Preface

Investigation of elliptic partial differential equations began more than two centuries ago with the mathematical theories of gravimetry, fluid dynamics, electrostatics and heat conduction due to works by Euler, Laplace, Lagrange, Poisson, Fourier, Green and Gauss. In the second half of the XIXth century, Schwarz, Neumann, Harnack, Poincaré and Picard developed efficient methods to study existence of solutions of boundary value problems for linear and nonlinear equations. The basis of the spectral theory was established by Schwarz, Poincaré, Steklov and Zaremba. Elliptic equations had important development in the XXth century. Its beginning can be related to the mathematical congress in Paris in 1900 where Hilbert had posed 23 problems, including the 18th and 19th devoted to elliptic boundary value problems. They stimulated works by Bernstein followed by Caccioppoli, Schauder, Leray and other authors who established the foundations of modern analysis. The method of Fredholm integral equations was developed in 1900–1903. It had important applications to elliptic partial differential equations.

Today's understanding of elliptic partial differential equations begins with a priori estimates of solutions of linear problems. They provide normal solvability, the Fredholm property and solvability conditions. Important achievements were related to index theories. The structure of the spectrum of linear elliptic problems provides their sectorial property. It allows introduction of analytic semi-groups and investigation of parabolic problems. The properties of linear operators play an important role for investigation of nonlinear problems. In particular, for construction of the topological degree, which is a powerful tool to study existence and bifurcations of solutions. Many of these methods and results are related to the Fredholm property of elliptic operators. Little can be done in elliptic partial differential equations without it. This determines the presentation of the material of the book around this fundamental property of elliptic problems.

Under the influence of numerous applications, the theory of elliptic partial differential equations continues to attract much attention. During the last several decades, travelling wave solutions of parabolic systems have been intensively studied in relation with combustion problems, population dynamics and many other applications. Travelling waves are solutions of elliptic problems in unbounded domains. These studies intensified the development of the theory of elliptic operators in unbounded domains.

It should be noted that the classical theory of elliptic problems has been developed in the case of bounded domains with a sufficiently smooth boundary. In this case, their Fredholm property is provided by the ellipticity condition, proper ellipticity and the Lopatinskii condition. In the case of unbounded domains, these conditions are no longer sufficient and the Fredholm property may not be satisfied. This is related to a lack of compactness. In order to satisfy the Fredholm property, we need to impose an additional condition, which can be formulated in terms of limiting problems. They characterize the behavior of the operator at infinity and determine the location of the essential spectrum.

In this book, we present a systematic investigation of general elliptic problems applicable both for bounded and unbounded domains. We pay more attention to the case of unbounded domains which is not yet sufficiently well presented in the existing literature. We introduce and use in essential ways some special function spaces well adapted for problems in unbounded domains. They generalize Sobolev spaces by specifying the behavior of functions at infinity. Another point we emphasize is related to limiting domains. This notion is necessary to define limiting operators. Moreover, there are two types of limiting domains. One of them determines the Fredholm property, another one some properties of the index. The proof of the Fredholm property is based on the properties of function spaces, on limiting domains and operators and on some special a priori estimates.

If the operator satisfies the Fredholm property, then its index is well defined. Computation of the index is well known in the case of bounded domains. It is also possible for some classes of operators in unbounded domains. We will develop a new method based on approximation of unbounded domains by a sequence of bounded domains. Under some conditions formulated in terms of limiting operators, the index in the sequence of bounded domains stabilizes to the index in the unbounded domain.

Elliptic operators with a parameter is an important class of operators essentially used in the theory of elliptic and parabolic problems. They are related to sectorial operators and to analytic semi-groups. We will extend the theory of elliptic operators with a parameter to general elliptic problems in unbounded domains.

If limiting problems have nonzero solutions, then the operator does not satisfy the Fredholm property, and the usual solvability conditions are not applicable. We introduce a class of operators, weakly non-Fredholm operators, for which it appears to be possible to formulate solvability conditions. In some cases, these conditions are similar to the usual ones and consist in orthogonality to solutions of the homogeneous adjoint problem. In some other cases, the solvability conditions are different.

Methods of nonlinear analysis, such as asymptotic methods, bifurcation theory or topological degree are based on the solvability conditions. We will define the topological degree for Fredholm and proper operators with the zero index and will discuss some of its applications including the Leray-Schauder method and bifurcations of solutions.

The book is organized as follows. In the first chapter we present a brief introduction to the classical theory of elliptic problems and to the theory of elliptic problems in unbounded domains. Chapters 2–8 are devoted to the theory of general linear elliptic problems (a priori estimates, normal solvability and the Fredholm property, operators with a parameter, index). In many cases we present it directly for unbounded domains, keeping in mind that it remains valid and even simpler for bounded domains. In the next chapter we deal with second-order operators in cylinders where some additional, more explicit methods can be used. Non-Fredholm operators are discussed in Chapter 10. The theory of linear operators will be used in Chapter 11 devoted to nonlinear Fredholm operators. Infinite-dimensional discrete operators are discussed in the supplement. Historical and bibliographical comments presented at the end of this monograph can help to acquire a general view of the theory of elliptic equations, its evolution and perspectives.

The presentation of the results is mostly self-contained. We should note that some proofs are rather technical. However, the formulations of the results and their application are quite clear. The results of the first volume will be used in the second volume to study reaction-diffusion problems.

Most of the results of this book devoted to linear and nonlinear elliptic problems in unbounded domains were obtained in our works with Aizik Volpert. We began to write this book together. It concludes many years of our collaboration.

http://www.springer.com/978-3-0346-0536-6

Elliptic Partial Differential Equations
Volume 1: Fredholm Theory of Elliptic Problems in Unbounded Domains
Volpert, V.
2011, XVIII, 642 p., Hardcover
ISBN: 978-3-0346-0536-6
A product of Birkhäuser Basel