

Preface

V názvu té knihy musí být slovo *struktura*.

Petr Vopěnka, 1975.

The title of this book must contain the word *structure*.

We must distinguish at least four periods – sometimes overlapping – in the history of investigation of the real numbers.

The first period, devoted to understanding the real numbers system, began as early as about 2000 B.C. in Mesopotamia, then in Egypt, India, China, Greece (Pythagorians showed that $\sqrt{2}$ is an irrational), through Eudoxus, Euclid, Archimedes, Fibonacci, Euler, Bolzano and many others, and continued till the second half of the nineteenth century. Moris Kline [1972], p. 979 says that “one of the most surprising facts in the history of mathematics is that the logical foundation of the real numbers system was not erected until the late nineteenth century”.

The rigorization of analysis forced the beginning of the second period, concentrated in the second half of the nineteenth century, and primarily devoted to an exact definition of the real numbers system. The investigations showed that mathematicians needed, as a framework for such a definition, a rigorous theory of infinity. The Zermelo – Fraenkel set theory **ZFC** was accepted as the best solution. Several independent definitions of reals in the framework of set theory turned out to be equivalent and, as usual in mathematics, that was a strong argument showing that the exact definition of an intuitive notion was established correctly.

Establishing an exact notion of the real numbers system, mathematicians began in the third period to intensively study the system. Essentially there were three possibilities: to study the algebraic structure of the field of reals, to study the subsets of the reals related to the topological and measure theoretic properties of reals induced by the order – in this case I speak about the real line instead of the system of real numbers – and finally, taking into account both the algebraic and topological (or measure theoretical) properties.

The fourth period started by arousing many open unanswered questions of the later one. It turned out that they are closely connected to set theoretical questions which were unanswered in set theory. By the invention of forcing as a

method for showing the non-provability of a statement in **ZFC**, mathematicians obtained a strong tool to show that many questions concerning the topological and the measure theoretical structure of the real line are undecidable, or at least non-provable, in the considered set theory. The forcing method concerns essentially questions connected with infinite sets. Since the algebraic structure (to be an algebraic number, to be linearly independent over rationals, to be a prime number, etc.) is usually connected with the properties of finite sets, forcing hardly can contribute to a solution of an algebraic problem.

Focusing on this historical point of view, in 1975, I began to work on the Slovak version of a book collecting the main results of the above mentioned second, third and mainly the fourth period. I had previously presented the design of the book to participants in the Prague Set Theory Seminar. After my presentation Petr Vopěnka spontaneously stated the sentence quoted above. And I immediately accepted this proposal that provided an appropriate emphasis in the title on the book's content.

When the book appeared, in 1979, I was pleasantly surprised by the interest it generated in countries where students understood Slovak (which included the former Czechoslovakia, Poland, and some other particular locations, e.g., Hungary). At the end of 1987 I was further surprised to receive permission¹ to publish an English edition abroad. Immediately I began to work on an English version of my book. After the political events in Czechoslovakia in 1989, I could not refuse the (at least moral) responsibility to contribute to academic politics and my work on the English language version of the book was interrupted. When the monograph by T. Bartoszyński and H. Judah [1995] appeared, I thought my rôle in writing about the structure of the real line was finished. However, at the beginning of the 21st Century, several colleagues, mainly Polish, urged me to prepare a new edition of the Slovak version of "The Structure of the Real Line". In March 2003, I was invited to participate in the Boise Extravaganza in Set Theory at Boise State University in Idaho. After the conference I visited Tomek Bartoszyński in his office and was surprised to find my Slovak book open on his desk. This started my reflection, finishing with the conclusion that a book with the intention of my Slovak edition is not a rival book to Bartoszyński and Judah's book but rather a complementary, maybe, useful monograph. This presented a final – convincing – reason for my decision to prepare a new – significantly revised – edition of "The Structure of the Real Line".

I tried to follow the spirit of the Slovak edition. I shall not discuss the history of the real system. Since I present the main consequences of the Axiom of Determinacy **AD**, that contradicts the Axiom of Choice **AC**, in the basic parts of the book I try to avoid any use of **AC**, if possible, or, at least to replace it by the Weak Axiom of Choice **wAC**, that is a consequence of **AD**. Moreover, for some readers this may be interesting. In Chapter 1, I briefly describe the framework

¹Let me remind the reader, that I lived in a country where everything was strongly controlled by a political party.

for the mathematical treatment of infinity – the Zermelo-Fraenkel set theory **ZF**. Then I sketch the main topological results in the above mentioned way. A precise definition of the real line is given in Chapter 2. It is shown that the definition leads to a unique object (up to a mathematical identification) and the set theoretical framework implies its existence. Chapters 3 to 8 contain mainly the rather classical theory of the structure of the real line – the study of properties of subsets of reals, which is important from the point of view of topology and measure theory. Chapter 7 is devoted to the phenomena of the measure – category duality. Finally, with overlapping topics, Chapters 7 to 10 deal with reductions of many important problems of the structure of the real line to the sentences of set theory, which the forcing method already proved are undecidable, or at least non-provable in set theory. To make the book self-contained, I wrote an Appendix containing miscellaneous, in my opinion, necessary material. I here recall all notions and basic facts of set theory and algebra that I use in the book. Then I present a short introduction to the metamathematics of set theory. Finally, I present the main results obtained (mainly) by the forcing method in the last fifty years, which I needed to answer important questions about the structure of the real line. I tried to attribute each result to its author. Any new notion or denotation can be found in the very large Index or Index of notations.

In the introduction of each chapter I try to explain its content and mainly, whether a reader should read a particular section immediately or before reading another sections. Especially, I suppose that a reader should start with reading Section 1.1 and then, she/he can begin to read any chapter and then go to the results needed for understanding of the presented material with the help of the indexes. Each section is supplemented by a series of exercises that contains a great deal of supplementary information concerning the topics of the section.

I am deeply satisfied that the book is being published in the series *Monografie Matematyczne*. For many years in Czechoslovakia we had found it difficult to obtain scientific information. The main source were the (often illegal) translations of English language books and articles in Russian and, in the topics of interest to me, Polish books that were mainly published as *Monografie Matematyczne*. So I consider it a great honor to contribute to this most prestigious series.

I would like to thank those who contributed to the preparation of the manuscript. Mirko Repický advised me as a \TeX specialist. Peter Eliaš helped me with presentation of results related to the thin sets of harmonic analysis. My postgraduate students, former or current, Jozef Haleš, Michal Staš and Jaroslav Šupina, read the manuscript, discovering and correcting many errors, both typographical and factual. They contributed significantly to the correctness of the exercises.

I wish to express my gratitude to the Institute of Mathematics of Pavol Jozef Šafárik University at Košice that created good conditions for me while I was writing the book. My work was supported by grants 1/3002/06 and 1/0032/09 of the Slovak Grant Agency VEGA.

My intention for the book was to survey progress in the study of the real line over the period encompassing the highly productive end of one century and the beginning of another. I hope that this effort will prove to be a source of useful and stimulating information for a wide variety of mathematicians.

Košice, October 1, 2010

Lev Bukovský



<http://www.springer.com/978-3-0348-0005-1>

The Structure of the Real Line

Bukovský, L.

2011, XIV, 542 p., Hardcover

ISBN: 978-3-0348-0005-1

A product of Birkhäuser Basel