$$\psi A_{\mathrm{loc}} u = \psi \sum_{i=1}^{m} \sum_{j=1}^{n} \theta_j A(\theta_i u) = \sum_{i=1}^{m} \left(\psi \sum_{j=1}^{\infty} \theta_j \right) A(\theta_i u) = \psi \sum_{i=1}^{m} A(\theta_i u),$$

where m depends on $\operatorname{supp}\psi$. Since $A(\theta_i u) \in F$, then $A_{\mathrm{loc}} u \in F_{\mathrm{loc}}$.

The fact that A_{loc} is a linear operator follows directly from (13.5). It remains to show that (13.3) does not depend on the partition of unity. From (13.5)

$$\langle A_{\mathrm{loc}} u, \phi \rangle = \sum_{i=1}^{m} \langle A(\theta_i u), \phi \rangle \tag{13.6}$$

for all m sufficiently large. Let $\tilde{\theta}_i$ be another partition of unity. Then from (13.6) we obtain

$$\langle A_{\mathrm{loc}} u, \phi \rangle = \sum_{i=1}^{m} \left\langle A \left(\sum_{j=1}^{\infty} \tilde{\theta}_j \theta_i u \right), \phi \right\rangle = \sum_{i=1}^{m} \left\langle A \left(\sum_{j=1}^{n} \tilde{\theta}_j \theta_i u \right), \phi \right\rangle, \tag{13.7}$$

where n depends on $\operatorname{supp}\phi$ and does not depend on i since $\operatorname{supp} A(\tilde{\theta}_j \theta_i u) \subset \operatorname{supp} \tilde{\theta}_j \theta_i$ and $\langle A(\tilde{\theta}_j \theta_i u), \phi \rangle = 0$ if $\operatorname{supp}\theta_j \cap \operatorname{supp}\phi = \emptyset$. From (13.7)

$$\langle A_{\mathrm{loc}} u, \phi \rangle = \sum_{j=1}^{n} \left\langle A \left(\tilde{\theta}_j \sum_{i=1}^{m} \theta_i u \right), \phi \right\rangle = \sum_{j=1}^{n} \left\langle A \left(\tilde{\theta}_j \sum_{i=1}^{\infty} \theta_i u \right), \phi \right\rangle = \sum_{j=1}^{n} \langle A(\tilde{\theta}_j u), \phi \rangle$$

for n sufficiently large. This equality together with (13.6) show that A_{loc} is independent of the partition of unity. The theorem is proved. □

Definition 13.4. Operator $A_{\mathrm{loc}} : E_{\mathrm{loc}} \to F_{\mathrm{loc}}$ is called the extension of $A : E \to F$ to E_{loc}. Operator $A_q (1 \le q \le \infty)$ is a restriction of A_{loc} to E_q.

Theorem 13.5. *Let $A : E \to F$ be a bounded local operator. Then A_q is a bounded operator from E_q to F_q.*

Proof. We begin with the case $q = \infty$. Let θ_i be a partition of unity, $u \in E_\infty$. We have

$$\theta_i A_{\mathrm{loc}} u = \theta_i \sum_{j=1}^{m} A(\theta_j u)$$

for all m sufficiently large. Since $\operatorname{supp} A(\theta_j u) \subset \operatorname{supp} \theta_j u \subset \operatorname{supp} \theta_j$, then

$$\theta_i A_\infty u = \theta_i A_{\mathrm{loc}} u = \theta_i \sum_{j'} A(\theta_{j'} u),$$

where j' are all those j for which $\operatorname{supp} \theta_i \cap \operatorname{supp} \theta_j \neq \oslash$. Therefore

$$\begin{aligned}\|\theta_i A_\infty u\|_F &\leq \sum_{j'} \|\theta_i A(\theta_{j'} u)\|_F \leq \sum_{j'} \|\theta_i\|_{M(F)} \|A\| \ \|\theta_{j'} u\|_E \\ &\leq N \|A\| \ \|\theta_i\|_{M(F)} \|u\|_{E_\infty}.\end{aligned}$$

Let $\kappa = \sup_i \|\theta_i\|_{M(F)}$. Then

$$\|A_\infty u\|_{F_\infty} \leq \kappa N \|A\| \ \|u\|_{E_\infty}.$$

Consider next $1 \leq q < \infty$. We have

$$\theta_i A_q u = \theta_i A_{\mathrm{loc}} u = \theta_i \sum_{j'} A(\theta_{j'} u),$$

and for any integer m,

$$\begin{aligned}\sum_{i=1}^{m} \|\theta_i A_q u\|_F^q &= \sum_{i=1}^{m} \|\theta_i \sum_{j'} A(\theta_{j'} u)\|_F^q \leq \sum_{i=1}^{m} N^{q-1} \sum_{j'} \|\theta_i A(\theta_{j'} u)\|_F^q \\ &= N^{q-1} \sum_{j=1}^{\infty} \sum_{i=1}^{m} \|\theta_i A(\theta_j u)\|_F^q = N^{q-1} \sum_{j=1}^{\infty} \sum_{i'}^{m} \|\theta_{i'} A(\theta_j u)\|_F^q \\ &\leq N^{q-1} \sum_{j=1}^{\infty} \sum_{i'}^{m} \|\theta_{i'}\|_{M(F)^q} \|A(\theta_j u)\|_F^q \leq N^q \kappa^q \sum_{j=1}^{\infty} \|A(\theta_j u)\|_F^q \\ &\leq N^q \kappa^q \|A\|^q \sum_{j=1}^{\infty} \|\theta_j u\|_E^q = N^q \kappa^q \|A\|^q \|u\|_{E_q}^q.\end{aligned}$$

Here i' are all those i for which $\operatorname{supp} \theta_i \cap \operatorname{supp} \theta_j \neq \oslash$. The number of such i is not greater than N. Passing to the limit as $m \to \infty$, we get

$$\|A_q u\|_{F_q}^q \leq N^q \kappa^q \|A\|^q \|u\|_{E_q}^q.$$

Therefore

$$\|A_q u\|_{F_q} \leq N \kappa \|A\| \ \|u\|_{E_q}.$$

The theorem is proved. $\square$

2. Operators in Ω. Let Ω be a domain in $\mathbb{R}^n$, E be the space in Definition 13.1. The space $E(\Omega)$ is defined in Definition 9.1.

Definition 13.6. The space $E_q(\Omega)$ is defined as the set of those generalized functions from D'_Ω that are restrictions to Ω of generalized functions from E_q, $(1 \le q \le \infty)$. The norm in $E_q(\Omega)$ is given by the equality

$$\|u\|_{E_q(\Omega)} = \inf \|u^c\|_{E_q},$$

where the infimum is taken over all those $u^c \in E_q$, whose restriction to Ω coincide with u (cf. Definition 9.2).

Definition 13.7. Let $A : E \to F$ be a local bounded operator. Operator $A_q(\Omega)$, $(1 \le q \le \infty)$ is the restriction of A_q to $E_q(\Omega)$.

We discuss the last definition in more detail. Let $u \in E_q(\Omega)$. Then there exists $u^c \in E_q$ such that

$$\langle u^c, \phi\rangle = \langle u, \phi\rangle, \ \ \forall \phi \in D_\Omega.$$

Then $A_q(\Omega)u$ is the restriction of $A_q u^c$ to Ω.

We show that $A_q(\Omega)u$ does not depend on the extension u^c. Indeed, let u_1 and u_2 be two extensions of u. Then

$$\langle u_1, \phi\rangle = \langle u_2, \phi\rangle = \langle u, \phi\rangle, \ \ \forall \phi \in D_\Omega.$$

Let $z = u_1 - u_2$. Then $\langle z, \phi\rangle = 0$, $\forall \phi \in D_\Omega$. This means that the support $\operatorname{supp} z$ of z belongs to the complement $C\Omega$ of the domain Ω. By the definition of local operators, $\operatorname{supp} Az \subset C\Omega$. Therefore $\langle Az, \phi\rangle = 0$, $\forall \phi \in D_\Omega$, that is

$$\langle Au_1, \phi\rangle = \langle Au_2, \phi\rangle = 0, \ \ \forall \phi \in D_\Omega.$$

Hence Au_1 and Au_2 coincide as elements of $F(\Omega)$. Thus $A_q(\Omega)$ acts from $E_q(\Omega)$ to $F(\Omega)$.

Theorem 13.8. *Operator $A_q(\Omega)$ is bounded as acting from $E_q(\Omega)$ to $F_q(\Omega)$.*

Proof. Let $u \in E_q(\Omega)$. By Definition 13.6 there exists $u^c \in E_q$ such that

$$\|u^c\|_{E_q} \le 2\|u\|_{E_q(\Omega)}.$$

Let $v^c = A_q u^c$. Then $v^c \in F_q$. We have

$$\|v^c\|_{F_q} = \|A_q u^c\|_{F_q} \le \|A_q\| \, \|u^c\|_{E_q} \le 2\|A_q\| \, \|u\|_{E_q(\Omega)}.$$

By definition, $A_q(\Omega)u$ is the restriction of v^c to Ω. Hence

$$\|A_q(\Omega)u\|_{F_q(\Omega)} \le \|v^c\|_{F_q} \le 2\|A_q\| \, \|u\|_{E_q(\Omega)}.$$

Therefore $A_q(\Omega) : E_q(\Omega) \to F_q(\Omega)$ is a bounded operator and $\|A_q(\Omega)\| \le 2\|A_q\|$. The theorem is proved. $\square$

3. Boundary operators. Let $\partial\Omega$ be a C^l manifold, where $l \geq 1$ is an integer. As in Section 8 we denote by D_l the space of all functions $\phi \in C^l(\mathbb{R}^n)$ with compact support. Let E be a local space and D_l be dense in E. We denote by $D_l(\Omega)$ the restriction of C^l to Ω. Since D_l is dense in E, then $D_l(\Omega)$ is dense in $E(\Omega)$.

For any $u \in E(\Omega)$ we can define its trace $\hat{u}$ on $\partial\Omega$. We first define the norm of traces of functions from $D_l(\Omega)$. Let $\phi \in D_l(\Omega)$. Then ϕ is defined on $\partial\Omega$. We put

$$\|\phi\|_{E(\partial\Omega)} = \inf \|\phi^c\|_{E(\Omega)}, \tag{13.8}$$

where the infimum is taken over all $\phi^c \in E(\Omega)$ such that $\phi^c(x) = \phi(x)$ for $x \in \partial\Omega$.

Definition 13.9. The space $E(\partial\Omega)$ is the closure of $D_l(\partial\Omega)$ in the norm (13.8), where $D_l(\partial\Omega)$ is the space of traces of $D_l(\Omega)$ on $\partial\Omega$.

Let $u \in E(\Omega)$. Then there exists a sequence $\phi_n \to u$ in $E(\Omega)$, $\phi_n \in D(\Omega)$. Then $\hat{u} = \lim_n \hat{\phi}_n$ in the norm (13.8), where $\hat{\phi}_n$ is the trace of ϕ_n. Obviously, $\hat{u}$ does not depend on the choice of ϕ_n.

Example 13.10. Let $E(\Omega) = W^{s,p}(\Omega)$. If $s > 1/p$, then $E(\partial\Omega) = W^{s-1/p,p}(\partial\Omega)$. Let $E(\Omega) = L^2(\Omega)$. Applying formally the definition we obtain $\|\phi\|_{E(\partial\Omega)} = 0$ for any ϕ. Therefore the norm of any function defined on $\partial\Omega$ equals zero. This means that we can formally define the space of traces if we consider equivalence classes of functions, but this definition has no sense because the space contains only the zero element. Nevertheless this definition can be useful since it allows us to consider the general case.

Definition 13.11. Linear operator $B : E \to F(\partial\Omega)$ is called local if for any $u \in E$ we have $\operatorname{supp} Bu \subset \operatorname{supp} u$, where $\operatorname{supp} Bu$ is taken in $\partial\Omega$.

It follows from the definition that if $\operatorname{supp} u \cap \partial\Omega = \oslash$, then $\operatorname{supp} Bu = \oslash$. Hence $Bu = 0$ as an element of $F(\partial\Omega)$. Consider the operator

$$B^* : (F(\partial\Omega))^* \to E^*.$$

We suppose that D is dense in E, and $D_l(\partial\Omega)$ is dense in $F(\partial\Omega)$.

Theorem 13.12. *Let $B : E \to F(\partial\Omega)$ be a bounded local operator. Then $B^* : (F(\partial\Omega))^* \to E^*$ is also a bounded local operator.*

The proof of this theorem is similar to the proof of Theorem 13.2. It follows from this theorem that for any $v \in (F(\partial\Omega))^*$, $\operatorname{supp} B^*v \in \partial\Omega$.

Definition 13.13. Linear operator $B : E(\Omega) \to F(\partial\Omega)$ is called local if for any $u \in E(\Omega)$ we have $\operatorname{supp} Bu \subset \operatorname{supp} u$.

Theorem 13.14. *Let $B : E(\Omega) \to F(\partial\Omega)$ be a bounded local operator. Then $B^* : (F(\partial\Omega))^* \to (E(\Omega))^*$ is also a bounded local operator.*

The proof of this theorem is similar to the proof of Theorem 13.2.

Elliptic Partial Differential Equations
Volume 1: Fredholm Theory of Elliptic Problems in Unbounded Domains
Volpert, V.
2011, XVIII, 642 p., Hardcover
ISBN: 978-3-0346-0536-6
A product of Birkhäuser Basel