

Chapter 1

Introduction to Poncelet Porisms

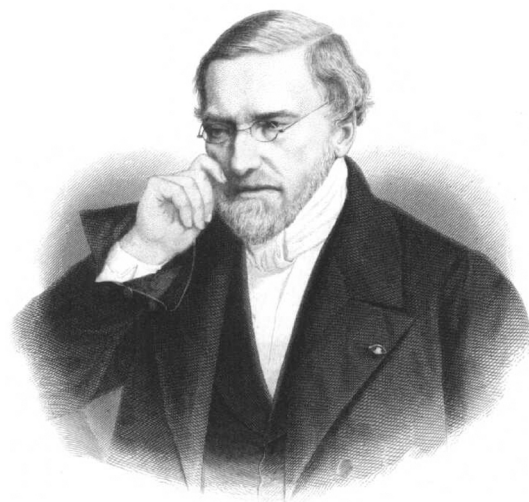


Figure 1.1: Jean Victor Poncelet

“One of the most important and also most beautiful theorems in classical geometry is that of Poncelet (...) His proof was synthetic and somewhat elaborate in what was to become the predominant style in projective geometry of last century. Slightly thereafter, Jacobi gave another argument based on the addition theorem for elliptic functions. In fact, as will be seen below, the Poncelet theorem and addition theorem are essentially equivalent, so that at least in principle Poncelet gave a synthetic derivation of the group law on an elliptic curve. Because of the appeal of the Poncelet theorem it seems reasonable to look for higher-dimensional analogues... Although this has not yet turned out to be the case in the Poncelet-type problems...”

These introductory words from [GH1977], written by Griffiths and Harris exactly 30 years ago, serve as a motto of the present book.

In a few years, we are going to reach a significant anniversary, the bicentennial of Jean Victor Poncelet's proof of one of the most beautiful and most important theorems of projective geometry. As is well known, he proved it during his captivity in Russia, in Saratov in 1813, after Napoleon's wars against Russia. The first proof was in a sense an analytic one. In 1822, Poncelet published another, purely geometric, synthetic proof in his *Traité des propriétés projectives des figures* [Pon1822]. Suppose that two ellipses are given in the plane, together with a closed polygonal line inscribed in one of them and circumscribed about the other one. Then, Poncelet's theorem states that infinitely many such closed polygonal lines exist – every point of the first ellipse is a vertex of such a polygon. Besides, all these polygons have the same number of sides. Later, using the addition theorem for elliptic functions, Jacobi gave another proof of the theorem in 1828 (see [Jac1884a]). Essentially, Poncelet's theorem is equivalent to the addition theorems for elliptic curves and his proof represents a synthetic way of deriving the group structure on an elliptic curve. Another proof of Poncelet's theorem, in a modern, algebro-geometrical manner, was done quite recently by Griffiths and Harris (see [GH1977]). There, they also presented an interesting generalization of the Poncelet theorem to the three-dimensional case, considering polyhedral surfaces both inscribed and circumscribed about two quadrics.

If we have in mind the geometric interpretation of the group structure on a cubic (see Figure 1.2), then the question of finding an analogous construction of the group structure in higher genera arises.

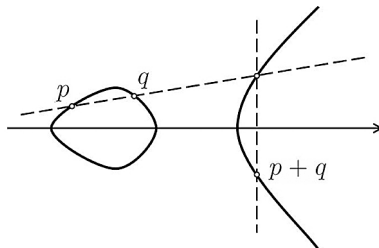


Figure 1.2: The group law on the cubic curve

Thus, thirty years ago, Griffiths and Harris announced a program of understanding higher-dimensional analogues of Poncelet-type problems and a synthetic approach to higher genera addition theorems.

The main aim of the present book is to report on progress made in settling and completing of this program. We will also present in a quite systematic way the most important results and ideas around Poncelet's theorem, both classical and modern, together with their historical origins and natural generalizations.

A natural question connected with Poncelet's theorem is to find an analytical condition determining, for two given conics, if an n -polygon inscribed in one and circumscribed about the second conic exists. In a short paper [Cay1854], Cayley derived such a condition in 1853, using the theory of Abelian integrals. He had dealt with Poncelet's porism in a number of other papers [Cay1853, Cay1855, Cay1857, Cay1858, Cay1861]. Inspired by [Cay1854], Lebesgue translated Cayley's proof to the language of geometry. Lebesgue's proof of Cayley's condition, derived by methods of projective geometry and algebra, can be found in his book *Les coniques* [Leb1942]. In modern settings, Griffiths and Harris derived Cayley's theorem by finding an analytical condition for points of finite order on an elliptic curve [GH1978a].

It is worth emphasizing that Poncelet, in fact, proved a statement that is much more general than the famous Poncelet theorem [Ber1987, Pon1822], then deriving the latter as a corollary. Namely, he considered $n + 1$ conics of a pencil in the projective plane. If there exists an n -polygon with vertices lying on the first of these conics and each side touching one of the other n conics, then infinitely many such polygons exist. We shall refer to this statement as the *Full Poncelet theorem* and call such polygons *Poncelet polygons*.

A nice historical overview of the Poncelet theorem, together with modern proofs and remarks is given in [BKOR1987]. Various classical theorems of Poncelet type with short modern proofs are reviewed in [BB1996], while the algebro-geometrical approach to families of Poncelet polygons via modular curves is given in [BM1993, Jak1993].



Figure 1.3: Elliptical billiard table

The Poncelet theorem has an important mechanical interpretation. An *Elliptical billiard* [KT1991, Koz2003] is a dynamical system where a material point of the unit mass is moving under inertia, or in other words, with a constant velocity inside an ellipse and obeying the reflection law at the boundary, i.e., having congruent impact and reflection angles with the tangent line to the ellipse at any bouncing point. It is also assumed that the reflection is absolutely elastic. It is

well known that any segment of a given elliptical billiard trajectory is tangent to the same conic, confocal with the boundary [CCS1993]. If a trajectory becomes closed after n reflections, then the Poncelet theorem implies that any trajectory of the billiard system, which shares the same caustic curve, is also periodic with the period n .

The Full Poncelet theorem also has a mechanical meaning. The configuration dual to a pencil of conics in the plane is a family of confocal second-order curves [Arn1978]. Let us consider the following, a little bit unusual billiard. Suppose n confocal conics are given. A particle is bouncing on each of these n conics respectively. Any segment of such a trajectory is tangent to the same conic confocal with the given n curves. If the motion becomes closed after n reflections, then, by the Full Poncelet theorem, any such trajectory with the same caustic is also closed.

The statement dual to the Full Poncelet theorem can be generalized to the d -dimensional space [CCS1993] (see also [Pre1999, Pre]). Suppose vertices of the polygon $x_1 x_2 \dots x_n$ are respectively placed on confocal quadric hypersurfaces Q_1, Q_2, \dots, Q_n in the d -dimensional Euclidean space, with consecutive sides obeying the reflection law at the corresponding hypersurface. Then all sides are tangent to some quadrics Q^1, \dots, Q^{d-1} confocal with $\{Q_i\}$; for the hypersurfaces $\{Q_i, Q^j\}$, an infinite family of polygons with the same properties exist.

But, more than one century before these quite recent results, in 1870, Darboux proved the generalization of Poncelet's theorem for a billiard within an ellipsoid in the three-dimensional space [Dar1870]. It seems that his work on this topic is completely forgotten nowadays.

Darboux was occupied by Poncelet's theorem for almost 50 years, and many of his results and ideas, in one way or another, are going to be incorporated throughout the book.

Let us mention that in the same year, 1870, appeared another very important work: [Wey1870] of Weyr. It can be treated as the historic origin of the modern Griffiths–Harris Space Poncelet Theorem. A few years later, Hurwitz used Weyr's results to get a new proof of the standard Poncelet theorem (see [Hur1879]).

It is natural to search for a Cayley-type condition related to generalizations of the Poncelet theorem. Such conditions for the billiard system inside an ellipsoid in the Euclidean space of arbitrary finite dimension were derived in [DR1998a, DR1998b]. In recent papers [DR2004, DR2005, DR2006b, DR2006a], algebro-geometric conditions for existence of periodical billiard trajectories within k quadrics in d -dimensional Euclidean space were derived. The second important goal of these papers, actually for the present book as well, was to offer a thorough historical overview of the subject with a special attention on the detailed analysis of ideas and contributions of Darboux and Lebesgue. While Lebesgue's work on this subject has been, although rarely, mentioned by experts, on the other hand, it seems to us that relevant Darboux's ideas are practically unknown in contemporary mathematics. We give natural higher-dimensional generalizations of the ideas

and results of Darboux and materials presented by Lebesgue, providing the proofs also in the low-dimensional cases if they were omitted in the original works. Besides other results, interesting new properties of pencils of quadrics are established – see Theorems 5.30 and 5.33. The latter gives a nontrivial generalization of the Basic Lemma from Lebesgue’s book.

In our presentation of the development connected with the Griffiths–Harris program, we follow the recent paper [DR2008]. We present a geometric construction generalizing a summation procedure on the elliptic curve for the case of hyperelliptic Jacobians. These ideas are continuations of those of Reid, Donagi and Knörrer, see [Rei1972], [Knö1980], [Don1980]. Further development, realization, simplification and visualization of their constructions is obtained by using the ideas of billiard dynamics on pencils of quadrics developed in [DR2004].

The projective geometry nucleus of that billiard dynamics is the Double Reflection Theorem, see Theorem 5.27 below. There are four lines belonging to a certain linear space and forming *the Double reflection configuration*: these four lines reflect to each other according to **the billiard law at some confocal quadrics**.

In higher genera, we construct the corresponding, more general, billiard configuration, again by using the Double Reflection Theorem. This configuration, which we call *s-brush*, is in one of the equivalent formulations, a certain billiard trajectory of length $s \leq g$ and the sum of s elements in the brush is, roughly speaking, the final segment of that billiard trajectory.

The milestones of this presentation are [Knö1980] and [DR2004] and the key observation, from [DR2008], giving a link between them is *that the correspondence $g \mapsto g'$ in Lemma 4.1 and Corollary 4.2 from [Knö1980] is the billiard map at the quadric Q_λ* .

Thus, after observing and understanding the billiard nature behind the constructions of [Rei1972], [Knö1980], [Don1980], we become able to use the billiard tools to construct and study hyperelliptic Jacobians, and particularly their real part. It may be realized as a set T of lines in \mathbf{R}^d simultaneously tangent to given $d - 1$ quadrics Q_1, \dots, Q_{d-1} of some confocal family. It is well known that such a set T is invariant under the billiard dynamics determined by quadrics from the confocal family. By using the Double Reflection Theorem and some other billiard constructions we construct a group structure on T , *a billiard algebra*. The usage of billiard dynamics in algebro-geometric considerations appears to be, as usual in such a situation, of a two-way benefit. We derive a fundamental property of T : *any two lines in T can be obtained from each other by at most $d - 1$ billiard reflections at some quadrics from the confocal family*. The last fact opens a possibility to introduce new hierarchies of notions: of *s-skew lines in T* , $s = -1, 0, \dots, d - 2$ and of *s-weak Poncelet trajectories of length n* . The last are natural quasi-periodic generalizations of Poncelet polygons. By using billiard algebra, we obtain complete analytical descriptions of them. These results are further generalizations of our recent description of Cayley’s type of Poncelet polygons in arbitrary dimension, see [DR2006b]. Let us emphasize that the method used in [DR2008], based on billiard

algebra, differs from the methods exposed in [DR2006b], see also [DR2010]. Both of the methods will be presented in the sequel.

The interrelations between billiard dynamics, subspaces of intersections of quadrics and hyperelliptic Jacobians developed in [DR2008], enable us to obtain higher-dimensional generalizations of several classical results. To demonstrate the power of the methods, generalizations of Weyr’s Poncelet theorem (see [Wey1870]) and also the Griffiths–Harris Space Poncelet theorem (see [GH1977]) in arbitrary dimension are derived and presented here. We also give an arbitrary-dimensional generalization of the Darboux theorem [Dar1914].

Let us mention at the end of a brief outline of main results which are going to be presented here, that the line we are going to establish and follow, is to demonstrate the deep intimate relationship between on one hand general hyperelliptic Jacobians and integrable billiard systems generated by pencils of quadrics on the other hand. This can be seen as a very simple and specialized level of general ideology of integrable systems which culminated with the so-called Novikov’s conjecture, solved by Shiota in 1985.

Let us recall that Novikov’s conjecture demonstrates the deepest relationship between the theory of integrable dynamical systems and theory of algebraic curves. It solved a century old, general and important Riemann–Schottky problem of description of period matrices of Jacobians among Riemannian matrices through the solutions of the Kadomtsev–Petviashvili integrable hierarchy.

There is another, very important connection of our subject with some of the most prominent parts of contemporary mathematics.

The Euler–Chasles correspondences, or symmetric (2-2)-correspondences play one of the main roles in our exposition. They were used by Jacobi, then by Trudi [Tru1853, Tru1863] and finally, Darboux extended their use in the theory of Poncelet porisms essentially.

One of the central objects in mathematical physics in the last 25 years is the R -matrix, or the solution $R(t, h)$ of the quantum Yang–Baxter equation

$$R^{12}(t_1 - t_2, h)R^{13}(t_1, h)R'^{23}(t_2, h) = R^{23}(t_2, h)R^{13}(t_1, h)R^{12}(t_1 - t_2, h),$$

as a paradigm of modern understanding of the addition relation. Here t is a so-called *spectral parameter* and h is the *Planck constant*. If the h dependence satisfies the quasi-classical property $R = I + hr + O(h^2)$, the classical r -matrix r satisfies the classical Yang–Baxter equation. Classification of the solutions of the classical Yang–Baxter equation was done by Belavin and Drinfeld in 1982 [BD1982]. The problem of classification of the quantum R -matrices is still open. However, some important results of classification have been obtained in the basic 4×4 case by Krichever in [Kri1981], and following his ideas in [Dra1992a, Dra1993].

Krichever in [Kri1981] applied the idea of “finite-gap” integration to the theory of the Yang equation:

$$R^{12}L^{13}L'^{23} = L'^{23}L^{13}R^{12}.$$

The principal objects that are considered are $2n \times 2n$ matrices L , understood as 2×2 matrices whose elements are $n \times n$ matrices; $L = L_{j\beta}^{i\alpha}$ is considered as a linear operator in the tensor product $\mathbf{C}^n \otimes \mathbf{C}^2$. The theorem from [Kri1981] uniquely characterizes them by the following spectral data:

1. the vacuum vectors, i.e., vectors of the form $X \otimes U$, which L maps to vectors of the same form $Y \otimes V$, where $X, Y \in \mathbf{C}^n$ and $U, V \in \mathbf{C}^2$;
2. the vacuum curve $\Gamma : P(u, v) = \det L = 0$, where $L_j^i = V^\beta L_{j\beta}^{i\alpha} U_\alpha$, $(V^\beta) = (1, -v)$, $X_n = Y_n = U_2 = V_2 = 1$; $U_1 = u$, $V_1 = v$;
3. the divisors of the vector-valued functions $X(u, v)$, $Y(u, v)$, $U(u, v)$, $V(u, v)$, which are meromorphic on the curve Γ .

It appeared that vacuum curves in 4×4 case are exactly Euler–Chasles correspondences. The Yang–Baxter equation itself provides the condition of commutation of the two Euler–Chasles correspondences. The classification follows by application of the Euler theorem in the general case, and by studying possible degenerations.

This is practically the same picture we meet in the study of the Poncelet theorem. The hope is that our study of higher-dimensional analogues of the Poncelet theorem could provide us the intuition that will help us in classification of higher-dimensional solutions of the Yang–Baxter equation.

Thus, we include the story about Krichever’s algebro-geometric approach to 4×4 solutions of the Quantum Yang–Baxter equation in the last chapter. We explained there the relationship between the Poncelet theorem for a triangle and the Darboux theorem from one side and Krichever’s commuting relation of vacuum curves from another side (see Theorem 10.12). We underline connection of classification results for 4×4 R -matrices to the classification of pencils of conics, see Theorem 10.12 and Proposition 10.13. Pencils of conics and their classification played a crucial role in previous chapters. Finally, we point out a sort of billiard construction within the Algebraic Bethe Ansatz associated to four-dimensional R -matrices, see Lemma 10.14 and Theorem 10.15.

The Poncelet theorem is usually called the Poncelet *porism*. Let us give some explanation of the meaning of the word **porism**. It has roots in ancient Greek mathematics, and it is usually translated in two ways. The first one is *lemma* or *corollary*. The second one goes deeper into the philosophy of ancient Greek mathematics. Scientists of that time used to divide mathematical statements into two categories:

- *Theorems* – where something has to be proven, and
- *Problems* – where something needs to be constructed.

Nevertheless, they recognized the third, intermediate, class as well, called *Porisms*, directed to finding what is proposed. The most famous collection of porisms of ancient times was the book *The Porisms* of Euclid. Unfortunately, this work is lost, and the trace which survived leads through *The Collection* of Pappus of

Alexandria. Even then, there was much discussion about the definition of the notion of porism as well as about Euclid's porisms. These discussions continue today. In the XVII century, important contributions were made by Albert Girard and Pierre Fermat. In the XVIII century, we can mention Robert Simson and John Playfair. Here is Simson's definition of a porism.

“Porisma est propositio in qua proponitur demonstrare rem aliquam vel plures Batas ease, cui vel quibus, ut et cuilibet ex rebus innumeris non quidem datis, sed quae ad ea quae data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam. Porisma etiam in forma problematis enuntiari potest, si nimirum ex quibus data demonstranda aunt, invenienda proponantur.”

Playfair, continuing the work of Simson, tried to understand the probable origin of porisms, to find out what led the ancient geometers to the discovery of them. He remarked that the careful investigation of all possible particular cases of a proposition would show that:

- (1) under certain conditions a problem becomes impossible;
- (2) under certain other conditions, indeterminate or capable of an infinite number of solutions.

For more details see [1911, E.B.].

This is exactly the situation we recognize in the Poncelet theorem. For two given conics, there are two possibilities. Either, a polygon inscribed in one of them and circumscribed about the other has an infinite number of sides, or the number of sides is finite. If it is finite, then the number of sides does not depend on an initial point. We want to stress here that the idea of porism of Poncelet type, in a very special case, existed almost 70 years before Poncelet. This case of Poncelet's theorem is the one with two circles, inscribed and circumscribed about the same triangle. We come to such a situation starting from an arbitrary triangle, and considering its inscribed and circumscribed circle. Denote by r and R their radii respectively, and by d the distance between the centers of the circles. The formula connecting these three values, sometimes referred as “Euler's formula” is well known:

$$d^2 = R^2 - 2rR.$$

However, this relation was discovered by English mathematician Chapple in 1746, and he caught sight of the *poristic* nature of the problem: if there are two circles satisfying the last *Chapple formula*, **then there are infinitely many triangles inscribed in one and circumscribed about the other circle**. Probably, this is the first known appearance of porisms of Poncelet type.

The Euler school was also interested in that subject. Nicolas Fuss, one of Euler's personal secretaries, and after Euler's death the secretary of St. Petersburg

Academy of Sciences, published several works on study of bicentric polygons. In 1797 he published the formula for bicentric quadrilaterals:

$$(R^2 - d^2)^2 = 2r^2(R^2 + d^2).$$

But, although it was 50 years after Chapple, Fuss did not understand the poristic nature of the problem.

It was Jacobi in 1828 who understood the relationship between Poncelet porism in general and study of bicentric polygons of Fuss, Steiner and others.

Some parts of the material presented here were used by the authors for graduate courses they taught: V. D. in 2002/2003 in the International School of Advanced Studies in Trieste [Dra2003], and M. R. in 2006 in the Weizmann Institute of Science in Rehovot. Both authors read mini-courses on the subject, M. R. in the Weizmann Institute of Sciences in 2005 and V. D. at the University of Lisbon in 2007. Also, both authors gave several lectures on seminars and conferences in Italy, France, Germany, Serbia, Spain, Portugal, Montenegro, Israel, Czechia, Poland, Hungary, Great Britain, Austria, Russia, Brazil, USA, Canada, and Bulgaria. One of our observations was that there was a visible division between the communities of Algebraic and Projective Geometry, although some 50 years ago these fields were quite a unified subject. Having this experience in mind, we decided to include introductions to both subjects in order to make the book self-contained as much as possible and usable for both communities and for the mathematical community at large.

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Dragović, V.; Radnović, M.

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